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Intitulée

Etude de la stabilité et de l'explosion en temps fini de certains systèmes hyperboliques non linéaires

Devant le jury composé de :

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Dédicace

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Contents

List of symbols

Ω: Bounded domain in $\mathbb{R}^n$.

Γ: Topological boundary of Ω.

$x = (x_1, x_1, ..., x_N)$:Generic point of $\mathbb{R}^n$.

$dx = dx_1 dx_1 ... dx_N$: Lebesgue measuring on Ω.

$\nabla u = \left(\frac{du}{dx_1}, \frac{du}{dx_2}, ..., \frac{du}{dx_n}\right)$: Gradient of u.

$\Delta u = \sum_{i=1}^{i=n} \frac{d^2 u}{d^2 x_i}$: Laplacien of u.

a.e: Almost everywhere.

$p^{'}$: Conjugate of p, i.e $\frac{1}{p} + \frac{1}{p'} = 1$.

$\partial_t^{\alpha,\eta}$: Generalized Caputo's fractional derivative dorder α.

$C(\Omega)$: Space of real continuous functions on Ω.

$C^k(\Omega), k \in \mathbb{N}$: Space of k times continuously differentiable functions on Ω.

$C_0^\infty(\Omega) = D(\Omega)$: Space of differentiable functions with compact support on Ω.

$D'(\Omega)$: Distribution space on Ω.

Chapter 1

Introduction

The thesis is devoted to the study of local existence and asymptotic behavior in time of solutions with the presence of an external force (polynomial source) to nonlinear of the wave equations . This polynomial source causes to prevent the global existence of solutions of the problem unless additional conditions have been used. More precisely, the solution of the problem tends to infinity when t tends to a finite value T. For this reason, the source term is called a blow up term. On the other hand, the terms of dissipation are terms that tend to stabilize the solution of the problem. There are several types of stabilization, we mention the most famous of them

1) Strong stabilization: $\lim_{t\to+\infty} E(t) = 0$.

2) Uniform stabilization: if $E(t) \leq Cexp(-\delta t)$, $\forall t > 0$, $(C, \delta > 0)$.

3) Polynomial stabilization: if $E(t) \leq Ct^{-\delta}$, $\forall t > 0$, $(C, \delta > 0)$.

4) Logarithmic stabilization: if $E(t) \leq C\left(\ln(t)\right)^{-\delta t}$, $\forall t > 0$, $(C, \delta > 0)$.

5) Weak Stabilization: $(u(t), u'(t)) \rightharpoonup (0, 0)$ when $t \to +\infty$ in an Hilbert space.

So, the central question is "which term wins over the other (term of dissipation or source term)"? This central question has been in many works and is still important. The interaction between the linear damping and the source terms was first considered by Levine [28, 30]. He proved that the solution blows up in finite time if the initial energy is negative. This interaction has been extended to the nonlinear-damping by many researchers, we mention them: Georgiev and Todorova [17], Messaoudi [22], Feng et al. [16], Guo et al. [19], Levine and Serrin [29], Vitillaro [47], Kafini et al. [23].

In this thesis, we will establish the existence, the uniqueness of solution using the semi-group theory .In order to prove the asymptotic behavior of the solution, we will introduce suitable Lyapunov functionals. Finally, to show the blow-up of the solution in finite time, we will use two methods: a diract methode in[32] and Georgiev and Todorova methode's in [17].

This Thesis is divided into 4 chapters.

CHAPTER 1: PRELIMINARIES

This Chapter contends to present some well known results on functional spaces and some basic definitions in addition to theorems. Furthermore, it intends to recall some results on Maximal monotone operators and semigroup. Moreover, it aims to display a brief historical introduction to fractional derivatives and define the fractional derivative operator as well as present some physical interpretations. Finally, the study attempts to present an appendix that contains almost all the secondary calculations used in this thesis.

CHAPTER 2: DECAY AND BLOW-UP OF SOLUTION FOR A NONLINEAR WAVE EQUATION WITH A FRACTIONAL BOUNDARY DAMPING

In this Chapter, we consider the following nonlinear wave equation with fractional derivative boundary and source terme:

$$\begin{cases} u_{tt} - \Delta u + a u_t = |u|^{p-2}u, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = -b\partial_t^{\alpha,\eta} u, & x \in \Gamma_0, t > 0, \\ u = 0, & x \in \Gamma_1, t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $a, b > 0$, $p > 2$, and Ω is a bounded domain in R^n,$n \geq 1$ with a smooth boundary $\partial\Omega$ of class C^2 and ν is the unit outward normal to $\partial\Omega$. We assume that $\partial\Omega = \Gamma_0 \cup \Gamma_1$, where Γ_0 and Γ_1 are closed subsets of $\partial\Omega$ with $\Gamma_0 \cap \Gamma_1 = \emptyset$. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α $(0 < \alpha < 1)$, with respect to the time variable(see [10, 11]). It is defined by the following formula:

$$\partial_t^{\alpha,\eta} u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} u_s(s) ds, \quad \eta \geq 0.$$

Under suitable conditions on the initial data , we establish the existence and uniqueness of solutions of the problem (1.1) and we prove a decay rate estimate for the energy. We also prove that the solution blows up in finit time.

CHAPTER 3:BLOW-UP AND ASYMPTOTIC BEHAVIOR FOR A WAVE EQUATION WITH A TIME DELAY CONDITION OF FRACTIONAL TYPE

In this Chapter, we consider the following wave equation with a time delay condition of fractional type and source terms:

$$\begin{cases} y_{tt} - \Delta y + a_1 \partial_t^{\alpha,\beta} y(t-s) + a_2 y_t = |y|^{p-2} y, & x \in \Omega,\ t > 0 \\ y = 0, & x \in \partial\Omega,\ t > 0 \\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), & x \in \Omega, \\ y_t(x,t-s) = f_0(x,t-s), & x \in \Omega,\ t \in (0,s), \end{cases} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, a_1 and a_2 are positive real numbers. The constant $s > 0$ is the time delay and $p > 2$. Moreover, (y_0, y_1, f_0) the initial data belong to a suitable function space. The notation $\partial_t^{\alpha,\beta}$ stands for the generalized Caputo's fractional derivative (see [10] and [11]) defined by the following formula:

$$\partial_t^{\alpha,\beta} u(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\beta(t-s)} u_s(s) ds, \quad 0 < \alpha < 1, \beta > 0.$$

Under appropriate conditions on a_1 and a_2 and suitable conditions on the initial data, we establish the existence of solutions of the problem (1.2). Furthermore, we prove a decay rate estimate for the energy. Finally, we show that the solution blows up in finite time.

CHAPTER 4: BLOW -UP OF SOLUTION FOR ELASTIC MEMBRANE EQUATION WITH FRACTIONAL BOUNDARY DAMPING

In this Chapter, we consider the following Kirchhoff equation with Balakrishnan-Taylor damping, fractional boundary condition and source terms:

$$\begin{cases} u_{tt} - \left(\xi_0 + \xi_1 \|\nabla u\|_2^2 + \xi_2 (\nabla u, \nabla u_t)\right) \Delta u = |u|^{p-1} u, & x \in \Omega, t > 0, \\ \left(\xi_0 + \xi_1 \|\nabla u\|_2^2 + \xi_2 (\nabla u, \nabla u_t)\right) \frac{\partial u}{\partial \nu} = -b \partial_t^{\alpha,\eta} u, & x \in \Gamma_0, t > 0, \\ u = 0, & x \in \Gamma_1, t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.3}$$

where Ω is a regular and bounded domain in $\mathbb{R}^n$, $(n \geq 1)$ with smooth boundary $\partial\Omega$ such that $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$ and $\Gamma_0,\ \Gamma_1$ have positive measure. $\partial\nu$ denotes the unit outer normal and $(.,.)$ the inner product with its corresponding norm$\|.\|_2$. The functions $u(x,t)$ is the plate transverse displacement. The viscoelastic structural damping terms $\xi_0+\xi_1\|\nabla u\|_2^2+\xi_2(\nabla u, \nabla u_t)$ is the nonlinear stiffness of the membrane. ξ_0, ξ_1, ξ_2 and b are positive constants. The initial data (u_0, u_1) are given functions. From the physical point of view, problem (5.1) is related to the panel flutter equation and to the spillover problem. The notation $\partial_t^{\alpha,\beta}$ stands for the generalized Caputo's fractional derivative (see [10] and [11]) defined by the following formula:

$$\partial_t^{\alpha,\beta} u(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\beta(t-s)} u_s(s) ds, \quad 0 < \alpha < 1, \beta > 0.$$

Under suitable conditions on the initial data, we establish the blow up result.

Chapter 2

Preliminaries

In this chapter, we will introduce and state without proofs some important materials needed in the proof of our results.

2.1 Functional Spaces

2.1.1 $L^p(\Omega)$ Spaces

Definition 2.1.1 *Let $1 \leq p < \infty$ and let Ω be an open domin in $\mathbb{R}^n$,$n \in \mathbb{N}$; we set*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |f(x)|^p dx < \infty \right\}. \tag{2.1}$$

Definition 2.1.2 *We set*

$$L^\infty(\Omega) = \left\{ f : \Omega \to \mathbb{R} \,\middle|\, \begin{array}{l} f \text{ is measurable and there is a constant } C \\ \quad \text{such that } |f(x)| \leq C \text{ a.e On } \Omega. \end{array} \right\}.$$

Lemma 2.1.1 *The space $L^p(\Omega)$ equipped with the norm*

$$\begin{array}{rcll} \|f\|_{L^p(\Omega)} & = & \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}}, & \text{for } p < +\infty \\ \text{and} & & & \\ \|f\|_{L^\infty(\Omega)} & = & inf\{C;\ |f(x)| \leq C \text{ a.e in } \Omega\}, & \text{for } p = +\infty \end{array}$$

is a Banach space. In particular, the space $L^2(\Omega)$ is a Hilbert space with respect to the inner product

$$\langle f, g \rangle_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx.$$

Definition 2.1.3 *We set*

$$L^1_{loc}(\Omega) = \left\{ f : \Omega \to \mathbb{R} : f \in L^1(A) \text{ for all compact } A \subset \Omega \right\}.$$

2.1.2 $L^p(a,b;X)$ Spaces

Let X be a Banach space and $a, b \in \mathbb{R}$ where $a < b$.

Definition 2.1.4 *Let $1 \leq p < \infty$; we set*

$$L^p(a,b;X) = \left\{ f :]a,b[\to X \quad \text{is mesurable and} \quad \int_a^b \|f\|_X^p \ dt < \infty \right\}.$$

Definition 2.1.5 *We set*

$$L^\infty(a,b;X) = \left\{ f :]a,b[\to X \;\middle|\; \begin{array}{l} f \text{ is measurable and there is a constant } C \\ \quad \text{such that } \sup_{t\in[a,b]} ess\|f\|_X \leq C \end{array} \right\}.$$

Lemma 2.1.2 *The space $L^p(a,b;X)$ equipped with the norm*

$$\begin{array}{rcll} & \|f\|_{L^p(a,b;X)} & = & \left(\int_a^b \|f\|_X^p \ dt\right)^{\frac{1}{p}}, \quad \text{for } p < +\infty \\ \text{and} & & & \\ & \|f\|_{L^\infty(a,b;X)} & = & \sup_{t\in[a,b]} ess\|f\|_X, \quad \text{for } p = +\infty \end{array}$$

is a Banach space. In particular, the space $L^2(a,b;X)$ is a Hilbert space with respect to the inner product

$$(f,g)_{L^2(a,b;X)} = \int_a^b (f(t), g(t))_X \ dt.$$

Notation 2.1.1 *Let $1 \leq p \leq \infty$; we denote by q the conjugate exposent,*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Notation 2.1.2 *We note that $L^\infty(a,b;X) = (L^1(a,b;X))'$.*

2.1.3 $W^{k,p}(\Omega)$ Spaces

Definition 2.1.6 *(Weak Derivative) A function $f \in L^1_{loc}(\Omega)$ has a weak derivative $g = D^\alpha f \in L^1_{loc}(\Omega)$ if*

$$\int_\Omega g\phi dx = -\int_\Omega f D^\alpha \phi dx, \ \text{for any } \phi \in C_0^\infty(\Omega).$$

Definition 2.1.7 *Let $k \in \mathbb{N}$, $1 \leq p \leq \infty$; we set*

$$W^{k,p}(\Omega) = \left\{ f \in L^p(\Omega) \text{ such that } \partial^\alpha f \in L^p(\Omega) \text{ for all } \alpha \in \mathbb{N}^k, \text{such that } |\alpha| = \sum_{i=1}^{j=n} \alpha_i \leq k \right\},$$

where $\partial^\alpha u = \partial_1^{\alpha_1} \partial_2^{\alpha_2} ... \partial_n^{\alpha_n} u$.

Lemma 2.1.3 *The Sobolev space $W^{k,p}(\Omega)$ equipped with the norm*

$$\begin{array}{rcll} & \|f\|_{W^{k,p}(\Omega)} & = & \left(\sum_{|\alpha|\le m}\|\partial^\alpha f\|^p_{L^p(\Omega)}\right)^{1/p}, \quad \textit{for } p<+\infty \\ \textit{and} & & & \\ & \|f\|_{W^{k,\infty}(\Omega)} & = & \sum_{|\alpha|\le m}\|\partial^\alpha f\|_{L^\infty(\Omega)}, \quad \textit{for } p=+\infty \end{array}$$

is a Banach space. In particular, the Sobolev space

$$W^{k,2}(\Omega) = H^k(\Omega)$$

is a Hilbert space with respect to the inner product

$$(f,g)_{H^k(\Omega)} = \sum_{|\alpha|\le k}(D^\alpha f, D^\alpha g)_{L^2(\Omega)} \ \forall f,g \in H^k(\Omega).$$

Theorem 2.1.1 *(Sobolev Embedding Theorem) Let Ω a bounded domain in $\mathbb{R}^n$, $(n\ge 1)$, with smooth boundary $\partial\Omega$, and $1\le p\le\infty$.*

$$W^{1,p}(\Omega)\subset\begin{cases} L^{\frac{np}{n-p}}(\Omega) & p<n \\ L^q(\Omega), q\in[p,\infty), & p=n \\ L^\infty(\Omega)\cap C^{0,\alpha}(\Omega), \alpha=\frac{p-n}{p}, & p>n. \end{cases}$$

Furthermore, those embeddings are continuous in the following sense: there exists $C(n,p,\Omega)$ such that for $u\in W^{1,p}(\Omega)$

$$\|u\|_{L^{\frac{np}{np}}(\Omega)} \le C\|\nabla u\|_{L^p(\Omega)}, \qquad \forall p<n$$

$$sup_\Omega|u| \le C'.Vol(\Omega)^{\frac{p-n}{np}}.\|\partial u\|_{L^p(\Omega)}, \quad \forall p>n.$$

2.1.4 $W^{k,p}(a,b;X)$ Space

Let X be a Banach space and $a,b\in\mathbb{R}$ where $a<b$.

Definition 2.1.8 *Let $k\in\mathbb{N}$, $1\le p\le\infty$; we set*

$$W^{k,p}(a,b;X) = \left\{v\in L^p(a,b;X); \frac{\partial v}{\partial x_i}\in L^p(a,b;X) \ \ \forall i\le k\right\}.$$

Lemma 2.1.4 *The Sobolev space $W^{k,p}(a,b;X)$ equipped with the norm*

$$\begin{array}{rcll} & \|f\|_{W^{k,p}(a,b;X)} & = & \left(\sum_{i=0}^k\|\frac{\partial f}{\partial x_i}\|^p_{L^p(a,b;X)}\right)^{1/p}, \quad \textit{for } p<+\infty \\ \textit{and} & & & \\ & \|f\|_{W^{k,\infty}(a,b;X)} & = & \sum_{i=0}^k\|\frac{\partial f}{\partial x_i}\|_{L^\infty(a,b;X)}, \quad \textit{for } p=+\infty \end{array}$$

is a Banach space. In particular, the Sobolev space

$$W^{k,2}(a,b;X) = H^k(a,b;X)$$

is a Hilbert space with respect to the inner product

$$(f,g)_{H^k(a,b;X)} = \sum_{i=0}^{k} \int_a^b \left(\frac{\partial f}{\partial x_i}(x), \frac{\partial g}{\partial x_i}(x) \right)_X dt.$$

2.2 Some Inequalities

We will give here some important inequalities. These inequalities play an important role in applied mathematics and also, itis very useful in our next chapters.

Lemma 2.2.1 *(Young's inequality) For $p, q \in \mathbb{R}$ and for all $p, q \in [1, \infty[$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have:*

$$|ab| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}.$$

Remark 2.2.1 *A simple case of Young's inequality is the inequality for $p = q = 2$:*

$$|ab| \leq \frac{(a)^2}{2} + \frac{(b)^2}{2}.$$

which also gives Young's inequality for all $\delta > 0$:

$$|ab| \leq \delta(a)^2 + \frac{(b)^2}{4\delta}.$$

Lemma 2.2.2 *(Holder's inequality) Assume that $f \in L^p$ and $g \in L^q$ with $1 \leq p \leq +\infty$. Then $fg \in L^1$ and*

$$\|fg\| \leq \|f\|_{L^p} \|g\|_{L^q}$$

when p =q = 2 one finds the Cauchy-Schwarz inequality.

Lemma 2.2.3 *Let $1 \leq p \leq r \leq q$, $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$ and $0 \leq \alpha \leq 1$. Then*

$$\|fg\|_{L^r} \leq \|f\|_{L^p}^{\alpha} \|g\|_{L^q}^{1-\alpha}.$$

2.3 Maximal Monotone Operators

In this section we recall some basic facts concerning bounded and unbounded linear operators acting in a Hilbert space.

Let $(E; \|.\|_E)$ and $(F; \|.\|_F)$ be two Banach spaces over $\mathbb{C}$, and H will always denote a Hilbert space equipped with the scalar product $\langle ., . \rangle_H$ and the corresponding norm $\|.\|_H$.

Definition 2.3.1 *A linear operator $T : E \to F$ is a transformation which maps linearly E in F, that is*

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v), \forall u, v \in E \text{ and } \alpha, \beta \in \mathbb{C}.$$

Definition 2.3.2 *A linear operator $T : E \to F$ is said to be bounded if there exists $C \geq 0$ such that*

$$\|Tu\|_F \leq C\|u\|_E \quad \forall u \in E.$$

The set of all bounded linear operators from E into F is denoted by $\mathcal{L}$(E,F).
Moreover, the set of all bounded linear operators from E into E is denoted by $\mathcal{L}$(E).

Definition 2.3.3 *An unbounded linear operator T from E into F is a pair (T,D(T)), consisting of a subspace $D(T) \subset E$ (called the domain of T) and a linear transformation.*

$$T : D(T) \subset E \to F.$$

In the case when $E = F$ then we say $(T, D(T))$ is an unbounded linear operator on E. If $D(T) = E$ then $T \in \mathcal{L}(E, F)$.

Definition 2.3.4 *Let $T : D(T) \subset E \to F$ be an unbounded linear operator. The graph of T is defined by*

$$G(T) = \{(u, Tu) : u \in D(T)\} \subset E \times F.$$

Definition 2.3.5 *The unbounded operator $T : D(T) \subset E \to F$ is closed if its graph G(T) is closed in $E \times F$.*

Remark 2.3.1 *The closedness of an unbounded linear operator T can be characterize as following if $u_n \in D(T)$ such that $u_n \to u$ in E and $Tu_n \to v$ in F, then $u \in D(T)$ and $Tu = v$.*

Definition 2.3.6 *An unbounded linear operator $A : D(A) \subset E \to F$ is said to be monotone (or accretive) if it satisfies*

$$(Av, v) \geq 0 \quad \forall v \in D(A).$$

Remark 2.3.2 *A is a monotone operator ⇔ -A is a dissipative operator*

Definition 2.3.7 *An unbounded linear operator $A : D(A) \subset E \to F$ is said to be maximal monotone if*

- *A is a monotone operator.*
- *$\forall f \in H \ \exists u \in D(A)$ such that $u + Au = f$.*

The first properties of maximal monotone operators are given in the result below.

Proposition 2.3.1 *Let A be a maximal monotone operator. Then*

- *D(A) is dense in H,*
- *A is a closed operator,*
- *For every $\lambda > 0$, $(I + \lambda A)$ is bijective from D(A) onto H, $(I + \lambda A)^{-1}$ is a bounded operator, and*

$$\|(I + \lambda A)^{-1}\|_{\mathcal{L}(H)} \leq 1.$$

2.4 Semigroups

Let $(X; \|.\|_X)$ be a Banach spaces and H be a Hilbert space equipped with the inner product $\langle ., . \rangle_H$ and the induced norm $\|.\|_H$.

Definition 2.4.1 . *Let X be a Banach space. and let $I : X \to X$ its identity operator.*

1.A one parameter family $(S(t))_{t\geq 0}$ of bounded linear operators from X into H is a semigroup of bounded linear operator on X if

- $S(0) = I$;
- $S(t+s) = S(t).S(s)$ *for every* $t, s \geq 0$.

2-A semigroup $(S(t))_{t\geq 0}$ of bounded linear operators is uniformly continuous if

$$\lim_{t\to 0} \|S(t) - I\| = 0.$$

3. A semigroup $(S(t))_{t\geq 0}$ of bounded linear operators is a strongly continuous semigroup (or a C_0-semigroup) if

$$\lim_{t\to 0} S(t) = x.$$

4.A strongly continuous contraction semigroup $(S(t))_{t\geq 0}$ on X is a strongly continuous semigroup on X such that

$$\|S(t) - I\|_{\mathcal{L}(X)} \leq 1 \quad \forall t \geq 0.$$

5.The linear operator A defined by

$$Ax = \lim_{t\to 0} \frac{S(t)x - x}{t}, \quad \forall x \in D(A)$$

where

$$D(A) = \left\{ x \in X; \lim_{t\to 0} \frac{S(t)x - x}{t} \quad exists \right\}$$

is the infinitesimal generator of the semigroup $(S(t))_{t\geq 0}$.

Theorem 2.4.1 *(Hille-Yosida Theorem: Lumer-Phillips from in Hilbert spaces)*
Let $A : D(A) \subset H \to H$ be a linear operator. Then A is maximal monoton if and only if -A is the infinitesimal generator of a C_0 semigroup of contraction on H.

Corollary 2.4.1 *Let H be a Hilbert space and let A be a linear operator defined from $A : D(A) \subset H \to H$. If A is maximal monotone then the initial value problem*

$$\begin{cases} u_t(t) + Au(t) = 0, \quad t > 0, \\ \\ u(0) = u_0 \end{cases} \tag{2.2}$$

has a unique solution

$$u(t) = S(t)u_0$$

such that

- *if* $u_0 \in H$ *then* $u \in C([0, \infty), H)$;
- *if* $u_0 \in D(A)$ *then* $u \in C([0, \infty), H) \cap C^1([0, \infty), D(A))$.

Corollary 2.4.2 *Let H be a Hilbert space and let* $f : H \times H \to H$ *be locally Lipschitz continuous in u. If A is maximal monotone then* $\exists T_* \in [0, \infty)$ *such that the initial value problem* $u_0 \in D(A)$ *the initial value problem*

$$\begin{cases} u_t(t) + Au(t) = f(t, u(t)), \quad t > 0, \\ \\ u(0) = u_0 \end{cases} \tag{2.3}$$

has a unique solution u on

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)ds \ \forall t \in [0, T_*[$$

. such that

- *if* $u_0 \in H$ *then* $u \in C([0, T_*), H)$;
- *if* $u_0 \in D(A)$ *then* $u \in C([0, T_*), H) \cap C^1([0, T_*), D(A))$.

2.5 Lax-Milgrame Theorem

Let H be a Hilbert space equipped with the inner product $\langle ., . \rangle_H$ and the induced norm $\|.\|_H$.

Definition 2.5.1 *A bilinear form*

$$a : H \times H \to R$$

is said to be

- *continuous if there is a constant C such that*

$$\|a(u, v)\| \leq C\|u\|\|v\|, \ \forall u, v \in H.$$

- *coercive if there is a constant* $\alpha > 0$ *such that*

$$|a(u, u)| \leq \alpha\|u\|^2, \ \forall u \in H.$$

Theorem 2.5.1 *(Lax-Milgrame Theorem) Assume that a(., .) is a continuous coercive bilinear form on H. Then, given any* $L \in \mathcal{L}(H, \mathbb{C})$*, there exists a unique element* $u \in H$ *such that*

$$a(u, v) = L(v), \ \forall v \in H.$$

2.6 Fractional Derivative Control

In this part, we introduce the necessary elements for the good understanding of this manuscript. It includes a brief reminder of the basic elements of the theory of fractional computation. The concept of fractional computation is a generalization of ordinary derivation and integration to an arbitrary order. Derivatives of non-integer order are now widely applied in many domains, for example in economics, electronics, mechanics, biology, probability and viscoelasticity. A particular interest for fractional derivation is related to the mechanical modeling of gums and rubbers. In short, all kinds of materials that preserve the memory of previous deformations in particular viscoelastic. Indeed, the fractional derivation is introduced naturally. The fractional calculus is an important developing field in both pure and applied mathematics. Many real world problems have been investigated within the fractional derivatives, particularly Caputo fractional derivative is extensively and successfully used in many branches of sciences and engineering.

2.6.1 Some history of fractional calculus:

In a letter dated September 30th, 1695 L'Hospital wrote to Leibniz asking him about the meaning of $\frac{d^n y}{dx^n}$ if $n = \frac{1}{2}$, that is what if n is fractional?. Leibniz response: An apparent paradox, from which one day useful consequences will be drawn In 1819 S. F. Lacroix , was the first to mention in some two pages a derivative of arbitrary order.Thus for $y = x^\alpha,\ \alpha \in \mathbb{R}_+$, he showed that

$$\frac{d^{\frac{1}{2}} y}{dx^{\frac{1}{2}}} = \frac{\Gamma(\alpha+1)}{\Gamma(1+\frac{1}{2})} x^{\frac{\alpha-1}{2}}.$$

In particular he had

$$(\frac{d}{dx})^{\frac{1}{2}} x = 2\sqrt{\frac{x}{\pi}}.$$

In 1822 J. B. J. Fourier derived an integral representation for f(x),

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} f(\alpha) d\alpha \int_{\mathbb{R}} cosp(x-\alpha) dp,$$

obtained (formally) the derivative version

$$\frac{d^\nu}{dx^\nu} f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} f(\alpha) d\alpha \int_{\mathbb{R}} p^\nu cos[p(x-\alpha) + \frac{\nu\pi}{2}] dp,$$

where "the number ν will be regarded as any quantity whatever, positive ornegative".
In 1823 Abel resolved the integral equation arising from the brachistochrone problem, namely

$$\frac{1}{\Gamma(\alpha)} \int_0^x \frac{g(u)}{(x-u)^{1-\alpha}} = f(x),\ 0 < \alpha < 1,$$

with the solution

$$g(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{f(u)}{(x-u)^\alpha} du.$$

Abel never solved the problem by fractional calculus but, in 1832 Liouville , did solve this integral equation. Perhaps the first serious attempt to give a logical definition of a fractional derivative is due to Liouville; he published nine papers on the subject between 1832 and 1837, the last in the field in 1855. They grew out of Liouville's early work on electromagnetism. There is further work of George Peacock (1833), D. F. Gregory (1841), Augustus de Morgan (1842), P. Kelland (1846), William Center (1848). Especially basic is Riemann's student paper of 1847 . After the participation of Riemann and the work of Cayley in 1880 , among the mathematicians spearheading research in the broad area of fractional calculus until 1941 were S.F. Lacroix, J.B.J. Fourier, N.H. Abel, J. Liouville, A. De Morgan, B. Riemann, Hj. Holmgren, K. Griinwald, A.V. Letnikov, N.Ya. Sonine, J. Hadamard, G.H. Hardy, H. Weyl, M. Riesz, H.T. Davis, A. Marchaud, J.E. Littlewood, E.L. Post, E.R. Love, B.Sz.-Nagy, A. Erdelyi and H. Kober. Fractional calculus has developed especially intensively since 1974 when the first international conference in the field took place.It was organized by Bertram Ross . Samko et al in their encyclopedic volume state and we cite: "We pay tribute to investigators of recent decades by citing the names of mathematicians who have made a valuable scientific contribution to fractional calculus development from 1941 until the present (1990). These are M.A. Al- Bassam, L.S. Bosanquet, P.L. Butzer, M.M. Dzherbashyan, A. Erdelyi, T.M. Flett, Ch. Fox, S.G. Gindikin, S.L. Kalla, LA. Kipriyanov, H. Kober, P.I. Lizorkin, E.R. Love, A.C. McBride, M. Mikolas, S.M. Nikol'skii, K. Nishimoto, LI. Ogievetskii, R.O. O'Neil, T.J. Osier, S. Owa, B. Ross, M. Saigo, I.N. Sneddon, H.M. Srivastava, A.F. Timan, U. Westphal, A. Zygmund and others". To this list must of course be added the names of the authors of Samko et al and many other mathematicians, particularly those of the younger generation. Books especially devoted to fractional calculus include K.B. Oldham and J. Spanier , S.G. Samko, A.A. Kilbas and O.I. Marichev , V.S. Kiryakova , K.S. Miller and B. Ross , B. Rubin . Books containing a chapter or sections dealing with certain aspects of fractional calculus include H.T. Davis , A. Zygmund , M.M.Dzherbashyan , I.N. Sneddon , P.L. Butzer and R.J. Nessel , P.L. Butzer and W. Trebels , G.O. Okikiolu , S. Fenyo and H.W. Stolle, H.M. Srivastava and H.L. Manocha , R. Gorenfio and S. Vessella.

2.6.2 Various approaches of fractional derivatives

There exists a many mathematical definitions of fractional order integration and derivation. These definitions do not always lead to identical results but are equivalent for a wide large of functions. We introduce the fractional integration operator as well as the two most definitions of fractional derivatives, used, namely that Riemann-Liouville, Caputo and Hadamard.

From the classical fractional calculus, we recall

Definition 2.6.1 *The left Riemann-Liouville fractional integral of order $\alpha > 0$ starting from a has the following form*

$$({}_aI^{\alpha}f)(x) = \int_a^x (x-t)^{\alpha-1}f(t)dt.$$

Definition 2.6.2 *The left Riemann-Liouville fractional derivative of order $\alpha > 0$ ending at $b > a$ is defined by*

$$(I_b^\alpha f)(x) = \int_x^b (x-t)^{\alpha-1} f(t)dt.$$

Definition 2.6.3 *The left Riemann-Liouville fractional derivative of order $\alpha > 0$ starting at a is given below*

$$({}_aD^\alpha f)(x) = (\frac{d}{dx})^n (aI^{n-\alpha} f)(x), \; n = [\alpha] + 1.$$

Definition 2.6.4 *The right Riemann-Liouville fractional derivative of order $\alpha > 0$ ending at b becomes*

$$(D_b^\alpha f)(x) = (-\frac{d}{dx})^n (I_b^{n-\alpha} f)(x).$$

Definition 2.6.5 *The left Caputo fractional of order $\alpha > 0$ sarting from a has the following form*

$$({}_aD^\alpha f)(x) = ({}_aI^{n-\alpha} f^{(n)})(x), \; n = [\alpha] + 1.$$

Definition 2.6.6 *The right Caputo fractional derivative of order $\alpha > 0$ ending at b becomes*

$$(D_b^\alpha f)(x) = (I_b^{n-\alpha} (-1)^n f^{(n)})(x).$$

The Hadamard type fractional integrals and derivatives were introduced in [**?**] as:

Definition 2.6.7 *The left Hadamard fractional integral of order $\alpha > 0$ starting from a has the following form*

$$({}_aI^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (lnx - lnt)^{\alpha-1} f(t)dt.$$

Definition 2.6.8 *The right Hadamard fractional integral of order $\alpha > 0$ ending at $b > a$ is defined by*

$$(I_b^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (lnt - lnx)^{\alpha-1} f(t)dt.$$

Definition 2.6.9 *The left Hadamard fractional derivative of orderr $\alpha > 0$ starting at a is given below*

$$({}_aD^\alpha f)(x) = (x\frac{d}{dx})^n (aI^{n-\alpha} f)(x), \; n = [\alpha] + 1.$$

Definition 2.6.10 *The right Hadamard fractional derivative of order $\alpha > 0$ ending at b becomes*

$$(D_b^\alpha f)(x) = (-x\frac{d}{dx})^n (I_b^{n-\alpha} f)(x).$$

Definition 2.6.11 *the fractional derivative of order α, $0 < \alpha < 1$, in sense of Caputo, is defined by*

$$(D^\alpha f)(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{df}{ds}(s)ds.$$

Definition 2.6.12 *The fractional integral of order α, $0 < \alpha < 1$, in sense Riemann-Liouville, is defined by*

$$(I^{\alpha}f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds.$$

Remark 2.6.1 *From the above definitions, clearly*

$$D^{\alpha}f = I^{\alpha-1}Df, \ 0 < \alpha < 1.$$

Lemma 2.6.1

$$I^{\alpha}D^{\alpha}f = f(t) - f(0), \ 0 < \alpha < 1.$$

Lemma 2.6.2 *If*

$$D^{\beta}f(0) = 0.$$

then

$$D^{\alpha}D^{\beta}f = D^{\alpha+\beta}f, \ 0 < \alpha < 1, \ 0 < \beta < 1.$$

Now, we give the definitions of the generalized Caputo's fractional derivative and the generalized fractional integral. These exponentially modified fractional integro-differential.

Definition 2.6.13 *The generalized Caputo's fractional derivative is given by*

$$(D^{\alpha,\eta}f)(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{df}{ds}(s)ds, \ 0 < \alpha < 1, \ \eta \geq 0.$$

Remark 2.6.2 *The operators D^{α} and $D^{\alpha,\eta}$ differ just by their kernels.*

Definition 2.6.14 *The generalized fractional integral is given by*

$$(I^{\alpha,\eta}f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\eta(t-s)} f(s)ds, \ 0 < \alpha < 1, \ \eta \geq 0.$$

2.7 Appendix

Lemma 2.7.1 *Let $\delta > 0$ and $B(t) \in C^2(0,\infty)$ ba a nonnegative function satisfying*

$$B''(t) - 4(\delta+1)B'(t) + 4(\delta+1)B(t) \geq 0. \tag{2.4}$$

If

$$B'(0) > r_2 B(0) + l_0, \tag{2.5}$$

then

$$B'(t) > l_0,$$

for $t > 0$, where l_0 is a constant, $r_2 = 2(\delta+1) - 2\sqrt{(\delta+1)\delta}$, is the smallest root of the equation

$$r^2 - 4(\delta+1)r + 4(\delta+1) = 0.$$

Proof Let r_1 be the largest root of $r^2 - 4(\delta+1)r + 4(\delta+1) = 0$. Then (2.4) is equivalent to

$$\left(\frac{d}{dt} - r_1\right)\left(\frac{d}{dt} - r_2\right)B(t) \geq 0. \tag{2.6}$$

By integrating (2.6) from 0 to t, we get

$$B'(t) \geq r_2 B(t) + (B'(0) - r_2 B(0))e^{r_1 t}.$$

By (2.5), we get

$$B'(t) > l_0 \text{ for } t > 0.$$

■

Lemma 2.7.2 *If $J(t)$ is a non-creasing function on $[t_0, \infty)$, $t_0 \geq 0$ and satisfies the differential inequality*

$$J'(t)^2 \geq a + bJ(t)^{2+\frac{1}{\delta}} \quad fort \geq t_0, \tag{2.7}$$

where $a > 0$, $b \in \mathbb{R}$, then there exist a finite time T^* such that

$$\lim_{t \to T^{*-}} J(t) = 0,$$

and the upper bound of T^* is estimated, respectively, by the following cases :

i) if $b < 0$ and $J(t_0) < \min\left\{1, \sqrt{a/(-b)}\right\}$ then

$$T^* \leq t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{\frac{a}{-b}}}{\sqrt{\frac{a}{-b}} - J(t_0)}. \tag{2.8}$$

ii) If $b = 0$, then

$$T^* \leq t_0 + \frac{J(t_0)}{\sqrt{a}}. \tag{2.9}$$

iii) If $b > 0$, then

$$T^* \leq \frac{J(t_0)}{\sqrt{a}}$$

or

$$T^* \leq t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\delta c}{\sqrt{a}} \left(1 - [1 + cJ(t_0)]^{\frac{1}{2\delta}}\right), \tag{2.10}$$

where

$$c = \left(\frac{b}{a}\right)^{\delta/(2+\delta)}.$$

Proof

i) Since $\sqrt{c^2 - d^2} \geq c - d$ for $c \geq d > 0$, we have from (2.7),

$$J'(t) \leq -\sqrt{a} + \sqrt{-b}J(t) \text{ for } t \geq t_0.$$

Thus we get

$$J(t) \leq \left(J(t_0) - \sqrt{-\frac{a}{b}} \right) e^{(t-t_0)\sqrt{-b}} + \sqrt{\frac{-a}{b}}.$$

Hance there exists a positive $T^* < \infty$ such that $\underset{t \to T^{*-}}{lim} J(t) = 0$, and an upper bound of T^* is given by (2.8)

ii) When $b = 0$, from (2.7), we get

$$J(t) \leq J(t_0) - \sqrt{a}(t - t_0) \text{ for } t \geq t_0.$$

Thus there exists $T^* < \infty$ such that $\underset{t \to T^{*-}}{lim} J(t) = 0$, and an upper bound of T^* is given by (2.9)

iii) When $b > 0$, we get from (2.7)

$$J'(t) \leq -\sqrt{a(1 + (cJ(t))^{2+\frac{1}{\delta}}},$$

where $c = \left(\frac{a}{b}\right)^{2+\frac{1}{\delta}}$.
By using the inequality

$$m^q + n^q \geq 2^{1-q}(m+n)^q \text{ for } m,\ n > 0 \text{ and } q \geq 1,$$

with $q = 2 + \frac{1}{\delta}$, we obtain

$$J'(t) \leq -\sqrt{a}2^{\frac{(-\delta-1)}{2\delta}}(1 + cJ(t))^{1+\frac{1}{\delta}}. \tag{2.11}$$

By solving the differential inequality (2.11), we get

$$J(t) \leq \frac{1}{c}\left\{ -1 + \left[(1 + cJ(t_0))^{\frac{-1}{2\delta}} + \frac{\sqrt{a}}{\delta c} 2^{\frac{-(3\delta+1)}{2\delta}} (t - t_0) \right]^{-2\delta} \right\}.$$

Hance there exists $T^* < \infty$ such that $\underset{t \to T^{*-}}{lim} J(t) = 0$, and an upper bound of T^* is given by (2.10)

■

Lemma 2.7.3 *We set the constant*

$$\varrho = \frac{2\sin(\alpha\pi)\Gamma(\frac{1}{2}+1)}{\pi^{\frac{1}{2}+1}}$$

and μ *be the function:*

$$\mu(\xi) = |\xi|^{\frac{(2\alpha-1)}{2}}, \quad \xi \in \mathbb{R},\ 0 < \alpha < 1. \tag{2.12}$$

Then the relationship between the "input" U and the "output" O of the system

$$\partial_t \phi(\xi,t) + (\xi^2+\eta)\phi(\xi,t) - U(t)\mu(\xi) = 0,\ \xi \in \mathbb{R},\ t > 0,\ \eta \geq 0, \tag{2.13}$$

$$\phi(\xi,0) = 0,\ \xi \in \mathbb{R}, \tag{2.14}$$

$$O(t) = \varrho \int_{\mathbb{R}} \phi(\xi,t)\mu(\xi)d\xi \tag{2.15}$$

is given by

$$O = I^{1-\alpha,\eta}U. \tag{2.16}$$

Proof Solving equation (2.13), we obtain

$$\phi(\xi,t) = \int_0^t \mu(\xi)e^{-(|\xi|^2+\eta)(t-\tau)}U(\tau)d\tau. \tag{2.17}$$

If follows from (2.15) that

$$O(t) = \varrho \int_0^t U(\tau) \int_{\mathbb{R}} |\xi|^{2\alpha-1} e^{-(|\xi|^2+\eta)(t-\tau)} d\xi d\tau. \tag{2.18}$$

Now using the fact that $\dfrac{sin(\alpha\pi)}{\pi} = \dfrac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}$ and $\Gamma(1+\frac{1}{2}) = \frac{\pi^{\frac{1}{2}}}{2}$, we obtain

$$\begin{aligned} O(t) &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t U(\tau) \int_{\mathbb{R}} |\xi|^{2\alpha-1} e^{-(|\xi|^2+\eta)(t-\tau)} d\xi d\tau \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^t U(\tau)(t-\tau)^{-\alpha} e^{-\eta(t-\tau)} d\tau \qquad \text{(with } \xi^2(t-\tau) = x\text{)} \\ &= I^{1-\alpha,\eta}U(t). \end{aligned} \tag{2.19}$$

This completes the proof ■

Lemma 2.7.4 *Let $\eta > 0$. For any real number $\lambda > -\eta$,we have*

$$\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda+\eta+\xi^2} d\xi = \frac{\pi}{\sin(\alpha\pi)} (\lambda+\eta)^{\alpha-1}.$$

Proof A direct computation gives

$$\begin{aligned}
\int_{-\infty}^{+\infty} \tfrac{\mu^2(\xi)}{\lambda+\eta+\xi^2} d\xi &= \int_{-\infty}^{+\infty} \tfrac{|\xi|^{2\alpha-1}}{\lambda+\eta+\xi^2} d\xi \\
&= \int_{0}^{+\infty} \tfrac{x^{\alpha-1}}{\lambda+\eta+x} dx && \text{with } \xi^2 = x \\
&= (\lambda+\eta)^{\alpha-1} \int_1^{+\infty} y^{-1}(y-1)^{\alpha-1} dy && (\text{with } \ y = \tfrac{x}{\lambda+\eta} + 1) \\
&= (\lambda+\eta)^{\alpha-1} \int_0^1 z^{-\alpha}(1-z)^{\alpha-1} dz && (\text{with } z = \tfrac{1}{y}) \\
&= (\lambda+\eta)^{\alpha-1} B(1-\alpha, \alpha) \\
&= (\lambda+\eta)^{\alpha-1} \Gamma(1-\alpha)\Gamma(\alpha) \\
&= (\lambda+\eta)^{\alpha-1} \frac{\pi}{sin(\alpha\pi)} && .
\end{aligned}$$

■

Chapter 3

General decay and blow-up of solution for a nonlinear wave equation with a fractional boundary damping

---

3.1 Introduction

Fractional calculus for partial differential equations has received great attention during the last two decades. Too many physical phenomena are successfully modeled by initial boundary value problems with fractional boundary conditions. Boundary dissipations of fractional order can be encountered in many fields of sciences and are widely applied in most instances chemical engineering, biological, ecological and physical phenomena related to electromagnetism. See Magin [34], Tarasov [42], and Valério et al [46].

In fact, most of the problems related to boundary dissipations of fractional order are about asymptotic stability by using the LaSalle's invariance principle and multiplier techniques combined with the frequency domain method, see [34, 3, 2, 14, 38]. Of course, the first step to do this is to write the equations as an augmented system as in [38]. In this context, Akil and Wehbe [3], discussed the following problem:

$$\begin{cases} u_{tt} - \Delta u = 0, & x \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial \nu} = -b\partial_t^{\alpha,\eta} u, & x \in \Gamma_0, \quad t > 0, \quad \eta \geq 0, \quad 0 < \alpha < 1, \\ u = 0, & x \in \Gamma_1, \ t > 0, \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), & x \in \Omega. \end{cases}$$

They proved the stability using the semigoup theory of linear operators and a result obtained by Borichev and Tomilov.

In this work [38], Mbodje studies the decay rate of the energy for the same problem. Using the energy methods, he proved the strong asymptotic stability under the condition $\eta = 0$ and a polynomial type decay rate $E(t) \leq \frac{c}{t}$, if $\eta > 0$. In this paper we first consider the following nonlinear wave equation

$$\begin{cases} u_{tt} - \Delta u + a u_t = |u|^{p-2}u, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = -b\partial_t^{\alpha,\eta} u, & x \in \Gamma_0, t > 0, \\ u = 0, & x \in \Gamma_1, t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{3.1}$$

where $a, b > 0$, $p > 2$, and Ω is a bounded domain in R^n,$n \geq 1$ with a smooth boundary $\partial\Omega$ of class C^2 and ν is the unit outward normal to $\partial\Omega$. We assume that $\partial\Omega = \Gamma_0 \cup \Gamma_1$, where Γ_0 and Γ_1 are closed subsets of $\partial\Omega$ with $\Gamma_0 \cap \Gamma_1 = \emptyset$. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α, $(0 < \alpha < 1)$, with respect to the time variable (see [10, 11]). It is defined as follows:

$$\partial_t^{\alpha,\eta} u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} u_s(s) ds, \quad \eta \geq 0,$$

We recall some results related to wave equation with a mild internal dissipation

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + a u_t(x,t) = g(x,t), & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu}(x,t) + \int_0^t K(x,t-s) u_s(x,s) ds = h(x,t), & x \in \Gamma_0, t > 0, \\ u_0(x,t) = 0 & x \in \Gamma_1, t > 0, \\ u(x,0) = u_0(x) \quad u_t(x,0) = u_1(x) & x \in \Omega. \end{cases}$$

In their study, Kirane and Tatar [24] considered and proved the above equation, the global existence and Exponential decay of the problem. In other work, the authors proved the global non-existence of the problem [26]. In particular Alabau and al [5]. studied the homogeneous case and established a polynomial stability result of the problem. Exponential decay of the problem was showed in Alabau [4]. When $\int_0^t K(x,t-s)u_s(x,s)ds$ is replaced by $\partial_t^{\alpha} u(x,t)$ and $h(x,t)$ is replaced by $|u|^{m-1}u(x,t)$, Dai and Zhang [14] proved the exponential growth of the problem. According to our last Knowledge, we are the first to prove the exponential stability and the blow up of solutions in finite time for the case of nonlinear wave equation with fractional boundary damping by suing the augment system.

In this paper, we prove under suitable conditions on the initial data the stability of wave equation with fractional damping we have based on the construction of a Lyapunov function. This technique of proof was recently used by Draifia and al [15] to study the exponential decay of a system of nonlocal singular viscoelastic equations. For some restrictions on the initial data that nonlinear source of polynomial type is able to force solutions to blow-up in finite time, here are three different cases on the sign of the initial energy are considered that have been recently used by Zarai and al [52] to study the blow up for a system of nonlocal singular viscoelastic equations.

The paper is organized as follows. In Sect. 2, we reformulate our problem (3.1) into an augmented system and give some lemmas and notations. In Section 3, we prove the existence and uniqueness of weak solutions using the Hille-Yosida Theorem. In Section 4, we prove the global existence using the potential well theory. In Sect. 4 we prove the general decay result. In Sect. 5, we state and prove blow up result that is also based on a direct method.

3.2 Preliminaries

In this section we give various notations and lemmas which will be desired in the proof of our results.

We introduce the set

$$H^1_{\Gamma_1}(\Omega) = \left\{ u \in H^1(\Omega), u|_{\Gamma_1} = 0 \right\},$$

where $u|_{\Gamma_1}$ is in the trace sense. And

$$\aleph = \{w \in H^1_0 | I(w) > 0\} \cup \{0\}.$$

Lemma 3.2.1 *(Sobolev-Poincaré Inequality.See [35]). If either* $1 \leq q \leq +\infty$ *(N=2) or* $1 \leq q \leq \frac{N+2}{N-2}$, *(*$N \geq 3$*). Then there is a constant* C_* *such that*

$$\|u\|_{q+1} \leq C_* \|\nabla u\|_2, \quad for \quad u \in H^1_0(\Omega).$$

Where

$$C_* = \sup \left\{ \frac{\|u\|_{q+1}}{\|\nabla u\|_2}, \backslash u \in H^1_0(\Omega), u \neq 0 \right\},$$

is positive and finite.

Lemma 3.2.2 *(See [1]) The trace -Sobolev embedding is given for*

$$2 < p \leq \frac{2(n-1)}{n-2} \tag{3.2}$$

by

$$H^1_{\Gamma_1}(\Omega) \hookrightarrow L^p(\Gamma_0).$$

It this case, the embedding constant is denoted by B_q, *i.e.,*

$$\|u\|_{p,\Gamma_0} \leq B_q \|u\|_2.$$

Lemma 3.2.3 (*See*[*32*]) *Let* $\delta > 0$ *and* $B(t) \in C^2(0, \infty)$ *be a nonnegative function satisfying*

$$B''(t) - 4(\delta + 1)B'(t) + 4(\delta + 1)B(t) \geq 0.$$

If

$$B'(0) > r_2 B(0) + l_0,$$

then

$$B'(t) \geq l_0.$$

For $t > 0$, *where* k_0 *is a constant,* $r_2 = 2(\delta + 1) - 2\sqrt{(\delta + 1)\delta}$, *is the smallest root of the equation*

$$r^2 - 4(\delta + 1)r + (\delta + 1) = 0.$$

Lemma 3.2.4 *(See[32]) If $J(t)$ is a non-creasing function on $[t_0,\infty)$, $t_0 \geq 0$ and satisfies the differential inequality*

$$J'(t)^2 \geq \alpha + bJ(t)^{2+\frac{1}{\delta}}, \quad t \geq t_0,$$

where $\alpha > 0$, $b \in \mathbb{R}$, then there exist a finite time T^* such that

$$\lim_{t\to T^{*-}} J(t) = 0,$$

and the upper bound of T^* is estimated, respectively, by the following cases :
(i) *if $b < 0$ and $J(t_0) < \min\left\{1, \sqrt{\alpha/(-b)}\right\}$ then*

$$T^* \leq t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{\frac{\alpha}{-b}}}{\sqrt{\frac{\alpha}{-b}} - J(t_0)}.$$

(ii) *If $b = 0$, then*

$$T^* \leq t_0 + \frac{J(t_0)}{\sqrt{\alpha}}.$$

(iii) *If $b > 0$, then*

$$T^* \leq \frac{J(t_0)}{\sqrt{\alpha}}$$

or

$$T^* \leq t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\delta c}{\sqrt{\alpha}} \left(1 - [1 + cJ(t_0)]^{\frac{1}{2\delta}}\right),$$

where

$$c = \left(\frac{b}{\alpha}\right)^{\delta/(2+\delta)}.$$

Definition 3.2.1 *A solution u of (3.1) is called blow-up if there exists a finite time T^* such that*

$$\lim_{t\to T^{*-}} \left(\|\nabla u\|_2^2\right)^{-1} = 0.$$

Theorem 3.2.1 *(See [38]) We set the constant*

$$\varrho = (\pi)^{-1} \sin(\alpha\pi),$$

and μ be the function:

$$\mu(\xi) = |\xi|^{\frac{(2\alpha-1)}{2}}, \quad \xi \in \mathbb{R}, \quad 0 < \alpha < 1.$$

Then the relationship between the "input" U and the "output" O of the system

$$\partial_t\phi(\xi,t)+(\xi^2+\eta)\phi(\xi,t)-U(L,t)\mu(\xi)=0,\ \xi\in\mathbb{R},t>0,\eta\geq 0, \tag{3.3}$$

$$\phi(\xi,0)=0,\ \xi\in\mathbb{R}, \tag{3.4}$$

$$O(t)=\varrho\int_{-\infty}^{+\infty}\phi(\xi,t)\mu(\xi)d\xi,\ \xi\in\mathbb{R},t>0,$$

is given by

$$O=I^{1-\alpha,\eta}U,$$

$$I^{\alpha,\eta}U=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}e^{-\eta(t-s)}u(s)ds.$$

where

Lemma 3.2.5 *[2] Let $\eta>0$. For any real number $\lambda>-\eta$,we have*

$$\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}d\xi=\frac{\pi}{\sin(\alpha\pi)}(\lambda+\eta)^{\alpha-1}.$$

We are now in a position to reformulate system (3.1). Indeed, by using Theorem 3.2.1, system (3.1) may be recast into the augmented model:

$$\begin{cases} u_{tt}-\Delta u+au_t=|u|^{p-2}u, & x\in\Omega,t>0,\\ \partial_t\phi(\xi,t)+(\xi^2+\eta)\phi(\xi,t)-u_t(x,t)\mu(\xi)=0, & x\in\Gamma_0,\xi\in\mathbb{R},t>0,\\ \frac{\partial u}{\partial\nu}=-b_1\int_{-\infty}^{+\infty}\phi(\xi,t)\mu(\xi)d\xi, & x\in\Gamma_0,\xi\in\mathbb{R},t>0,\\ u=0, & x\in\Gamma_1,t>0,\\ u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x), & x\in\Omega,\\ \phi(\xi,0)=0, & \xi\in\mathbb{R}, \end{cases} \tag{3.5}$$

where $b_1=\varrho b$. We define the energy associated to the solution of the problem (3.5) by the following formula:

$$E(t)=\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|\nabla u\|_2^2-\frac{1}{p}\|u\|_p^p+\frac{b_1}{2}\int_{\Gamma_0}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2d\xi d\rho. \tag{3.6}$$

Lemma 3.2.6 *Let (u,ϕ) be a regular solution of the problem (3.5). Then, the energy functional defined by (3.6) satisfies*

$$\frac{d}{dt}E(t)=-a\|u_t\|_2^2-b_1\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,t)|^2d\xi d\rho\leq 0. \tag{3.7}$$

Proof Multiplying the first equation in (3.5) by u_t, integrating over Ω and using integration by parts, we get

$$\frac{1}{2}\frac{d}{dt}\|u_t\|_2^2-\int_\Omega\Delta uu_tdx+a\|u_t\|_2^2=\int_\Omega|u|^{p-2}uu_tdx.$$

Then

$$\begin{aligned}&\tfrac{d}{dt}\left[\tfrac{1}{2}\|u_t\|_2^2+\tfrac{1}{2}\|\nabla u\|_2^2-\tfrac{1}{p}\|u\|_p^p\right]\\&+a\|u_t\|_2^2+b_1\textstyle\int_{\Gamma_0}u_t(x,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi,t)d\xi d\rho=0.\end{aligned}\tag{3.8}$$

Multiplying the second equation in (3.5) by $b_1\phi$ and integrating over $\Gamma_0\times(-\infty,+\infty)$, to obtain:

$$\begin{aligned}&\tfrac{b_1}{2}\tfrac{d}{dt}\textstyle\int_{\Gamma_0}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2d\xi d\rho+b_1\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,t)|^2d\xi d\rho\\&-b_1\textstyle\int_{\Gamma_0}u_t(x,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi,t)d\xi d\rho=0.\end{aligned}\tag{3.9}$$

From (3.6), (3.8) and (3.9) we get

$$\frac{d}{dt}E(t)=-a\|u_t\|_2^2-b_1\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,t)|^2d\xi d\rho\leq 0.$$

This completes the proof of the lemma. ■

3.3 Well-posedness

In this section, we give an existence and uniqueness result for problem(3.5) using the semigroup theory. In traducing the vector function $U=(u,v,\phi)^T$ where $v=u_t$ and let $J(U)=(0,|u|^{p-2}u,0)^T$, system (3.5) is equivalent to:

$$(P'')\begin{cases}U_t(t)+AU(t)=J\left(U(t)\right),\\ U_0=(u_0,u_1,\phi_0)^T,\end{cases}$$

where the operator A is defined by

$$AU=\begin{pmatrix}-v\\ -\Delta u+av\\ (\xi^2+\eta)\phi(x,\xi)-v(x)\mu(\xi)\end{pmatrix}.\tag{3.10}$$

We denote by $\mathcal{H}$ the energy space associated to system:

$$\mathcal{H}=H^1_{\Gamma_1}(\Omega)\times L^2(\Omega)\times L^2(\Gamma_0\times(-\infty,+\infty)),$$

where

$$H^1_{\Gamma_1}(\Omega)=\left\{u\in H^1(\Omega),u/\Gamma_1=0\right\}.$$

For $U=(u,v,\phi)^T\in\mathcal{H}$ and $\bar{U}=(\bar{u},\bar{v},\bar{\phi})^T\in\mathcal{H}$, we define the following inner product in $\mathcal{H}$

$$\left\langle U,\bar{U}\right\rangle_{\mathcal{H}}=\int_\Omega[\nabla u.\nabla\bar{u}+v\bar{v}]\,dx+b_1\int_{\Gamma_0}\int_{-\infty}^{+\infty}\phi(x,\xi)\bar{\phi}(x,\xi)d\xi d\rho.$$

The domain of the operator A is then

$$D(A)=\left\{\begin{array}{l} U\in\mathcal{H}: u\in H^2(\Omega)\cap H^1_{\Gamma_1}(\Omega),\quad v\in H^1_{\Gamma_1}(\Omega),\\ (\xi^2+\eta)\phi-v(x)\mu(\xi)\in L^2(\Gamma_0\times(-\infty,+\infty)),\\ \frac{\partial u}{\partial\nu}+b_1\int_{-\infty}^{+\infty}\phi(\xi,t)\mu(\xi)d\xi=0,\ \text{on }\Gamma_0,\\ |\xi|\phi\in L^2(\Gamma_0\times(-\infty,+\infty)). \end{array}\right\}. \tag{3.11}$$

Then, we have the following local existence result.

Theorem 3.3.1 *Suppose that (3.2) holds. Then for any $U_0\in\mathcal{H}$, problem (3.5) has a unique weak solution $U\in C([0,T),\mathcal{H})$, where T is small.*

Proof First, for all $U\in D(A)$, using (3.10)and (3.7), we have

$$\langle AU,U\rangle_{\mathcal{H}}=a\|v\|_2^2+b\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(x,\xi)|^2d\xi d\rho\geq 0.$$

Therefore, A is a monotone operator.

To show that A is maximal operator, we prove that for each $F=(f_1,f_2,f_3)^T\in\mathcal{H}$, there exists $U=(y,u,\phi)^T\in D(A)$ such that $(I+A)U=F$. That is,

$$\begin{cases} u-v=f_1,\\ \\ (1+a)v-\Delta u=f_2,\\ \\ \phi+(\xi^2+\eta)\phi-v(x)\mu(\xi)=f_3(\xi). \end{cases} \tag{3.12}$$

Using equations $(3.12)_3$, $(3.12)_1$ and the fact that $\eta\geq 0$, we have

$$\phi(\xi)=\frac{f_3(\xi)}{\xi^2+\eta+1}+\frac{u(x)\mu(\xi)}{\xi^2+\eta+1}-\frac{f_1(x)\mu(\xi)}{\xi^2+\eta+1},\forall x\in\Gamma_0. \tag{3.13}$$

Inserting the equation $(3.12)_1$ into $(3.12)_2$, we get

$$(1+a)u-\Delta u=f_2+(1+a)f_1, \tag{3.14}$$

Now, solving equation (3.14) is equivalent to finding $u\in H^2(\Omega)\cap H^1_{\Gamma_1}(\Omega)$ such that

$$\int_\Omega[(1+a)u-\Delta u]\,wdx=\int_\Omega[f_2+(1+a)f_1]\,wdx, \tag{3.15}$$

for all $w\in H^1_{\Gamma_1}(\Omega)$. By using (3.15),$(3.11)_3$ and (3.13) the function u satisfying the following system

$$\begin{array}{rl} & \int_\Omega[(1+a)uw+\nabla u\nabla w]\,dx+b_2\int_{\Gamma_0}uwd\rho\\ \\ = & \int_\Omega(f_2+(1+a)f_1)\,wdx+b_2\int_{\Gamma_0}f_1wd\rho\\ \\ & -b_1\int_{\Gamma_0}w\int_{-\infty}^{+\infty}\frac{\mu(\xi)f_3(\xi)}{\xi^2+\eta+1}d\xi d\rho, \end{array} \tag{3.16}$$

where $b_2 = b_1 \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2+\eta+1} d\xi$. Consequently, problem (3.16) is equivalent to the problem

$$B(u,w) = L(w). \tag{3.17}$$

where the sesquilinear form $B : H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega) \to \mathbb{R}$ and the antilinear form $L : H^1_{\Gamma_1}(\Omega) \to \mathbb{R}$ are defined by

$$B(u,w) = \int_\Omega [(1+a)uw + \nabla u \nabla w]\, dx + b_2 \int_{\Gamma_0} uw d\rho$$

and

$$\begin{aligned} L(w) = \ & \textstyle\int_\Omega (f_2 + (1+a) f_1)\, w dx + b_2 \int_{\Gamma_0} f_1 w d\rho \\ & -b_1 \textstyle\int_{\Gamma_0} w \int_{-\infty}^{+\infty} \frac{\mu(\xi) f_3(\xi)}{\xi^2+\eta+1} d\xi d\rho. \end{aligned}$$

It is easy to verify that B is continuous and coercive, and L is continuous. Consequently, So applying the Lax-Milgram theorem, we deduce that for all $w \in H^1_{\Gamma_1}(\Omega)$ system (3.17) admits a unique solution $u \in H^1_{\Gamma_1}(\Omega)$. In particular, setting $w \in D(\Omega)$ in (3.17), we get

$$(1+a)u - \Delta u = f_2 + (1+a) f_1 \in D'(\Omega), \tag{3.18}$$

As $f_2 + (1+a) f_1 \in L^2(\Omega)$,using (3.18), we deduce that

$$(1+a)u - \Delta u = f_2 + (1+a) f_1 \in L^2(\Omega).$$

Due to the fact that $u \in H^1_{\Gamma_1}(\Omega)$ we get $\Delta u \in L^2(\Omega)$, and we deduce tha $u \in H^1_{\Gamma_1}(\Omega) \cap H^2(\Omega)$. Consequently, defining $v = u - f_1 \in H^1_{\Gamma_0}(\Omega)$ and ϕ by (3.13), we deduce that $U \in D(A)$. Consequently, $I + A$ is surjective and then A is maximal.

Finally, we show that $J : \mathcal{H} \to \mathcal{H}$ is locally Lipschitz. So,

$$\begin{aligned} \|\, J(U) - J(\bar{U}) \,\|^2_{\mathcal{H}} &= \|(0, u|u|^{p-2} - \bar{u}|\bar{u}|^{p-2}, 0)\|^2_{\mathcal{H}} \\ &= \|u|u|^{p-2} - \bar{u}|\bar{u}|^{p-2}\|^2_{L^2(\Omega)} \\ &= \textstyle\int_\Omega \left||u|^{p-2}u - |\bar{u}|^{p-2}\bar{u}\right|^2 dx. \end{aligned}$$

As a consequence of the mean value theorem, we have, for $0 \le \theta \le 1$

$$\begin{aligned} \|J(U) - J(\bar{U})\|^2_{\mathcal{H}} &= \|J'(\theta u + (1-\theta)\bar{u})(u - \bar{u})\|^2_{L^2(\Omega)} \\ &= (p-1)^2 \textstyle\int_\Omega |\theta u + (1-\theta)\bar{u}|^{2(p-2)} |u - \bar{u}|^2 dx. \end{aligned}$$

Using Hölder's inequality, we have

$$\|\, J(U) - J(\bar{U}) \,\|^2_{\mathcal{H}} \le (p-1)^2 \left(\int_\Omega (|u - \bar{u}|^{2\gamma}) \right)^{\frac{1}{\gamma}} \left(\int_\Omega |\theta u + (1-\theta)\bar{u}|^{2(p-2)\delta} dx \right)^{\frac{1}{\delta}}, \ \frac{1}{\gamma} + \frac{1}{\delta} = 1$$

with $\gamma = \frac{n}{n-2}$ and $\delta = \frac{n}{2}$. So,

$$\begin{aligned}
\left\| J(U) - J(\bar{U}) \right\|_{\mathcal{H}} &\leq (p-1)^2 \left(\int_\Omega |u - \bar{u}|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \left(\int_\Omega |\theta u + (1-\theta)\bar{u}|^{n(p-2)} dx \right)^{\frac{2}{n}} \\
&\leq (p-1)^2 \|u - \bar{u}\|^2_{L^{\frac{2n}{n-2}}(\Omega)} \|\theta u + (1-\theta)\bar{u}\|^{2(p-2)}_{L^{n(p-2)}(\Omega)} \qquad (3.19) \\
&\leq C \|u - \bar{u}\|^2_{L^{\frac{2n}{n-2}}(\Omega)} \left(\|u\|_{L^{n(p-2)}(\Omega)} + \|\bar{u}\|_{L^{n(p-2)}(\Omega)} \right)^{2(p-2)}.
\end{aligned}$$

As $u, \bar{u} \in H^1_{\Gamma_1}(\Omega)$, then by using the Sobolev embedding, we get

$$\|u - \bar{u}\|_{L^{\frac{2n}{n-2}}(\Omega)} \leq C \|\nabla u - \nabla \bar{u}\|_{L^2(\Omega)} \leq C \left\| U - \bar{U} \right\|_{\mathcal{H}}. \qquad (3.20)$$

since $p \leq \frac{2(n-1)}{n-2}$,then we have $n(p-2) \leq \frac{2n}{n-2}$. So, by using the Sobolev embedding, we get

$$\|u\|_{L^{n(p-2)}(\Omega)} \leq C \|u\|_{H^1_{\Gamma_1}(\Omega)}. \qquad (3.21)$$

Therefore, by combining (3.19)–(3.21), we obtain

$$\left\| J(U) - J(\bar{U}) \right\|^2_{\mathcal{H}} \leq C(\|u\|_{H^1_{\Gamma_1}(\Omega)} + \|\bar{u}\|_{H^1_{\Gamma_1}(\Omega)})^{2(p-2)} \left\| U - \bar{U} \right\|^2_{\mathcal{H}}.$$

Therefore, J is locally Lipchitz. Thanks to the theorems in Komornik [27] (See also Pazy [41]), the proof is completed. ■

3.4 Global existence

This section is concerned with the proof of the global existence of the solution of problem (3.5). We introduce the following functionals:

$$I(t) = \|\nabla u\|_2^2 - \|u\|_p^p + b_1 \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi(\xi, t)|^2 d\xi d\rho. \qquad (3.22)$$

and

$$J(t) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p + \frac{b_1}{2} \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi(\xi, t)|^2 d\xi d\rho.$$

Lemma 3.4.1 *Suppose that (3.2) holds. Then for any $(u_0, u_1, \phi_0) \in D(A)$, satisfying*

$$\begin{cases} \beta = C_*^p \left(\frac{2p}{p-2} E(0) \right)^{\frac{p-2}{2}} < 1 \\ I(u_0) > 0, \end{cases} \qquad (3.23)$$

we get $u(t) \in \aleph, \quad \forall t \in [0, T]$.

Proof Because $I(u_0) > 0$, then there exists $T^* \leq T$, such that $I(u) \geq 0$, for all $t \in [0, T^*)$. This implies:

$$\begin{aligned} \|\nabla u\|_2^2 &\leq \tfrac{2p}{p-2} J(t), \quad \forall t \in [0, T^*) \\ &\leq \tfrac{2p}{p-2} E(0). \end{aligned} \tag{3.24}$$

Using (3.23), (3.24) and the Poincare inequality, we get

$$\begin{aligned} \|u\|_p^p &\leq C_*^p \|\nabla u\|_2^p \\ &\leq C_*^p \left(\tfrac{2p}{p-2} E(0)\right)^{\frac{p-2}{2}} \|\nabla u\|_2^2. \end{aligned} \tag{3.25}$$

Hence $\|\nabla u\|_2^2 - \|u\|_p^p > 0, \forall t \in [0, T^*)$ this shows that $u \in \aleph, \forall t \in [0, T^*)$. By repeating this procedure, T^* is extended to T. ■

We are now ready to prove our global existence result.

Theorem 3.4.1 *Suppose that (3.2) holds. Then for any $(u_0, u_1, \phi_0) \in D(A)$, satisfying (3.23), the solution of system (3.5) is bounded and global.*

Proof By (3.7), we have

$$\begin{aligned} E(0) \geq E(t) &= \tfrac{1}{2}\|u_t\|_2^2 + \tfrac{1}{2}\|\nabla u\|_2^2 - \tfrac{1}{p}\|u\|_p^p + \tfrac{b_1}{2}\int_{\Gamma_0}\int_{-\infty}^{+\infty} |\phi(\xi, t)|^2 d\xi d\rho \\ &= \tfrac{1}{2}\|u_t\|_2^2 + \tfrac{p-2}{2p}\|\nabla u\|_2^2 + \tfrac{1}{p} I(t) + \tfrac{b_1(p-2)}{2p}\int_{\Gamma_0}\int_{-\infty}^{+\infty} |\phi(\xi, t)|^2 d\xi d\rho. \end{aligned} \tag{3.26}$$

Since $I(t) > 0$, therefore

$$\|u_t\|_2^2 + \|\nabla u\|_2^2 + b_1 \int_{\Gamma_0}\int_{-\infty}^{+\infty} |\phi(\xi, t)|^2 d\xi d\rho \leq C_1 E(0),$$

where $C_1 = \max\{2, \frac{2p}{p-2}, \frac{2p}{b_1(p-2)}\}$. ■

3.5 Decay of solutions

In order to establish the energy decay result. Let us constructing a suitable Lyapunov functional as follows:

$$L(t) = \epsilon_1 E(t) + \epsilon_2 \psi_1(t) + \frac{\epsilon_2 b_1}{2}\psi_2(t), \tag{3.27}$$

where ϵ_1 and ϵ_2 are positive constants and

$$\begin{aligned} \psi_1(t) &= \int_\Omega u_t u dx, \\ \psi_2(t) &= \int_{\Gamma_0}\int_{-\infty}^{+\infty} (\xi^2 + \eta)\left(\int_0^t \phi(\xi, s) ds\right)^2 d\xi d\rho. \end{aligned}$$

Lemma 3.5.1 *Let (u,ϕ) be a regular solution of the problem (3.5). Then the equality*

$$\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi(\xi,t)\int_0^t\phi(\xi,s)dsd\xi d\rho =$$

$$\int_{\Gamma_0}u(x,t)\int_{-\infty}^{+\infty}\phi(\xi,t)\mu(\xi)d\xi d\rho-\int_{\Gamma_0}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2d\xi d\rho,$$

holds.

Proof It is clear that by using $(3.5)_2$, we get

$$(\xi^2+\eta)\phi(\xi,t)=u_t(x,t)\mu(\xi)-\partial_t\phi(\xi,t),\quad \forall x\in\Gamma_0. \tag{3.28}$$

A simple integration of (3.28) between 0 and t, and use the equation 3 and 6 of the system (3.5), leads to

$$\int_0^t(\xi^2+\eta)\phi(\xi,s)ds=u(x,t)\mu(\xi)-\phi(\xi,t),\quad \forall x\in\Gamma_0,$$

thus,

$$(\xi^2+\eta)\int_0^t\phi(\xi,s)ds=u(x,t)\mu(\xi)-\phi(\xi,t),\quad \forall x\in\Gamma_0. \tag{3.29}$$

Multiplying (3.29) by ϕ and integrating over $\Gamma_0\times(-\infty,+\infty)$, we obtain

$$\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi(\xi,t)\int_0^t\phi(\xi,s)dsd\xi d\rho =$$

$$\int_{\Gamma_0}u(x,t)\int_{-\infty}^{+\infty}\phi(\xi,t)\mu(\xi)d\xi d\rho-\int_{\Gamma_0}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2d\xi d\rho.$$

■

Lemma 3.5.2 *Let (u,ϕ) be a regular solution of the problem (3.5), then there exists two positive constants C_1,C_2 such that*

$$|\psi_2(t)|\le C_1\int_{\Gamma_0}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2d\xi d\rho+C_2\|\nabla u\|_2^2. \tag{3.30}$$

Proof Using (3.29), we get

$$\int_0^t\phi(\xi,s)ds=\frac{-\phi(\xi,t)}{\xi^2+\eta}+\frac{u(x,t)\mu(\xi)}{\xi^2+\eta},\quad \forall x\in\Gamma_0.$$

Then

$$\left(\int_0^t\phi(\xi,s)ds\right)^2=\frac{|\phi(\xi,t)|^2}{(\xi^2+\eta)^2}+\frac{|u(x,t)|^2\mu^2(\xi)}{(\xi^2+\eta)^2}-2\frac{\phi(\xi,t)u(x,t)\mu(\xi)}{(\xi^2+\eta)^2}. \tag{3.31}$$

Multiplying (3.31) by $\xi^2+\eta$ and integrating over $\Gamma_0\times(-\infty,+\infty)$, we easily get

$$\begin{aligned}|\psi_2(t)| &\le \int_{\Gamma_0}\int_{-\infty}^{+\infty}\frac{|\phi(\xi,t)|^2}{\xi^2+\eta}d\xi d\rho+\int_{\Gamma_0}|u(x,t)|^2\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta}d\xi d\rho\\ &+2\int_{\Gamma_0}\int_{-\infty}^{+\infty}\frac{|\phi(\xi,t)u(x,t)\mu(\xi)|}{\xi^2+\eta}d\xi d\rho.\end{aligned} \tag{3.32}$$

To estimate the last term in (3.32), we use Young's inequality, we arrive to:

$$\begin{aligned}\int_{\Gamma_0}\int_{-\infty}^{+\infty}\frac{|\phi(\xi,t)u(x,t)\mu(\xi)|}{\xi^2+\eta}d\xi d\rho &= \int_{\Gamma_0}\int_{-\infty}^{+\infty}\frac{|\phi(\xi,t)|}{(\xi^2+\eta)^{\frac{1}{2}}}\frac{|u(x,t)\mu(\xi)|}{(\xi^2+\eta)^{\frac{1}{2}}}d\xi d\rho\\ &\leq \frac{1}{2}\int_{\Gamma_0}\int_{-\infty}^{+\infty}\frac{|\phi(\xi,t)|^2}{\xi^2+\eta}d\xi d\rho \qquad (3.33)\\ &+\frac{1}{2}\int_{\Gamma_0}|u(x,t)|^2\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta}d\xi d\rho.\end{aligned}$$

Inserting (3.33) in (3.32), we get

$$|\psi_2(t)| \leq 2\int_{\Gamma_0}\int_{-\infty}^{+\infty}\frac{|\phi(\xi,t)|^2}{\xi^2+\eta}d\xi d\rho + 2\int_{\Gamma_0}|u(x,t)|^2\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta}d\xi d\rho. \qquad (3.34)$$

Using the fact $\frac{1}{\xi^2+\eta} \leq \frac{1}{\eta}$. Then (3.34), becomes

$$|\psi_2(t)| \leq \frac{2}{\eta}\int_{\Gamma_0}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2 d\xi d\rho + 2\int_{\Gamma_0}|u(x,t)|^2\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta}d\xi d\rho.$$

Applying Lamma 3.2.2 and Lemma 3.2.5 we have

$$|\psi_2(t)| \leq C_1\int_{\Gamma_0}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2 d\xi d\rho + C_2\|\nabla u\|_2^2.$$

■

Lemma 3.5.3 *For ϵ_1 large and ϵ_2 small enough, we have*

$$\frac{\epsilon_1}{2}E(t) \leq L(t) \leq 2\epsilon_1 E(t). \qquad (3.35)$$

Proof Using Young's inequality and Poincaré's inequality, we obtain

$$\begin{aligned}L(t) \leq\ & \epsilon_1 E(t) + \frac{\epsilon_2}{2}\|u_t\|_2^2 + \frac{\epsilon_2 C_*^2}{2}\|\nabla u\|_2^2\\ &+\frac{b_1\epsilon_2}{2}\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\beta)\left(\int_0^t\phi(\xi,s)ds\right)^2 d\xi d\rho.\end{aligned}$$

Using (3.6) and Lemma 3.5.2, we get

$$\begin{aligned}L(t) \leq\ & \frac{1}{2}\left\{\epsilon_1+\epsilon_2\right\}\|u_t\|_2^2 - \frac{\epsilon_1}{p}\|u\|_p^p\\ &+\frac{1}{2}\left(\epsilon_1+\epsilon_2 b_1 C_2\right)\|\nabla u\|_2^2\\ &+\frac{b_1}{2}\left(\epsilon_1+\epsilon_2 C_1\right)\int_{\Gamma_0}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2 d\xi d\rho.\end{aligned}$$

So, by using (3.22), we get

$$\begin{aligned} 2\epsilon_1 E(t) - L(t) &\geq \tfrac{1}{2}\left\{\epsilon_1 - \epsilon_2\right\}\|u_t\|_2^2 + \tfrac{\epsilon_1}{p} I(t) \\ &+\tfrac{1}{2}\left\{\tfrac{(p-2)\epsilon_1}{p} - \epsilon_2 b_1 C_2\right\}\|\nabla u\|_2^2 \\ &+\tfrac{b_1}{2}\left\{\tfrac{(p-2)\epsilon_1}{p} - \epsilon_2 C_1\right\}\int_{\Gamma_0}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2 d\xi d\rho. \end{aligned}$$

Similarly, we have

$$\begin{aligned} L(t) - \tfrac{\epsilon_1}{2} E(t) &\geq \tfrac{1}{2}\left\{\tfrac{\epsilon_1}{2} - \epsilon_2\right\}\|u_t\|_2^2 + \tfrac{\epsilon_1}{2p} I(t) \\ &+\tfrac{1}{2}\left\{\tfrac{(p-2)\epsilon_1}{2p} - \epsilon_2 b_1 C_2\right\}\|\nabla u\|_2^2 \\ &+\tfrac{b_1}{2}\left\{\tfrac{(p-2)\epsilon_1}{2p} - \epsilon_2 C_1\right\}\int_{\Gamma_0}\int_{-\infty}^{+\infty}|\phi(x,\xi,t)|^2 d\xi d\rho. \end{aligned}$$

By fixing ϵ_2 small and ϵ_1 large enough, we obtain $L(t) - \frac{\epsilon_1}{2}E(t) \geq 0$ and $2\epsilon_1 E(t) - L(t) \geq 0$. The proof is completed. ■

Now, we state and prove our main theorem.

Theorem 3.5.1 *Suppose that (3.2) and (3.23) holds. Then there exist positive constants k and K such that the global solution of (3.5) satisfies*

$$E(t) \leq K e^{-kt}. \tag{3.36}$$

Proof We differentiate (3.27) to obtain

$$\begin{aligned} L'(t) &= \epsilon_1 E'(t) + \epsilon_2\|u_t\|_2^2 + \epsilon_2\int_\Omega u_{tt} u dx \\ &+\epsilon_2 b_1\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi(\xi,t)\int_0^t\phi(\xi,s)ds d\xi d\rho. \end{aligned}$$

Using problem (3.5), we have

$$\begin{aligned} L'(t) &= \epsilon_1 E'(t) + \epsilon_2\left[\|u_t\|_2^2 - \|\nabla u\|_2^2 + \|u\|_p^p - a\int_\Omega u u_t dx\right] \\ &-b_1\epsilon_2\int_{\Gamma_0} u(x,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi,t)d\xi d\rho \\ &+b_1\epsilon_2\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi(\xi,t)\int_0^t\phi(\xi,s)ds d\xi d\rho. \end{aligned}$$

Applying Lemma 3.5.1 we obtain:

$$\begin{aligned} L'(t) &= \epsilon_1 E'(t) + \epsilon_2\|u_t\|_2^2 - \epsilon_2\|\nabla u\|_2^2 + \epsilon_2\|u\|_p^p \\ &-b_1\epsilon_2\int_{\Gamma_0}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2 d\xi d\rho - a\epsilon_2\int_\Omega u u_t dx. \end{aligned} \tag{3.37}$$

Using Young's inequality and Poincare-type inequality to estimate the last term in (3.37) as follows, for any $\delta' > 0$

$$\int_\Omega u u_t dx \leq \frac{1}{4\delta'}\|u_t\|_2^2 + C_*^2\delta'\|\nabla u\|_2^2. \tag{3.38}$$

Inserting (3.38) into (3.37), and by (3.7), we obtain:

$$\begin{aligned} L'(t) &\leq \left[-a\epsilon_1 + \epsilon_2(1+\tfrac{a}{4\delta'})\right]\|u_t\|_2^2 + \epsilon_2\left[-1+\delta' C_*^2 a\right]\|\nabla u\|_2^2 \\ &+\epsilon_2\|u\|_p^p - b_1\epsilon_2\int_{\Gamma_0}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2 d\xi d\rho. \end{aligned}$$

By (3.25), we get:

$$\begin{aligned} L'(t) &\leq \left[-a\epsilon_1 + \epsilon_2(1+\tfrac{a}{4\delta'})\right]\|u_t\|_2^2 + \epsilon_2\left[-1+\delta' C_*^2 a + C_*^p(\tfrac{2p}{p-2})^{\frac{p-2}{2}}\right]\|\nabla u\|_2^2 \\ &-b_1\epsilon_2\int_{\Gamma_0}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2 d\xi d\rho. \end{aligned}$$

From (3.23), we have

$$-1 + C_*^p(\frac{2p}{p-2})^{\frac{p-2}{2}} < 0.$$

Now, we choose δ' such that:

$$-1 + \delta' C_*^2 a + C_*^p(\frac{2p}{p-2})^{\frac{p-2}{2}} < 0.$$

Then we find $d > 0$, which depends only on δ', such that:

$$\begin{aligned} L'(t) &\leq \left[-a\epsilon_1 + \epsilon_2(1+\tfrac{a}{4\delta'})\right]\|u_t\|_2^2 - \epsilon_2 d\|\nabla u\|_2^2 \\ &-b_1\epsilon_2\int_{\Gamma_0}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2 d\xi d\rho. \end{aligned} \tag{3.39}$$

For any positive constant M, (3.39) is equivalent to:

$$\begin{aligned} L'(t) &\leq \left[-a\epsilon_1 + \epsilon_2(1+\tfrac{a}{4\delta'}+\tfrac{M}{2})\right]\|u_t\|_2^2 + \epsilon_2\left[\tfrac{M}{2} - d\right]\|\nabla u\|_2^2 \\ &+b_1\epsilon_2\left[\tfrac{M}{2}-1\right]\int_{\Gamma_0}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2 d\xi d\rho - M\epsilon_2 E(t). \end{aligned} \tag{3.40}$$

At this point we choose $M < \min\{2, 2d\}$, and ϵ_1 such that

$$\epsilon_1 > \frac{\epsilon_2(1+\frac{a}{4\delta'}+\frac{M}{2})}{a}.$$

Consequently (3.40) yields

$$L'(t) \leq -M\epsilon_2 E(t) \leq \frac{-M\epsilon_2}{2\epsilon_1}L(t), \tag{3.41}$$

by virtue of (3.35). A simple integration of (3.41) then the last inequality becomes:

$$L(t) \leq L(0)e^{-kt},$$

where $k = \frac{M\epsilon_2}{2\epsilon_1}$, Again using (3.35) we have (3.36). ■

3.6 Blow up

In this section, we consider the property of blowing up of the solution of problem (3.5).

Remark 3.6.1 *A simple integration of (3.7) over (0,t) leads to*

$$\begin{aligned} E(t) &= E(0) - a\int_0^t \|u_s\|_2^2 ds \\ &- b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,s)|^2 d\xi d\rho ds. \end{aligned} \tag{3.42}$$

Now, introduce the functional F defined as follows:

$$F(t) = \|u\|_2^2 + a\int_0^t \|u\|_2^2 ds + b_1 H(t), \tag{3.43}$$

where

$$H(t) = \int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\left(\int_0^s \phi(\xi,z)dz\right)^2 d\xi d\rho ds.$$

Lemma 3.6.1 *Suppose that (3.2) holds. Then we have:*

$$\begin{aligned} &F''(t) \geq (p+2)\|u_t\|_2^2 \\ &+2p\left\{-E(0) + a\int_0^t \|u_s\|_2^2 ds + b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,s)|^2 d\xi d\rho ds\right\}. \end{aligned} \tag{3.44}$$

Proof Differentiating relation (3.43) with respect to t, we have

$$\begin{aligned} F'(t) &= 2\int_\Omega u u_t dx + a\|u\|_2^2 \\ &+2b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi(\xi,s)\int_0^s \phi(\xi,z)dz d\xi d\rho ds. \end{aligned} \tag{3.45}$$

By (3.5) and divergence theorem, we get

$$\begin{aligned} F''(t) &= 2\|u_t\|_2^2 - 2\|\nabla u\|_2^2 + 2\|u\|_p^p + 2b_1\int_{\Gamma_0} u(x,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi,t)d\xi d\rho \\ &+2b_1\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi(\xi,t)\int_0^t \phi(\xi,s)ds d\xi d\rho. \end{aligned} \tag{3.46}$$

To estimate the third term, we use the definition of the energy (3.6), and by (3.42) we have

$$\begin{aligned} 2\|u\|_p^p &= p\|u_t\|_2^2 + p\|\nabla u\|_2^2 + pb_1\int_{\Gamma_0}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2 d\xi d\rho - 2pE(0) \\ &+2p\left[a\int_0^t \|u_s\|_2^2 ds + b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,s)|^2 d\xi d\rho ds\right]. \end{aligned} \tag{3.47}$$

It's clear that using Lemma 3.5.1, the last term in (3.46) can be evaluated as follows:

$$\begin{aligned} &\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi(\xi,t)\int_0^t \phi(\xi,s)ds d\xi d\rho = \\ &\int_{\Gamma_0} u(x,t)\int_{-\infty}^{+\infty}\phi(\xi,t)\mu(\xi)d\xi d\rho - \int_{\Gamma_0}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2 d\xi d\rho. \end{aligned} \tag{3.48}$$

By combining (3.47) and (3.48) in (3.46), we obtain

$$F''(t) \geq (p+2)\|u_t\|_2^2 + (p-2)\|\nabla u\|_2^2 + b_1(p-2)\int_{\Gamma_0}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2 d\xi d\rho$$

$$+2p\left[-E(0) + a\int_0^t \|u_s\|_2^2 ds + b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,s)|^2 d\xi d\rho ds\right].$$

Now, choosing $p > 2$ we get:

$$\begin{aligned} F''(t) &\geq (p+2)\|u_t\|_2^2 \\ &\quad +2p\left\{-E(0) + a\int_0^t \|u_s\|_2^2 ds + b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,s)|^2 d\xi d\rho ds\right\}. \end{aligned}$$

■

Now, we prove the following lemma.

Lemma 3.6.2 *Suppose that (3.2) holds and that either one the following conditions is satisfied*

(i) $E(0) < 0$.

(ii) $E(0) = 0$, *and*

$$F'(0) > a\|u_0\|_2^2. \tag{3.49}$$

(iii) $E(0) > 0$, *and*

$$F'(0) > r\left[F(0) + l_0\right] + a\|u_0\|_2^2, \tag{3.50}$$

where

$$r = 2p - 2\sqrt{p^2 - p}$$

and

$$l_0 = a\|u_0\|_2^2 + 2E(0).$$

Then $F'(t) > a\|u_0\|_2^2$, *for* $t > t_0$, *where*

$$t^* > \max\left\{0, \frac{F'(0) - a\|u_0\|_2^2]}{2pE(0)}\right\}, \tag{3.51}$$

where $t_0 = t^*$ *in case(i), and* $t_0 = 0$ *in case(ii) and (iii)*

Proof (i) If $E(0) < 0$, then from (3.44), we get

$$F''(t) \geq -2pE(0),$$

we easily obtain :

$$F'(t) \geq F'(0) - 2pE(0)t.$$

Then

$$F'(t) > a\|u_0\|_2^2, \quad \forall t \geq t^*,$$

where t^*, is defined in (3.51).

(ii) If $E(0)=0$ then from (3.44) we fined

$$F''(t)\geq 0, \quad \forall t\geq 0.$$

We easily obtain :

$$F'(t)\geq F'(0), \quad \forall t\geq 0.$$

Using (3.49) we have

$$F^{'}(t)>a\|u_0\|_2^2, \quad \forall t\geq 0.$$

(iii) For the last case of $E(0)>0$ then from (3.45) we have

$$\begin{aligned} F'(t) &= 2\int_\Omega uu_t dx + a\|u\|_2^2 \\ &+2b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi(\xi,s)\int_0^s\phi(\xi,z)dzd\xi d\rho ds. \end{aligned} \tag{3.52}$$

Applying Young's inequality to estimate the last term in (3.52) we get

$$\begin{aligned} &\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi(\xi,s)\int_0^s\phi(\xi,z)dzd\xi d\rho ds \\ &\leq \tfrac{1}{2}\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,s)|^2 d\xi d\rho ds \\ &+\tfrac{1}{2}\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\left(\int_0^s\phi(\xi,z)dz\right)^2 d\xi d\rho ds \end{aligned} \tag{3.53}$$

and we note that

$$2\int_0^t\int_\Omega u_s u dx ds = \int_0^t \frac{d}{ds}\|u_s\|_2^2 ds = \|u\|_2^2-\|u_0\|_2^2.$$

Applying Young's inequality

$$\|u\|_2^2\leq \int_0^t\|u_s\|_2^2 ds+\int_0^t\|u\|_2^2 ds+\|u_0\|_2^2. \tag{3.54}$$

By combining (3.53) and(3.54) in (3.52), we obtain

$$\begin{aligned} F'(t) &\leq \|u\|_2^2+\|u_t\|_2^2+a\int_0^t\|u_s\|_2^2 ds+a\int_0^t\|u\|_2^2 ds+a\|u_0\|_2^2 \\ &+b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,s)|^2 d\xi d\rho ds \\ &+b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\left(\int_0^s\phi(\xi,z)dz\right)^2 d\xi d\rho ds. \end{aligned} \tag{3.55}$$

Using the definition of the function F in (3.43), then (3.55) becomes

$$\begin{aligned} F'(t)\leq\ & F(t)+\|u_t\|_2^2+b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,s)|^2 d\xi d\rho ds \\ &+a\int_0^t\|u_s\|_2^2 ds+a\|u_0\|_2^2. \end{aligned}$$

Hence by (3.44), we obtain

$$\begin{aligned} F''(t) - p\left\{F'(t) - F(t)\right\} &\geq 2\|u_t\|_2^2 + ap\int_0^t \|u_s\|_2^2 ds - pa\|u_0\|_2^2 - 2pE(0) \\ &+pb_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,s)|^2 d\xi d\rho ds. \end{aligned}$$

Thus, we get

$$F''(t) - pF'(t) + pF(t) + pl_0 \geq 0,$$

where

$$l_0 = a\|u_0\|_2^2 + 2E(0).$$

Now let

$$B(t) = F(t) + l_0.$$

Then $B(t)$ satisfies

$$B''(t) - pB'(t) + pB(t) \geq 0. \tag{3.56}$$

Using Lemma 3.2.3 in (3.56) for $p = \delta + 1$ then if

$$B'(0) > (2p - 2\sqrt{p^2 - p})B(0) + a\|u_0\|_2^2.$$

Then

$$F'(t) = B'(t) > a\|u_0\|_2^2 \quad \forall t \geq 0.$$

■

Theorem 3.6.1 *Suppose that (3.2) holds and that either one of the following conditions is satisfied*
(i)$E(0) < 0$.
(ii) $E(0) = 0$ *and (3.49) holds.*
(iii) $0 < E(0) < \frac{(2p-4)\left(F'(t_0)-a\|u_0\|_2^2\right)^2 J(t_0)^{\frac{1}{\gamma_1}}}{16p}$ *and (3.50) holds. Then the solutions* (u,ϕ) *blows up in finite time* T^* *in the sense of (3.2.1).*
In case (i):

$$T^* \leq t_0 - \frac{J(t_0)}{J'(t_0)}.$$

Furthermore, if $J(t_0) < \min\left\{1, \sqrt{\frac{\sigma}{-b}}\right\}$, *then we have*

$$T^* \leq t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{\frac{\sigma}{-b}}}{\sqrt{\frac{\sigma}{-b}} - J(t_0)}.$$

In case (ii):

$$T^* \leq t_0 - \frac{J(t_0)}{J'(t_0)},$$

or

$$T^* \leq t_0 + \frac{J(t_0)}{J'(t_0)}.$$

In case (iii):

$$T^* \leq \frac{J(t_0)}{\sqrt{\sigma}},$$

or

$$T^* \leq t_0 + 2^{\frac{3\gamma_1+1}{2\gamma_1}} \frac{\gamma_1 c}{\sqrt{\sigma}}\{1 - [1 - cJ(t_0)]^{\frac{1}{2\gamma_1}}\},$$

where $c = (\frac{b}{\sigma})^{\frac{\gamma_1}{2+\gamma_1}}$, $\gamma_1 = \frac{p-4}{4}$, *and* $J(t)$, σ *and* b *are given in (3.57) and (3.66) respectively. Note that in case (i),* $t_0 = t^*$ *is given in (3.51) and* $t_0 = 0$ *in case (ii) and (iii).*

Proof Let

$$J(t) = \left[F(t) + a(T-t)\|u_0\|_2^2\right]^{-\gamma_1}, \quad t \in [t_0, T]. \tag{3.57}$$

Differentiating $J(t)$ twice, we obtain

$$J^{'}(t) = -\gamma_1 J(t)^{1+\frac{1}{\gamma_1}} \left[F'(t) - a\|u_0\|_2^2\right]$$

and

$$J^{''}(t) = -\gamma_1 J(t)^{1+\frac{2}{\gamma_1}} G(t), \tag{3.58}$$

where

$$G(t) = F^{''}(t)\left[F(t) + a(T-t)\|u_0\|_2^2\right] - (1+\gamma_1)\left\{F^{'}(t) - a\|u_0\|_2^2\right\}^2. \tag{3.59}$$

We get, from (3.44)

$$\begin{aligned} F''(t) &\geq (p+2)\|u_t\|_2^2 \\ &+2p\left\{-E(0) + a\int_0^t \|u_s\|_2^2 ds + b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,s)|^2 d\xi d\rho ds\right\}. \end{aligned}$$

Therefore, we find

$$\begin{aligned} &F''(t) \geq -2pE(0) \\ &p\left\{\|u_t\|_2^2 + a\int_0^t \|u_s\|_2^2 ds + b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,s)|^2 d\xi d\rho ds\right\} \end{aligned} \tag{3.60}$$

and since $\|u\|_2^2 - \|u_0\|_2^2 = 2\int_0^t\int_\Omega u_s u dx ds$ Then, from (3.45) we get

$$\begin{aligned} F'(t) - a\|u_0\|_2^2 &= 2\int_\Omega u u_t dx + 2a\int_0^t\int_\Omega u_s u dx ds \\ &+2b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi(\xi,s)\int_0^s \phi(\xi,z)dz d\xi d\rho ds. \end{aligned} \tag{3.61}$$

By combining (3.60) and (3.61) in (3.59), we get

$$
\begin{aligned}
&G(t) \geq -2pE(0)J(t)^{\frac{-1}{\gamma_1}}\\
&+p\left\{\|u_t\|_2^2 + a\int_0^t \|u_s\|_2^2 ds + b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,s)|^2 d\xi d\rho ds\right\}\\
&\times\left[\|u\|_2^2 + a\int_0^t \|u\|_2^2 ds + b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\left(\int_0^s \phi(\xi,z)dz\right)^2 d\xi d\rho ds\right]\\
&-4(1+\gamma_1)\left\{\int_\Omega uu_t dx + a\int_0^t\int_\Omega u_s u dx ds + b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi(\xi,s)\int_0^s\phi(\xi,z)dz d\xi d\rho ds\right\}^2.
\end{aligned}
$$

For simplicity of calculations, we make the notations

$$
\begin{aligned}
\mathbf{A} &= \|u\|_2^2 + a\int_0^t \|u\|_2^2 ds + b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\left(\int_0^s\phi(\xi,z)dz\right)^2 d\xi d\rho ds,\\
\mathbf{B} &= \int_\Omega uu_t dx + a\int_0^t\int_\Omega u_s u dx ds + b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi(\xi,s)\int_0^s\phi(\xi,z)dz d\xi d\rho ds,\\
\mathbf{C} &= \|u_t\|_2^2 + a\int_0^t \|u_s\|_2^2 ds + b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,s)|^2 d\xi d\rho ds.
\end{aligned}
$$

Thus, we obtain

$$G(t) \geq -2pE(0)J(t)^{\frac{-1}{\gamma_1}} + p\left\{\mathbf{AC}-\mathbf{B}^2\right\}. \tag{3.62}$$

Now we observe that, for all $w \in R$ and $t > 0$,

$$
\begin{aligned}
\mathbf{A}w^2 + 2\mathbf{B}w + \mathbf{C} &= \left[w^2\|u\|_2^2 + 2w\int_\Omega uu_t dx + \|u_t\|_2^2\right]\\
&+a\int_0^t\left[w^2\|u\|_2^2 + 2w\int_\Omega uu_s dx + \|u_s\|_2^2\right]ds\\
&+b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\left[w^2\left(\int_0^s\phi(\xi,z)dz\right)^2\right.\\
&\left.+2w\phi(\xi,s)\int_0^s\phi(\xi,z)dz + |\phi(\xi,s)|^2\right]d\xi d\rho ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathbf{A}w^2 &+2\mathbf{B}w + \mathbf{C} = \|wu+u_t\|_2^2 + a\int_0^t \|wu+u_s\|_2^2 ds\\
&+b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\left[w\int_0^s\phi(\xi,z)dz + |\phi(\xi,s)|\right]^2 d\xi d\rho ds.
\end{aligned}
$$

It is easy to see that

$$\mathbf{A}w^2 + 2\mathbf{B} + \mathbf{C} \geq 0$$

and

$$\mathbf{B}^2 - \mathbf{AC} \leq 0. \tag{3.63}$$

Hence, by (3.62) and (3.63), we get

$$G(t) \geq -2pE(0)J(t)^{\frac{-1}{\gamma_1}}, \quad t \geq t_0. \tag{3.64}$$

Therefore, by (3.58) and (3.64), we get

$$J''(t) \leq \frac{p^2-4p}{2}E(0)J(t)^{1+\frac{1}{\gamma_1}}, \quad t \geq t_0. \tag{3.65}$$

Note that using Lemma 3.6.2, $J^{'}(t) < 0$ for $t \geq t_0$. Multiplying (3.6) by $J^{'}(t)$ and integrating it from t_0 to t, we have

$$J^{'}(t)^2 \geq \sigma + bJ(t)^{2+\frac{1}{\gamma_1}},$$

where

$$\begin{cases} \sigma &= \left[\frac{(p-4)^2}{16}\left(F^{'}(t_0) - \|u_0\|_2^2\right)^2 - \frac{p(p-4)^2}{2p-4}E(0)J(t_0)^{\frac{-1}{\gamma_1}}\right]J(t_0)^{2+\frac{2}{\gamma_1}} \\ \\ b &= \frac{p(p-4)^2}{2p-4}E(0). \end{cases} \tag{3.66}$$

Then by Lemma 3.2.4 the proof of theorem is completed.
Hence, there exists a finite time T such that $\underset{t\to T^{*-}}{lim} J(t) = 0$ and the upper bounds of T^* are estimated according to the sign of $E(0)$ (see Lemma 3.2.4).

■

Chapter 4

Blow up and asymptotic behavior for a wave equation with a time delay condition of fractional type

4.1 Introduction

In this Chapter, we consider the following wave equation with a time delay condition of fractional type and source terms:

$$(P)\begin{cases} y_{tt} - \Delta y + a_1 \partial_t^{\alpha,\beta} y(t-s) + a_2 y_t = |y|^{p-2}y, & x \in \Omega,\ t > 0 \\ y = 0, & x \in \partial\Omega,\ t > 0 \\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), & x \in \Omega, \\ y_t(x,t-s) = f_0(x,t-s), & x \in \Omega,\ t \in (0,s), \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, a_1 and a_2 are positive real numbers such that $a_1\beta^{\alpha-1} < a_2$. The constant $s > 0$ is the time delay and $p > 2$. Moreover, (y_0, y_1, f_0) the initial data belong to a suitable function space. The notation $\partial_t^{\alpha,\beta}$ stands for the generalized Caputo's fractional derivative (see [10] and [11]) defined by the following formula:

$$\partial_t^{\alpha,\beta} u(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\beta(t-s)} u_s(s)ds, \quad 0 < \alpha < 1, \beta > 0.$$

In the absence of the fractional time delay term ($a_1 = 0$), problem (P) has been extensively studied and many results concerning well-posedness and stability or instability have been established. For instance, for the equation

$$y_{tt} - \Delta y(t) + h(y_t) = |y|^{p-2}y, \text{ in } \Omega \times (0,\infty),$$

it is well known that, when $h \equiv 0$, the source term $|y|^{p-2}y, (p > 2)$ causes finite time blow up of solutions with negative initial energy (see [7]). The interaction between the damping and the source terms was first considered by Levine [28, 30] in the linear damping case $h(y_t) = y_t$. Using a concavity argument, he proved that solutions with negative initial energy blow up in finite time. For $h(y_t) = |y_t|^{m-2}y_t, (m > 2)$ and $E(0) < 0$, Georgiev and Todorova in [17] introduced a different method and proved the global existence when $p \leq m$ and the blow-up properties when $p > m$ with negative initial energy. The different method of Georgiev and Todorova has become an important to prove the blow-up in finite time for nonlinear evolution equations (see[16, 19, 22]).

When $h(y_t) = \partial_t^\alpha y$, problems related to (P) have been treated by several authors. In [25], Kirane and Tatar studied the following problem:

$$\begin{cases} y_{tt} - \Delta y + \partial_t^\alpha y = |y|^{p-2}y, & \text{in } \Omega \times (0, \infty), \\ y = 0, & \text{on } \partial\Omega \times (0, \infty), \\ y(x, 0) = y_0(x), \quad y_t(x, 0) = y_1(x), & \text{in } \Omega. \end{cases}$$

They proved the exponential growth in the L_p-norm by using some techniques based on Fourier transforms and some inequalities such as the Hardy-Littlewood inequality. Later, Tatar [44] extended this result for larger initial energy. A blow-up result for sufficiently large data has been obtained in [43].

On the other hand, in the presence of the time delay with $\alpha = 1$ the equation

$$y_{tt} - \Delta y + a_1 y_t(t - s) + a_2 y_t = f(y), \tag{4.1}$$

has been considered by Nicaise and Pignotti [39] for $f = 0$. The authors proved that the energy is exponentially stable when $a_2 < a_1$. Later, Kafini and Messaoudi in [23], investigated (4.1) for $f(u) = |y|^{p-2}y \ln |y|^k$. Under the same conditions in [39], They established the local existence result using the semi-group theory and they proved the blow-up properties of solutions for negative initial energy.

In [38], by describing the fractional damping by means of a suitable diffusion equation, we can transform the problem (P) into an augmented model which can be easily tackled by the energy method.

To the best of our knowledge, the decay estimates of energy and blow-up of solutions for wave equations with the presence of a time delay condition of fractional type in the internal feedback have not been studied yet.

In the present paper, we shall consider the problem (P). Under a suitable condition on the damping, the delay functions and the initial data, we give several results concerning the well-posedness, the decay estimates of energy and blow-up of solutions to problem (P).

The paper is organized as follows. In Section 2, we reformulate the problem (P) into an augmend system. In Section 3, the local existence result is proved. In Section 4, global existence and decay estimates of energy are discussed. Finally, in Section 5, we prove the blow-up of solutions for negative initial energy.

4.2 Preliminaries

This section is concerned with the reformulation of the problem (P) into an augmend system. For that, we need the following claims.

Lemma 4.2.1 *[38] Let η be the function:*

$$\eta(\xi) := |\xi|^{\frac{(2\alpha-1)}{2}}, \quad \xi \in \mathbb{R}, \quad 0 < \alpha < 1.$$

Then the relationship between the "input" U and the "output" O of the system

$$\begin{cases} \phi_t(x,\xi,t) + (\xi^2+\beta)\phi(x,\xi,t) - U(x,t)\eta(\xi) = 0, \quad \xi \in \mathbb{R}, t > 0, \beta > 0, \\ \phi(x,\xi,0) = 0, \\ O(t) := (\pi)^{-1}\sin(\alpha\pi)\int_{-\infty}^{+\infty}\phi(x,\xi,t)\eta(\xi)d\xi \end{cases} \tag{4.2}$$

is given by

$$O := I^{1-\alpha,\beta}U,$$

where

$$I^{\alpha,\beta}u(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e^{-\beta(t-s)}u(s)ds.$$

Lemma 4.2.2 *[9] Let $\eta > 0$. For any real number $\lambda > -\eta$,we have*

$$\int_{-\infty}^{+\infty}\frac{\eta^2(\xi)}{\lambda+\beta+\xi^2}d\xi = \frac{\pi}{\sin(\alpha\pi)}(\lambda+\beta)^{\alpha-1}.$$

We make the following hypotheses on the damping and the delay functions:

$$a_1\beta^{\alpha-1} < a_2. \tag{4.3}$$

Now, we introduce, as in [13], the new variable

$$z(x,\rho,t) = y_t(x, t-s\rho), \quad x \in \Omega,\ \rho \in (0,1),\ \mathrm{t} \in \mathbb{R}_+. \tag{4.4}$$

Then, we have

$$z_t(x,\rho,t) = \frac{-1}{s}z_\rho(x,\rho,t), \quad x \in \Omega,\ \rho \in (0,1),\ \mathrm{t} \in \mathbb{R}_+. \tag{4.5}$$

Therefore, by (4.4)-(4.5) and using Lemma 4.2.1, problem (P) is equivalent to

$$(P')\begin{cases} y_{tt} - \Delta y + b\int_{-\infty}^{+\infty}\phi(x,\xi,t)\eta(\xi)d\xi + a_2 y_t = |y|^{p-2}y, & x\in\Omega, \mathrm{t}>0,\\ \phi_t(x,\xi,t) + (\xi^2+\beta)\phi(x,\xi,t) - z(x,1,t)\eta(\xi) = 0, & x\in\Omega, \xi\in\mathbb{R},\ \mathrm{t}>0,\\ sz_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, & x\in\Omega, \rho\in(0,1), \mathrm{t}>0,\\ y=0, & x\in\partial\Omega, t>0,\\ z(x,0,t) = y_t(x,t), & x\in\Omega, t>0,\\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), & x\in\Omega,\\ \phi(x,\xi,0) = 0, & x\in\Omega, \xi\in\mathbb{R},\\ z(x,\rho,0) = f_0(x,-\rho s), & x\in\Omega, \rho\in(0,1), \end{cases}$$

where $b := (\pi)^{-1}\sin(\alpha\pi)a_1$.

Lemma 4.2.3 *For $z\in L^2(\Omega)$ and $\xi\phi\in L^2(\Omega\times(-\infty,+\infty))$, we have*

$$\left|\int_\Omega z(x,\rho,t)\int_{-\infty}^{+\infty}\eta(\xi)\phi(x,\xi,t)d\xi dx\right| \le A_0\int_\Omega |z(x,\rho,t)|^2 dx + \frac{1}{4}\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)|\phi(x,\xi,t)|^2 d\xi dx$$

for a positive constant A_0 .

Proof Applying the Cauchy-Schwarz inequality, we get

$$\left|\int_{-\infty}^{+\infty}\eta(\xi)\phi(x,\xi,t)d\xi\right| \le \left(\int_{-\infty}^{+\infty}\frac{\eta^2(\xi)}{\xi^2+\beta}d\xi\right)^{\frac{1}{2}}\left(\int_{-\infty}^{+\infty}(\xi^2+\beta)|\phi(x,\xi,t)|^2 d\xi\right)^{\frac{1}{2}}.$$

Using Young's inequality, we have

$$\left|\int_\Omega z(x,\rho,t)\int_{-\infty}^{+\infty}\eta(\xi)\phi(x,\xi,t)d\xi dx\right| \le A_0\int_\Omega |z(x,\rho,t)|^2 dx + \frac{1}{4}\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)|\phi(x,\xi,t)|^2 d\xi dx$$

with

$$A_0 := \int_{-\infty}^{+\infty}\frac{\eta^2(\xi)}{\xi^2+\beta}d\xi.$$

This completes the proof. ■

We define the energy associated to the solution of the problem (P') by

$$E(t): \quad = \tfrac{1}{2}\|y_t\|_2^2 + \tfrac{b}{2}\int_\Omega \int_{-\infty}^{+\infty} |\phi(x,\xi,t)|^2 d\xi dx + \tfrac{1}{2}\|\nabla y\|_2^2$$
$$-\tfrac{1}{p}\|y\|_p^p + \nu s \int_\Omega \int_0^1 |z(x,\rho,t)|^2 d\rho dx, \tag{4.6}$$

where ν is a positive constant verifying

$$bA_0 < \nu < a_2 - bA_0. \tag{4.7}$$

Lemma 4.2.4 *Assume that (4.3) holds and*

$$2 < p < \infty, \ \textit{if}\ n = 1,2; \quad 2 < p \leq \frac{2n}{n-2}, \ \textit{if}\ n \geq 3. \tag{4.8}$$

Then, the energy functional defined by (4.6) satisfies

$$\frac{dE(t)}{dt} \leq \ -C\int_\Omega \left(|z(x,1,t)|^2 + |z(x,0,t)|^2\right) dx$$
$$-\tfrac{b}{2}\int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\beta)|\phi(x,\xi,t)|^2 d\xi dx, \tag{4.9}$$

for a positive constant C.

Proof Multiplying the first equation of (P') by y_t, integrating over Ω and using integration by parts, we get

$$\tfrac{d}{dt}\left\{\tfrac{1}{2}\|y_t\|_2^2 + \tfrac{1}{2}\|\nabla y\|_2^2 - \tfrac{1}{p}\|y\|_p^p\right\} + a_2\|y_t\|_2^2$$
$$+b\int_\Omega y_t \int_{-\infty}^{+\infty} \eta(\xi)\phi(x,\xi,t) d\xi dx = 0. \tag{4.10}$$

Multiplying the second equation of (P') by $b\phi$ and integrating over $\Omega \times (-\infty, +\infty)$, we obtain:

$$\tfrac{d}{dt}\left\{\tfrac{b}{2}\int_\Omega \int_{-\infty}^{+\infty} \phi(x,\xi,t)|^2 d\xi dx\right\} + b\int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\beta)|\phi(x,\xi,t)|^2 d\xi dx$$
$$-b\int_\Omega z(x,1,t)\int_{-\infty}^{+\infty} \eta(\xi)\phi(x,\xi,t) d\xi dx = 0. \tag{4.11}$$

Multiplying the third equation of (P') by $2\nu z$ and integrating over $\Omega \times (0,1)$, we get:

$$\frac{d}{dt}\left\{s\nu \int_\Omega \int_0^1 |z(x,\rho,t)|^2 d\rho dx\right\} + \nu \int_\Omega \left[|z(x,1,t)|^2 - |z(x,0,t)|^2\right] dx = 0. \tag{4.12}$$

By summing (4.10), (4.11), (4.12) and using $y_t = z(x,0,t)$, we arrive at

$$\frac{dE(t)}{dt} \quad = -\left(a_2 - \nu\right)\int_\Omega |z(x,0,t)|^2 dx - \nu\int_\Omega |z(x,1,t)|^2 dx$$
$$-b\int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\beta)|\phi(x,\xi,t)|^2 d\xi dx$$
$$-b\int_\Omega z(x,0,t)\int_{-\infty}^{+\infty} \eta(\xi)\phi(x,\xi,t) d\xi dx$$
$$+b\int_\Omega z(x,1,t)\int_{-\infty}^{+\infty} \eta(\xi)\phi(x,\xi,t) d\xi dx. \tag{4.13}$$

Using Lemma 4.2.3, we get

$$\begin{aligned}\frac{dE(t)}{dt} \leq \quad & -C\int_{\Omega}\left(|z(x,1,t)|^2+|z(x,0,t)|^2\right)dx \\ & -\frac{b}{2}\int_{\Omega}\int_{-\infty}^{+\infty}(\xi^2+\beta)|\phi(x,\xi,t)|^2 d\xi dx\end{aligned}$$

with

$$C=\min\left\{(\nu-bA_0),(a_2-bA_0-\nu)\right\}. \tag{4.14}$$

Since ν is chosen satisfying assumption (4.7), the constant C is positive. The proof is completed

■

4.3 Well-posedness

In this section, we give an existence and uniqueness result for problem(P') using the semi-group theory. In traducing the vector function $U=(y,u,\phi,z)^T$ where $u=y_t$ and let $J(U)=(0,|y|^{p-2}y,0,0)^T$, system (P') is equivalent to:

$$\begin{cases} U_t(t)+AU(t)=J\left(U(t)\right), \\ U_0=(y_0,y_1,0,f_0(-\rho s))^T, \end{cases} \tag{4.15}$$

where the operator A is defined by:

$$AU:=\begin{pmatrix} -u \\ -\Delta y+b\int_{-\infty}^{+\infty}\phi(x,\xi)\eta(\xi)d\xi+a_2u \\ (\xi^2+\beta)\phi(x,\xi)-z(x,1)\eta(\xi) \\ \frac{1}{s}z_\rho(x,\rho) \end{pmatrix}.$$

We denote by $\mathcal{H}$ the energy space associated to system:

$$\mathcal{H}:=H_0^1(\Omega)\times L^2(\Omega)\times L^2(\Omega\times(-\infty,+\infty))\times L^2(\Omega\times(0,1)).$$

For $U=(y,u,\phi,z)^T\in\mathcal{H}$ and $\bar{U}=(\bar{y},\bar{u},\bar{\phi},\bar{z})^T\in\mathcal{H}$, we define the following inner product in $\mathcal{H}$

$$\begin{aligned}\left\langle U,\bar{U}\right\rangle_{\mathcal{H}}= \quad & \int_{\Omega}\left[\nabla y.\nabla\bar{y}+u\bar{u}\right]dx+b\int_{\Omega}\int_{-\infty}^{+\infty}\phi(x,\xi)\bar{\phi}(x,\xi)d\xi dx \\ & +2\nu s\int_{\Omega}\int_0^1 z(x,\rho)\bar{z}(x,\rho)dxd\rho.\end{aligned}$$

The domain of the operator A is then

$$D(A)=\left\{\begin{array}{ll} U\in\mathcal{H}: & y\in H^2(\Omega),\quad u\in H_0^1(\Omega),\quad z_\rho\in L^2(\Omega\times(0,1)), \\ & u=z(.,0),\quad \xi\phi\in L^2(\Omega\times(-\infty,+\infty)), \\ & (\xi^2+\beta)\phi-z(x,1,t)\eta(\xi)\in L^2(\Omega\times(-\infty,+\infty)). \end{array}\right\}.$$

Then, we have the following local existence result.

Theorem 4.3.1 *Assume that (4.7) and (4.8) hold. Then for any* $U_0 \in \mathcal{H}$, *problem (4.15) has a unique weak solution* $U \in C([0,T),\mathcal{H})$, *where* T *is small.*

Proof First, for all $U \in D(A)$, using (4.15) and (4.9), then

$$\begin{aligned}\langle AU, U\rangle_{\mathcal{H}} \geq\ & C\int_\Omega \left[|z(x,1)|^2 + |z(x,0)|^2\right] dx \\ & + \tfrac{b}{2}\int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\beta)|\phi(x,\xi)|^2 d\xi dx.\end{aligned}$$

Therefore, A is a monotone operator.

To show that A is maximal operator, we prove that for each $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$, there exists $U = (y, u, \phi, z)^T \in D(A)$ such that $(I + A)U = F$. That is,

$$\begin{cases} y - u = f_1, \\ (1+a_2)u - \Delta y + b\int_{-\infty}^{+\infty} \phi(x,\xi)\eta(\xi)d\xi = f_2, \\ \phi(\xi) + (\xi^2+\beta)\phi(\xi) - z(x,1)\eta(\xi) = f_3(\xi), \\ z(\rho) + \frac{1}{s}z_\rho(\rho) = f_4(\rho). \end{cases} \tag{4.16}$$

Suppose y is found with the appropriate regularity. Therefore, the first and third equations in (4.16) give

$$u = y - f_1 \tag{4.17}$$

and

$$\phi = \frac{f_3(\xi) + z(1)\eta(\xi)}{\xi^2+\beta+1}, \quad \xi \in \mathbb{R}. \tag{4.18}$$

On the other hand, the fourth equation of (4.16) with $z(x,0) = y - f_1$ has a unique solution

$$z(\rho) = (y - f_1)\, e^{-s\rho} + se^{-s\rho}\int_0^\rho e^{s\tau} f_4(\tau)d\tau, \quad \rho \in (0,1). \tag{4.19}$$

Substituting (4.17) in the second equation of (4.16), we get

$$(1+a_2)y - \Delta y + b\int_{-\infty}^{+\infty} \phi(\xi)\eta(\xi)d\xi = f_2 + (1+a_2)f_1. \tag{4.20}$$

Solving equation (4.20) is equivalent to finding $y \in H^2(\Omega)$ such that

$$\begin{aligned}&\int_\Omega \left[(1+a_2)y - \Delta y\right] w dx + b\int_\Omega w \int_{-\infty}^{+\infty} \phi(\xi)\eta(\xi)d\xi dx \\ &= \int_\Omega \left[f_2 + (1+a_2)f_1\right] w dx, \quad w \in H_0^1(\Omega).\end{aligned} \tag{4.21}$$

By using (4.21), (4.19) and (4.18), we have

$$\begin{aligned}\int_\Omega (\sigma y - \Delta y)\, w \;&= \int_\Omega (f_2 + \sigma f_1)\, w dx \\ &-b\int_\Omega w \int_{-\infty}^{+\infty} \tfrac{\eta(\xi) f_3(\xi)}{\xi^2+\beta+1} d\xi dx. \\ &-bse^{-s} A_1 \int_\Omega w \int_0^1 e^{s\tau} f_4(\tau) d\tau dx, \quad w \in H_0^1(\Omega),\end{aligned} \tag{4.22}$$

where

$$\sigma = 1 + a_2 + be^{-s} A_1 > 0, \quad A_1 = \int_{-\infty}^{+\infty} \frac{\eta^2(\xi)}{\xi^2+\beta+1} d\xi.$$

Consequently, problem (4.22) is equivalent to the problem

$$B(y, w) = L(w), \tag{4.23}$$

where the bilinear from $B : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ defined by

$$B(y, w) = \sigma \int_\Omega y w dx + \int_\Omega \nabla y . \nabla w dx$$

and the linear form $L : H_0^1(\Omega) \to \mathbb{R}$ by

$$\begin{aligned}L(w) = \;& \int_\Omega (f_2 + \sigma f_1)\, w dx - b\int_\Omega w \int_{-\infty}^{+\infty} \tfrac{\eta(\xi) f_3(\xi)}{\xi^2+\beta+1} d\xi dx \\ & -bse^{-s} A_1 \int_\Omega w \int_0^1 e^{s\tau} f_4(\tau) d\tau dx.\end{aligned}$$

It is easy to verify that B is continuous and coercive, and L is continuous. Consequently, So applying the Lax-Milgram theorem, we deduce that for all $w \in H_{\Gamma_1}^1(\Omega)$ system (4.23) admits a unique solution $y \in H_{\Gamma_1}^1(\Omega)$. Applying the classical elliptic regularity, it follows from (4.23) that $y \in H_0^2(\Omega)$. Using the second equation of (4.16) and Green's formulla, we get

$$\int_\Omega \left[(1 + a_2)u - \Delta y + b\int_{-\infty}^{+\infty} \phi(\xi)\eta(\xi) d\xi - f_2\right] w = 0, \quad w \in H_0^1(\Omega).$$

Hence,

$$(1 + a_2)u - \Delta y + b\int_{-\infty}^{+\infty} \phi(\xi)\eta(\xi) d\xi = f_2 \in L^2(\Omega).$$

Using the third equation of (4.16), we get

$$\int_\Omega \int_{-\infty}^{+\infty} \left[\phi(\xi) + (\xi^2 + \beta)\phi(\xi) - z(1)\eta(\xi) - f_3(\xi)\right] w d\xi = 0, \quad w \in H_0^1(\Omega).$$

Hence,

$$\phi(\xi) + (\xi^2 + \beta)\phi(\xi) - z(x, 1)\eta(\xi) = f_3(\xi) \in L^2(\Omega \times (-\infty, +\infty)).$$

Therefore,

$$U \in D(A).$$

So, the operator $I + A$ is surjective.

Finally, we demonstrate that $J : \mathcal{H} \to \mathcal{H}$ is locally Lipschitz. For $U \in \mathcal{H}$, we have

$$\begin{aligned} \|J(u) - J(\bar{u})\|^2_{\mathcal{H}} &= \|(0, y|y|^{p-2} - \bar{y}|\bar{y}|^{p-2}, 0, 0)\|^2_{\mathcal{H}} \\ &= \|y|y|^{p-2} - \bar{y}|\bar{y}|^{p-2}\|^2_2 . \end{aligned}$$

It is easilly to verify that

$$\left\| y|y|^{p-2} - \bar{y}|\bar{y}|^{p-2} \right\|^2_2 \le C\|y - \bar{y}\|^2_{H^1_0(\Omega)}.$$

Therefore, J is locally Lipchitz. Thanks to the theorems in Komornik [27] (See also Pazy [41]), the proof is completed. ■

4.4 Global existence

This section is concerned with the proof of the global existence of the solution of problem (P'). We introduce the following functionals:

$$\begin{aligned} I(t) &:= b\int_\Omega \int_{-\infty}^{+\infty} |\phi(x,\xi,t)|^2 d\xi dx + \|\nabla y\|^2_2 \\ &-\|y\|^p_p + \nu s \int_\Omega \int_0^1 |z(x,\rho,t)|^2 d\rho dx \end{aligned} \tag{4.24}$$

and

$$\begin{aligned} J(t) &:= \frac{b}{2}\int_\Omega \int_{-\infty}^{+\infty} |\phi(x,\xi,t)|^2 d\xi dx + \frac{1}{2}\|\nabla y\|^2_2 \\ &-\frac{1}{p}\|y\|^p_p + \nu s \int_\Omega \int_0^1 |z(x,\rho,t)|^2 d\rho dx. \end{aligned} \tag{4.25}$$

Lemma 4.4.1 *Assume that (4.3) and (4.8) hold. Then, for $U_0 \in \mathcal{H}$ satisfying*

$$\begin{cases} \bar{\beta} := C_*^p \left(\frac{2p}{(p-2)} E(0) \right)^{\frac{p-2}{2}} < 1, \\ I(0) > 0, \end{cases} \tag{4.26}$$

we have

$$I(t) > 0, \quad \forall t > 0.$$

Proof Since $I(0) > 0$, then there exists (by continuity of $y(t)$) $T^* < T$ such that $I(t) \ge 0$, for all $t \in [0, T^*]$. This implies:

$$\begin{aligned} \|\nabla y\|^2_2 &\le \frac{2p}{p-2} J(t), \quad \forall t \in [0, T^*) \\ &\le \frac{2p}{p-2} E(0). \end{aligned} \tag{4.27}$$

Using (4.26), (4.27) and the Poincare inequality, we get

$$\begin{aligned} \|y\|_p^p &\le C_*^p \|\nabla y\|_2^p \\ &\le C_*^p \left(\tfrac{2p}{(p-2)} E(0)\right)^{\frac{p-2}{2}} \|\nabla y\|_2^2 \end{aligned} \tag{4.28}$$

Whereupon

$$\begin{aligned} I(t) &= b \int_\Omega \int_{-\infty}^{+\infty} |\phi(x,\xi,t)|^2 d\xi dx + \|\nabla y\|_2^2 \\ &-\|y\|_p^p + \nu s \int_\Omega \int_0^1 |z(x,\rho,t)|^2 d\rho dx > 0, \quad \forall t \in [0, T^*]. \end{aligned}$$

By repeating this procedure and using the fact that

$$\lim_{t \to T^*} C_*^p \left(\frac{2p}{(p-2)} E(0)\right)^{\frac{p-2}{2}} < 1,$$

we can take $T^* = T$. ■

We are now ready to prove our global existence result.

Theorem 4.4.1 *Assume that (4.7), (4.8) hold and $U_0 \in D(A)$ satisfying (4.26). Then the solution of system (P') is bounded and global.*

Proof Using (4.10), we have

$$\begin{aligned} E(0) &\ge E(t) = \tfrac{1}{2}\|y_t\|_2^2 + J(t) \\ &\ge \tfrac{1}{2}\|y_t\|_2^2 + \tfrac{(p-2)}{2p}\|\nabla y\|_2^2. \end{aligned}$$

Therefore,

$$\|y_t\|_2^2 + \|\nabla y\|_2^2 \le \xi_1 E(0),$$

where ξ_1 is a positive constant, which depends only on the parameter p. ■

4.5 Decay of solutions

In order to establish the energy decay result. For $N > 0$ and $\epsilon_1 > 0$, we define a perturbed modified energy by

$$L(t) := NE(t) + \epsilon_1 K_1(t) + K_2(t), \tag{4.29}$$

where

$$K_1(t) := \int_\Omega y_t y dx + \frac{b}{2} \int_\Omega \int_{-\infty}^{+\infty} (\xi^2 + \beta) |M(x,\xi,t)|^2 d\xi dx, \tag{4.30}$$

$$K_2(t) := s \int_\Omega \int_0^1 e^{-s\rho} |z(x,\rho,t)|^2 d\rho dx, \tag{4.31}$$

where

$$M(x,\xi,t) := \int_0^t \phi(x,\xi,\tau) d\tau - \frac{s\eta(\xi)}{\xi^2+\beta} \int_0^1 f_0(x,-\rho s) d\rho + \frac{y_0(x)\eta(\xi)}{\xi^2+\beta}.$$

Lemma 4.5.1 *Let (y, ϕ, z) be a regular solution of the problem (P'), then*

$$\begin{aligned}
&\int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\beta)\phi(x,\xi,t)M(x,\xi,t)d\xi dx \\
&= \int_\Omega y(x,t) \int_{-\infty}^{+\infty} \phi(x,\xi,t)\eta(\xi)d\xi dx \\
&-s \int_\Omega \int_0^1 z(x,\rho,t) \int_{-\infty}^{+\infty} \eta(\xi)\phi(x,\xi,t)d\xi d\rho dx \\
&- \int_\Omega \int_{-\infty}^{+\infty} |\phi(x,\xi,t)|^2 d\xi dx.
\end{aligned} \tag{4.32}$$

Proof Using the second equation in (P'), to obtain

$$\begin{aligned}
(\xi^2+\beta)\phi(x,\xi,t) &= z(x,1,t)\eta(\xi) - \phi_t(x,\xi,t) \\
&= \eta(\xi)\left[z(x,1,t) - z(x,0,t)\right] \\
&+ y_t(x,t)\eta(\xi) - \phi_t(x,\xi,t).
\end{aligned}$$

Observe that

$$-s\int_0^1 z_t(x,\rho,t)d\rho = \int_0^1 z_\rho(x,\rho,t)d\rho = z(x,1,t) - z(x,0,t).$$

Whereupon

$$\begin{aligned}
(\xi^2+\beta)\phi(x,\xi,t) = &-s\eta(\xi)\int_0^1 z_t(x,\rho,t)d\rho \\
&+ y_t(x,t)\eta(\xi) - \phi_t(x,\xi,t).
\end{aligned}$$

Integrating the last equality over $[0,t]$, to get

$$\begin{aligned}
\int_0^t (\xi^2+\beta)\phi(x,\xi,\tau)d\tau = &-s\eta(\xi)\int_0^1 z(x,\rho,t)d\rho \\
&+ s\eta(\xi)\int_0^1 f_0(x,-\rho s)d\rho \\
&+ y(x,t)\eta(\xi) - y_0(x)\eta(\xi) - \phi(x,\xi,t).
\end{aligned}$$

Therefore,

$$\begin{aligned}
(\xi^2+\beta)M(x,\xi,t) = &-s\eta(\xi)\int_0^1 z(x,\rho,t)d\rho \\
&+ y(x,t)\eta(\xi) - \phi(x,\xi,t).
\end{aligned} \tag{4.33}$$

Multiplying (4.33) by ϕ and integrating over $\Omega \times (-\infty, +\infty)$, we obtain (4.32) . ■

Lemma 4.5.2 *Let (y, ϕ, z) be a regular solution of the problem (P'), then*

$$\begin{aligned}
&\left| \int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\beta)\left|M(x,\xi,t)\right|^2 d\xi dx \right| \\
&\leq 3s^2 A_0 \int_\Omega \int_0^1 |z(x,\rho,t)|^2 d\rho dx + 3A_0 C_*^2 \|\nabla y\|_2^2 \\
&+ \frac{3}{\beta} \int_\Omega \int_{-\infty}^{+\infty} |\phi(x,\xi,t)|^2 d\xi dx.
\end{aligned} \tag{4.34}$$

Proof Invoking (4.33), to obtain

$$\begin{aligned}
&\left|\int_\Omega \int_{-\infty}^{+\infty}(\xi^2+\beta)|M(x,\xi,t)|^2 d\xi dx\right| \\
&\le s^2 A_0 \int_\Omega \left(\int_0^1 z(x,\rho,t)d\rho\right)^2 dx \\
&+A_0\|y\|_2^2 + \int_\Omega \int_{-\infty}^{+\infty} \tfrac{|\phi(x,\xi,t)|^2}{\xi^2+\beta} d\xi dx \\
&+2\int_\Omega \int_{-\infty}^{+\infty} \tfrac{|\phi(x,\xi,t)y(x,t)\eta(\xi)|}{\xi^2+\beta} d\xi dx \\
&+2sA_0 \int_\Omega \left|y(x,t)\int_0^1 z(x,\rho,t)d\rho\right| dx \\
&+2s\int_\Omega \int_{-\infty}^{+\infty} \tfrac{|\phi(x,\xi,t)\eta(\xi)\int_0^1 z(x,\rho,t)d\rho|}{\xi^2+\beta} d\xi dx.
\end{aligned} \tag{4.35}$$

Now, we will estimate the right hand side of (4.35). First by Hölder's inequality, we get

$$\int_0^1 z(x,\rho,t)d\rho \le \left(\int_0^1 |z(x,\rho,t)|^2 d\rho\right)^{\frac{1}{2}}. \tag{4.36}$$

For the fourth and fifth terms, we use Young's inequality to obtain

$$\begin{aligned}
&\int_\Omega \int_{-\infty}^{+\infty} \tfrac{|\phi(x,\xi,t)y(x,t)\eta(\xi)|}{\xi^2+\beta} d\xi dx \\
\le\ & \tfrac{A_0}{2}\|y\|_2^2 + \tfrac{1}{2}\int_\Omega \int_{-\infty}^{+\infty} \tfrac{|\phi(x,\xi,t)|^2}{\xi^2+\beta} d\xi dx
\end{aligned} \tag{4.37}$$

and

$$s\int_\Omega \left|y(x,t)\int_0^1 z(x,\rho,t)d\rho\right| dx \le \frac{s^2}{2}\int_\Omega \int_0^1 |z(x,\rho,t)|^2 d\rho dx + \frac{1}{2}\|y\|_2^2. \tag{4.38}$$

For the last term, we use Young's inequality, (4.36) and Lemma 4.2.3 to get

$$\begin{aligned}
&s\int_\Omega \int_{-\infty}^{+\infty} \tfrac{|\phi(x,\xi,t)\eta(\xi)\int_0^1 z(x,\rho,t)d\rho|}{\xi^2+\beta} d\xi dx \\
\le\ & \tfrac{s^2 A_0}{2}\int_\Omega \int_0^1 |z(x,\rho,t)|^2 d\rho dx \\
&+\tfrac{1}{2}\int_\Omega \int_{-\infty}^{+\infty} \tfrac{|\phi(x,\xi,t)|^2}{\xi^2+\beta} d\xi dx.
\end{aligned} \tag{4.39}$$

Consequently, we arrive at

$$\begin{aligned}
&\left|\int_\Omega \int_{-\infty}^{+\infty}(\xi^2+\beta)|M(x,\xi,t)|^2 d\xi dx\right| \\
&\le 3s^2\int_\Omega \int_0^1 |z(x,\rho,t)|^2 d\rho dx + 3A_0\|y\|_2^2 \\
&+3\int_\Omega \int_{-\infty}^{+\infty} \tfrac{|\phi(x,\xi,t)|^2}{\xi^2+\beta} d\xi dx.
\end{aligned} \tag{4.40}$$

Using the fact that $\frac{1}{\xi^2+\beta} \leq \frac{1}{\beta}$ and Poincaré's inequality, then (4.34) is established. ■

Lemma 4.5.3 *For ϵ_1 small and N large enough, we have*

$$\frac{N}{2}E(t) \leq L(t) \leq 2NE(t), \quad \forall t \geq 0. \tag{4.41}$$

Proof Using Young's inequality and Poincaré's inequality, we obtain

$$\begin{aligned} L(t) \leq \quad & NE(t) + \tfrac{\epsilon_1}{2}\|y_t\|_2^2 + s\int_\Omega \int_0^1 e^{-s\rho}|z(x,\rho,t)|^2 d\rho dx \\ & + \tfrac{\epsilon_1 C_*^2}{2}\|\nabla y\|_2^2 + \tfrac{b\epsilon_1}{2}\int_\Omega \int_{-\infty}^{+\infty}(\xi^2+\beta)|M(x,\xi,t)|^2 d\xi dx. \end{aligned}$$

Using (4.6) and Lemma 4.5.2, we get

$$\begin{aligned} L(t) \leq \quad & \tfrac{1}{2}\left\{N+\epsilon_1\right\}\|y_t\|_2^2 - \tfrac{N}{p}\|y\|_p^p \\ & + s\left\{N\nu + \tfrac{3\nu s\epsilon_1}{2}\right\}\int_\Omega \int_0^1 |z(x,\rho,t)|^2 d\rho dx \\ & + s\int_0^1 e^{-s\rho}|z(x,\rho,t)|^2 d\rho dx \\ & + \tfrac{1}{2}\left(N + \epsilon_1 C_*^2\left\{1+3v\right\}\right)\|\nabla y\|_2^2 \\ & + \tfrac{b}{2}\left(N + \tfrac{3\epsilon_1}{\beta}\right)\int_\Omega \int_{-\infty}^{+\infty}|\phi(\xi,t)|^2 d\xi dx. \end{aligned}$$

So, by using (4.24), we get

$$\begin{aligned} 2NE(t) - L(t) \quad & \geq \tfrac{1}{2}\left\{N-\epsilon_1\right\}\|y_t\|_2^2 + \tfrac{N}{p}I(t) \\ & + s\left\{\tfrac{(p-1)\nu N}{p} - 1 - \tfrac{3s\nu\epsilon_1}{2}\right\}\int_\Omega \int_0^1 |z(x,\rho,t)|^2 d\rho dx \\ & + \tfrac{1}{2}\left\{\tfrac{(p-2)N}{p} - \epsilon_1 C_*^2\left\{1+3\nu\right\}\right\}\|\nabla y\|_2^2 \\ & + \tfrac{b}{2}\left\{\tfrac{(p-2)N}{p} - \tfrac{3\epsilon_1}{\beta}\right\}\int_\Omega \int_{-\infty}^{+\infty}|\phi(\xi,t)|^2 d\xi dx. \end{aligned}$$

Similarly, we have

$$\begin{aligned} L(t) - \tfrac{N}{2}E(t) \quad & \geq \tfrac{1}{2}\left\{\tfrac{N}{2}-\epsilon_1\right\}\|y_t\|_2^2 + \tfrac{N}{2p}I(t) \\ & + s\left\{\tfrac{(p-1)\nu N}{2p} - e^{-s} - \tfrac{3s\nu\epsilon_1}{2}\right\}\int_\Omega \int_0^1 |z(x,\rho,t)|^2 d\rho dx \\ & + \tfrac{1}{2}\left\{\tfrac{(p-2)N}{2p} - \epsilon_1 C_*^2\left\{1+3\nu\right\}\right\}\|\nabla y\|_2^2 \\ & + \tfrac{b}{2}\left\{\tfrac{(p-2)N}{2p} - \tfrac{3\epsilon_1}{\beta}\right\}\int_\Omega \int_{-\infty}^{+\infty}|\phi(x,\xi,t)|^2 d\xi dx. \end{aligned}$$

By fixing ϵ_1 small and N large enough, we obtain $L(t)-\frac{N}{2}E(t)\geq 0$ and $2NE(t)-L(t)\geq 0$. The proof is completed. ■

Lemma 4.5.4 *Assume that (4.3) and (4.8) hold. Then, the functional K_1 defined by (4.30) satisfies*

$$\begin{aligned} K_1'(t) &\leq C_1\|y_t\|_2^2-\tfrac{1}{2}\|\nabla y\|_2^2-b\int_\Omega\int_{-\infty}^{+\infty}|\phi(x,\xi,t)|^2d\xi dx\\ &+\tfrac{b}{4}\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)|\phi(x,\xi,t)|^2d\xi dx\\ &+s^2\nu\int_\Omega\int_0^1|z(x,\rho,t)|^2d\rho dx+\|y\|_p^p, \end{aligned} \tag{4.42}$$

for some positive constant C_1.

Proof A direct differentiation of K_1 using Lemma 4.5.1, gives

$$\begin{aligned} K_1'(t) &= \|y_t\|_2^2+\int_\Omega y_{tt}y dx+b\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)\phi(x,\xi,t)M(x,\xi,t)d\xi dx\\ &= \|y_t\|_2^2-\|\nabla y\|_2^2-bs\int_\Omega\int_0^1 z(x,\rho,t)\int_{-\infty}^{+\infty}\eta(\xi)\phi(x,\xi,t)d\xi d\rho dx\\ &+\|y\|_p^p-b\int_\Omega\int_{-\infty}^{+\infty}|\phi(x,\xi,t)|^2d\xi dx-a_2\int_\Omega y_t y dx. \end{aligned} \tag{4.43}$$

Using Young's inequality and Lemma 4.2.3, we get

$$\begin{aligned} K_1'(t)\leq\ & (1+\eta_1 a_2)\|y_t\|_2^2-\left(1-\frac{a_2C_*^2}{4\eta_1}\right)\|\nabla y\|_2^2\\ &+\tfrac{b}{4}\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)|\phi(x,\xi,t)|^2d\xi dx\\ &+s^2\nu\int_\Omega\int_0^1|z(x,\rho,t)|^2d\rho dx+\|y\|_p^p\\ &-b\int_\Omega\int_{-\infty}^{+\infty}|\phi(x,\xi,t)|^2d\xi dx. \end{aligned}$$

By choosing $\eta_1:=\frac{a_2C_*^2}{2}$, then (4.42) is established. ■

Lemma 4.5.5 *Assume that (4.3) and (4.8) hold. Then the functional K_2 defined by (4.31) satisfies*

$$K_2'(t)\leq -se^{-s}\int_\Omega\int_0^1|z(x,\rho,t)|^2d\rho dx+\|y_t\|_2^2. \tag{4.44}$$

Proof By taking a time derivative of K_2 and using the third equation in (P'), we arrive at

$$\begin{aligned} K_2'(t)=\ & -2s\int_\Omega\int_0^1 e^{-s\rho}z(x,\rho,t)z_t(x,\rho,t)d\rho dx\\ &= -2\int_\Omega\int_0^1 e^{-s\rho}z(x,\rho,t)z_\rho(x,\rho,t)d\rho dx\\ &= -\int_\Omega\int_0^1\frac{d}{d\rho}\left[e^{-s\rho}|z(x,\rho,t)|^2\right]d\rho dx-s\int_\Omega\int_0^1 e^{-s\rho}|z(x,\rho,t)|^2d\rho dx\\ &= -s\int_\Omega\int_0^1 e^{-s\rho}|z(x,\rho,t)|^2d\rho dx-e^{-s}\int_\Omega|z(x,1,t)|^2dx+\|y_t\|_2^2. \end{aligned} \tag{4.45}$$

Then (4.44) is established. ■

Theorem 4.5.1 *Assume that (4.3), (4.8) hold and $U_0 \in \mathcal{H}$ satisfying (4.26). Then any solution of (P') satisfies*

$$E(t) \leq Ke^{-wt}, \quad t \geq 0, \tag{4.46}$$

for some positive constants K and w independent of t.

Proof By using (4.42) and (4.44), we get, for all $t \geq 0$,

$$\begin{aligned} L'(t) \leq \; & -(NC - C_1\epsilon_1 - 1)\|y_t\|_2^2 - \tfrac{\epsilon_1}{2}\|\nabla y\|_2^2 \\ & +\epsilon_1\|y\|_p^p - b\epsilon_1 \int_\Omega \int_{-\infty}^{+\infty} |\phi(x,\xi,t)|^2 d\xi dx \\ & -\tfrac{b}{2}\left\{N - \tfrac{\epsilon_1}{2}\right\} \int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\beta)|\phi(x,\xi,t)|^2 d\xi dx \\ & -s(e^{-s} - \nu s\epsilon_1) \int_\Omega \int_0^1 |z(x,\rho,t)|^2 d\rho dx. \end{aligned} \tag{4.47}$$

At this point, we choose ϵ_1 small enough such that

$$e^{-s} - \nu s\epsilon_1 > 0,$$

then pick N large enough such that

$$N > max\left\{\frac{C_1\epsilon_1 + 1}{C}, \frac{\epsilon_1}{2}\right\}.$$

Consequently, from the above, we deduce that there exists a positive constant m such that (4.47) becomes

$$L'(t) \leq -mE(t), \quad \text{for all } t \geq 0. \tag{4.48}$$

By using (4.41), we have

$$L'(t) \leq -wL(t), \quad \text{for all } t \geq 0. \tag{4.49}$$

A simple integration of (4.49) over $(0,t)$ leads to

$$L(t) \leq L(0)e^{-wt}, \quad t \geq 0.$$

As $L(t)$ and $E(t)$ are equivalent, we obtain

$$E(t) \leq ke^{-wt}, \quad t \geq 0. \tag{4.50}$$

■

4.6 blow-up result

In this section, we consider the property of blowing up of the solution of problem (P'). Let (y,ϕ,z) be a solution of (P') and define

$$\begin{aligned} H(t) := -E(t) = & \ \tfrac{1}{p}\|y\|_p^p - \tfrac{1}{2}\|y_t\|_2^2 - \tfrac{b}{2}\int_\Omega\int_{-\infty}^{+\infty}|\phi(x,\xi,t)|^2 d\xi dx \\ & -\tfrac{1}{2}\|\nabla y\|_2^2 - \nu s\int_\Omega\int_0^1 |z(x,\rho,t)|^2 d\rho dx. \end{aligned} \tag{4.51}$$

Lemma 4.6.1 *Suppose that (4.8) holds. Then there exists a positive constant $C_2 > 1$, depending on Ω only, such that*

$$\|y\|_p^l \le C_2\left[\|y\|_p^p + \|\nabla y\|_2^2\right]$$

for any $y \in H_0^1(\Omega)$ and $2 \le l \le p$.

Proof If $\|y\|_p \le 1$ then $\|y\|_p^l \le \|y\|_p^2 \le C_*\|\nabla y\|_2^2$ by Sobolev embedding theorems. If $\|y\|_p \ge 1$ then $\|y\|_p^l \le \|y\|_p^p$. ■

Therefore the result follows.

Theorem 4.6.1 *Let the conditions of the Theorem 4.3.1 be fulfilled. Assume further that*

$$E(0) = \frac{1}{p}\|y_0\|_p^p - \frac{1}{2}\|y_1\|_2^2 - \frac{1}{2}\|\nabla y_0\|_2^2 - \nu s\int_\Omega\int_0^1 |f_0(x,-\rho s)|^2 d\rho dx < 0. \tag{4.52}$$

Then the solution of system (P') blows up in finite time.

Proof From (4.51), we have

$$H'(t) = -E'(t) \ge \frac{b}{2}\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)|\phi(x,\xi,t)|^2 d\xi dx \ge 0, \tag{4.53}$$

hence

$$0 < H(0) \le H(t) \le \frac{1}{p}\|y\|_p^p. \tag{4.54}$$

We then define

$$A(t) = H^{1-\gamma}(t) + \epsilon\int_\Omega y y_t dx + \frac{a_2\epsilon}{2}\|y\|_2^2 \tag{4.55}$$

for $\epsilon > 0$ small to be chosen later and

$$0 < \gamma < \frac{p-2}{2p}. \tag{4.56}$$

By taking a derivative of (4.55), using Eq. (P') and Using Young's inequality, we obtain for $\delta > 0$,

$$\begin{aligned} & -\int_\Omega y\int_{-\infty}^{+\infty}\eta(\xi)\phi(x,\xi,t)d\xi dx \\ \ge & \ -\delta A_0\|y\|_2^2 - \tfrac{1}{4\delta}\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)|\phi(x,\xi,t)|^2 d\xi dx, \end{aligned} \tag{4.57}$$

which yields, by substitution in (4.57),

$$\begin{aligned} A'(t) \geq & \quad (1-\gamma)H^{-\gamma}H'(t) + \epsilon\|y_t\|_2^2 - \epsilon\|\nabla y\|_2^2 - \delta\nu\epsilon\|y\|_2^2 \\ & -\tfrac{b\epsilon}{4\delta}\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)|\phi(x,\xi,t)|^2 d\xi dx + \epsilon\|y\|_p^p. \end{aligned} \tag{4.58}$$

Using (4.53), we have

$$\begin{aligned} A'(t) \geq & \quad \left[(1-\gamma)H^{-\gamma} - \tfrac{\epsilon}{2\delta}\right] H'(t) \\ & +\epsilon\|y_t\|_2^2 - \epsilon\|\nabla y\|_2^2 - \delta\nu\epsilon\|y\|_2^2 + \epsilon\|y\|_p^p. \end{aligned} \tag{4.59}$$

Therefore by taking δ so that $\frac{1}{2\delta} = kH^{-\gamma}(t)$, for large k to be specified later, and substituting in (4.59) we arrive at

$$\begin{aligned} A'(t) \geq & \quad \left[(1-\gamma) - \epsilon k\right] H^{-\gamma}(t)H'(t) + \epsilon\|y_t\|_2^2 \\ & -\epsilon\|\nabla y\|_2^2 - \tfrac{\nu\epsilon}{2k}H^{\gamma}(t)\|y\|_2^2 + \epsilon\|y\|_p^p. \end{aligned} \tag{4.60}$$

Consequently, using (4.51), we have for some $0 < r < 1$

$$\begin{aligned} A'(t) \geq & \quad \left[(1-\gamma) - \epsilon k\right] H^{-\gamma}(t)H'(t) + \epsilon\tfrac{p(1-r)+2}{2}\|y_t\|_2^2 \\ & +\epsilon\tfrac{p(1-r)-2}{2}\|\nabla y\|_2^2 - \tfrac{\nu\epsilon}{2k}\|y\|_2^2 H^{\gamma}(t) + \epsilon r\|y\|_p^p \\ & +p(1-r)\epsilon H(t) + \epsilon\tfrac{p(1-r)b}{2}\int_\Omega\int_{-\infty}^{+\infty}|\phi(x,\xi,t)|^2 d\xi dx \\ & +\epsilon p(1-r)\nu s\int_\Omega\int_0^1 |z(x,\rho,t)|^2 d\rho dx. \end{aligned} \tag{4.61}$$

By exploiting (4.54) and the inequality $\|y\|_2 \leq C_*\|y\|_p$, we obtain

$$H^{\gamma}(t)\|y\|_2^2 \leq \left(\frac{1}{p}\right)^{\gamma}\|y\|_p^{p\gamma}\|y\|_2^2 \leq C_3\|y\|_2^{p\gamma+2}.$$

Exploiting (4.56), we have

$$2 < p\gamma + 2 \leq p.$$

So, Lemma 4.6.1 yields

$$H^{\gamma}(t)\|y\|_2^2 \leq C_4\left[\|\nabla y\|_2^2 + \|y\|_p^p\right]. \tag{4.62}$$

Inserting (4.62) in (4.61), we obtain

$$\begin{aligned} A'(t) \geq & \quad \left[(1-\gamma) - \epsilon k\right] H^{-\gamma}(t)H'(t) + \epsilon\tfrac{p(1-r)+2}{2}\|y_t\|_2^2 \\ & +\epsilon\left[\tfrac{p(1-r)-2}{2} - \tfrac{C_4\nu}{2k}\right]\|\nabla y\|_2^2 + \epsilon\left[r - \tfrac{C_4\nu}{2k}\right]\|y\|_p^p \\ & +p(1-r)\epsilon H(t) + \epsilon\tfrac{p(1-r)b}{2}\int_\Omega\int_{-\infty}^{+\infty}|\phi(x,\xi,t)|^2 d\xi dx \\ & +\epsilon p(1-r)\nu s\int_\Omega\int_0^1 |z(x,\rho,t)|^2 d\rho dx. \end{aligned} \tag{4.63}$$

At this point, we choose r small enough such that

$$p(1-r)-2>0$$

and k large enough such that

$$r-\frac{C_4\nu}{2k}>0, \quad \frac{p(1-r)-2}{2}-\frac{C_4\nu}{2k}>0.$$

When r and k are fixed, we pick ϵ small enough such that

$$(1-\gamma)-\epsilon k>0, \quad H(0)+\epsilon\int_\Omega y_0y_1dx>0.$$

So, (4.63) becomes, for some $C_5>0$

$$\begin{aligned} A'(t)\geq\ & C_5\left[H(t)+\|y_t\|_2^2+\|\nabla y\|_2^2+\|y\|_p^p\right.\\ & \left.+b\int_\Omega\int_{-\infty}^{+\infty}|\phi(x,\xi,t)|^2d\xi dx+\nu s\int_\Omega\int_0^1|z(x,\rho,t)|^2d\rho dx\right] \end{aligned} \tag{4.64}$$

and

$$A(t)\geq A(0)>0, \quad t\geq 0. \tag{4.65}$$

On the other hand, by Hölder's inequality and the embedding $L\|y\|_2\leq C_*\|y\|_p$, we get

$$\int_\Omega yy_tdx\leq\|y\|_2\|y_t\|_2\leq C_*\|y\|_p\|y_t\|_2.$$

Using Young's inequality and Lemma 4.6.1, we have

$$\begin{aligned} \left(\int_\Omega yy_tdx\right)^{\frac{1}{1-\gamma}} &\leq C_6\left[\|y\|_p^l+\|y_t\|_2^2\right]\\ &\leq C_7\left[\|y_t\|_2^2+\|\nabla y\|_2^2+\|y\|_p^p\right], \end{aligned} \tag{4.66}$$

where C_6, C_7 are positive constants and $2\leq l=\frac{2}{1-2\gamma}\leq p$.
Therefore,

$$\begin{aligned} A^{\frac{1}{1-\gamma}}(t) &\leq C\left[H(t)+\left(\int_\Omega yy_tdx\right)^{\frac{1}{1-\gamma}}+\|y\|_2^{\frac{2}{1-\gamma}}\right]\\ &\leq C_8\left[H(t)+\|y_t\|_2^2+\|\nabla y\|_2^2+\|y\|_p^p\right], \quad t\geq 0, \end{aligned} \tag{4.67}$$

where C_8 is a positive constant. Combining (4.66) and (4.67), we arrive at

$$A'(t)\geq C_9A^{\frac{1}{1-\gamma}}(t), \quad t\geq 0. \tag{4.68}$$

A simple integration of (4.68) over $(0,t)$, we get

$$A(t)\geq\left[A^{\frac{-\gamma}{1-\gamma}}(0)-\frac{\gamma}{1-\gamma}C_9t\right]^{-\frac{1-\gamma}{\gamma}}. \tag{4.69}$$

So, $A(t)$ blows up in time

$$T\leq T^*=\frac{1-\gamma}{C_9\gamma A^{\frac{\gamma}{1-\gamma}}(0)}.$$

This completes the proof. ■

Chapter 5

Blow-up of solution for elastic membrane equation with fractional boundary damping

5.1 Introduction

Partial differential equations with a fractional derivatives have attracted the attention of many researchers in mathematical, biological and physical fields ([34, 42, 46]). In the recent years, they have been widely applied in electronics , relaxation vibrations and viscoelasticity etc.e.g.,[18, 33].

In this paper, we consider the following Kirchhoff equation with Balakrishnan-Taylor damping, fractional boundary condition and source terms:

$$\begin{cases} u_{tt} - (\xi_0 + \xi_1\|\nabla u\|_2^2 + \xi_2(\nabla u, \nabla u_t))\,\Delta u = |u|^{p-1}u, & x \in \Omega, t > 0, \\ (\xi_0 + \xi_1\|\nabla u\|_2^2 + \xi_2(\nabla u, \nabla u_t))\,\frac{\partial u}{\partial \nu} = -b\partial_t^{\alpha,\eta}u, & x \in \Gamma_0, t > 0, \\ u = 0, & x \in \Gamma_1, t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{5.1}$$

where Ω is a regular and bounded domain in $\mathbb{R}^n$, $(n \geq 1)$ with smooth boundary $\partial\Omega$ such that $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$ and Γ_0, Γ_1 have positive measure. $\partial\nu$ denotes the unit outer normal and $(.,.)$ the inner product with its corresponding norm$\|.\|_2$. The functions $u(x,t)$ is the plate transverse displacement. The viscoelastic structural damping terms $\xi_0+\xi_1\|\nabla u\|_2^2+\xi_2(\nabla u, \nabla u_t)$ is the nonlinear stiffness of the membrane. ξ_0, ξ_1, ξ_2 and b are positive constants. The initial data (u_0, u_1) are given functions. From the physical point of view, problem (5.1) is related to the panel flutter equation and to the spillover problem. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative (see [10] and [11]) defined by the following formula:

$$\partial_t^{\alpha,\eta}u(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}e^{-\eta(t-s)}u_s(s)ds, \quad \eta \geq 0,$$

where Γ is the usual Euler gamma function and $0 < \alpha < 1$. We recall some results related to Kirchhoff equation with Balakrishnan-Taylor damping

$$u_{tt} - \left(\xi_0 + \xi_1 \|\nabla u\|_2^2 + \xi_2(\nabla u, \nabla u_t)\right) \Delta u + \int_0^t h(t-s)\Delta u ds = |u|^p u.$$

In [54], Tatar and Zaraï considered the above equation and proved the global existence and polynomial decay of the problem. Exponential decay and blow up of solution to the problem were established in Tatar and Zaraï [45]. Park [40] studied the homogeneous case, and established a general decay result of the problem without imposing the usual relation between the relaxation function h and its derivative. Recently, Ha [20] proved a general decay result of energy without imposing any restrictive growth assumption on the damping term and weakening the usual assumptions on the relaxation function. Balakrishnan-Taylor damping $\xi_2(\nabla u, \nabla u_t)$, was initially proposed by Balakrishnan and Taylor in 1989 [6] and Bass and Zes [8]. For more results concerning Kirchhoff equation with Balakrishnan-Taylor damping, one can refer to Clark [12], Ha[20, 21], Tatar and Zaraï [54, 55, 53], Wu [48, 49, 50] and You [51].

Since very little attention has been paid to boundary condition of fractional derivative type with source term, motivated by above scenario, we prove under suitable conditions on the initial data that the nonlinear source of polynomial type is able to force solutions to blow-up in finite time. Here, three different cases on the sign of the initial energy are considered.

In [38], by redescribing the fractional boundary condition by means of a suitable diffusion equation, we can be transformed the problem (5.1) into an augmented model which can be easily tackled.

In the present paper, we shall consider the problem (5.1). Under a suitable condition on the initial data, we give a several results concerning the blow up results to problem (5.1) for both positive and nonpositive initial energy.

The paper is organized as follows. In Section 2, we Present the preliminaries and some lemmas.In section 3, we prove the blow-up of solutions on different cases of the sing of the initial energy.

5.2 Preliminaries

In this section, we provide some materails for the proof of our results.

Lemma 5.2.1 *[38] Let μ be the function:*

$$\mu(\xi) := |\xi|^{\frac{(2\alpha-1)}{2}}, \quad \xi \in \mathbb{R}, \quad 0 < \alpha < 1.$$

Then the relationship between the "input" U and the "output" O of the system

$$\begin{cases} \partial_t\phi(\xi,t)+(\xi^2+\eta)\phi(\xi,t)-U(x,t)\mu(\xi)=0, \quad \xi\in\mathbb{R}, t>0, \eta\geq 0,\\ \phi(\xi,0)=0,\\ O(t):=(\pi)^{-1}\sin(\alpha\pi)\int_{-\infty}^{+\infty}\phi(\xi,t)\mu(\xi)d\xi \end{cases}$$

is given by

$$O:=I^{1-\alpha,\eta}U,$$

where

$$I^{\alpha,\eta}u(t):=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}e^{-\eta(t-s)}u(s)ds.$$

Definition 5.2.1 *A solution u of (P) is called a blow-up solution, if there exists a finit time $T^*>0$, such that*

$$\lim_{t\to T^{*-}}\left(\|\nabla u(t)\|_2^2\right)^{-1}=0.$$

Lemma 5.2.2 *[32] Let $\delta>0$ and $B\in C^2(0,\infty)$ be a nonnegative function satisfying*

$$B''(t)-4(\delta+1)B'(t)+4(\delta+1)B(t)\geq 0.$$

If

$$B'(0)>r_2B(0)+K_0, \text{ with } r_2:=2(\delta+1)-2\sqrt{(\delta+1)\delta},$$

then

$$B'(t)>K_0 \text{ for } t>0, \text{where } K_0 \text{ is a constant.}$$

Lemma 5.2.3 *[32] If J is a nonincreasing function on $[t_0,\infty)$ and satisfies the differential inequality*

$$J'(t)^2\geq a+bJ(t)^{2+\frac{1}{\delta}}, \quad t\geq t_0, \tag{5.2}$$

where $a>0$, $b\in\mathbb{R}$, then there exists a finite time T^* such that

$$\lim_{t\to T^{*-}}J(t)=0. \tag{5.3}$$

and T^* is such that:
(i) *If $b<0$, then*

$$T^*\leq t_0+\frac{1}{\sqrt{-b}}\ln\frac{\sqrt{\frac{a}{-b}}}{\sqrt{\frac{a}{-b}}-J(t_0)}.$$

(ii) *If $b=0$, then*

$$T^*\leq t_0+\frac{J(t_0)}{J'(t_0)}.$$

(iii) *If $b > 0$, then*

$$T^* \leq \frac{J(t_0)}{\sqrt{a}}$$

or

$$T^* \leq t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\delta c}{\sqrt{a}} \left(1 - [1 + cJ(t_0)]^{\frac{1}{2\delta}}\right),$$

where $c = \left(\frac{b}{a}\right)^{\delta/(2+\delta)}$.

5.3 Blow up of solution

In this section, we use the method in [32] to consider the property of bolwing up of the solution of problem (5.1). By using Lemma 5.2.1, system (5.1) can be rewritten as :

$$\begin{cases} u_{tt} - (\xi_0 + \xi_1 \|\nabla u\|_2^2 + \xi_2(\nabla u, \nabla u_t))\, \Delta u = |u|^{p-1}u, \\ \partial_t \phi(\xi, t) + (\xi^2 + \eta)\phi(\xi, t) - u_t(x, t)\mu(\xi) = 0, \\ (\xi_0 + \xi_1 \|\nabla u\|_2^2 + \xi_2(\nabla u, \nabla u_t))\, \frac{\partial u}{\partial \nu} = -b_1 \int_{-\infty}^{+\infty} \phi(\xi, t)\mu(\xi) d\xi, \quad , \\ u = 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \\ \phi(\xi, 0) = 0, \end{cases} \tag{5.4}$$

where $b_1 = (\pi)^{-1} \sin(\alpha\pi) b$. The energy functional is then given by:

$$\begin{aligned} E(t) \quad &:= \tfrac{1}{2}\|u_t\|_2^2 + \tfrac{\xi_0}{2}\|\nabla u\|_2^2 + \tfrac{\xi_1}{4}\|\nabla u\|_2^4 \\ &-\tfrac{1}{p+1}\|u\|_{p+1}^{p+1} + \tfrac{b_1}{2}\int_{\Gamma_0}\int_{-\infty}^{+\infty} |\phi(\xi, t)|^2 d\xi d\rho. \end{aligned} \tag{5.5}$$

Lemma 5.3.1 *Let (u, ϕ) be a regular solution of the problem (5.4). Then, the energy functional defined by (5.5) satisfies*

$$\begin{aligned} \tfrac{dE(t)}{dt} \quad &= -b_1 \int_{\Gamma_0}\int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi(\xi, t)|^2 d\xi d\rho \\ &-\tfrac{\xi_2}{4}\left(\tfrac{d}{dt}\|\nabla u\|_2^2\right)^2 \leq 0. \end{aligned} \tag{5.6}$$

Proof Multiplying the first equation in (5.4) by u_t, integrating over Ω and using integration by parts, we get

$$\frac{1}{2}\|u_t\|_2^2 - \left(\xi_0 + \xi_1\|\nabla u\|_2^2 + \xi_2(\nabla u, \nabla u_t)\right) \int_\Omega \Delta u u_t dx = \int_\Omega |u|^{p-1} u u_t dx.$$

Then

$$\begin{aligned} \frac{d}{dt} & \left[\frac{1}{2}\|u_t\|_2^2 + \frac{\xi_0}{2}\|\nabla u\|_2^2 + \frac{\xi_1}{4}\|\nabla u\|_2^4 - \frac{1}{p+1}\|u\|_{p+1}^{p+1}\right] \\ & +b_1 \int_{\Gamma_0} u_t(x,t) \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi,t) d\xi d\rho + \frac{\xi_2}{4}\left(\frac{d}{dt}\|\nabla u\|_2^2\right)^2 = 0. \end{aligned} \tag{5.7}$$

Now multiplying the second equation in (5.4) by $b_1\phi$ and integrating over $\Gamma_0 \times (-\infty, +\infty)$, we get

$$\begin{aligned} \frac{b_1}{2} & \frac{d}{dt} \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi(\xi,t)|^2 d\xi d\rho + b_1 \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi,t)|^2 d\xi d\rho \\ & -b_1 \int_{\Gamma_0} u_t(x,t) \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi,t) d\xi d\rho = 0. \end{aligned} \tag{5.8}$$

By combining (5.5), (5.7) and (5.8), we get (5.6). The Lemma is proved. ■

Remark 5.3.1 *After integration of (5.6) over (0,t), we get*

$$\begin{aligned} E(t) = & \ E(0) - b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi,s)|^2 d\xi d\rho ds \\ & -\frac{\xi_2}{4} \int_0^t \left(\frac{d}{ds}\|\nabla u\|_2^2\right)^2 ds. \end{aligned} \tag{5.9}$$

Now, we define

$$\begin{aligned} H(t) & = \|u\|_2^2 + \frac{\xi_2}{2} \int_0^t \|\nabla u\|_2^4 ds \\ & +b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta) \left(\int_0^s \phi(\xi,z)dz\right)^2 d\xi d\rho ds. \end{aligned} \tag{5.10}$$

Lemma 5.3.2 *Let (u,ϕ) be a regular solution of the problem (5.4). Then*

$$\begin{aligned} & \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)\phi(\xi,t) \int_0^t \phi(\xi,s) ds d\xi d\rho = \\ & \int_{\Gamma_0} u(x,t) \int_{-\infty}^{+\infty} \phi(\xi,t)\mu(\xi) d\xi d\rho - \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi(\xi,t)|^2 d\xi d\rho. \end{aligned} \tag{5.11}$$

Proof Using the second equation in (5.4), to obtain

$$(\xi^2+\eta)\phi(\xi,t) = u_t(x,t)\mu(\xi) - \partial_t\phi(\xi,t). \tag{5.12}$$

Then, integrate the last equality over [0, t] to get

$$\int_0^t (\xi^2+\eta)\phi(\xi,s) ds = u(x,t)\mu(\xi) - \phi(\xi,t). \tag{5.13}$$

Multiplying (5.13) by ϕ and integrating over $\Gamma_0 \times (-\infty, +\infty)$, we obtien (5.11). ■

Lemma 5.3.3 *Assume that $p > 3$ then we have:*

$$\begin{aligned} H''(t) - (p+3)\|u_t\|_2^2 & \geq -2(p+1)E(0) + \frac{\xi_2(p+1)}{2} \int_0^t \left(\frac{d}{ds}\|\nabla u\|_2^2\right)^2 ds \\ & +2(p+1)b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi,s)|^2 d\xi d\rho ds. \end{aligned} \tag{5.14}$$

Proof From (5.10), we have

$$\begin{aligned} H'(t) &= 2\int_\Omega uu_t dx + \tfrac{\xi_2}{2}\|\nabla u\|_2^4 \\ &+2b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi(\xi,s)\int_0^s\phi(\xi,z)dzd\xi d\rho ds, \end{aligned} \tag{5.15}$$

and

$$\begin{aligned} H''(t) &= 2\|u_t\|_2^2 + 2\int_\Omega uu_{tt}dx + 2\xi_2(\nabla u,\nabla u_t)\|\nabla u\|_2^2 \\ &+2b_1\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi(\xi,t)\int_0^t\phi(\xi,s)dsd\xi d\rho. \end{aligned}$$

Employing the divergence theorem and Lamma 5.3.2 , we get

$$\begin{aligned} H''(t) = \quad & (p+3)\|u_t\|_2^2 + \xi_0(p-1)\|\nabla u\|_2^2 + \xi_1(\tfrac{p+1}{2}-2)\|\nabla u\|_2^4 \\ &+b_1(p-1)\int_{\Gamma_0}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2 d\xi d\rho - 2(p+1)E(0) \\ &+2(p+1)b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,s)|^2 d\xi d\rho ds \\ &+\tfrac{\xi_2(p+1)}{2}\int_0^t\left(\tfrac{d}{ds}\|\nabla u\|_2^2\right)^2 ds. \end{aligned} \tag{5.16}$$

Since $p > 3$, then (5.14) holds. ■

Lemma 5.3.4 *Assume that $p > 3$ holds and that either one the following conditions is satisfied*
(i)$E(0) < 0$,
(ii) $E(0) = 0$, and

$$H'(0) > \frac{\xi_2}{2}\|\nabla u_0\|_2^4. \tag{5.17}$$

(iii)$E(0) > 0$, and

$$H'(0) > r\left[H(0) + k_0\right] + \frac{\xi_2}{2}\|\nabla u_0\|_2^4, \tag{5.18}$$

where

$$r = \frac{(p+1) - \sqrt{(p+1)^2 - 2(p+1)}}{2},$$

and

$$k_0 = \frac{\xi_2}{2}\|\nabla u_0\|_2^4 + 2E(0). \tag{5.19}$$

Then $H'(t) > \frac{\xi_2}{2}\|\nabla u_0\|_2^4$, for $t > t_0$, for $t > t_0$ where

$$t^* = \max\left\{0, \frac{2H'(0) - \xi_2\|\nabla u_0\|_2^4}{4(p+1)E(0)}\right\}, \tag{5.20}$$

where $t_0 = t^*$ in cas(i), and $t_0 = 0$ in case(ii) and (iii)

Proof If $E(0) < 0$, from (5.14), we have

$$H^{''}(t) \geq -2(p+1)E(0),$$

which gives

$$H^{'}(t) \geq H^{'}(0) - 2(p+1)E(0)t,$$

Thus, we obtain

$$H^{'}(t) > \frac{\xi_2}{2}\|\nabla u_0\|_2^4, \quad t \geq t^*$$

where t^* is defined in (5.20).

(ii) If $E(0) = 0$, then from (5.14), we obtien

$$H''(t) \geq 0, \quad t \geq 0.$$

Furthermore, if (5.17) holds, then

$$H'(t) > \frac{\xi_2}{2}\|\nabla u_0\|_2^4, \quad t > 0.$$

(iii) For the case that $E(0) > 0$, we first note that

$$2\int_0^t \|\nabla u\|_2^2 \frac{d}{dt}\|\nabla u\|_2^2 ds = \|\nabla u\|_2^4 - \|\nabla u_0\|_2^4. \tag{5.21}$$

By Young's inequality, we have

$$\|\nabla u\|_2^4 \leq \|\nabla u_0\|_2^4 + \int_0^t \|\nabla u\|_2^4 ds + \int_0^t \left(\frac{d}{dt}\|\nabla u\|_2^2\right)^2 ds. \tag{5.22}$$

Combination of (5.15) and (5.22), shows that

$$\begin{aligned} H'(t) \quad &\leq \|u\|_2^2 + \|u_t\|_2^2 + \tfrac{\xi_2}{2}\|\nabla u_0\|_2^4 + \tfrac{\xi_2}{2}\int_0^t \|\nabla u\|_2^4 ds \\ &+\tfrac{\xi_2}{2}\int_0^t \left(\tfrac{d}{dt}\|\nabla u\|_2^2\right)^2 ds + b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,s)|^2 d\xi d\rho ds \\ &+b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\left(\int_0^s \phi(\xi,z)dz\right)^2 d\xi d\rho. \end{aligned} \tag{5.23}$$

By (5.10), (5.14) and (5.23), we get

$$H''(t) - (p+1)H'(t) + (p+1)\left\{H(t) + k_0\right\} \geq 0,$$

where k_0 is defined in (5.19). Now let

$$B(t) = H(t) + k_0.$$

Then $B(t)$ satisfies

$$B''(t)-(p+1)B'(t)+(p+1)B(t)\geq 0. \tag{5.24}$$

Using Lemma 5.2.2 in (5.24) and (5.18), we get

$$H'(t)>\frac{\xi_2}{2}\|\nabla u_0\|_2^4, \quad t\geq 0.$$

■

Theorem 5.3.1 *Assume that $p>3$. Then the solution (u,ϕ) blows up in finit time T^* in the sense of (5.2.1). and T^* is such that:*
(i) If $E(0)<0$, then

$$T^*\leq t_0-\frac{J(t_0)}{J'(t_0)}.$$

Furthemore, if $J(t_0)<\min\left\{1,\sqrt{\frac{a}{-b'}}\right\}$, then we have

$$T^*\leq t_0+\frac{1}{\sqrt{-b'}}\ln\frac{\sqrt{\frac{a}{-b'}}}{\sqrt{\frac{a}{-b'}}-J(t_0)}$$

(ii) If $E(0)=0$ and (5.17) holds, then

$$T^*\leq-\frac{J(0)}{J'(0)}$$

or

$$T^*\leq\frac{J(0)}{J'(0)}$$

(iii) If

$$0<E(0)<\frac{(p-1)\left[H'(t_0)-\frac{\xi_2}{2}\|\nabla u\|_2^4\right]^2 J(t_0)^{\frac{1}{\gamma_1}}}{4(p+1)}$$

and (5.18) holds, then

$$T^*\leq\frac{J(0)}{\sqrt{a}}$$

or

$$T^*\leq 2^{\frac{3\gamma_1+1}{2\gamma_1}}\frac{\gamma_1 c}{\sqrt{a}}\{1-[1+cJ(0)]^{\frac{-1}{2\gamma_1}}\},$$

where $c=(\frac{b'}{a})^{2+\frac{1}{\gamma_1}}$, $\gamma_1=\frac{p-3}{4}$ and J(t), a and b' are given in (5.25), (5.34) and (5.35) respectively.

Proof Let

$$J(t)=\left\{H(t)+(T-t)\frac{\xi_2}{2}\|\nabla u_0\|_2^4\right\}^{-\gamma_1}. \tag{5.25}$$

Differentiating $J(t)$ twice, we obtain

$$J^{'}(t) = -\gamma_1 J(t)^{1+\frac{1}{\gamma_1}} \left\{ H'(t) - \frac{\xi_2}{2}\|\nabla u_0\|_2^4 \right\}, \tag{5.26}$$

and

$$J^{''}(t) = -\gamma_1 J(t)^{1+\frac{2}{\gamma_1}} Q(t), \tag{5.27}$$

where

$$\begin{aligned} Q(t) = \quad & H^{''}(t)\left\{H(t) + (T-t)\tfrac{\xi_2}{2}\|\nabla u_0\|_2^4\right\} \\ & -(1+\gamma_1)\left\{H'(t) - \tfrac{\xi_2}{2}\|\nabla u_0\|_2^4\right\}^2. \end{aligned} \tag{5.28}$$

It follows from (5.15) that

$$\begin{aligned} & H'(t) - \tfrac{\xi_2}{2}\|\nabla u_0\|_2^4 = 2\int_\Omega u u_t dx + \xi_2 \int_0^t \|\nabla u\|_2^2 \tfrac{d}{ds}\|\nabla u\|_2^2 ds \\ & + 2b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)\phi(\xi,s) \int_0^s \phi(\xi,z) dz d\xi d\rho ds. \end{aligned} \tag{5.29}$$

Hence, taking (5.10), (5.14), (5.28), and (5.29) into account, we obtain

$$Q(t) \geq -2(p+1)E(0)J(t)^{\frac{-1}{\gamma_1}} + (p+1)\left\{\mathbf{A}\mathbf{C} - \mathbf{B}^2\right\}, \tag{5.30}$$

where

$$\mathbf{A} = \|u\|_2^2 + \tfrac{\xi_2}{2}\int_0^t \|\nabla u\|_2^4 ds + b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)\left(\int_0^s \phi(\xi,z)dz\right)^2 d\xi d\rho ds,$$

$$\mathbf{B} = \int_\Omega u u_t dx + \tfrac{\xi_2}{2}\int_0^t \|\nabla u\|_2^2 \tfrac{d}{ds}\|\nabla u\|_2^2 ds + b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)\phi(\xi,s)\int_0^s \phi(\xi,z) dz d\xi d\rho ds,$$

$$\mathbf{C} = \|u_t\|_2^2 + b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi,s)|^2 d\xi d\rho ds + \tfrac{\xi_2}{2}\int_0^t \left(\tfrac{d}{ds}\|\nabla u\|_2^2\right)^2 ds.$$

Thus, we obtain Now we observe that, for all $\rho_1 \in \mathbb{R}$ and $t > 0$,

$$\begin{aligned} \mathbf{A}\rho_1^2 \quad & + 2\mathbf{B}\rho_1 + \mathbf{C} = \|\rho_1 u + u_t\|_2^2 \\ & + b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)\left[\rho_1 \int_0^s \phi(\xi,z)dz + |\phi(\xi,s)|\right]^2 d\xi d\rho ds \\ & + \tfrac{\xi_2}{2}\int_0^t \left[\rho_1\|\nabla u\|_2^2 + \left(\tfrac{d}{ds}\|\nabla u\|_2^2\right) ds\right]^2. \end{aligned}$$

It is easy to see that

$$\mathbf{A}\rho_1^2 + 2\mathbf{B}\rho_1 + \mathbf{C} \geq 0,$$

and

$$\mathbf{B}^2 - \mathbf{A}\mathbf{C} \leq 0. \tag{5.31}$$

Hence, taking (5.30) and (5.31) into account, we get

$$Q(t) \geq -2(p+1)E(0)J(t)^{\frac{-1}{\gamma_1}}, \quad t \geq t_0. \tag{5.32}$$

Therefore, by (5.27), and (5.32), we get

$$J''(t) \leq \frac{(p+1)(p-3)}{4} E(0) J(t)^{1+\frac{1}{\gamma_1}}, \quad t \geq t_0. \tag{5.33}$$

Note that using Lemma 5.3.4, $J^{'}(t) < 0$ for $t \geq t_0$. Multiplying (5.33) by $J^{'}(t)$ and integrating it from t_0 to t, we have

$$J^{'}(t)^2 \geq a + b' J(t)^{2+\frac{1}{\gamma_1}},$$

where

$$\begin{aligned} a = \ & \Big[\tfrac{(p-3)^2}{16} \left(H'(t_0) - \tfrac{\xi_2}{2}\|\nabla u_0\|_2^4\right)^2 \\ & -\tfrac{(p-3)^2(p+1)}{4(p-1)} E(0) J(t_0)^{\frac{-1}{\gamma_1}}\Big] J(t_0)^{2+\frac{2}{\gamma_1}}, \end{aligned} \tag{5.34}$$

and

$$b' = \frac{(p-3)^2(p+1)}{4(p-1)} E(0). \tag{5.35}$$

Then by Lemma 5.2.3 the proof of theorem is completed.

Hence, there exists a finite time T such that $\lim\limits_{t \to T^{*-}} J(t) = 0$ and the upper bounds of T^* are estimated according to the sign of $E(0)$ (see Lemma 5.2.3).

■

Conclusion

In this thesis, we study the interaction between the polynomial source and the term of dissipation: Balakrishnan-Taylor or fractional dissipation. Several results of existence, global existence, exponential decay and blow-up in finite time are proved. The methods used are mainly: the semi-group theory, Lyapunov method the method of Georgiev and Todorova

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الملخص (بالعربية) :

ندرس .خارجية قوة زوايا مختلفة من كسرية أشكال ذات للتبديد آليات بوجود و لمعادلات الرياضية المسائل بعض اقترحنا الأطروحة هذه في

ودراسة الحلول وجود على دراستنا ركزنا الشروط الحدية و و الابتدائية الشروط على الفرضيات بعض تحت الإرسال موجات جمل خاصة

محدود وقت في والانفجار الطاقة تناقص طريقة حول نتائج عدة توصلنا لإيجاد أين الزمنية اللانهاية عند الموجودة للحلول المقارب السلوك

الكلمات المفتاحية : المعادلات التفاضلية الكسرية – الوجود والوحدانية –انفجار الحل – استقرار الحل

تصنيف ماجستير: 93D15; 35L35; 35L20

Résumé (en Français) :

Cette thèse est consacrée à l'étude de l'existence et l'unicité de la solution et le comportement asymptotique de quelques problèmes aux dérivées partielles de type hyperbolique. Commençons alors par la question la plus importante qui est le comportement asymptotique. Cela veut dire: est-ce qu'elle explose en temps fini ? Est-ce qu'elle existe pour tout temps? Et quel est son comportement à temps grand (non-existence de solutions, décroissance exponentielle, décroissance polynômiale, décroissance logarithmique, …etc)? Mais avant d'étudier le comportement asymptotique, il faut prouver l'existence et l'unicité de la solution.

Cette thèse se compose de 4 chapitres dont:

Le premier chapitre est consacré pour les notions de la théorie des espaces fonctionnels et de certaines notions utilisées tout au long de cette thèse.

Dans le deuxième chapitre, nous considérons une équation des ondes non-linéaire soumise à un contrôle frontière de type fractionnaire. Nous montrons l'existence et l'unicité de la solution par la théorie des semi-groupes et nous étudions la stabilité exponentielle et la stabilité polynômiale. La preuve que nous avons établi est basée sur la construction d'une fonction de Lyapunov appropriée et équivalente à l'énergie de la solution considérée. Enfin, sous quelques hypothèses sur les données initiales et aux bords, nous avons prouvé l'explosion de la solution en temps fini.

Dans le troisième chapitre, nous considérons une équation des ondes non-linéaire avec condition retard de type fractionnaire. Dans le même contexte et moyennant les mêmes méthodes du deuxième chapitre, nous prouvons l'existence locale et étudions la stabilité exponentielle ou stabilité polynômiale. Enfin, nous avons prouvé l'explosion de la solution en temps fini si l'énergie initiale est négative.

Dans le quatrième chapitre, nous considérons une équation non linéaire de Kirchhoff soumise à un contrôle frontière de type fractionnaire. Dans le même contexte et moyennant les mêmes méthodes du quatrième chapitre, nous prouvons l'explosion de la solution en temps fini

Les mots clés : Équation des ondes non-linéaire, Équation non linéaire de Kirchhoff, Dérivée Fractionnaire, Stabilité exponentielle, Stabilité polynômiale, Éxplosion de la solution en temps fini, C_0 semi-groupe, Fonction de Lyapunov.
CLASSIFICATION MSC: 93D15; 35L35; 35L20

Abstract (en Anglais) :

After we prove the existence and uniqueness of the solution it crosses our minds the most important question which is asymptotic behaviour. That means: does it blow up in finite time? Does it exist for all time? And what is its behaviour in big time (non-existence of solutions, exponential decay, polynomial decay, logarithmic decay, ... etc)?

This thesis is composed of four chapters including:

The first chapter is devoted to the notions of the theory of functional spaces and of certain notions used throughout this thesis. In the second chapter, we consider a nonlinear wave equation with a fractional boundary damping. Using the semi-group theory, we establish the existence of the solution and we prove a decay rate estimate for the energy by introducing suitable Lyapunov functionals . We also prove that the solution blows up in finite time if the initial energy is non-positive combined with a positive initial energy.

The third chapter, we consider a nonlinear wave equation with a time delay condition of fractional type.

In the same context, we will establish the existence of the solutions and we will prove a decay rate estimate for the energy. We also prove that the solution blows up in finite time if the initial energy is negative.

In fourth chapter, we consider the Kirchhoff equation with Balakrishnan-Taylor damping and fractional boundary condition. In the same context, we will prove that the solution blows up in finite time.

Keywords : Nonlinear wave equation, Kirchhoff equation, Fractional derivative, Exponential decay, Polynomial decay, Blow up in finite time , C_0 semi-group, Lyapunov functional.

MSC CLASSIFICATION: 93D15; 35L35; 35L20