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Contribution à l'Étude de Certaines Équations Différentielles Fractionnaires

Soutenue le :

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Dédicaces

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particulièrement
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ABSTRACT

Our main purpose in this thesis, is devoted to study the existence of solutions for various types of fractional differential equations and inclusions with non instantaneous impulses with the Caputo fractional derivative in a Banach space of infinite dimension. The arguments are based upon Mönch's fixed point theorem and the technique of measures of noncompactness.

Key words and phrases: Initial value problem, impulses, measure of noncompactness, Caputo fractional derivative, Differential inclusions, multivalued jump, fixed point, Banach space.

AMS Subject Classification : 26A33, 34A37, 34G20, 34A60, 47H08.

RÉSUMÉ

Notre but principal, dans cette thèse, est de présenter plusieurs résultats d'existence pour certaines classes d'équations et inclusion différentielles d'ordre fractionnaire au sens de Caputo dans des espaces de Banach de dimension infinie. Ces résultats ont été obtenus par l'utilisation de théorème de point fixe de Mönch combiné avec la mesure de non compacité de Kuratowski.

Mots clé: Problème à valeur initiale, impulsions, mesure de non compacité, dérivé fractionnaire de Caputo, inclusions différentielles, saut à valeurs multivoque, point fixe, espace de Banach.

Classification AMS: 26A33, 34A37, 34G20, 34A60, 47H08.

CONTENTS

INTRODUCTION

The theory of fractional differential equations and inclusions is an important branch of differential equation theory, which has an extensive physical, chemical, biological, and engineering background, and hence has been emerging as an important area of investigation in the last few decades; see the monographs of Abbas *et al.* [1–3], Atangana [11], Kilbas *et al.* [36], Podlubny [44], and Zhou [48], and the references therein.

On the other hand, the theory of impulsive differential equations has undergone rapid development over the years and played a very important role in modern applied mathematical models of real processes rising in phenomena studied in physics, population dynamics, chemical technology, biotechnology and economics; see for instance the monographs by Bainov and Simeonov [17], Benchohra *et al.* [18], Hilfer [34], Lakshmikantham *et al.* [37], and Samoilenko and Perestyuk [45] and references therein. Moreover, fractional differential equations and inclusions present a natural framework for mathematical modeling of several real-world problems.

In pharmacotherapy, instantaneous impulses cannot describe the dynamics of certain evolution processes. For example, when one considers the hemodynamic equilibrium of a person, the introduction of the drugs in the bloodstream and the consequent absorption for the body are a gradual and continuous process. In [4, 5, 7, 33, 43] the authors initially studied some new classes of abstract fractional differential equations with non instantaneous impulses in Banach spaces.

However, the theory for fractional differential equations in Banach spaces has yet been sufficiently developed. Recently, Benchohra [19] applied the measure of noncompactness to a class of Caputo fractional differential equations of order $r \in (0,1]$ in a Banach space. Let E be a Banach space with norm $\|\cdot\|$.

In what follows, we will give a brief description of each chapter of this thesis.

Chapter 1 entitled *"Preliminaries"* contains notations and preliminary results, definitions, theorems and other auxiliary results which will be needed in this thesis, in ***section 1.1*** we give some generalities, in ***section 1.2*** we present some properties of Measures of noncompactness, in ***section 1.3*** we give brief on fractional calculus, in ***section 1.4*** we present some properties of set-valued maps and in ***section 1.5*** we cite some fixed point theorems.

The chapter 2 that's titled *"Impulsive Fractional Differential Equations In Banach Spaces"*.
We present in ***section 2.2*** the existence of solutions for a class of initial value problems for impulsive fractional differential equations involving the Caputo fractional derivative in a Banach space

$$^cD^r y(t) = f(t, y(t)), \ \text{ for a.e. } t \in [0,T], t \neq t_k, \ k = 1, \dots, m, \ 0 < r \leq 1,$$

$$\Delta y|_{t_k} = I_k(y(t_k^-)), \ k = 1, \dots, m,$$

$$y(0) = y_0.$$

where $^cD^r$ is the Caputo fractional derivative, $f : J \times E \to E$ is a given function, $I_k : E \to E, \ k = 1, \dots, m$, and $y_0 \in E, 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T, \Delta y|_{t_k} = y(t_k^+) - y(t_k^-), y(t_k^+) = \lim_{h\to 0^+} y(t_k + h), y(t_k^-) = \lim_{h \to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k, k = 1, \dots, m$.
In ***section 2.3***, we shall present some existence for the following nonlocal problem

$$^cD^r y(t) = f(t, y(t)), \ \text{ for a.e. } t \in [0,T], t \neq t_k, \ k = 1, \dots, m, \ 0 < r \leq 1,$$

$$\Delta y|_{t_k} = I_k(y(t_k^-)), \ k = 1, \dots, m,$$

$$y(0) + \psi(y) = y_0,$$

where $\psi : PC'(J,E) \to E$ is a continuous function.
In ***section 2.4*** we give an example to illustrate the usefulness of our main results.

The chapter 3 that's titled *"Nonlinear Fractional Differential Equations With Non instantaneous Impulses In Banach Spaces"*.
We establish in ***section 3.2*** the existence of solutions for a class of initial value problems for non instantaneous impulsive fractional differential equations involving the Caputo fractional derivative in a Banach space.

$$^cD^r y(t) = f(t, y(t)), \text{ for a.e. } t \in (s_k, t_{k+1}],\ k = 0, \dots, m,\ 0 < r \leq 1,$$

$$y(t) = g_k(t, y(t)),\ t \in (t_k, s_k],\ k = 1, \dots, m,$$

$$y(0) = y_0,$$

where $^cD^r$ is the Caputo fractional derivative, $f : J \times E \to E,\ g_k : (t_k, s_k] \times E \to E,\ k = 1, \dots, m,$ are given functions, $J = [0,T]$ and $y_0 \in E, 0 = s_0 < t_1 < s_1 < \cdots < t_m < s_m < t_{m+1} = T$.
In ***section 3.3*** we deal with the following nonlocal case:

$$^cD^r y(t) = f(t, y(t)), \text{ for a.e. } t \in (s_k, t_{k+1}],\ k = 0, \dots, m,\ 0 < r \leq 1,$$

$$y(t) = g_k(t, y(t)),\ t \in (t_k, s_k],\ k = 1, \dots, m,$$

$$y(0) + \psi(y) = y_0,$$

where $\psi : PC(J,E) \to E$ is a continuous function.
In ***section 3.3*** we present an example to illustrate the above results.

The chapter 4, that's titled *"Nonlocal Fractional Differential Inclusions with Non Instantaneous Impulses"*.
In ***section 4.2*** we prove results on the existence of solutions for a class of fractional differential inclusions with non instantaneous impulses involving the Caputo fractional derivative in a Banach space.

$$^cD^r y(t) \in F(t, y(t)), \text{ for a.e. } t \in (s_k, t_{k+1}],\ k = 0, \dots, m,\ 0 < r \leq 1,$$

$$y(t) = g_k(t, y(t)), t \in (t_k, s_k],\ k = 1, \dots, m,$$

$$y(0) = y_0,$$

where ${}^cD^r$ is the Caputo fractional derivative, $F : [0,T] \times E \to \mathcal{P}(E)$ is a multivalued map, $g_k : (t_k, s_k] \times E \to E,\ k = 1, \ldots, m,$ is a given function, and $y_0 \in E, 0 = s_0 < t_1 < s_1 < \cdots < t_m < s_m < t_{m+1} = T.$
In ***section 4.3*** we present the following nonlocal problem:

$$
\begin{gathered}
{}^cD^r y(t) \in F(t, y(t)),\ \text{for a.e. } t \in (s_k, t_{k+1}],\ k = 0, \ldots, m,\ 0 < r \leq 1, \\
y(t) = g_k(t, y(t)), t \in (t_k, s_k],\ k = 1, \ldots, m, \\
y(0) + \psi(y) = y_0,
\end{gathered}
$$

where $\psi : PC(J, E) \to E$ is a continuous function.
In ***section 4.4***, we give an example

The chapter 5, that's titled *" Fractional Differential Inclusions with Non Instantaneous Impulses and Multivalued Jump"*.
In ***section 5.2***, we study the existence of solutions for a class of fractional differential inclusions with non instantaneous impulses and multivalued jump in a Banach space, exactly the problems of the following form:

$$
\begin{gathered}
{}^cD^r y(t) \in F(t, y(t)),\ \text{for a.e. } t \in J_k := (s_k, t_{k+1}],\ k = 0, \ldots, m,\ 0 < r \leq 1, \\
y(t) \in G_k(t, y(t)),\ t \in J'_k := (t_k, s_k],\ k = 1, \ldots, m, \\
y(0) = y_0.
\end{gathered}
$$

where ${}^cD^r$ is the Caputo fractional derivative, $F : [0,T] \times E \to \mathcal{P}(E)$ is a multivalued map, $G_k : (t_k, s_k] \times E \to \mathcal{P}(E),\ k = 1, \ldots, m,$ is a given multivalued map, and $y_0 \in E, 0 = s_0 < t_1 < s_1 < \cdots < t_m < s_m < t_{m+1} = T.$
In ***section 5.3*** for more generalization we present the following problem:

$$
\begin{gathered}
{}^cD^r y(t) \in F(t, y(t)),\ \text{ a.e. } t \in J_k := (s_k, t_{k+1}],\ k = 0, \ldots, m,\ 0 < r \leq 1, \\
y(t) \in G_k(t, y(t)),\ t \in J'_k := (t_k, s_k],\ k = 1, \ldots, m, \\
y(0) + \psi(y) = y_0,
\end{gathered}
$$

where $\psi : PC(J, E) \to E$ is a continuous function.
In ***section 5.4*** we end this chapter with an example

CHAPTER 1

PRELIMINARIES

Contents

In this chapter, we collect some notations, definitions, theorems, lemmas and facts concerning measures of noncompactness, phase spaces and multivalued analysis and other auxiliary results which will be needed in the sequel.

1.1 Generalities

Let E be a Banach space with the norm $\|\cdot\|$ and $\ddot{}J := [0,T]$.

Definition 1.1.1. *A function $f : J \to E$ is called strongly measurable if there exists a sequence of simple functions $(f_n)_n$ such that*

$$\lim_{n\to\infty} \|f_n(t) - f(t)\| = 0.$$

Definition 1.1.2. *A function $f : J \to E$ is Bochner integrable on J if it is strongly measurable and*

$$\lim_{n\to\infty} \int_0^T \|f_n(t) - f(t)\| dt = 0$$

for any sequence of simple functions $(f_n)_n$,
then

$$\left\|\int_0^T f(t)dt\right\| \le \int_0^T \|f(t)\| dt.$$

Definition 1.1.3. *A function $f : J \to E$, is said absolutely continuous if for every $\varepsilon > 0$ there is $\delta > 0$ such that, for every finite collection of pairwise disjoint intervals $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n) \subset I$ with $\sum_i (b_i - a_i) < \delta$, we have*

$$\sum_i \|f(b_i) - f(a_i)\| < \varepsilon.$$

Propertie:
If f is an absolutely continuous function, then the following properties hold:

(I) f is continuous.

(II) f is almost differentiable.

Denote by

- $C(J, E)$ the Banach space of all continuous functions from J into E with the norm
$$\|y\|_\infty = \sup\{\|y(t)\| : t \in J\}.$$

- $L^1(J, E)$ the Banach space of measurable functions $y : J \to E$ which are Bochner integrable, equipped with the norm
$$\|y\|_{L^1} = \int_0^T \|y(t)\|dt.$$

- $L^\infty(J, E)$ be the Banach space of measurable functions $y : J \to E$ which are essentially bounded.

- $PC(J, E)$ and $PC^{\text{'}}(J, E)$ are the Banach spaces defined by
$$\begin{aligned}&PC(J, E) = \{y : J \to E : y \in C([0, t_1] \cup (t_k, s_k] \cup (s_k, t_{k+1}], E),\\ &k = 1, \dots, m \text{ and there exist } y(t_k^-),\ y(t_k^+), y(s_k^-) \text{and } y(s_k^+)\\ &k = 1, \dots, m \text{ with } y(t_k^-) = y(t_k) \text{and } y(s_k^-) = y(s_k)\}\end{aligned}$$
and
$$\begin{aligned}&PC^{\text{'}}(J, E) = \{y : J \to E : y \in C((t_k, t_{k+1}], E),\ k = 1, \dots, m,\\ &\text{and there exist } y(t_k^-), \text{and } y(t_k^+)\ k = 1, \dots, m \text{ with } y(t_k^-) = y(t_k)\},\end{aligned}$$
with the norm
$$\|y\|_{PC} = \|y\|_{PC^{\text{'}}} = \sup_{t\in J} \|y(t)\|.$$

- $AC(J, E)$ is the space of absolutely continuous functions.

- $AC^1(J, E)$ is the space of continuously differentiable functions whose first derivative is absolutely continuous.

- $AC^n(J, E)$ is the space of continuously n differentiable functions whose the derivative of order n is absolutely continuous.

- $J^{\text{'}}, J^{\text{''}}$ are the sets defined by
$$J^{\text{'}} = [0, T] \setminus \cup_{k=1}^m (t_k, s_k].$$
$$J^{\text{''}} = [0, T] \setminus \{t_1, \dots, t_m\}.$$

Moreover, for a given set V of functions $v : J \to E$ let us denote by

$$V(t) = \{v(t),\ v \in V\},\ t \in J$$

and

$$V(J) = \{v(t),\ v \in V,\ t \in J\}.$$

Definition 1.1.4. *A map $f : J \times E \to E$ is Carathéodory if*

(i) $t \longmapsto f(t, y)$ *is measurable for all $y \in E$, and*

(ii) $y \longmapsto f(t, y)$ *is continuous for almost each $t \in J$.*

If, in addition,

(iii) *for each $r > 0$, there exists $g_r \in L^1(J, \mathbb{R}_+)$ such that*

$$|f(t, y) \leq g_r(t) \text{ for all } |y| \leq r \text{ and almost each } t \in J,$$

then we say that the map is L^1-Carathéodory.

We refer to [40, 47] for more details.

1.2 Some properties of measure of noncompactness

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness.

Definition 1.2.1. *([14]). Let X be a Banach space and Ω_X the bounded subsets of X. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_X \to [0, \infty]$ defined by*

$$\alpha(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^{n} B_i \ and\ diam(B_i) \leq \epsilon\}; hereB \in \Omega_X.$$

The Kuratowski measure of noncompactness satisfies the following properties:
Properties: Let X be a Banach space. Then for all bounded subsets $A,\ B$ of X the following assertions hold (for more details see [14])

(a) $\alpha(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact).

(b) $\alpha(B) = \alpha(\overline{B})$.

(c) $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$.

(d) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$.

(e) $\alpha(cB) = |c|\alpha(B); c \in \mathbb{R}$.

(f) $\alpha(convB) = \alpha(B)$.

1.3 Brief on Fractional Calculus

For completeness we recall the definition of Caputo derivative of fractional order.

Definition 1.3.1. *([36]). The fractional (arbitrary) integral of the function $h \in L^1([0,T],E)$ of order $r \in \mathbb{R}_+$ is defined by*

$$I^r h(t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} h(s) ds,$$

where Γ is the Euler gamma function defined by

$$\Gamma(r) = \int_0^\infty t^{r-1} e^{-t} dt, \; r > 0.$$

Definition 1.3.2. *([36]). For a function $h \in AC^n(J,E)$, the Caputo fractional derivative of order r of h is defined by*

$$({}^c D_0^r h)(t) = \frac{1}{\Gamma(n-r)} \int_0^t (t-s)^{n-r-1} h^{(n)}(s) ds,$$

where $n = [r] + 1$.

We need the following auxiliary lemmas [36].

Lemma 1.3.1. *[36] Let $r > 0$ and $h \in AC^n(J,E)$. Then the differential equation*

$${}^c D_0^r h(t) = 0, \quad \textit{for a.e. } t \in J$$

has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$, $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [r] + 1$.

Lemma 1.3.2. *[36] Let $r > 0$ and $h \in AC^n(J, E)$. Then*

$$I^{rc}D_0^r h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \quad \textit{for a.e. } t \in J$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [r] + 1$.

1.4 Some Properties of Set-Valued Maps

Let $(X, \|\cdot\|)$ be a Banach space and A be a subset of X. We denote:

$$\mathcal{P}(X) = \{A \subset X : A \neq \emptyset\}$$

and
$\mathcal{P}_b(X) = \{A \subset X : A \text{ bounded } \}$,

$\mathcal{P}_{cl}(X) = \{A \subset X : A \text{ closed } \}$,

$\mathcal{P}_{cp}(X) = \{A \subset X : A \text{ compact } \}$,

$P_{cv}(X) = \{A \subset X : A \text{ convexe } \}$,

$\mathcal{P}_{cv,cp}(X) = \mathcal{P}_{cv}(X) \cap \mathcal{P}_{cp}(X)$.

Definition 1.4.1. *Let X and Y be Banach spaces. A set-valued map F from X to Y is characterized by its $graph(F)$, the subset of the product space $X \times Y$ defined by*

$$graph(F) := \{(x, y) \in X \times Y : y \in F(x).\}$$

Definition 1.4.2. *1. A measurable multivalued function $F : J \to \mathcal{P}_{b,cl}(X)$ is said to be integrably bounded if there exists a function $g \in L^1(\mathbb{R}_+)$ such that $\|f\| \leq g(t)$ for almost $t \in J$ for all $f \in F(t)$.*

2. *F is bounded on bounded sets if $F(\mathcal{W}) = \bigcup_{x \in \mathcal{W}} F(x)$ is bounded in X for all $\mathcal{W} \in \mathcal{P}_b(X)$, i.e. $\sup_{x \in \mathcal{W}}\{\sup\{\|y\| : y \in F(x)\}\} < \infty$.*

3. *A set-valued map F is called upper semi-continuous (u.s.c. for short) on X if for each $x_0 \in X$ the set $F(x_0)$ is a nonempty, closed subset of X and for each open set U of X containing $F(x_0)$, there exists an*

open neighborhood V of x_0 such that $F(V) \subset U$.
A set-valued map F is said to be upper semi-continuous if it is so at every point $x_0 \in X$.

4. *A set-valued map F is called lower semi-continuous (l.s.c) at $x_0 \in X$ if for any $y_0 \in F(x_0)$ and any neighborhood V of y_0 there exists a neighborhood U of x_0 such that*

$$F(x_0) \cap V \neq \quad \text{for all } \; x_0 \in U.$$

A set-valued map F is said to be lower semi-continuous if it is so at every point $x_0 \in X$.

5. *F is said to be completely continuous if $F(B)$ is relatively compact for every $B \in P_b(X)$. If the multivalued map f is completely continuous with nonempty compact values, then f is upper semi-continuous if and only if f has closed graph.*

Proposition 1.4.1. *Let $F : X \to Y$ be an u.s.c map with closed values. Then $graph(F)$ is closed.*

Definition 1.4.3. *Let E be a Banach space. A multivalued map $F : J \times E \to E$ is said to be L^1-Carathéodory if*

(i) *$t \longmapsto F(t, y)$ is measurable for all $y \in E$,*

(ii) *$y \longmapsto F(t, y)$ is upper semicontinuous for almost each $t \in J$,*

(iii) *for each $\rho > 0$, there exists $\psi_\rho \in L^1(J, \mathbb{R}_+)$ such that*

$$\|F(t, y)\|_P \leq \psi_\rho(t), \quad \text{for all } |y| \leq \rho \text{ and e.a. } \; t \in J,$$

such that $\|F(t, y)\| = \sup\{\|f\| : f \in F(t, y)\}$.

Definition 1.4.4. *Let X, Y be nonempty sets and $F : X \to \mathcal{P}(Y)$. The single-valued operator $f : X \to Y$ is called a selection of F if and only if $f(x) \in F(x)$, for each $x \in X$. The set of all selection functions for F is denoted by S_F.*

► For each $y \in C(J, E)$, define the set of selections of F by

$$S_{F,y} = \{f \in L^1(J, E), f(t) \in F(t, y(t)) \;\; a.e.\; t \in J\}.$$

The following lemmas is very important to prove our result.

Lemma 1.4.1. *[38] Let J be a compact real interval. Let F be a multivalued be a Carathéodory multivalued map and let Θ be a linear continuous map from $L^1(J,E) \to C(J,E)$. Then the operator*

$$\Theta \circ S_{F,y} : C(J,E) \to \mathcal{P}_{KC}\ (\mathsf{C}(J,E)), y \mapsto (\Theta \circ S_{F,y})(y) = \Theta(S_{F,y})$$

is a closed graph operator in $C(J,E) \times C(J,E)$.

Definition 1.4.5. Let $\mathcal{T} : X \to \mathcal{P}(X)$ be a multi-valued map. An element $x \in X$ is said to be a fixed point of $\mathcal{T}$ if $x \in \mathcal{T}(x)$.

For more details on multivalued maps and the proof of the known results cited in this section we refer interested reader to the
books [2,3,12,13,29,31,35].

1.5 Fixed Point Theorems

Fixed point theory plays an important role in our existence results, therefore we state the following fixed point theorems.

Theorem 1.5.1. *([8,41]) (**Mönch's** fixed point theorem). Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv} N(V) \text{ or } V = N(V) \cup \{0\} \Rightarrow \alpha(V) = 0$$

holds for every subset V of D, then Nhas a fixed point.

Lemma 1.5.1. *([32]) If $V \subset C([0,T];E)$ is a bounded and equicontinuous set, then*

(i) *the function $t \to \alpha(V(t))$ is continuous on J, and*

$$\alpha_c(V) = \sup_{0 \le t \le T} \alpha(V(t)),$$

(ii) $\alpha(\int_0^T x(s)ds : x \in V) \le \int_0^T \alpha(V(s))ds,$

where

$$V(s) = \{x(s) : x \in V\},\ s \in J.$$

Theorem 1.5.2. *([8,41]). Let E be a Banach space and $f \in L^1(J,E)$ countable with $|u(t)| \leq h(t)$ for a.e.t $\in J$, and each $u \in C$; where $h \in L^1(J,\mathbb{R}_+)$ then the function $\phi(t) = \alpha(C(t))$ belongs to $L^1(J,\mathbb{R}_+)$ and satisfies*

$$\alpha\left(\left\{\int_0^T u(s)ds : u \in C\right\}\right) \leq 2\int_0^T \alpha(C(s)ds.$$

Theorem 1.5.3. *([8,41]) (the set-valued analog of **Mönch's** fixed point theorem). Let K be a closed, convex subset of a Banach space E ; U a relatively open subset of K, and $N : \bar{U} \to \mathcal{P}_C\,(K)$ Assume graph (N) is closed, N maps compact sets into relatively compact sets, and that for some $0 \in U$; the following two conditions are satisfied:*

$$\left(\begin{array}{c} M \subset U, M \subset conv(\{0\} \bigcup N(M)) \\ and\ \overline{M} = \overline{C}\ with\ C \subset M\ countable \end{array}\right) \Rightarrow \overline{M} compact.$$

$$x \notin \lambda N(x) for\ all\ x \in \overline{U} \backslash U, \lambda \in (0,1).$$

Then there exists $x \in U$ with $x \in N(x)$.

CHAPTER 2

IMPULSIVE FRACTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACES [1]

Contents

[1] [20] M. Benchohra, J. Henderson and D. Seba, Measure of noncompactness and fractional differential equations in Banach spaces, *Commun. Appl. Anal.* **12** (4) (2008), 419-428.

2.1 Introduction

This chapter is organized as follows. In section **2.2** we give results for a class of initial value problems for impulsive fractional differential equations in Banach spaces

$$^cD^r y(t) = f(t, y(t)), \text{ for a.e. } t \in [0, T], t \neq t_k, \; k = 1, \dots, m, \; 0 < r \leq 1, \tag{2.1}$$

$$\Delta y|_{t_k} = I_k(y(t_k^-)), \; k = 1, \dots, m, \tag{2.2}$$

$$y(0) = y_0, \tag{2.3}$$

where $^cD^r$ is the Caputo fractional derivative, $f : J \times E \to E$ is a given function, $I_k : E \to E, \; k = 1, \dots, m$, and $y_0 \in E, 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T, \Delta y|_{t_k} = y(t_k^+) - y(t_k^-), y(t_k^+) = \lim_{h \to 0^+} y(t_k + h), y(t_k^-) = \lim_{h \to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k, k = 1, \dots, m.$ In section **2.3**, we shall present some results for the following nonlocal problem

$$^cD^r y(t) = f(t, y(t)), \text{ for a.e. } t \in [0, T], t \neq t_k, \; k = 1, \dots, m, \; 0 < r \leq 1,$$

$$\Delta y|_{t_k} = I_k(y(t_k^-)), \; k = 1, \dots, m,$$

$$y(0) + \psi(y) = y_0,$$

where $\psi : PC'(J, E) \to E$ is a continuous function.
This results is based on Mönch's fixed point theorem combined with the technique of measures of noncompactness,which is an important method for seeking solutions of differential equations. See Akhmerov *et al.* [9], Alvàrez [10], Banaś *et al.* [14–16], Guo *et al.* [32], M¨onch [41], Mönch , Von Harten [42] and Szufla [46]. An example is given in section **2.4** to demonstrate the application of our main results.

2.2 Existence of solution

First of all, we define what we mean by a solution of the problem (2.1)-(2.3).

Definition 2.2.1. A function $y \in PC^{‘}(J, E)$ is said to be a solution of (2.1)-(2.3) if y satisfies $y(0) = y_0$, $^cD^r y(t) = f(t, y(t))$, for a.e. $t \in J^{‘’}$, and $\Delta y|_{t_k} = I_k(y(t_k^-)), \; k = 1, \dots, m.$

To prove the existence of solutions to (2.1)-(2.3), we need the following auxiliary lemmas.

Lemma 2.2.1. *Let* $0 < r \leq 1$ *and let* $h : J \to E$ *be integrable. Then linear problem*

$$^cD^r y(t) = h(t),\ t \in J'',\quad k = 0, \dots, m, \tag{2.4}$$

$$y(t) = I_k(y(t_k^-)),\ k = 1, \dots, m, \tag{2.5}$$

$$y(0) = y_0, \tag{2.6}$$

has a unique solution which is given by :

$$y(t) = \begin{cases} y_0 + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}h(s)ds & \text{if } t \in [0, t_1], \\ y_0 + \frac{1}{\Gamma(r)}\sum_1^k \int_{t_{i-1}}^{t_i} (t_i - s)^{r-1}h(s)ds & \\ +\frac{1}{\Gamma(r)}\int_{t_k}^t (t-s)^{r-1}h(s)ds & \\ +\sum_i^k I_i(y(t_i^-)), & \text{if } t \in (t_k, t_{k+1}] \quad k = 1, \dots, m. \end{cases} \tag{2.7}$$

We are now in a position to state and prove our existence result for the problem (2.1)–(2.3) based on Mönch's fixed point. Let us list some conditions on the functions involved in the problem (2.1)–(2.3):

(H1) The function $f : J \times E \to E$ satisfies the Carathéodory conditions.

(H2) There exists $p \in L^1(J, \mathbb{R}_+) \bigcap C(J, \mathbb{R}_+)$ such that

$$\|f(t, y)\| \leq p(t)\|y\| \text{ for any } y \in E \text{ and } t \in J.$$

(H3) There exists c > 0 such that

$$\|I_k(y)\| \leq c\|y\|, \text{ for each } y \in E \text{ and } k = 1, \dots, m.$$

(H4) For each bounded set $B \subset E$ we have

$$\alpha(I_k(B)) \leq c\alpha(B).$$

(H5) For each bounded set $B \subset E$ we have

$$\alpha(f(t, B)) \leq p(t)\alpha(B),\ t \in J.$$

Let
$$p^* = \sup_{t\in J} p(t).$$

Theorem 2.2.1. *Assume that assumptions (H1)-(H5) hold. If*
$$\frac{(m+1)p^*T^r}{\Gamma(r+1)} + mc < 1, \tag{2.8}$$
then the problem (2.1)–(2.3) has at least one solution defined on J.

Proof. Transform the problem (2.1)–(2.3) into a fixed point problem. Consider the operator $N : PC^r(J,E) \to PC^r(J,E)$ defined by
$$\begin{aligned} N(y)(t) = y_0 &+ \frac{1}{\Gamma(r)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t-s)^{r-1} f(s,y(s))ds \\ &+ \frac{1}{\Gamma(r)} \int_{t_k}^{t} (t-s)^{r-1} f(s,y(s))ds \\ &+ \sum_{0<t_k<t} I_k(y(t_k^-)). \end{aligned}$$

Clearly, the fixed points of operator N are solutions of problem (2.1)–(2.3).
Let
$$r_0 \geq \frac{\|y_0\|}{1 - mc - \frac{(m+1)p^*T^r}{\Gamma(r+1)}}, \tag{2.9}$$
and consider the set
$$D_{r_0} = \{y \in PC^r(J,E) : \|y\|_\infty \leq r_0\}.$$
Clearly, the subset D_{r_0} is closed, bounded and convex. We shall show that N satisfies the assumptions of Theorem 1.5.1. The proof will be given in a three of steps.

Step 1: *N is continuous.*
Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $PC^r(J,E)$.

Then for each $t \in J$, we have

$$\begin{aligned}\|N(y_n)(t) - N(y)(t)\| &= y_0 + \frac{1}{\Gamma(r)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t-s)^{r-1} \|f(s, y_n(s)) - f(s, y(s))\| ds \\ &+ \frac{1}{\Gamma(r)} \int_{t_k}^{t} (t-s)^{r-1} \|f(s, y_n(s)) - f(s, y(s))\| ds \\ &+ \sum_{0<t_k<t} \|I_k(y(t_k^-)) - I_k(y_n(t_k^-))\|.\end{aligned}$$

Since I_k is continuous and f is of Carathéodory type, the Lebesgue dominated convergence theorem implies

$$\|N(y_n) - N(y)\|_\infty \to 0 \; as \; n \to \infty.$$

Consequently, N is continuous.

Step 2: *N maps D_{r_0} into itself.*
For each $y \in D_{r_0}$, by (H2), (2.8) and (2.9)we have for each $t \in J$,

$$\begin{aligned}\|N(y)(t)\| &\leq \|y_0\| + \frac{1}{\Gamma(r)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t-s)^{r-1} \|f(s, y(s))\| ds \\ &+ \frac{1}{\Gamma(r)} \int_{t_k}^{t} (t-s)^{r-1} \|f(s, y(s))\| ds \\ &+ \sum_{0<t_k<t} \|I_k(y(t_k^-))\| \\ &\leq \|y_0\| + \frac{1}{\Gamma(r)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t-s)^{r-1} p(t) \|y\| ds \\ &+ \frac{1}{\Gamma(r)} \int_{t_k}^{t} (t-s)^{r-1} p(t) \|y\|| ds \\ &+ \sum_{0<t_k<t} c\|y\| \\ &\leq \|y_0\| + r_0 \left(\frac{(m+1)p^* T^r}{\Gamma(r+1)} + mc \right) \\ &\leq r_0.\end{aligned}$$

Step 3: $N(D_{r_0})$ *is bounded and equicontinuous.*
By Step 2, it is obvious that $N(D_{r_0}) \subset PC(J, E)$ is bounded.
For the equicontinuous of $N(D_{r_0})$, let $\tau_1, \tau_2 \in J, \tau_1 < \tau_2$ and $y \in D_{r_0}$. Then

$$\begin{aligned}
\|N(y)(\tau_2) - N(y)(\tau_1)\| &= y_0 \\
&+ \frac{1}{\Gamma(r)} \int_0^{\tau_1} |(\tau_2 - s)^{r-1} - (\tau_1 - s)^{r-1}| \|f(s, y(s))\| ds \\
&+ \frac{1}{\Gamma(r)} \int_{\tau_1}^{\tau_2} |(\tau_2 - s)^{r-1} - (\tau_1 - s)^{r-1}| \|f(s, y(s))\| ds \\
&+ \sum_{0<t_k<\tau_2-\tau_1} I_k(y(t_k^-)) \\
&\leq \frac{r_0 p^*}{\Gamma(r+1)} [\tau_2^r - \tau_1^r] \\
&+ \sum_{0<t_k<(\tau_2-\tau_1)} I_k(y(t_k^-)).
\end{aligned}$$

As $\tau_1 \to \tau_2$, the right-hand side of the above inequality tens to zero.

Now let V be a subset of D_{r_0} such that $V \subset \overline{conv}(N(V) \cup \{0\})$. Then V is bounded and equicontinuous and therefore the function $t \to v(t) = \alpha(V(t))$ is continuous on J. By (H4),(H5), Lemma 1.5.1 and the properties of the measure α we have for each $t \in J$

$$\begin{aligned}
v(t) &\leq \alpha(N(V)(t) \cup \{0\}) \\
&\leq \alpha(N(V)(t)) \\
&\leq \frac{1}{\Gamma(r)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t-s)^{r-1} p(s) \alpha\left(V(s)\right) ds \\
&+ \frac{1}{\Gamma(r)} \int_{t_k}^{t} (t-s)^{r-1} p(s) \alpha\left(V(s)\right) ds \\
&+ \sum_{0<t_k<t} \alpha\left(V(s)\right) \\
&\leq \frac{1}{\Gamma(r)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t-s)^{r-1} p(s) \alpha(v(s)) ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(r)}\int_{t_k}^{t}(t-s)^{r-1}p(s)\alpha(v(s))ds \\
&+\sum_{0<t_k<t} c\alpha(v(s)) \\
&\leq \|v\|_\infty\left(\frac{(m+1)p^*T^r}{\Gamma(r+1)}+mc\right).
\end{aligned}
$$

This means that

$$\|v\|_\infty\left[1-\left(\frac{(m+1)p^*T^r}{\Gamma(r+1)}\right)+mc\right]\leq 0.$$

By (2.8) it follows that $\|v\|_\infty = 0$; that is, $v(t) = 0$ for each $t \in J$, and then $V(t)$ is relatively compact in E. In view of the Arzéla-Ascoli theorem, V is relatively compact in D_{r_0}. Applying now Theorem 1.5.1 we conclude that N has a fixed point which is a solution of the problem (2.1)-(2.3).

□

2.3 Nonlocal impulsive differential equations

This section is concerned with a generalization of the results presented in the previous section to nonlocal impulsive fractional differential equations. More precisely we shall present some existence results for the following nonlocal problem

$$^cD^r y(t) = f(t, y(t)), \text{ for a.e. } t \in [0,T], t \neq t_k,\ k = 1,\dots,m,\ 0 < r \leq 1, \tag{2.10}$$

$$\Delta y|_{t_k} = I_k(y(t_k^-)),\ k = 1,\dots,m, \tag{2.11}$$

$$y(0) + \psi(y) = y_0, \tag{2.12}$$

where $f, I_k, k = 1,\dots,m$ are as in section 2.2 and $\psi : PC'(J,E) \to E$ is a continuous function.

Nonlocal conditions were initiated by Byszewski [28] when he proved the existence and uniqueness of mild and classical solutions of nonlocal

Cauchy problems. As remarked by Byszewski [26, 27], the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena. For example, in [30],the author used

$$\psi(y) = \sum_{i=1}^{p} e_i y(x_i), \tag{2.13}$$

where $e_i, i = 1, \ldots, p$, are given constants and $0 < x_1 < \ldots < x_p \leq T$, to describe the diffusion phenomenon of a small amount of gas in a transparent tube. In this case, 2.13 allows the additional measurements at $x_i, i = 1, \ldots, p$.

Let us introduce the following set of conditions.

(H6) There exists a constant $M^* > 0$ such that

$$\|\psi(u)\| \leq M^* \; for \; each \; u \in PC'(J, E).$$

(H7) For each bounded set $B \subset PC'(J, E)$ we have

$$\alpha(\psi(B)) \leq M^* \alpha(B).$$

Theorem 2.3.1. *Assume that assumptions (H1)-(H7) hold.If*

$$\frac{(m+1)p^* T^r}{\Gamma(r+1)} + mc + M^* < 1, \tag{2.14}$$

then the nonlocal problem (2.10)–(2.12) has at least one solution defined on J.

Proof. Transform the problem (2.10)–(2.12) into a fixed point problem. Consider the operator $\tilde{N} : PC'(J, E) \to PC'(J, E)$ defined by

$$\begin{aligned}
\tilde{N}(y)(t) = y_0 - \psi(y) &+ \frac{1}{\Gamma(r)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t-s)^{r-1} f(s, y(s)) ds \\
&+ \frac{1}{\Gamma(r)} \int_{t_k}^{t} (t-s)^{r-1} f(s, y(s)) ds \\
&+ \sum_{0<t_k<t} I_k(y(t_k^-)).
\end{aligned}$$

Clearly, the fixed points of the operator $\tilde{N}$ are solution of the problem (2.10)–(2.12). We can easily show the conditions of Theorem 1.5.1 are satisfied by $\tilde{N}$.

□

2.4 An Example

Let us consider the following infinite system of impulsive fractional initial value problem,

$$^{c}D^{r}y_n(t)=\frac{1}{9+e^t}y_n(t),\ \text{for a.e. } t\in[0,1],t\neq\frac{1}{2},0<r\leq 1 \tag{2.15}$$

$$y_n|_{(t=\frac{1}{2})}=\frac{1}{5}y_n(t)\left(\frac{1}{2}^{-}\right), \tag{2.16}$$

$$y_n(0)=0. \tag{2.17}$$

Set

$$E=l^1=\{y=(y_1,y_2,\dots,y_n,\dots,),\sum_{n=1}^{\infty}|y_n|<\infty\},$$

E is a Banach space with the norm

$$\|y\|=\sum_{n=1}^{\infty}|y_n|.$$

Let

$$f(t,y)=(f_1(t,y),f_2(t,y),\dots,f_n(t,y),\dots),$$

$$f_n(t,y_n)=\frac{1}{9+e^t}y_n(t),$$

and

$$I_k(y_n)=\frac{1}{5}y_n.$$

Clearly conditions (H2) and (H3) hold with

$$p(t)=\frac{1}{9+e^t},\ \text{and } c=\frac{1}{5}.$$

We shall check that condition (2.8) is satisfied with $T=1,p^*=\frac{1}{10}$ and $m=1$. Indeed

$$\left(\frac{(m+1)p^*T^r}{\Gamma(r+1)}+\frac{1}{5}\right)<1\Leftrightarrow\Gamma(r+1)>\frac{1}{4}. \tag{2.18}$$

Then by Theorem 2.2.1 the problem (2.15)-(2.17) has at least one solution for values of r satisfying (2.18).

CHAPTER 3

NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH NON INSTANTANEOUS IMPULSES IN BANACH SPACES [2]

Contents

[2] [21] M. Benchohra and M. Slimane, Nonlinear fractional differential equations with non instantaneous impulses in Banach spaces.*Journal of Mathematics and Applications*. **41** (2018), 39-51.

3.1 Introduction

In this chapter, we offer a generalization of the problem presented in the previous chapter in particular in the first section we are interested with the existence of the following differential equations with non instantaneous impulses in Banach spaces

$$^cD^r y(t) = f(t, y(t)), \text{ for a.e. } t \in (s_k, t_{k+1}], \ k = 0, \dots, m, \ 0 < r \leq 1, \quad (3.1)$$

$$y(t) = g_k(t, y(t)), \ t \in (t_k, s_k], \ k = 1, \dots, m, \quad (3.2)$$

$$y(0) = y_0, \quad (3.3)$$

where $^cD^r$ is the Caputo fractional derivative, $f : J \times E \to E$, $g_k : (t_k, s_k] \times E \to E$, $k = 1, \dots, m$, are given functions, $J = [0, T]$ and $y_0 \in E, 0 = s_0 < t_1 < s_1 < \dots < t_m < s_m < t_{m+1} = T$. The second section is concerned with a generalization of the results presented in the previous section to non instantaneous impulsive fractional differential equations with nonlocal conditions. More precisely we shall present some existence and uniqueness results for the following nonlocal problem

$$^cD^r y(t) = f(t, y(t)), \text{ for a.e. } t \in (s_k, t_{k+1}], \ k = 0, \dots, m, \ 0 < r \leq 1,$$

$$y(t) = g_k(t, y(t)), \ t \in (t_k, s_k], \ k = 1, \dots, m,$$

$$y(0) + \psi(y) = y_0,$$

where $\psi : PC(J, E) \to E$ is a continuous function.

3.2 Existence of solution

First of all, we define what we mean by a solution of the problem (3.1)-(3.3).

Definition 3.2.1. A function $y \in PC(J, E) \cap AC(J', E)$ is said to be a solution of (3.1)-(3.3) if y satisfies $y(0) = y_0$, $^cD^r y(t) = f(t, y(t))$, for a.e. $t \in (s_k, t_{k+1}]$, and each $k = 0, \dots, m$, and $y(t) = g_k(t, y(t))$, for all $t \in (t_k, s_k]$, and every $k = 1, \dots, m$.

To prove the existence of solutions to (3.1)-(3.3), we need the following auxiliary lemmas.

Lemma 3.2.1. *Let $0 < r \leq 1$ and let $h : J \to E$ be Bochner integrable. Then linear problem*

$$^cD^r y(t) = h(t),\ t \in J_k := (s_k, t_{k+1}], \quad k = 0, \dots, m, \tag{3.4}$$
$$y(t) = g_k(t),\ t \in J'_k := (t_k, s_k] \quad k = 1, \dots, m, \tag{3.5}$$
$$y(0) = y_0, \tag{3.6}$$

has a unique solution which is given by :

$$y(t) = \begin{cases} y_0 + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}h(s)ds & \textit{if } t \in [0, t_1], \\ g_k(t), & \textit{if } t \in J'_k \quad k = 1, \dots, m, \\ g_k(s_k) + \frac{1}{\Gamma(r)}\int_{s_k}^t (t-s)^{r-1}h(s)ds, & \textit{if } t \in J_k \quad k = 1, \dots, m. \end{cases} \tag{3.7}$$

Proof. Assume that y satisfies (3.4)-(3.6).
If $t \in [0, t_1]$ then

$$^cD^r y(t) = h(t).$$

Lemma 1.3.2 implies

$$y(t) = y_0 + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}h(s)ds.$$

If $t \in J'_1 = (t_1, s_1]$ we have $y(t) = g_1(t)$.
If $t \in J_1 = (s_1, t_2]$, then Lemma 1.3.2 implies

$$\begin{aligned} y(t) &= y(s_1^+) + \frac{1}{\Gamma(r)}\int_{s_1}^t (t-s)^{r-1}h(s)ds \\ &= g_1(s_1) + \frac{1}{\Gamma(r)}\int_{s_1}^t (t-s)^{r-1}h(s)ds. \end{aligned}$$

If $t \in J'_2 = (t_2, s_2]$ we have $y(t) = g_2(t)$.
If $t \in J_2 = (s_2, t_3]$ then again Lemma 1.3.2 implies

$$\begin{aligned} y(t) &= y(s_2^+) + \frac{1}{\Gamma(r)}\int_{s_2}^t (t-s)^{r-1}h(s)ds \\ &= g_2(s_2) + \frac{1}{\Gamma(r)}\int_{s_2}^t (t-s)^{r-1}h(s)ds. \end{aligned}$$

If $t \in J'_k = (t_k, s_k]$ we have $y(t) = g_k(t)$.
If $t \in J_k = (s_k, t_{k+1}]$ then Lemma 1.3.2 implies

$$\begin{aligned} y(t) &= y(s_k^+) + \frac{1}{\Gamma(r)} \int_{s_k}^{t} (t-s)^{r-1} h(s) ds \\ &= g_k(s_k) + \frac{1}{\Gamma(r)} \int_{s_k}^{t} (t-s)^{r-1} h(s) ds. \end{aligned}$$

Conversely, assume that y satisfies equation (3.7).
If $t \in [0, t_1]$, then $y(0) = y_0$ and, using the fact that ${}^cD^r$ is the left inverse of I^r, we get

$$ {}^cD^r y(t) = h(t), \ t \in (0, t_1]. $$

If $t \in J_k := (s_k, t_{k+1}], \quad k = 1, \ldots, m$, and using the fact that ${}^cD^r C = 0$, where C is a constant, we get

$$ {}^cD^r y(t) = h(t), \ \ t \in J_k := (s_k, t_{k+1}], \quad k = 1, \ldots, m. $$

Also, we have easily that

$$ y(t) = g_k(t), \ \ t \in J'_k := (t_k, s_k], \quad k = 1, \ldots, m. $$

We are now in a position to state and prove our existence result for the problem (3.1)–(3.3) based on Mönch's fixed point. Let us list some conditions on the functions involved in the problem (3.1)–(3.3).

(H1) The function $f : J \times E \to E$ satisfies the Carathéodory conditions.

(H2) There exists $p \in C(J, \mathbb{R}_+)$ such that

$$ \|f(t, y)\| \leq p(t)\|y\| \text{ for any } y \in E \text{ and } t \in J. $$

(H3) g_k are uniformly continuous functions and there exists $c_k \in C(J, \mathbb{R}_+)$ such that

$$ \|g_k(t, y)\| \leq c_k(t)\|y\|, \text{ for each } y \in E \text{ and } t \in J, \ k = 1, \ldots, m. $$

(H4) For each bounded set $B \subset E$ we have

$$ \alpha(g_k(t, B)) \leq c_k(t)\alpha(B), \ t \in J. $$

(H5) For each bounded set $B \subset E$ we have

$$\alpha(f(t,B)) \le p(t)\alpha(B),\ t \in J.$$

Let

$$p^* = \sup_{t\in J} p(t),\ c^* = \max_{k=1,\dots,m}(\sup_{t\in J}(c_k(t))).$$

Theorem 3.2.1. *Assume that assumptions (H1)-(H5) hold. If*

$$\frac{p^*T^r}{\Gamma(r+1)} + c^* < 1, \tag{3.8}$$

then the problem (3.1)–(3.3) has at least one solution defined on J.

Proof. Transform the problem (3.1)–(3.3) into a fixed point problem. Consider the operator $N : PC(J,E) \to PC(J,E)$ defined by

$$N(y)(t) = \begin{cases} y_0 + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} f(s,y(s))ds & \text{if } t \in [0,t_1], \\ g_k(t,y(t)), & \text{if } t \in J'_k, k = 1,\dots,m, \\ g_k(s_k,y(s_k)) + \frac{1}{\Gamma(r)}\int_{s_k}^t (t-s)^{r-1} f(s,y(s))ds, & \text{if } t \in J_k, k = 1,\dots,m. \end{cases} \tag{3.9}$$

Clearly, the fixed points of operator N are solutions of problem (3.1)–(3.3). Let

$$r_0 \ge \frac{\|y_0\|}{1 - \frac{p^*T^r}{\Gamma(r+1)} - c^*}, \tag{3.10}$$

and consider the set

$$D_{r_0} = \{y \in PC(J,E) : \|y\|_\infty \le r_0\}.$$

Clearly, the subset D_{r_0} is closed, bounded and convex. We shall show that N satisfies the assumptions of Theorem 1.5.1. The proof will be given in a three of steps.

Step 1: *N is continuous.*
Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $PC(J,E)$. Then

for $t \in J_k$, we have

$$\begin{aligned} \|N(y_n)(t) - N(y)(t)\| &\le \|g_k(t,y_n(t)) - g_k(t,y(t))\| \\ &\quad + \frac{1}{\Gamma(r)}\int_{s_k}^t (t_k - s)^{r-1}\|f(s,y_n(s)) - f(s,y(s))\|ds, \end{aligned}$$

for $t \in [0, t_1]$, we have

$$\|N(y_n)(t) - N(y)(t)\| \leq \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} \|f(s, y_n(s)) - f(s, y(s))\| ds,$$

and for $t \in J'_k$, we have

$$\|N(u_n)(t) - N(u)(t)\| \leq \|g_k(t, y_n(t)) - g_k(t, y(t))\|.$$

Since g_k is continuous and f is of Carathéodory type, the Lebesgue dominated convergence theorem implies

$$\|N(u_n) - N(u)\|_\infty \to 0 \; as \; n \to \infty.$$

Consequently, N is continuous.

Step 2: *N maps D_{r_0} into itself.*
For each $y \in D_{r_0}$, by (H2),(H3),(3.8) and (3.10) we have:
for each $t \in [0, t_1]$,

$$\begin{aligned} \|N(y)(t)\| &\leq \|y_0\| + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} \|f(s, y(s))\| ds \\ &\leq \|y_0\| + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} p(t) \|y(t)\| ds \\ &\leq \|y_0\| + r_0 \left(\frac{p^* T^r}{\Gamma(r+1)} + c^* \right) \\ &\leq r_0. \end{aligned}$$

For each $t \in J'_k$,

$$\begin{aligned} \|N(y)(t)\| &\leq \|g_k(t, y(t))\| \\ &\leq \|y_0\| + r_0 \left(\frac{p^* T^r}{\Gamma(r+1)} + c^* \right) \\ &\leq r_0. \end{aligned}$$

For each $t \in J_k$,

$$\begin{aligned}
\|N(y)(t)\| &\leq \|g_k(s_k, y(s_k))\| + \frac{1}{\Gamma(r)}\int_{s_k}^{t}(t-s)^{r-1}\|f(s,y(s))\|ds \\
&\leq c_k\|y(s_k)\| + \frac{1}{\Gamma(r)}\int_{s_k}^{t}(t-s)^{r-1}p(t)\|y(t)\|ds \\
&\leq \|y_0\| + r_0\left(\frac{p^*T^r}{\Gamma(r+1)} + c^*\right) \\
&\leq r_0.
\end{aligned}$$

Step 3: $N(D_{r_0})$ *is bounded and equicontinuous.*
By Step2, it is obvious that $N(D_{r_0}) \subset PC(J,E)$ is bounded.
For the equicontinuous of $N(D_{r_0})$, let $\tau_1, \tau_2 \in J, \tau_1 < \tau_2$ and $y \in D_{r_0}$. Then, for $\tau_1, \tau_2 \in J_k$, we have

$$\begin{aligned}
\|N(y)(\tau_2) - N(y)(\tau_1)\| &= \frac{1}{\Gamma(r)}\int_{\tau_1}^{\tau_2} |(\tau_2 - s)^{r-1} - (\tau_1 - s)^{r-1}|\|f(s,y(s)))\|ds \\
&\leq 2\frac{r_0p^*}{\Gamma(r+1)}[\tau_2^r - \tau_1^r],
\end{aligned}$$

for $\tau_1, \tau_2 \in [0, t_1]$, we have

$$\begin{aligned}
\|N(y)(\tau_2) - N(y)(\tau_1)\| &= \frac{1}{\Gamma(r)}\int_{\tau_1}^{\tau_2} |(\tau_2 - s)^{r-1} - (\tau_1 - s)^{r-1}|\|f(s,y(s))\|ds. \\
&\leq 2\frac{r_0p^*}{\Gamma(r+1)}[\tau_2^r - \tau_1^r],
\end{aligned}$$

and for $\tau_1, \tau_2 \in J'_k$, we have

$$\|N(y)(\tau_2) - N(y)(\tau_1)\| = \|g_k(\tau_2, y(\tau_2)) - g_k(\tau_1, y(\tau_1))\|.$$

As $\tau_1 \to \tau_2$, the right-hand side of the above inequality tens to zero.

Now let V be a subset of D_{r_0} such that $V \subset \overline{conv}(N(V) \cup \{0\})$. Then V is bounded and equicontinuous and therefore the function $t \to v(t) = \alpha(V(t))$ is continuous on J. By (H4),(H5), Lemma 1.5.1 and the properties of the measure α we have for each $t \in J$

$$\begin{aligned}
v(t) &\leq \alpha(N(V)(t) \cup \{0\}) \\
&\leq \alpha(N(V)(t)).
\end{aligned}$$

If $t \in J_k$,

$$\begin{aligned}
v(t) &\leq \alpha(g_k(s_k, V(s_k)) + \frac{1}{\Gamma(r)} \int_{s_k}^{t} (t-s)^{r-1} f(s, V(s))ds) \\
&\leq c_k(t)\alpha(V(s)) + \frac{1}{\Gamma(r)} \int_{s_k}^{t} (t-s)^{r-1} p(t)\alpha(V(s))ds) \\
&\leq c_k(t)v(s) + \frac{1}{\Gamma(r)} \int_{s_k}^{t} (t-s)^{r-1} p(t)v(s)ds) \\
&\leq \|v\|_\infty \left(c^* + \frac{p^* T^r}{\Gamma(r+1)} \right),
\end{aligned}$$

if $t \in [0, t_1]$

$$\begin{aligned}
v(t) &\leq \alpha(\frac{1}{\Gamma(r)} \int_{0}^{t} (t-s)^{r-1} f(s, V(s))ds) \\
&\leq \frac{1}{\Gamma(r)} \int_{0}^{t} (t-s)^{r-1} p(t)\alpha(V(s))ds) \\
&\leq \frac{1}{\Gamma(r)} \int_{0}^{t} (t-s)^{r-1} p(t)v(s)ds) \\
&\leq \|v\|_\infty \left(\frac{p^* T^r}{\Gamma(r+1)} \right) \\
&\leq \|v\|_\infty \left(c^* + \frac{p^* T^r}{\Gamma(r+1)} \right),
\end{aligned}$$

if $t \in J'_k$

$$\begin{aligned}
v(t) &\leq \alpha(g_k(s_k, V(s_k)) \\
&\leq c_k(t)\alpha(V(s)) \\
&\leq c_k(t)v(s) \\
&\leq \|v\|_\infty c^* \\
&\leq \|v\|_\infty \left(c^* + \frac{p^* T^r}{\Gamma(r+1)} \right).
\end{aligned}$$

This means that

$$\|v\|_\infty \left[1 - \left(c^* + \frac{p^* T^r}{\Gamma(r+1)} \right) \right] \leq 0.$$

By (3.8) it follows that $\|v\|_\infty = 0$; that is, $v(t) = 0$ for each $t \in J$, and then $V(t)$ is relatively compact in E. In view of the Arzéla-Ascoli theorem, V is relatively compact in D_{r_0}. Applying now Theorem 1.5.1 we conclude that N has a fixed point which is a solution of the problem (3.1)-(3.3).

□

3.3 Nonlocal Fractional Differential Equations With Non instantaneous Impulses

This section is concerned with a generalization of the results presented in the previous section to nonlocal impulsive fractional differential equations. More precisely we shall present some existence results for the following nonlocal problem

$$^cD^r y(t) = f(t, y(t)), \text{ for a.e. } t \in (s_k, t_{k+1}],\ k = 0, \ldots, m,\ 0 < r \leq 1, \tag{3.11}$$

$$y(t) = g_k(t, y(t)),\ t \in (t_k, s_k],\ k = 1, \ldots, m, \tag{3.12}$$

$$y(0) + \psi(y) = y_0, \tag{3.13}$$

where $f, g_k, k = 1, \ldots, m$ are as in section 3.2 and $\psi : PC(J, E) \to E$ is a continuous function. Let us introduce the following set of conditions.

(H6) There exists a constant $M^* > 0$ such that

$$\|\psi(u)\| \leq M^* \|u\|_{PC}\ for\ each\ u \in PC(J, E).$$

(H7) For each bounded set $B \subset PC(J, E)$ we have

$$\alpha(\psi(B)) \leq M^* \alpha(B).$$

Theorem 3.3.1. *Assume that assumptions (H1)-(H7) hold.If*

$$\frac{p^* T^r}{\Gamma(r+1)} + c^* + M^* < 1, \tag{3.14}$$

then the nonlocal problem (3.11)–(3.13) has at least one solution defined on J.

Proof. Transform the problem (3.11)–(3.13) into a fixed point problem. Consider the operator $\tilde{N} : PC(J,E) \to PC(J,E)$ defined by

$$\tilde{N}(y)(t) \begin{cases} y_0 - \psi(y) + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f(s,y(s))ds & \text{if } t \in [0,t_1], \\ g_k(t,y(t)), & \text{if } t \in J_k', k = 1,\dots,m, \\ g_k(s_k,y(s_k)) + \frac{1}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} f(s,y(s))ds, & \text{if } t \in J_k k = 1,\dots,m. \end{cases} \quad (3.15)$$

Clearly, the fixed points of the operator $\tilde{N}$ are solution of the problem (3.11)–(3.13). We can easily show the conditions of Theorem 1.5.1 are satisfied by $\tilde{N}$.

□

3.4 An Example

Let us consider the following infinite system of impulsive fractional initial value problem,

$$^cD^{\frac{1}{2}} y_n(t) = \frac{1}{9+n+e^t} \ln(1+|y_n(t)|), \text{ for a.e. } t \in \left(0, \frac{1}{3}\right] \cup \left(\frac{1}{2}, 1\right], \quad (3.16)$$

$$y_n(t) = \frac{1}{4+n+e^t} \sin|y_n(t)|, \ t \in \left(\frac{1}{3}, \frac{1}{2}\right], \quad (3.17)$$

$$y_n(0) = 0. \quad (3.18)$$

Set

$$E = l^1 = \{y = (y_1, y_2, \dots, y_n, \dots,), \sum_{n=1}^{\infty} |y_n| < \infty\},$$

E is a Banach space with the norm

$$\|y\| = \sum_{n=1}^{\infty} |y_n|.$$

Let

$$f(t,y) = (f_1(t,y), f_2(t,y), \dots, f_n(t,y), \dots),$$

$$f_n(t,y) = \frac{\ln(1+|y_n(t)|)}{9+n+e^t},$$

and

$$g_1(t,y) = (g_{1_1}(t,y), g_{1_2}(t,y), \ldots, g_{1_n}(t,y), \ldots),$$

$$g_{1_n}(t,y) = \frac{\sin|y_n(t)|}{4+n+e^t}.$$

Clearly conditions (H2) and (H3) hold with

$$p(t) = \frac{1}{9+e^t}, \text{ and } c_1(t) = \frac{1}{4+e^t}.$$

We shall check that condition (3.8) is satisfied with $r = \frac{1}{2}, T = 1, P^* = \frac{1}{10}$ and $c^* = \frac{1}{5}$. Indeed

$$\left(\frac{p^*T^r}{\Gamma(r+1)} + c^*\right) = \frac{1}{5\sqrt{\pi}} + \frac{1}{5} < 1.$$

Then by Theorem 3.2.1 the problem (3.16)-(3.18) has at least one solution.

CHAPTER 4

FRACTIONAL DIFFERENTIAL INCLUSIONS WITH NON INSTANTANEOUS IMPULSES IN BANACH SPACES [3]

Contents

(3) [22] M. Benchohra and M.Slimane,Fractional differential inclusions with non instantaneous impulses in Banach spaces. *Results in Nonlinear Analysis* **2** (1) (2019), 36-47

4.1 Introduction

In this chapter, we will continue to generalize the results obtained in chapter 3 by giving the results of the existence of solutions to a inclusions problem and will be organized as follows: in Section 2, we deal with the existence of solutions of fractional differential inclusions with non instantaneous impulses

$$^cD^r y(t) \in F(t, y(t)), \text{ for each } t \in (s_k, t_{k+1}],\ k = 0, \dots, m,\ 0 < r \leq 1, \quad (4.1)$$

$$y(t) = g_k(t, y(t)), t \in (t_k, s_k],\ k = 1, \dots, m, \quad (4.2)$$

$$y(0) = y_0, \quad (4.3)$$

where $^cD^r$ is the Caputo fractional derivative, $F : [0, T] \times E \to \mathcal{P}(E)$ is a multivalued map, $g_k : (t_k, s_k] \times E \to E,\ k = 1, \dots, m$, is a given function, and $y_0 \in E, 0 = s_0 < t_1 < s_1 < \cdots < t_m < s_m < t_{m+1} = T$.
in section 3,we study the fractional differential inclusions with non instantaneous impulses and nonlocal initial conditions, of the form

$$^cD^r y(t) \in F(t, y(t)), \text{ for each } t \in (s_k, t_{k+1}],\ k = 0, \dots, m,\ 0 < r \leq 1,$$

$$y(t) = g_k(t, y(t)), t \in (t_k, s_k],\ k = 1, \dots, m,$$

$$y(0) + \psi(y) = y_0,$$

where $\psi : PC(J, E) \to E$ is a continuous function.
We use analogize Mönch's fixed point theorem (multivalued's version) combined with the technique of measures of noncompactness.

4.2 Existence of solution

First of all, we define what we mean by a solution of the problem (4.1)-(4.3).

Definition 4.2.1. A function $y \in PC(J, E) \cap AC(J', E)$ is said to be a solution of (4.1)-(4.3) if there exists a function $f \in L^1(J, E)$ with $f(t) \in F(t, y(t))$, for a.e. $t \in J$, such that

$$^cD^r y(t) = f(t), for\ a.e.\ t \in J, 0 < r \leq 1,$$

and the function y satisfies conditions (4.2)-(4.3).

We are now in a position to state and prove our existence result for the problem (4.1)–(4.3) based on Mönch's fixed point. Let us list the following conditions.

(H1) $F : J \times E \to \mathcal{P}_{kc}(E)$ is a Carathéodory multi-valued map.

(H2) There exists a function $p \in L^\infty(J, \mathbb{R}_+)$ such that

$$\|F(t,y)\|_{\mathcal{P}} = sup\{\|v\|_\infty : v(t) \in F(t,y)\} \leq p(t), \text{ for each } (t,y) \in J \times E.$$

(H3) For each bounded and measurable set $B \subset C(J,E)$ and for each $t \in J$, we have

$$\alpha(F(t,B(t)) \leq p(t)\alpha(B(t)),$$

where $B(t) = \{u(t) : u \in B\}$.

(H4) g_k are uniformly continuous functions and there exists $c_k \in C(J, \mathbb{R}_+)$ such that

$$\|g_k(t,y)\| \leq c_k(t)\|y\|, \ for\ each\ y \in E\ and\ t \in J, \ k = 1, \dots, m.$$

(H5) The function $\phi \equiv 0$ is the unique solution in $PC(J;E)$ of the inequality

$$\Phi(t) \leq \frac{2p^*}{1-c^*} \int_{s_k}^t \Phi(s)ds, k = 0, \dots, m.$$

(H6) For each bounded set $B \subset E$ we have

$$\alpha(g_k(t,B)) \leq c_k(t)\alpha(B), t \in J.$$

Let

$$p^* = esssup_{t \in J} p(t),$$

$$c^* = \max\{\sup_{t \in J} c_k(t) : k = 1, \dots, m\} < 1. \tag{4.4}$$

Remark 4.2.1. *In (H3) and (H6), α is the Kuratowski measure of noncompactness on the space E.*

Theorem 4.2.1. *Assume that assumptions (H1) - (H6) hold. Then the problem (4.1)–(4.3) has at least one solution defined on J.*

Proof. Transform the problem (4.1)–(4.3) into a fixed point problem. Consider the multi-valued map $N : PC(J,E) \to \mathcal{P}\left(PC(J,E)\right)$ defined by
$N(y)(t) = \{h \in PC(J,E) :$

$$h(t) = \begin{cases} y_0 + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} f(s)ds & \text{if } t \in [0,t_1], \\ g_k(t, y(t)), & \text{if } t \in J'_k, k = 1,\dots,m, \\ g_k(s_k, y(s_k)) + \frac{1}{\Gamma(r)}\int_{s_k}^t (t-s)^{r-1} f(s))ds, & \text{if } t \in J_k, k = 1,\dots,m, \end{cases}$$

$, f \in S_{F,y}\}$

Clearly, the fixed points of operator N are solutions of problem (4.1)–(4.3). The proof will be given in a couple of steps.

Step 1: *N is convex for each $y \in PC(J,E)$.*
If h_1, h_2 belong to $N(y)$, then there exist $f_1, f_2 \in S_{F,y}$ such that for each $t \in J$ we have

$$h_i(t) = \begin{cases} y_0 + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} f_i(s)ds & \text{if } t \in [0,t_1], \\ g_k(t, y(t)), & \text{if } t \in J'_k, k = 1,\dots,m, \; i = 1,2 \\ g_k(s_k, y(s_k)) + \frac{1}{\Gamma(r)}\int_{s_k}^t (t-s)^{r-1} f_i(s))ds, & \text{if } t \in J_k, k = 1,\dots,m. \end{cases}$$

Let $0 \le \lambda \le 1$. For each $t \in J$, we have
$(\lambda h_1 + (1-\lambda)h_2)(t) =$

$$\begin{cases} y_0 + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} (\lambda f_1 + (1-\lambda)f_2)(s)ds & \text{if } t \in [0,t_1], \\ g_k(t, y(t)), & \text{if } t \in J'_k, \\ g_k(s_k, y(s_k)) + \frac{1}{\Gamma(r)}\int_{s_k}^t (t-s)^{r-1} (\lambda f_1 + (1-\lambda)f_2)(s))ds, & \text{if } t \in J_k. \end{cases}$$

Since $S_{F,y}$ is convex (because F has convex values), we have

$$(\lambda h_1 + (1-\lambda)h_2) \in N(y).$$

Step 2: *For each compact $M \in \bar{U}$, $N(M)$ is relatively compact .*
To prove this, let $M \in \bar{U}$ be a compact set and let (h_n) be any sequence of elements $N(M)$. We show that (h_n) has a convergent subsequence by using Arzéla-Ascoli criterion of noncompactness in $PC(J,E)$. Since $(h_n) \in$

$N(M)$ there exist $(y_n) \in M$ and $(f_n) \in S_{F,y_n}$ such that

$$h_n(t) = \begin{cases} y_0 + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f_n(s)ds & \text{if } t \in [0, t_1], \\ g_k(t, y_n(t)), & \text{if } t \in J_k', k = 1, \dots, m, \\ g_k(s_k, y_n(s_k)) + \frac{1}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} f_n(s))ds, & \text{if } t \in J_k, k = 1, \dots, m. \end{cases}$$

Using Theorem 1.5.2, the properties of measure of α and (H5), we obtain that

$$\alpha(\{h_n(t)\}) = \alpha \left(\begin{cases} y_0 + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f_n(s)ds & \text{if } t \in [0, t_1], \\ g_k(t, y_n(t)), & \text{if } t \in J_k', k = 1, \dots, m, \\ g_k(s_k, y_n(s_k)) + \frac{1}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} f_n(s))ds, & \text{if } t \in J_k, k = 1, \dots, m. \end{cases} \right)$$

If $t \in [0, t_1]$

$$\begin{aligned} \alpha(\{h_n(t)\}) &\leq \frac{2}{\Gamma(r)} \int_0^t \alpha\{(t-s)^{r-1} f_n(s)ds\} \\ &= \frac{2}{\Gamma(r)} \int_0^t (t-s)^{r-1} \alpha\{f_n(s)ds\}. \end{aligned} \tag{4.5}$$

If $t \in J_k$, we have

$$\begin{aligned} \alpha(\{h_n(t)\}) &\leq \alpha\{g_k(s_k, y_n(s_k))\} + \frac{2}{\Gamma(r)} \int_{s_k}^t \alpha\{(t-s)^{r-1} f_n(s)\}ds \\ &\leq c_k(t)\alpha\{y_n(s_k)\} + \frac{2}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} \alpha\{f_n(s)\}ds \\ &\leq c^* \alpha\{y_n(t)\} + \frac{2}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} \alpha\{f_n(s)\}ds. \end{aligned} \tag{4.6}$$

If $t \in J_k'$

$$\begin{aligned} \alpha(\{h_n(t)\}) &= \alpha(g_k(t, y_n(t)) \\ &\leq c_k(t)\alpha\{y_n(t)\} \\ &\leq c^* \alpha\{y_n(t)\}. \end{aligned} \tag{4.7}$$

On the other hand, since M is compact in $\overline{U}$, the sets $\{f_n(s), n \geq 1\}$, $\{y_n(t), n \geq 1\}$ are compact. Consequently, $\alpha\{f_n(s), n \geq 1\} = 0$ for a.e.$s \in J$ and $\alpha\{y_n(t), n \geq 1\} = 0$ for a.e. $t \in J$. we conclude that $\{h_n(t), n \geq 1\}$ is relatively compact in E , for each $t \in J$. In addition let τ_1 and τ_2 from J , $\tau_1 < \tau_2$. Then, for $\tau_1, \tau_2 \in J_k$, we have

$$\begin{aligned}\|h_n(\tau_2) - |h_n(\tau_1)\| &= \|\tfrac{1}{\Gamma(r)}\int_{\tau_1}^{\tau_2}((\tau_2 - s)^{r-1} - (\tau_1 - s)^{r-1})f_n(s)\|ds \\ &\le \tfrac{1}{\Gamma(r)}\int_{\tau_1}^{\tau_2}|(\tau_2 - s)^{r-1} - (\tau_1 - s)^{r-1}|p(s)ds,\end{aligned} \tag{4.8}$$

for $\tau_1, \tau_2 \in [0, t_1]$, we have

$$\begin{aligned}\|h_n(\tau_2) - |h_n(\tau_1)\| &= \|\tfrac{1}{\Gamma(r)}\int_{\tau_1}^{\tau_2}((\tau_2 - s)^{r-1} - (\tau_1 - s)^{r-1})f_n(s)\|ds. \\ &\le \tfrac{1}{\Gamma(r)}\int_{\tau_1}^{\tau_2}|(\tau_2 - s)^{r-1} - (\tau_1 - s)^{r-1}|p(s)ds,\end{aligned} \tag{4.9}$$

and for $\tau_1, \tau_2 \in J'_k$, we have

$$\|h_n(\tau_2) - |h_n(\tau_1)\| = \|g_k(\tau_2, y_n(\tau_2)) - g_k(\tau_1, y_n(\tau_1))\|. \tag{4.10}$$

As $\tau_2 \to \tau_2$, the right hand side of the above inequality tends to zero. This shows that $\{h_n(t), n \ge 1\}$ is equicontinuous. Consequently, $\{h_n, n \ge 1\}$ is relatively compact in $PC(J, E)$.

Step 3: *N has a closed graph.*
Let $(y_n, h_n) \in graph(N), n \ge 1\}$, with $\|y_n - y\|, \|h_n - h\| \to 0$ as $n \to \infty$. We must show that $(y, h) \in graph(N)$. $(y_n, h_n) \in graph(N)$ means that $h_n \in N(y_n)$ which means that there exists $f_n \in S_{F,y_n}$, such that for each $t \in J$,

$$h_n(t) = \begin{cases} y_0 + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} f_n(s)ds & \text{if } t \in [0, t_1], \\ g_k(t, y_n(t)), & \text{if } t \in J'_k, k = 1, \dots, m, \\ g_k(s_k, y_n(s_k)) + \frac{1}{\Gamma(r)}\int_{s_k}^t (t-s)^{r-1} f_n(s))ds, & \text{if } t \in J_k, k = 1, \dots, m. \end{cases}$$

Let

$$a_n(t) = \begin{cases} y_0 & \text{if } t \in [0, t_1], \\ g_k(t, y_n(t)), & \text{if } t \in J'_k, k = 1, \dots, m, \\ g_k(s_k, y_n(s_k)), & \text{if } t \in J_k, k = 1, \dots, m. \end{cases}$$

$$a(t) = \begin{cases} y_0 & \text{if } t \in [0, t_1], \\ g_k(t, y(t)), & \text{if } t \in J'_k, k = 1, \dots, m, \\ g_k(s_k, y(s_k)), & \text{if } t \in J_k, k = 1, \dots, m. \end{cases}$$

We have $\|a_n - a\| \to 0$ as $n \to \infty$.
Consider the continuous linear operator

$$\Theta : L^{\infty}(J,E) \longrightarrow C(J,E)$$
$$f \longmapsto \Theta(f)(t) = \begin{cases} \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f(s)ds & \text{if } t \in [0,t_1], \\ 0, & \text{if } t \in J'_k, k = 1,\dots,m, \\ \frac{1}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} f(s))ds, & \text{if } t \in J_k, k = 1,\dots,m. \end{cases}$$

We have

$$\begin{aligned} \|(h_n - a_n)(t) - (h-a)(t)\| &= \|(h_n - h)(t) + (a - a_n)(t)\|. \\ &\leq \|(h_n - h)(t)\| + \|(a - a_n)(t)\|, \end{aligned}$$

implies that

$$\|(h_n - a_n)(t) - (h-a)(t)\| \longrightarrow 0 \text{ as } n \to \infty.$$

From Lemma 1.4.1 it follows that $\Theta \circ S_F$ is a closed graph operator. Moreover, we have

$$(h_n - a_n)(t) \in \Theta(S_{F,y_n})$$

Since $y_n \longrightarrow y$, Lemma 1.4.1 implies that

$$(h_n - a_n)(t) = \begin{cases} \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f(s)ds & \text{if } t \in [0,t_1], \\ 0, & \text{if } t \in J'_k, k = 1,\dots,m, \\ \frac{1}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} f(s))ds, & \text{if } t \in J_k, k = 1,\dots,m, \end{cases}$$

for some $f \in S_{F,y}$.

Step 4: *M relatively compact in $PC(J,E)$.*

Let $M \subset U$, where $M \subset$conv$(\{0\} \cup N(M))$ and for some countable set $C \subset M$ let $\overline{M} = \overline{C}$. Taking into account (4.8)-(4.10) , it is easily seen that $N(M)$ is equicontinuous. Therefore, $M \subset$conv$(\{0\} \cup N(M))$ implies that M is equicontinuous. It remains to apply the Arzéla-Ascoli theorem to show that for each $t \in I$ the set $M(t)$ is relatively compact. By taking into account that C is countable and $C \subset M \subset$conv$(0 \cup N(M))$, we can find a countable set $H = \{h_n : n \geq 1\} \subset N(M)$ such that $C \subset \text{conv}(\{0\} \cup H)$. Then, there are $y_n \in M$ and $f_n \in S_{F,y_n}$ with

$$h_n(t) = \begin{cases} y_0 + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f_n(s)ds & \text{if } t \in [0, t_1], \\ g_k(t, y_n(t)), & \text{if } t \in J'_k, k = 1, \ldots, m, \\ g_k(s_k, y_n(s_k)) + \frac{1}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} f_n(s))ds, & \text{if } t \in J_k, k = 1, \ldots, m. \end{cases}$$

Taking into account Theorem 1.5.2 and the fact that $M \subset \overline{C} \subset \overline{conv}(\{0\} \cup H)$, we obtain

$$\alpha(M(t)) \leq (\alpha(\overline{C}(t)) \leq \alpha(H(t)) = \alpha\{h_n(t) : n \leq 1\}).$$

Using (4.5)-(4.7), we obtain
if $t \in [0, t_1]$

$$\alpha(\{M(t)\}) \leq \frac{2}{\Gamma(r)} \int_0^t (t-s)^{r-1} \alpha\{f_n(s)ds, n \geq 1\},$$

if $t \in J_k$, we have

$$\alpha(\{M(t)\}) \leq c^* \alpha\{y_n(t)\} + \frac{2}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} \alpha\{f_n(s), n \geq 1\} ds,$$

if $t \in J'_k$

$$\alpha(\{M(t)\}) \leq c^* \alpha\{y_n(t), n \geq 1\}.$$

Also, since $f_n \in S_{F,y_n}$ and $y_n(s) \in M(s)$, then from (H3) we have

$$(t-s)^{r-1} \alpha\{f_n(s)ds, n \geq 1\} = (t-s)^{r-1} p(s) \alpha(M(s))ds.$$

It follows that
if $t \in [0, t_1]$

$$\begin{aligned} \alpha(\{M(t)\}) &\leq \frac{2p^*}{\Gamma(r)} \int_0^t (t-s)^{r-1} \alpha(M(s))ds, \\ &\leq \frac{2p^*}{(1-c^*)\Gamma(r)} \int_0^t (t-s)^{r-1} \alpha(M(s))ds, \end{aligned}$$

if $t \in J_k$, we have

$$\alpha(\{M(t)\}) \leq \frac{2p^*}{(1-c^*)\Gamma(r)} \int_{s_k}^{t} (t-s)^{r-1} \alpha(M(s)) ds,$$

if $t \in J'_k$

$$\alpha(\{M(t)\}) \leq c^* \alpha(M(t)) \Rightarrow (1-c^*)\alpha(M(t)) \leq 0.$$

Consequently, if $t \in [0, t_1] \cup J_k$, we have by (H5)and condition(4.4), the function Φ given by $\Phi(t) = \alpha(M(t))$ satisfies $\Phi \equiv 0$; that is, $\alpha(M(t)) = 0$ for all $t \in J$. Now, by the Arzéla-Ascoli theorem, M is relatively compact in $PC(J, E)$.

Step 5: *A priori estimate.*
Let $y \in PC(J, E)$ be such that $y \in \lambda N(y)$ for some $\lambda \in (0, 1)$. Then for each $t \in J$ we have

$$y(t) = \begin{cases} \lambda y_0 + \frac{\lambda}{\Gamma(r)} \int_0^t (t-s)^{r-1} f(s) ds & \text{if } t \in [0, t_1], \\ \lambda g_k(t, y(t)), & \text{if } t \in J'_k, k = 1, \dots, m, \\ \lambda g_k(s_k, y(s_k)) + \frac{\lambda}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} f(s) ds, & \text{if } t \in J_k, k = 1, \dots, m, \end{cases}$$

for some $f \in S_{F,y}$. On the other hand we have,

$$\begin{aligned} \|y(t)\| &\leq \|g_k(t, y(t))\| + \|y_0\| + \frac{1}{\Gamma(r)} \int_{s_k}^{t_{k+1}} (t-s)^{r-1} \|f(s)\| ds \\ &\leq c_k(t)\|y(t)\| + \|y_0\| + \frac{1}{\Gamma(r)} \int_{s_k}^{t_{k+1}} (t-s)^{r-1} p(s) ds \\ &\leq c_*\|y(t)\| + \|y_0\| + \frac{p^* T^r}{\Gamma(r+1)} \\ &\leq c^*\|y(t)\| + \|y_0\| + \frac{p^* T^r}{\Gamma(r+1)}. \end{aligned}$$

Then

$$\|y\| \leq \frac{1}{1-c^*} \left(\|y_0\| + \frac{p^* T^r}{\Gamma(r+1)} \right) := d.$$

Set

$$U = \{y \in PC(J, E) : \|y\| < d + 1\}.$$

Condition (1.5.3) is satisfied by our choice of the open set U. From Theorem 1.5.3, we conclude that N has at least one fixed point $y \in PC(J,E)$ being a solution of problem (4.1)-(4.3).

□

4.3 Nonlocal Fractional Differential Inclusions with Non Instantaneous Impulses

In this section we consider the following class of impulsive differential inclusion:

$$^{c}D^{r}y(t) \in F(t,y(t)),\ \text{for a.e. } t \in (s_k,t_{k+1}],\ k=0,\dots,m,\ 0<r\leq 1, \quad (4.11)$$

$$y(t) = g_k(t,y(t)), t \in (t_k,s_k],\ k=1,\dots,m, \quad (4.12)$$

$$y(0)+\psi(y) = y_0, \quad (4.13)$$

where $f, g_k, k=1,\dots,m$ are as in section 4.2 and $\psi : PC(J,E) \to E$ is a continuous function.

Let us introduce the following set of conditions.

(H7) There exists a constant $M^* > 0$ such that

$$\|\psi(u)\| \leq M^*\|u\|_{PC}\ for\ each\ u \in PC(J,E).$$

(H8) For each bounded set $B \subset PC(J,E)$ we have

$$\alpha(\psi(B)) \leq M^*\alpha(B).$$

Theorem 4.3.1. *Assume that assumptions (H1)-(H8) hold, then the nonlocal problem (4.11)–(4.13) has at least one solution defined on J.*

Proof. Transform the problem (4.11)–(4.13) into a fixed point problem. Consider the multi-valued map $\tilde{N} : PC(J,E) \to \mathcal{P}(PC(J,E))$ defined by
$\tilde{N}(y)(t) = \{h \in PC(J,E) :$

$$h(t) = \begin{cases} y_0 - \psi(y) + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}f(s)ds & \text{if } t \in [0,t_1], \\ g_k(t,y(t)), & \text{if } t \in J'_k, k=1,\dots,m, \\ g_k(s_k,y(s_k)) + \frac{1}{\Gamma(r)}\int_{s_k}^t (t-s)^{r-1}f(s))ds, & \text{if } t \in J_k, k=1,\dots,m. \end{cases} , f \in S_{F,y}\}$$

Clearly, the fixed points of operator $\tilde{N}$ are solutions of problem (4.11)–(4.13). We can easily show the conditions of Theorem 1.5.3 are satisfied by $\tilde{N}$.

□

4.4 An Example

Let us consider the following problem fractional differential inclusions with non instantaneous Impulses,

$$^{c}D^{\frac{1}{2}}y_n(t) \in \frac{1}{(9+n+e^t)(1+\|y(t)\|)}[y_n(t)-1, y_n(t)], \text{ for each } t \in \left(0, \frac{1}{3}\right] \cup \left(\frac{1}{2}, 1\right], \tag{4.14}$$

$$y_n(t) = \frac{1}{4+n+e^t}\sin|y_n(t)|, t \in \left(\frac{1}{3}, \frac{1}{2}\right], \tag{4.15}$$

$$y_n(0) = 0. \tag{4.16}$$

Set

$$E = l^1 = \{y = (y_1, y_2, \ldots, y_n, \ldots,), \sum_{n=1}^{\infty} |y_n| < \infty\},$$

E is a Banach space with the norm

$$\|y\| = \sum_{n=1}^{\infty} |y_n|.$$

Let

$$F(t, y) = (F_1(t, y), F_2(t, y), \ldots, F_n(t, y), \ldots),$$

$$F_n(t, y) = \frac{1}{(9+n+e^t)(1+\|y(t)\|)}[y_n(t) - 1, y_n(t)],$$

and

$$g_1(t, y) = (g_{1_1}(t, y), g_{1_2}(t, y), \ldots, g_{1_n}(t, y), \ldots),$$

$$g_{1_n}(t, y) = \frac{\sin|y_n(t)|}{4+n+e^t}.$$

Clearly F is closed and convex valued. For each $y \in E$ and $t \in [0,1]$, we have

$$\|F(t,y)\|_{\mathcal{P}} \leq \frac{1}{9+e^t}.$$

Hence, the hypothesis (H2) is satisfied with $p(t) = \frac{1}{9+e^t}$ and $c_1(t) = \frac{1}{4+e^t}$.

Since all conditions of Theorem 4.2.1 are satisfied with $p^* = \frac{1}{10}$ and $c^* = \frac{1}{5}$, problem (4.14)-(4.16) has at least one solution.

CHAPTER 5

FRACTIONAL DIFFERENTIAL INCLUSIONS WITH NON INSTANTANEOUS IMPULSES AND MULTIVALUED JUMP [(4)]

Contents

[(4)] [23] M. Benchohra and M. Slimane, Fractional differential inclusions with non instantaneous impulses and multivalued jump, *Libertas Mathematica* (accepted).

5.1 Introduction

There are very few results for impulsive differential inclusions with multivalued jump operator see [6,24,25,39]. In order to give more general results of the above in chapter 4 in this chapter, we study the following fractional differential inclusions with non instantaneous impulses and multivalued jump operator

$$^cD^r y(t) \in F(t,y(t)), \text{ a.e. } t \in J_k := (s_k, t_{k+1}],\ k = 0,\ldots,m,\ 0 < r \leq 1, \tag{5.1}$$

$$y(t) \in G_k(t,y(t)),\ t \in J'_k := (t_k, s_k],\ k = 1,\ldots,m, \tag{5.2}$$

$$y(0) = y_0, \tag{5.3}$$

where $^cD^r$ is the Caputo fractional derivative, $F : [0,T] \times E \to \mathcal{P}(E)$ is a multivalued map, $G_k : (t_k, s_k] \times E \to \mathcal{P}(E),\ k = 1,\ldots,m,$ is a given multivalued map, and $y_0 \in E, 0 = s_0 < t_1 < s_1 < \cdots < t_m < s_m < t_{m+1} = T$. Then we study the nonlocal case exactly the problem

$$^cD^r y(t) \in F(t,y(t)), \text{ a.e. } t \in J_k := (s_k, t_{k+1}],\ k = 0,\ldots,m,\ 0 < r \leq 1,$$

$$y(t) \in G_k(t,y(t)),\ t \in J'_k := (t_k, s_k],\ k = 1,\ldots,m,$$

$$y(0) + \psi(y) = y_0,$$

where $\psi : PC(J,E) \to E$ is a continuous function.
These results are obtained by the same method as in the previous chapter.

5.2 Existence of solution

First of all, we define what we mean by a solution of the problem (5.1)-(5.3).

Definition 5.2.1. A function $y \in PC(J,E)$ is said to be a solution of (5.1)-(5.3) if there exists a function $f \in L^1(J,E)$ with $f(t) \in F(t,y(t))$, for a.e. $t \in J$ and a function $g_k \in L^1(J_k,E)$ with $g_k(t) \in G_k(t,y(t))$, for a.e. $t \in J_k^{'}$ such that

$$^cD^r y(t) = f(t), for\ a.e.\ t \in J_k, k = 1,\ldots,m, 0 < r \leq 1,$$

$$y(t) = g_k(t), for\ a.e.\ t \in J'_k, k = 1,\ldots,m,$$

and the function y satisfies conditions (5.3).

To prove the existence of solutions to (5.1)-(5.3), we need the following auxiliary lemmas.

Lemma 5.2.1. *Let $0 < r \leq 1$ and let $h : J \to E$ be measurable. Then linear problem*

$$^cD^r y(t) = h(t), \quad t \in J_k, \quad k = 0, \ldots, m, \tag{5.4}$$

$$y(t) = \sigma_k(t), \quad t \in J_k^{'}, \quad k = 1, \ldots, m, \tag{5.5}$$

$$y(0) = y_0, \tag{5.6}$$

has a unique solution which is given by :

$$y(t) = \begin{cases} y_0 + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} h(s) ds & \text{if } t \in [0, t_1], \\ \sigma_k(t), & \text{if } t \in J_k', \ k = 1, \ldots, m, \\ \sigma_k(t) + \frac{1}{\Gamma(r)} \int_{t_k}^t (t-s)^{r-1} h(s) ds, & \text{if } t \in J_k, \ k = 1, \ldots, m. \end{cases} \tag{5.7}$$

We are now in a position to state and prove our existence result for the problem (5.1)–(5.3) based on Mönch's fixed point. Let us list some conditions on the functions involved in the (5.1)–(5.3).

(H1) $F : J \times E \to \mathcal{P}_{kc}(E)$ is a Carathéodory multivalued map.

(H2) There exists a function $p \in L^\infty(J, \mathbb{R}_+)$ such that

$$\|F(t,y))\|_{\mathcal{P}} = sup\{\|v\| : v(t) \in F(t,y)\} \leq p(t).$$

for each $(t, y) \in J \times E$.

(H3) For each bounded set $B \subset PC(J, E)$ and for each $t \in J$, we have

$$\alpha(F(t, B(t)) \leq p(t)\alpha(B(t)),$$

where $B(t) = \{u(t) : u \in B\}$.

(H4) $G_k : J \times E \to \mathcal{P}_{kc}(E)$ and there exists $c_k \in C(J, \mathbb{R}_+)$ such that

$$\|G_k(t,y))\|_{\mathcal{P}} = sup\{|u| : u(t) \in G_k(t,y)\} \leq c_k(t)\|y\|,$$

$for\ each\ y \in E\ and\ t \in J,\ k = 1, \ldots, m.$

(H5) For each constant $d > 0$, the function $\phi \equiv 0$ is the unique solution in $PC(J;E)$ of the inequality

$$\Phi(t) \le d \int_{s_k}^{t} (t-s)^{r-1}\Phi(s)ds, \ k = 0, \dots, m.$$

(H6) For each $y \in PC(J;E)$, the selection functions $g_k \in S_{G_k,y}$ are uniformly continuous on J.

(H7) For each bounded set $B \subset E$ we have

$$\alpha(G_k(t,B)) \le c_k(t)\alpha(B), \ t \in J.$$

Let

$$\begin{aligned} p^* &= esssup_{t\in J} p(t), \\ c^* &= \max_{k=1,\dots,m}(\sup_{t\in J}(c_k(t))) < 1. \end{aligned} \tag{5.8}$$

Remark 5.2.1. *In (H3) and (H7), α is the Kuratowski measure of noncompactness on the space E.*

Theorem 5.2.1. *Assume that assumptions (H1) - (H7) hold. Then the problem (5.1)–(5.3) has at least one solution defined on J.*

Proof. Transform the problem (5.1)–(5.3) into a fixed point problem. Consider the multi-valued map $N : PC(J,E) \to \mathcal{P}(PC(J,E))$ defined by

$$N(y)(t) = \{h \in PC(J,E) : h(t) = \begin{cases} y_0 + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} f(s)ds & \text{if } t \in [0,t_1], \\ g_k(t), & \text{if } t \in J'_k, k = 1,\dots,m \\ g_k(s_k) + \frac{1}{\Gamma(r)}\int_{s_k}^t (t-s)^{r-1} f(s))ds, & \text{if } t \in J_k, k = 1,\dots,m, \end{cases}$$

$$f \in S_{F,y}, g_k \in S_{G_k,y}\}.$$

Clearly, the fixed points of operator N are solutions of problem (5.1)–(5.3). The proof will be given in a couple of steps.

Step 1: *N is convex for each $y \in PC(J,E)$.*
If h_1, h_2 belong to $N(y)$, then there exist $f_1, f_2 \in S_{F,y}$ and $g_{k_1}, g_{k_2} \in S_{G_k,y}$

such that for a.e.$t \in J$ we have

$$h_i(t) = \begin{cases} y_0 + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f_i(s) ds & \text{if } t \in [0, t_1], \\ g_{k_i}(t), & \text{if } t \in J'_k, \qquad i = 1, 2 \\ g_{k_i}(s_k) + \frac{1}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} f_i(s)) ds, & \text{if } t \in J_k. \end{cases}$$

Let $0 \le \lambda \le 1$. For each $t \in J$, we have

$$(\lambda h_1 + (1-\lambda) h_2)(t) =$$

$$\begin{cases} y_0 + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} (\lambda f_1 + (1-\lambda) f_2)(s) ds & \text{if } t \in [0, t_1], \\ (\lambda g_{k_1} + (1-\lambda) g_{k_2})(t), & \text{if } t \in J'_k, \\ (\lambda g_{k_1} + (1-\lambda) g_{k_2})(s_k) + \frac{1}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} (\lambda f_1 + (1-\lambda) f_2)(s)) ds, & \text{if } t \in J_k. \end{cases}$$

Since $S_{F,y}, S_{G_k,y}$ are convex (because F, G_k has convex values),we have

$$(\lambda h_1 + (1-\lambda) h_2) \in N(y).$$

Step 2: *For each compact $M \in \bar{U}$ $N(M)$ is relatively compact .*
To prove this, let $M \in \bar{U}$ be a compact set and let (h_n) be any sequence of elements $N(M)$. By using Arzéla-Ascoli criterion of noncompactness in $PC(J, E)$. Since $(h_n) \in N(M)$ there exist $(y_n) \in M$, $(f_n) \in S_{F,y_n}$ and $g_{k_n} \in S_{G_k,y_n}$ such that

$$h_n(t) = \begin{cases} y_0 + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f_n(s) ds & \text{if } t \in [0, t_1], \\ g_{k_n}(t), & \text{if } t \in J'_k, \\ g_{k_n}(s_k) + \frac{1}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} f_n(s)) ds, & \text{if } t \in J_k. \end{cases}$$

Using Theorem 1.5.2,the properties of measure α, we obtain that

$$\alpha(\{h_n(t)\}) = \alpha \left(\begin{cases} y_0 + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f_n(s) ds & \text{if } t \in [0, t_1], \\ g_{k_n}(t), & \text{if } t \in J'_k, \\ g_{k_n}(s_k) + \frac{1}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} f_n(s)) ds, & \text{if } t \in J_k. \end{cases} \right)$$

If $t \in [0, t_1]$ we have

$$\begin{aligned} \alpha(\{h_n(t)\}) &\le \frac{2}{\Gamma(r)} \int_0^t \alpha\{(t-s)^{r-1} f_n(s) ds\} \\ &= \frac{2}{\Gamma(r)} \int_0^t (t-s)^{r-1} \alpha\{f_n(s) ds\}, \end{aligned} \tag{5.9}$$

if $t \in J_k$, we have

$$\begin{aligned}
\alpha(\{h_n(t)\}) & \leq \alpha\{g_{k_n}(s_k)\} + \tfrac{2}{\Gamma(r)} \int_{s_k}^{t} \alpha\{(t-s)^{r-1} f_n(s)\} ds \\
& \leq \alpha\{G_k(s_k, y_n(s_k))\} + \tfrac{2}{\Gamma(r)} \int_{s_k}^{t} \alpha\{(t-s)^{r-1} f_n(s)\} ds \\
& \leq c_k(t)\alpha\{y_n(s_k)\} + \tfrac{2}{\Gamma(r)} \int_{s_k}^{t} (t-s)^{r-1} \alpha\{f_n(s)\} ds \\
& \leq c^{*}\alpha\{y_n(t)\} + \tfrac{2}{\Gamma(r)} \int_{s_k}^{t} (t-s)^{r-1} \alpha\{f_n(s)\} ds,
\end{aligned} \tag{5.10}$$

if $t \in J'_k$ we obtain that

$$\begin{aligned}
\alpha(\{h_n(t)\}) & = \alpha\{g_{k_n}(t)\} \\
& \leq \alpha\{G_k(t, y_n(t)\} \\
& \leq c_k(t)\alpha\{y_n(t)\} \\
& \leq c^{*}\alpha\{y_n(t)\}.
\end{aligned} \tag{5.11}$$

On the other hand, by (H1)and since M is compact in $\overline{U}$, the sets $\{f_n(s), n \geq 1\}, \{y_n(t), n \geq 1\}$ are compact. Consequently, $\alpha\{f_n(s), n \geq 1\} = 0$ for a.e. $s \in J$ and $\alpha\{y_n(t), n \geq 1\} = 0$ for a.e. $t \in J$. we conclude that $\{h_n(t), n \geq 1\}$ is relatively compact in E , for each $t \in J$. In addition let τ_1 and τ_2 from J , $\tau_1 < \tau_2$. Then, for $\tau_1, \tau_2 \in J_k$, we have

$$\begin{aligned}
\|h_n(\tau_2) - |h_n(\tau_1)\| & = \| \tfrac{1}{\Gamma(r)} \int_{\tau_1}^{\tau_2} ((\tau_2 - s)^{r-1} - (\tau_1 - s)^{r-1}) f_n(s) \| ds \\
& \leq \tfrac{1}{\Gamma(r)} \int_{\tau_1}^{\tau_2} |(\tau_2 - s)^{r-1} - (\tau_1 - s)^{r-1}| p(s) ds.
\end{aligned} \tag{5.12}$$

For $\tau_1, \tau_2 \in [0, t_1]$, we have

$$\begin{aligned}
\|h_n(\tau_2) - |h_n(\tau_1)\| & = \| \tfrac{1}{\Gamma(r)} \int_{\tau_1}^{\tau_2} ((\tau_2 - s)^{r-1} - (\tau_1 - s)^{r-1}) f_n(s) \| ds. \\
& \leq \tfrac{1}{\Gamma(r)} \int_{\tau_1}^{\tau_2} |(\tau_2 - s)^{r-1} - (\tau_1 - s)^{r-1}| p(s) ds,
\end{aligned} \tag{5.13}$$

and for $\tau_1, \tau_2 \in J'_k$, we have

$$\|h_n(\tau_2) - |h_n(\tau_1)\| = \|g_{k_n}(\tau_2) - g_{k_n}(\tau_1)\|. \tag{5.14}$$

By (H6) as $\tau_1 \to \tau_2$, the right hand side of the above inequality tends to zero. This shows that $\{h_n(t), n \geq 1\}$ is equicontinuous. Consequently, $\{h_n, n \geq 1\}$ is relatively compact in $PC(J, E)$.

Step 3: *N has a closed graph.*
Let $(y_n, h_n) \in graph(N), n \geq 1\}$, with $\|y_n - y\|, \|h_n - h\| \to 0$ as $n \to \infty$. We must show that $(y, h) \in graph(N)$. $(y_n, h_n) \in graph(N)$ means that $h_n \in N(y_n)$ which means that there exists $f_n \in S_{F,y_n}$ and $g_{k_n} \in S_{G_k,y_n}$ such that, such that for each $t \in J$,

$$h_n(t) = \begin{cases} y_0 + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f_n(s) ds & \text{if } t \in [0, t_1], \\ g_{k_n}(t), & \text{if } t \in J'_k, \\ g_{k_n}(s_k) + \frac{1}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} f_n(s)) ds, & \text{if } t \in J_k. \end{cases}$$

Let

$$a_n(t) = \begin{cases} y_0 & \text{if } t \in [0, t_1], \\ g_{k_n}(t), & \text{if } t \in J'_k, \\ g_{k_n}(s_k), & \text{if } t \in J_k, \end{cases}$$

and

$$a(t) = \begin{cases} y_0 & \text{if } t \in [0, t_1], \\ g_k(t), & \text{if } t \in J'_k, \\ g_k(s_k), & \text{if } t \in J_k. \end{cases}$$

We have $\|a_n - a\| \to 0$ as $n \to \infty$.
Consider the continuous linear operator

$$\Theta : L^\infty(J, E) \longrightarrow C(J, E)$$

$$f \longmapsto \Theta(f)(t) = \begin{cases} \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f(s) ds & \text{if } t \in [0, t_1], \\ 0, & \text{if } t \in J'_k, \\ \frac{1}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} f(s)) ds, & \text{if } t \in J_k. \end{cases}$$

We have

$$\begin{aligned} \|(h_n - a_n)(t) - (h - a)(t)\| &= \|(h_n - h)(t) + (a - a_n)(t)\| \\ &\leq \|(h_n - h)(t)\| + \|(a - a_n)(t)\|. \end{aligned}$$

Thus

$$\|(h_n - a_n)(t) - (h - a)(t)\| \longrightarrow 0 \text{ as } n \to \infty.$$

From Lemma 1.4.1 it follows that $\Theta \circ S_F$ is a closed graph operator. Moreover, we have

$$(h_n - a_n)(t) \in \Theta(S_{F,y_n}).$$

Since $y_n \longrightarrow y$, Lemma 1.4.1 implies that

$$(h_n - a_n)(t) = \begin{cases} \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f(s) ds & \text{if } t \in [0, t_1], \\ 0, & \text{if } t \in J_k', \\ \frac{1}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} f(s)) ds, & \text{if } t \in J_k. \end{cases}$$

for some $f \in S_{F,y}$.

Step 4: *M relatively compact in $PC(J, E)$*

Let$M \subset U$, where $M \subset$conv$(\{0\} \cup N(M))$ and for some countable set $C \subset M$ let $\overline{M} = \overline{C}$. Taking into account (5.12)-(5.14) , it is easily seen that $N(M)$ is equicontinuous. Therefore, $M \subset$conv$(\{0\} \cup N(M))$ implies that M is equicontinuous. It remains to apply the Arzéla-Ascoli theorem to show that for each $t \in I$ the set $M(t)$ is relatively compact. By taking into account that C is countable and $C \subset M \subset$conv$(0 \cup N(M))$, we can find a countable set $H = \{h_n : n \geq 1\} \subset N(M)$ such that $C \subset$ conv$(\{0\} \cup H)$. Then, there are $y_n \in M$, $g_{k_n} \in S_{G_k, y_n}$ and $f_n \in S_{F,y_n}$ with

$$h_n(t) = \begin{cases} y_0 + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f_n(s) ds & \text{if } t \in [0, t_1], \\ g_{k_n}(t), & \text{if } t \in J_k', \\ g_{k_n}(s_k) + \frac{1}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} f_n(s)) ds, & \text{if } t \in J_k. \end{cases}$$

Taking into account Theorem 1.5.2 and the fact that $M \subset \overline{C} \subset \overline{conv}(\{0\} \cup H)$, we obtain

$$\alpha(M(t)) \leq (\alpha(\overline{C}(t)) \leq \alpha(H(t)) = \alpha\{h_n(t) : n \leq 1\}).$$

Using (5.9)-(5.11), we obtain for $t \in [0, t_1]$

$$\alpha(\{M(t)\}) \leq \frac{2}{\Gamma(r)} \int_0^t (t-s)^{r-1} \alpha\{f_n(s) ds, n \geq 1\},$$

for $t \in J_k$, we have

$$\alpha(\{M(t)\}) \leq c^* \alpha\{y_n(t)\} + \frac{2}{\Gamma(r)} \int_{s_k}^{t} (t-s)^{r-1} \alpha\{f_n(s), n \geq 1\} ds$$

,

and for $t \in J'_k$

$$\alpha(\{M(t)\}) \leq c^* \alpha\{y_n(t), n \geq 1\}.$$

Also, since $f_n \in S_{F,y_n}$ and $y_n(s) \in M(s)$, then from (H3) we have

$$(t-s)^{r-1} \alpha\{f_n(s) ds, n \geq 1\} = (t-s)^{r-1} p(s) \alpha(M(s)) ds.$$

It follows that if $t \in [0, t_1]$, we have

$$\begin{aligned}
\alpha(\{M(t)\}) &\leq \frac{2p^*}{\Gamma(r)} \int_0^t (t-s)^{r-1} \alpha(M(s)) ds, \\
&\leq \frac{2p^*}{(1-c^*)\Gamma(r)} \int_0^t (t-s)^{r-1} \alpha(M(s)) ds,
\end{aligned}$$

if $t \in J_k$, we have

$$\alpha(\{M(t)\}) \leq \frac{2p^*}{(1-c^*)\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} \alpha(M(s)) ds,$$

if $t \in J'_k$

$$\alpha(\{M(t)\}) \leq c^* \alpha(M(t)) \Rightarrow (1-c^*) \alpha(M(t)) \leq 0.$$

Consequently, if $t \in [0, t_1] \cup J_k$, we have by (H5)and condition(5.8), the function Φ given by $\Phi(t) = \alpha(M(t))$ satisfies $\Phi \equiv 0$; that is, $\alpha(M(t)) = 0$ for all $t \in J$. Now, by the Arzéla-Ascoli theorem, M is relatively compact in $PC(J, E)$.

Step 5: *A priori estimate.*
Let $y \in PC(J, E)$ be such that $y \in \lambda N(y)$ for some $\lambda \in (0, 1)$. Then for each $t \in J$ we have

$$y(t) = \begin{cases} \lambda y_0 + \frac{\lambda}{\Gamma(r)} \int_0^t (t-s)^{r-1} f(s) ds & \text{if } t \in [0, t_1], \\ \lambda g_{n_k}(t), & \text{if } t \in J'_k, \\ \lambda g_{n_k}(s_k) + \frac{\lambda}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} f(s) ds, & \text{if } t \in J_k, \end{cases}$$

for some $f \in S_{F,y}$. On the other hand we have,

$$\begin{aligned}
\|y(t)\| &\leq \|g_{n_k}(t)\| + \|y_0\| + \frac{1}{\Gamma(r)} \int_{s_k}^{t_{k+1}} (t-s)^{r-1} \|f(s)\| ds \\
&\leq c_k(t)\|y(t)\| + \|y_0\| + \frac{1}{\Gamma(r)} \int_{s_k}^{t_{k+1}} (t-s)^{r-1} p(s) ds \\
&\leq c_* \|y(t)\| + \|y_0\| + \frac{p^* T^r}{\Gamma(r+1)} \\
&\leq c^* \|y(t)\| + \|y_0\| + \frac{p^* T^r}{\Gamma(r+1)}.
\end{aligned}$$

Then

$$\|y\| \leq \frac{1}{1-c^*}\left(\|y_0\| + \frac{p^* T^r}{\Gamma(r+1)}\right) := d.$$

Set

$$U = \{y \in PC(J,E) : \|y\| < d+1\}.$$

Condition (1.5.3) is satisfied by our choice of the open set U. From Theorem 1.5.3, we conclude that N has at least one fixed point $y \in PC(J,E)$ being a solution of problem (5.1)-(5.3).

□

5.3 Nonlocal Fractional Differential Inclusions with Non Instantaneous Impulses and Multivalued Jump

In this section we consider the following class of impulsive differential inclusion:

$$^cD^r y(t) \in F(t, y(t)), \ \text{ a.e. } t \in J_k := (s_k, t_{k+1}], \ k = 0, \ldots, m, \ 0 < r \leq 1, \tag{5.15}$$

$$y(t) \in G_k(t, y(t)), \ t \in J'_k := (t_k, s_k], \ k = 1, \ldots, m, \tag{5.16}$$

$$y(0) + \psi(y) = y_0, \tag{5.17}$$

where $f, G_k, k = 1, \ldots, m$ are as in section 4.2 and $\psi : PC(J,E) \to E$ is a continuous function.

Let us introduce the following conditions.

(H8) There exists a constant $M^* > 0$ such that

$$\|\psi(u)\| \leq M^*\|u\|_{PC} \; for \; each \; u \in PC(J,E).$$

(H9) For each bounded set $B \subset PC(J,E)$ we have

$$\alpha(\psi(B)) \leq M^*\alpha(B).$$

Theorem 5.3.1. *Assume that assumptions (H1)-(H9) hold, then the nonlocal problem (5.15)–(5.17) has at least one solution defined on J.*

Proof.

Transform the problem (5.15)–(5.17) into a fixed point problem. Consider the multi-valued map $N : PC(J,E) \to \mathcal{P}\left(PC(J,E)\right)$ defined by

$$N(y)(t) = \{h \in PC(J,E) : h(t) = \begin{cases} y_0 - \psi(y) + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} f(s)ds & \text{if } t \in [0,t_1], \\ g_k(t), & \text{if } t \in J'_k, \\ g_k(s_k) + \frac{1}{\Gamma(r)}\int_{s_k}^t (t-s)^{r-1} f(s))ds, & \text{if } t \in J_k, \end{cases}$$

$$f \in S_{F,y}, g_k \in S_{G_k,y}\}.$$

Clearly, the fixed points of operator N are solutions of problem (5.15)–(5.17). We can easily show the conditions of Theorem 1.5.3 are satisfied by $\tilde{N}$.

□

5.4 An Example

Let us consider the following problem ,

$$^{c}D^{\frac{1}{2}} y_n(t) \in \frac{1}{(9+n+e^t)(1+\|y(t)\|)}[y_n(t)-1, y_n(t)], \text{ for each } t \in \left(0, \frac{1}{3}\right] \cup \left(\frac{1}{2}, 1\right], \tag{5.18}$$

$$y_n(t) \in \frac{\sin|y_n(t)|}{4+n+e^t}[y_n(t), y_n(t)+1], t \in \left(\frac{1}{3}, \frac{1}{2}\right], \tag{5.19}$$

$$y_n(0) = 0. \tag{5.20}$$

Set

$$E = l^1 = \{y = (y_1, y_2, \ldots, y_n, \ldots,), \sum_{n=1}^{\infty} |y_n| < \infty\}.$$

E is a Banach space with the norm

$$\|y\| = \sum_{n=1}^{\infty} |y_n|.$$

Let

$$F(t,y) = (F_1(t,y), F_2(t,y), \ldots, F_n(t,y), \ldots),$$

$$F_n(t,y) = \frac{1}{(9+n+e^t)(1+\|y(t)\|)}[y_n(t)-1, y_n(t)],$$

and

$$G_1(t,y) = (G_{11}(t,y), G_{12}(t,y), \ldots, G_{1n}(t,y), \ldots),$$

$$G_{1n}(t,y) = \frac{\sin|y_n(t)|}{4+n+e^t}[y_n(t), y_n(t)+1].$$

Clearly F is closed and convex valued. For each $y \in E$ and $t \in [0,1]$, we have

$$\|F(t,y))\|_{\mathcal{P}} \leq \frac{1}{9+e^t}.$$

Clearly G is closed and convex valued. For each $y \in E$ and $t \in [0,1]$, we have

$$\|G_1(t,y))\|_{\mathcal{P}} \leq \frac{1}{4+e^t}y.$$

Hence, (H2) and (H4) are satisfied with $p(t) = \frac{1}{9+e^t}$ and $c_1(t) = \frac{1}{4+e^t}$.

Therefore all conditions of Theorem 1.5.3 are satisfied with $p^* = \frac{1}{10}$ and $c^* = \frac{1}{5}$. Hence, the problem (5.18)-(5.20) has at least one solution.

CONCLUSION AND PERSPECTIVE

Our main purpose in this thesis, is to present some results of the existence of solutions for some classes of fractional differential equations and inclusions with the Caputo fractional derivative in infinite dimensional Banach spaces. These results were obtained by using the Mönch fixed point theorem combined with the Kuratowski's measure of noncompactness.

For the perspective, we will study the global existence of solutions together with stability for more classes of fractional differential equations and inclusions.

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Abstrac

Our main purpose in this thesis, is devoted to study the existence of solutions for various types of fractional differential equations and inclusions with non instantaneous impulses with the Caputo fractional derivative in a Banach space.

The arguments are based upon Mönch's fixed point theorem and the technique of measures of noncompactness..

....

Résumé

Notre but principal, dans cette these, est de presenter plusieurs résultats d'existence pour certaines classes d' équations et d' inclusions différentielles avec impulsions non instantanées d'ordre fractionnaire au sens de Caputo dans des espaces de Banach de dimension infinie.

Ces résultats ont été obtenus par l'utilisation du théorème de point fixe de Mönch combiné avec la mesure de non compacité de Kuratowski.

....

ملخص

هدفنا الرئيسي ، في هذه الأطروحة ، هو تقديم العديد من نتائج اثبات وجود حلول لبعض فئات المعادلات والاحتواءات التفاضلية ذات رتبة كسرية بمفهوم **كبيتو** و بتقطعات غير لحظية في فضاءات بناخ ذات بعد لانهائي.

تم الحصول على هذه النتائج باستخدام نظرية النقطة الثابتة **لـ مونك** ومترافقة مع قياس غير المتراص لـ **كيروفتسكي** .