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Une contribution à l'étude de certaines classes d'équations différentielles fractionnaires avec retard et anticipation

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Publications

1. M. Boumaaza, M. Benchohra, F. Berhoun, Nonlinear implicit Caputo type modification of the Erdélyi-Kober fractional differential equations with retarded and advanced Arguments, *Panam. Math. J* **30** (2020), 21-36,

2. M. Benchohra, F. Berhoun, M. Boumaaza, S. Sivasundaram, Caputo type modification of the Erdelyi-Kober fractional implicit differential equations with retarded and advanced arguments in Banach Spaces, *Nonlinear Stud.* **27** (1) (2020), 285-296.

3. M. Boumaaza, M. Benchohra, Caputo type modification of the Erdélyi-Kober fractional differential inclusions with retarded and advanced arguments, *Adv. Dyn. Syst. Appl.* **15** (2) (2020), 63-78.

4. M. Boumaaza, M. Benchohra, and C. Tunç, Erdélyi-Kober fractional differential inclusions in Banach spaces with retarded and advanced arguments, *Discus. Mathem. Differ. Incl., Contr. Optim.* **40** (2020), 75-92.

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Abstract

Functional differential equations occur in a variety of areas of biological, physical, and engineering applications, and such equations have received much attention in recent years. This thesis discusses the existence of solutions and random solutions for some implicit fractional differential equations, coupled systems and inclusions involving both retarded and advanced arguments, with Caputo-type modification of the Erdélyi-Kober fractional derivative. Our results will be obtained by means of fixed points theorems and by the technique of measures of noncompactness.

Key words and phrases : Functional differential equations, coupled systems, inclusions Caputo-type modification of the Erdélyi-Kober, existence, solutions, random solutions, retarded arguments, advanced arguments, measure of noncompactness, fixed point.

AMS Subject Classification : 34A08, 34A60, 34G20, 34K32, 34K37, 34F05.

Contents

Introduction

Fractional calculus is a generalization of differentiation and integration to arbitrary order (non-integer) fundamental operator $D^{\alpha}_{a^+}$ where $a, \alpha \in \mathbb{R}$. Several approaches to fractional derivatives exist : Riemann-Liouville (RL), Hadamard, Erdélyi-Kober, Grunwald-Letnikov (GL), Weyl and Caputo etc. The Caputo fractional derivative is well suitable to the physical interpretation of initial conditions and boundary conditions. We refer readers, for example, to the books [21, 65, 67, 75, 90] and the references therein. In this thesis, we always use the Caputo type modification of the Erdélyi-Kober derivative.

Fractional differential equations and inclusions appear in several areas such as engineering, mathematics, bio-engineering, physics, viscoelasticity, electrochemistry, control, etc. For current advances of fractional calculus, we refer the reader to the monographs [4, 5, 6, 62, 67, 81, 84, 92] and the references therein. In particular, time fractional differential equations are used when attempting to describe transport processes with long memory. Recently, considerable attention has been given to the existence of solutions of boundary value problem and boundary conditions for implicit fractional differential equations and integral equations with Caputo and Caputo type modification of the Erdélyi-Kober derivative. See for example [15, 16, 17, 18, 24, 33, 34, 66, 89] and references therein.

The differential equation with delay is a special type of functional differential equations. Delay differential equations arise in many biological and physical applications and it often forces us to consider variable or state-dependent delays. The functional differential equations with state-dependent delay have many important applications in mathematical models of real phenomena and the study of this type of equations has received much attention in recent years. We refer the reader to the monographs [7, 12, 26, 27, 32, 37, 39, 40, 41, 45, 57].

Recently there have been special situations in decision making, organizational transformation, chaotic equations, wavelet theory and so on, where specific equations with anticipation as well as retardation and anticipation appear in modeling [20, 54, 55]. This lead to the initiation of the study of the general theory of differential equations involving anticipation as well as retardation and anticipation in [72, 86] and continued in [47, 48]. The authors studied the existence and uniqueness of solutions for boundary value problems of Hadamard-type fractional functional differential equations and inclusions involving both retarded and advanced arguments;see [14, 32, 37] and the references therein.

Coupled systems of fractional differential equations arise in various problems of applied nature. In recent years, some authors have investigated the existence and uniqueness of solutions for coupled systems of nonlinear fractional differential equations; see [11, 12, 19] and the references therein.

The measure of noncompactness which is one of the fundamental tools in the theory of nonlinear analysis was initiated by the pioneering articles of Kuratowski [70], Darbo [51] and was developed by Bana's and Goebel [23] and many researchers in the literature. The applications of the measure of noncompactness (for the weak case, the measure of weak noncompactness developed by De Blasi [52]) can be seen in the wide range of applied mathematics: theory of differential equations (see [11, 12, 13, 37, 38] and references therein).

Implicit differential equations involving the regularized fractional derivative were analyzed by many authors, in the last year ; see for instance[8, 25, 31, 37, 88] and the references therein.

Probabilistic functional analysis is an important mathematical area of research due to its applications to probabilistic models in applied problems. Random differential equations, used in many on cases, to describe phenomena in biology, physics, engineering, and systems sciences contain certain parameters or coefficients which have specific interpretations, but whose values are unknown. We refer the reader to the monographs [49, 71, 85], the papers [1, 2, 3, 9, 10] and references therein.

In the following we give an outline of our thesis organization consisting of six chapters. The first chapter gives some notations, definitions, lemmas and fixed point theorems which are used throughout this thesis.

In Chapter 2, we establish the existence of solutions for a class of problems for nonlinear implicit Caputo type modification of the Erdélyi-Kober fractional differential equations involving both retarded and advanced arguments. Here two results are discussed, the first is based on the Banach contraction principle, Schauder's and Schaefer's fixed point theorems, the second is based on the method associated with the technique of measures of non compactness and the fixed point theorems of Darbo and Mönch.

In Section 2.2, we discuss existence and uniqueness of solutions for a class of problem for nonlinear implicit fractional differential equations (NIFDE for short) involving both retarded and advanced arguments.

$$\begin{cases} {}_c^\rho D_{a^+}^\alpha y(t) = f(t, y^t, {}_c^\rho D_{a^+}^\alpha y(t)), \text{ for } & t \in I := [a, T],\ 1 < \alpha \le 2, \\ y(t) = \phi(t), & t \in [a-r, a],\ r > 0 \\ y(t) = \psi(t), & t \in [T, T+\beta],\ \beta > 0, \end{cases} \tag{1}$$

where ${}_c^\rho D_{a^+}^\alpha$ is the Caputo type modification of the Erdélyi-Kober fractional derivative, $f : I \times C([-r, \beta], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ is a given function, $\phi \in C([a-r, a], \mathbb{R})$ with $\phi(a) = 0$ and $\psi \in C([T, T+\beta], \mathbb{R})$ with $\psi(T) = 0$. We denote by y^t the element of $C([-r, \beta])$ defined by:

$$y^t(s) = y(t+s) : s \in [-r, \beta]$$

here $y^t(\cdot)$ represents the history of the state from time $t-r$ up to time $t+\beta$.

In Section 2.4, we discuss existence and uniqueness of solutions for the following problem of nonlinear implicit fractional differential equations in Banach space with retarded and advanced arguments

$$\begin{cases} {}^{\rho}_{c}D^{\nu}_{a^+}y(t) = f(t, y^t, {}^{\rho}_{c}D^{\nu}_{a^+}y(t)), & t \in I := [a, T],\ 1 < \nu \leq 2, \\ y(t) = \phi(t), & t \in [a-r, a],\ r > 0 \\ y(t) = \psi(t), & t \in [T, T+\beta],\ \beta > 0, \end{cases} \tag{2}$$

where ${}^{\rho}_{c}D^{\nu}_{a^+}$ is the Caputo type modification of the Erdélyi-Kober fractional derivative, $(E, \|\cdot\|)$ is a real Banach space and $f : I \times C([-r, \beta], E) \times E \to E$ is a given function, $\phi \in C([a-r, a], E)$ with $\phi(a) = 0$ and $\psi \in C([T, T+\beta], E)$ with $\psi(T) = 0$.
We denote by y^t the element of $C([-r, \beta])$ defined by

$$y^t(s) = y(t+s) : s \in [-r, \beta].$$

In Chapter 3, we establish the existence of solutions to the following coupled system nonlinear implicit of Caputo type modification of the Erdélyi-Kober fractional differential equations involving both retarded and advanced arguments. Here two results are discussed, the first is based on the Banach contraction principle and Schauder's fixed point theorem, the second is based our investigation relies upon Mönch fixed point theorem combined with the technique of measures of weak non compactness.

In Section 3.2 we deal with the existence and uniqueness of solutions to the following coupled system nonlinear implicit of Caputo type modification of the Erdélyi-Kober fractional differential equations involving both retarded and advanced arguments

$$\begin{cases} {}^{\rho}_{c}D^{\alpha}_{a^+}u(t) = f_1(t, u^t, v^t, {}^{\rho}_{c}D^{\alpha}_{a^+}u(t), {}^{\rho}_{c}D^{\alpha}_{a^+}v(t)) \\ {}^{\rho}_{c}D^{\alpha}_{a^+}v(t) = f_2(t, u^t, v^t, {}^{\rho}_{c}D^{\alpha}_{a^+}u(t), {}^{\rho}_{c}D^{\alpha}_{a^+}v(t)) \end{cases} \quad t \in I := [a, T], \tag{3}$$

$$\begin{cases} (u(t), v(t)) = (\phi_1(t), \phi_2(t)), & t \in [a-r, a],\ r > 0 \\ (u(t), v(t)) = (\psi_1(t), \psi_2(t)), & t \in [T, T+\beta],\ \beta > 0, \end{cases}$$

where ${}^{\rho}_{c}D^{\alpha}_{a^+}$ is the Caputo type modification of the Erdélyi-Kober fractional derivative and $f_i : I \times C([-r, \beta], \mathbb{R})^2 \times \mathbb{R}^2 \to \mathbb{R}$ is a given function, $\phi_i \in C([a-r, a], \mathbb{R})$ with $\phi_i(a) = 0$ and $\psi_i \in C([T, T+\beta], \mathbb{R})$ with $\psi_i(T) = 0$,$i = 1, 2$.

In Section 3.4, we prove the existence of weak solutions to the following Coupled system nonlinear implicit of Caputo type modification of the Erdélyi-Kober fractional differential equations involving both retarded and advanced arguments:

$$\begin{cases} {}^{\rho}_{c}D^{\alpha}_{a^+}u(t) = f_1(t, u^t, v^t, {}^{\rho}_{c}D^{\alpha}_{a^+}u(t), {}^{\rho}_{c}D^{\alpha}_{a^+}v(t)) \\ {}^{\rho}_{c}D^{\alpha}_{a^+}v(t) = f_2(t, u^t, v^t, {}^{\rho}_{c}D^{\alpha}_{a^+}u(t), {}^{\rho}_{c}D^{\alpha}_{a^+}v(t)) \end{cases} \quad t \in I := [a, T], \tag{4}$$

$$\begin{cases} (u(t),v(t)) = (\phi_1(t),\phi_2(t)), \ t \in [a-r,a], \ r>0 \\ (u(t),v(t)) = (\psi_1(t),\psi_2(t)), \ t \in [T,T+\beta], \ \beta>0, \end{cases}$$

where ${}_c^{\rho}D_{a+}^{\alpha}$ is the Caputo type modification of the Erdélyi-Kober fractional derivative and E is a real (or complex) Banach space with norm $\|\cdot\|_E$ and dual E^*, such that E is the dual of a weakly compactly generated Banach space X, $f_i : I \times C([-r,\beta],E)^2 \times E^2 \to E$ is a given function, $\phi_i \in C([a-r,a],E)$ with $\phi_i(a) = 0$ and $\psi_i \in C([T,T+\beta],E)$ with $\psi_i(T) = 0, \ i = 1,2.$

In Chapter 4, we study the existence and uniqueness of Random solutions to the following coupled nonlinear implicit system of Caputo type modification of the Erdélyi-Kober fractional differential equations involving both retarded and advanced arguments:

$$\begin{cases} ({}_c^{\rho}D_{a^+}^{\alpha_1}u)(t,w) = f_1(t,u^t(w),v^t(w),({}_c^{\rho}D_{a^+}^{\alpha_1}u)(t,w),w) \\ ({}_c^{\rho}D_{a^+}^{\alpha_2}v)(t,w) = f_2(t,u^t(w),v^t(w),({}_c^{\rho}D_{a^+}^{\alpha_2}v)(t,w),w) \end{cases} \quad t \in I := [a,T], w \in \Omega, \tag{5}$$

$$\begin{cases} (u(t,w),v(t,w)) = (\phi_1(t,w),\phi_2(t,w)), \ t \in [a-r,a], \ r>0, \\ (u(t,w),v(t,w)) = (\psi_1(t,w),\psi_2(t,w)), \ t \in [T,T+\beta], \ \beta>0, \end{cases} \quad w \in \Omega$$

where $\alpha_i \in (1,2]$, ${}_c^{\rho}D_{a^+}^{\alpha_i}, i = 1,2$, is the Caputo type modification of the Erdélyi-Kober fractional derivative and $f_i : I \times C([-r,\beta],\mathbb{R}^n) \times C([-r,\beta],\mathbb{R}^n) \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ is a given function, $\phi_i \in C([a-r,a],\mathbb{R}^n)$ with $\phi_i(a,w) = 0$ and $\psi_i \in C([T,T+\beta],\mathbb{R}^n)$ with $\psi_i(T,w) = 0$, $i = 1,2$.

In Chapter 5, we study the existence of solutions for a class of problem for nonlinear Caputo type modification of the Erdélyi-Kober fractional differential inclusions (FDI for short) involving both retarded and advanced arguments.
Here two results are discussed, the first present the existence results for convex and non-convex multi-valued maps involved which, respectively, rely on the nonlinear alternative of Leray-Schauder type and a fixed point theorem for contractive multi-valued maps due to Covitz and Nadler. The second is based on the method associated with the technique of measures of non compactness and the fixed point theorems of Darbo and Mönch. In Section 5.2, we prove the existence of solutions for a class of problems for Caputo type modification of the Erdélyi-Kober fractional differential inclusions with retarded and advanced arguments given by:

$$\begin{cases} {}_c^{\rho}D_{a+}^{\alpha}y(t) = F(t,y^t), \ t \in I := [a,T], \ 1<\alpha\le 2, \\ y(t) = \phi(t), \qquad t \in [a-r,a], \ r>0 \\ y(t) = \psi(t), \qquad t \in [T,T+\beta], \ \beta>0, \end{cases} \tag{6}$$

where ${}_c^{\rho}D_{a+}^{\alpha}$ is the Caputo type modification of the Erdélyi-Kober fractional derivative, $F : I \times C([-r,\beta],\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ is a given function, $\phi \in C([a-r,a],\mathbb{R})$ with $\phi(a) = 0$ and $\psi \in C([T,T+\beta],\mathbb{R})$ with $\psi(T) = 0$.

In Section 5.4, we discuss the existence of solutions for a class of problem for nonlinear Caputo type modification of the Erdélyi-Kober fractional differential inclusions

$$\begin{cases} {}^{\rho}_{c}D^{\alpha}_{a^+}y(t) \in F(t, y^t), \text{ for } \; t \in I := [a, T], \; 1 < \alpha \leq 2, \\ y(t) = \phi(t), \; t \in [a-r, a], \; r > 0 \\ y(t) = \psi(t), \; t \in [T, T+\beta], \; \beta > 0, \end{cases} \tag{7}$$

where ${}^{\rho}_{c}D^{\alpha}_{a^+}$ is the Caputo type modification of the Erdélyi-Kober fractional derivative, $F : I \times C([-r, \beta], E) \to \mathcal{P}(E)$ is a given function, $\phi \in C([a-r, a], E)$ with $\phi(a) = 0$ and $\psi \in C([T, T+\beta], E)$ with $\psi(T) = 0$.

In Chapter 6, we study the existence of weak solutions for Caputo type modification of the Erdélyi-Kober fractional differential system inclusions. with retarded and advanced arguments in Banach space given by:

$$\begin{cases} ({}^{\rho}_{c}D^{\alpha_1}_{a^+}u)(t) \in F_1(t, u^t, v^t) \\ ({}^{\rho}_{c}D^{\alpha_2}_{a^+}v)(t) \in F_2(t, u^t, v^t) \end{cases} \; ; \; t \in I := [a, T], \tag{8}$$

$$\begin{cases} (u(t), v(t)) = (\phi_1(t), \phi_2(t)), \; t \in [a-r, a], \; r > 0 \\ (u(t), v(t)) = (\psi_1(t), \psi_2(t)), \; t \in [T, T+\beta], \; \beta > 0, \end{cases}$$

where $\beta > 0, \alpha_i \in (1, 2]$ $(E, \|\cdot\|)$ is a real Banach space and ${}^{\rho}_{c}D^{\alpha}_{a^+}$ is the Caputo type modification of the Erdélyi-Kober fractional derivative, $F : I \times C([-r, \beta], E) \times C([-r, \beta], E) \to \mathcal{P}(E)$ is a given function, $\phi_i \in C([a-r, a], E)$ with $\phi_i(a) = 0$ and $\psi_i \in C([T, T+\beta], E)$ with $\psi_i(T) = 0, i = 1, 2$. The main result of the chapter is based on the fixed point theorem of Mönch's type and the technique of measure of weak noncompactness.

Chapter 1

Preliminaries

In this chapter, we review some fundamental concepts, notations, definitions, fixed point theorems and properties required to establish our main results.

1.1 Notations and Definitions

Let $C([-r,\beta],E)$ be the Banach space of all continuous functions from $[-r,\beta]$ into Banach space E equipped with the norm

$$\|y\|_{[-r,\beta]} = \sup\{\|y(t)\| : -r \le t \le \beta\}, \quad r,\beta > 0,$$

and $C([a,T],E)$ is the Banach space endowed with the norm

$$\|y\|_{[a,T]} = \sup\{\|y(t)\| : a \le t \le T\}.$$

Also, let $E_1 = C([a-r,a],E)$, $E_2 = C([T,T+\beta],E)$, $a,T \in \mathbb{R}_+$
and let the space

$$AC^1(I) := \{w : I \longrightarrow E : w' \in AC(I)\},$$

where

$$w'(t) = t\frac{d}{dt}w(t),\ t \in I = [a,T],$$

$AC(I,E)$ is the space of absolutely continuous functions on I,
$\mathcal{C} = \{y : [a-r,T+\beta] \longmapsto E : y\,|_{[a-r,a]} \in C([a-r,a]), y\,|_{[a,T]} \in AC^1([a,T])$

$$\text{and } y\,|_{[T,T+\beta]} \in C([T,T+\beta])\}$$

be the spaces endowed, respectively, with the norms

$$\|y\|_{[a-r,a]} = \sup\{\|y(t)\| : a-r \le t \le a\},$$

and

$$\|y\|_{[T,T+\beta]} = \sup\{\|y(t)\| : T \le t \le T+\beta\},$$

$$\|y\|_{\Omega} = \sup\{\|y(t)\| : a - r \leq t \leq T + \beta\}.$$

Let $L^1(I)$, be the Banach space of measurable functions $v : I \longrightarrow E$ which are Bochner integrable, equipped with the norm

$$\|v\|_{L^1} = \int_a^T \|v(t)\|dt.$$

Consider the space $X_c^p(a,b)$, $(c \in \mathbb{R}, 1 \leq p \leq \infty)$ of those complex-valued Bochner measurable functions f on $[a,b]$ for which $\|f\|_{X_c^p} < \infty$, where the norm is defined by :

$$\|f\|_{X_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t}\right)^{\frac{1}{p}}, \quad (1 \leq p < \infty, c \in \mathbb{R}).$$

In particular, where $c = \frac{1}{p}$ the space $X_c^p(a,b)$ coincides $L^p(a,b)$ space, i.e., $X_{\frac{1}{p}}^p(a,b) = L^p(a,b)$.

Denote by $L^\infty(I, \mathbb{R})$, the Banach space of essentially bounded measurable functions $u : I \longrightarrow \mathbb{R}$ equipped with the norm

$$\|f\|_{L^\infty} = \inf\{c \geq 0; \quad |f(x)| \leq c \text{ a.e. on } \quad I\}.$$

Definition 1.1.1 *A Banach space X is said to be weakly compactly generated (WCG) if it contains a weakly compact set whose linear span is dense in X.*

Definition 1.1.2 *A function $h : E \longrightarrow E$ is said to be weakly sequentially continuous if h takes each weakly convergent sequence in E to weakly convergent sequence in E (i.e. for any $(x_n)_n$ in E with $x_n \longrightarrow x$ in (E, ω), $h(x_n) \longrightarrow h(x)$ in (E, ω)).*

Definition 1.1.3 *([80]) The function $x : J \longrightarrow E$ is said to be Pettis integrable on J if and only if there is an element $u_I \in E$ corresponding to each $I \subset J$ such that $\varphi(u_I) = \int_I \varphi(u(s))ds$ for all $\varphi \in E^*$, where the integral on the right is supposed to exist in the sense of Lebesgue. We have $u_I = \int_I \varphi(u(s))ds$. Let $P(J,E)$ be the space of all E-valued Pettis integrable functions in the interval J, and let $L_1(I,E)$ be the Banach space of Bochner-integrable measurable functions $u : I \longrightarrow E$. Define the class*

$$P_1(J,E) = \{u \in P(J,E) : \varphi(u) \in L_1(I, \mathbb{R}) \quad for \quad every \quad \varphi \in E^*\}$$

The space $P_1(J,E)$ is normed by

$$\|u\|_{P_1} = \sup_{\varphi \in E^*, \|\varphi\| \leq 1} \int_a^T |\varphi(u(x))| d\lambda x,$$

where λ is the Lebesgue measure on J.

Proposition 1.1.1 *([80]) If $u \in P_1(I,E)$ and h is a measurable and essentially bounded real-valued function, then $uh \in P(I,E)$. In what follows,the symbol " $\int$ " denotes the Pettis integral.*

1.2 Fractional Calculus

Definition 1.2.1 *([65, 68, 69]): **(Erdélyi-Kober fractional integral)** Let $\alpha \in \mathbb{R}, c \in \mathbb{R}$ and $g \in X_c^p(a,b)$, the Erdélyi-Kober fractional integral of order α is defined by :*

$$({}^\rho I_{a+}^\alpha g)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{\rho-1} (t^\rho - s^\rho)^{\alpha-1} g(s)ds, \qquad t > a, \rho > 0 \tag{1.1}$$

where Γ is the Euler gamma function defined by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt, \ \alpha > 0.$$

Definition 1.2.2 *([64]) The generalized fractional derivative, corresponding to the fractional integral (1.1), is defined, for $0 \leq a < t$, by:*

$$\begin{aligned} {}^\rho D_{a+}^\alpha g(t) &= \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)} \left(t^{1-\rho}\frac{d}{dt}\right)^n \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-n+\alpha}} g(s)ds \\ &= \delta_\rho^n ({}^\rho I_{a^+}^{n-\alpha} g)(t), \end{aligned} \tag{1.2}$$

where $\delta_\rho^n = \left(t^{1-\rho}\frac{d}{dt}\right)^n$.

Definition 1.2.3 *([64, 74]) The Caputo-type generalized fractional derivative ${}_c^\rho D_{a+}^\alpha$ is defined via the above generalized fractional derivative (1.2) as follows*

$$({}_c^\rho D_{a+}^\alpha g)(t) = \left({}^\rho D_{a+}^\alpha \left[g(t) - \sum_{k=0}^{n-1} \frac{g^{(k)}(a)}{k!} (s-a)^k\right]\right). \tag{1.3}$$

Lemma 1.2.1 *([64]) Let $\alpha, \rho \in \mathbb{R}^+$, then*

$$({}^\rho I_{a+}^\alpha {}_c^\rho D_{a+}^\alpha g)(t) = g(t) - \sum_{k=0}^{n-1} c_k \left(\frac{t^\rho - a^\rho}{\rho}\right)^k, \tag{1.4}$$

for some $c_k \in \mathbb{R}$, $n = [\alpha] + 1$.

1.3 Multi-valued analysis

Let $(E, \|\cdot\|)$ be a Banach space. We define the following subsets of $\mathcal{P}(E)$:

$$P_{cl}(E) = \{Y \in \mathcal{P}(E) : Y \text{ is closed}\},$$

$$P_b(E) = \{Y \in \mathcal{P}(E) : Y \text{ is bounded}\},$$

$$P_{cp}(E) = \{Y \in \mathcal{P}(E) : Y \text{ is compact}\}$$

$P_{cv}(E) = \{Y \in \mathcal{P}(E) : Y \text{ is convex}\}$

$P_{cp,cv}(E) = P_{cp}(E) \cap P_{cv}(E)$.

Definition 1.3.1 *A multivalued map $G : E \to \mathcal{P}(E)$ is said to be convex (closed) valued if $G(x)$ is convex (closed) for all $x \in E$. A multivalued map G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in E for all $B \in P_b(E)$ (i.e. $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}$ exists).*

Definition 1.3.2 *A multivalued map $G : E \to \mathcal{P}(E)$ is called upper semi-continuous (u.s.c.) on E if for each $x_0 \in E$, the set $G(x_0)$ is a nonempty closed subset of E, and for each open set N of E containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subset N$. G is said to be completely continuous if $G(B)$ is relatively compact for every $B \in P_b(\mathbb{R})$.*

Definition 1.3.3 *Let $G : X \to \mathcal{P}(E)$ be completely continuous with nonempty compact values. Then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \to x_*, y_n \to y_*, y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in E$ such that $x \in G(x)$.*

We denote by $FixG$ the fixed point set of the multivalued operator G.

Definition 1.3.4 *A multivalued map $G : J \to P_{cl}(E)$ is said to be measurable if for every $y \in E$, the function:*

$$t \to d(y, G(t)) = \inf\{\|y - z\| : z \in G(t)\}$$

is measurable.

Lemma 1.3.1 *[78] Let G be a completely continuous multivalued map with nonempty compact values, then G is u.s.c. if and only if G has a closed graph.*

Definition 1.3.5 *A multivalued map $F : I \times E \to \mathcal{P}(E)$ is said to be Carathéodory if:*

(1) $t \to F(t, u)$ is measurable for each $u \in E$

(2) $u \to F(t, u)$ is upper semicontinuous for almost all $t \in I$.

F is said to be L^1-Carathéodory if $(1), (2)$ and the following condition holds:

(3) For each $q > 0$, there exists $\varphi_q \in L^1(I, \mathbb{R}^+)$ such that

$$\|F(t, u)\|_{\mathcal{P}} = \sup\{\|v\| : v \in F(t, u)\} \leq \varphi_q \text{ for all } \|u\| \leq q \text{ and for a.e. } t \in I.$$

For each $y \in C(I)$, define the set of selections of F by

$$S_{F\circ y} = \{v \in L^1(I) : v(t) \in F(t, y(t)) \text{ a.e. } t \in I\}.$$

Let (E, d) be a metric space induced from the normed space $(|\cdot|)$. The function $H_d : \mathcal{P}(E) \times \mathcal{P}(E) \to \mathbb{R}_+ \cup \{\infty\}$ given by:

$$H_d(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\}$$

is known as the Hausdorff-Pompeiu metric. For more details on multivalued maps see the books of Hu and Papageorgiou [78].

1.4 Measure of Noncompactness and Auxiliary Results

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness and De Blasi measure of weak noncompactness.

Definition 1.4.1 *([23]) Let E be a Banach space and Ω_E the family of bounded subsets of E. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_E \to [0, \infty)$ defined by*

$$\alpha(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^{n} B_i \text{ and } diam(B_i) \leq \epsilon\}; \text{ here } B \in \Omega_E,$$

where

$$diam(B_i) = sup\{||x - y|| : x, y \in B_i\}.$$

Properties 1.4.1 *The Kuratowski measure of noncompactness satisfies the following properties (for more details see [23]).*

(a_1) $\alpha(B) = 0 \Longleftrightarrow \overline{B}$ *is compact (B is relatively compact).*

(b_1) $\alpha(B) = \alpha(\overline{B})$.

(c_1) $A \subset B \Longrightarrow \alpha(A) \leq \alpha(B)$.

(d_1) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$.

(e_1) $\alpha(cB) = |c|\alpha(B); c \in \mathbb{R}$

(f_1) $\alpha(convB) = \alpha(B)$.

Theorem 1.4.1 *[61] Let E be a Banach space. Let $C \subset L^1(I, E)$ be a countable set with $\|u(t)\| \leq h(t)$ for a.e. $t \in J$ and every $u \in C$, where $h \in L^1(I, \mathbb{R}_+)$. Then $\phi(t) = \mu(C(t)) \in L^1(I, \mathbb{R}_+)$ and verifies*

$$\mu\left(\left\{\int_a^T u(s)\,ds : u \in C\right\}\right) \leq 2\int_a^T \mu(C(s))\,ds,$$

where μ is the Kuratowski measure of noncompactness on the set E.

Lemma 1.4.1 *[73] Let I be a compact real interval. Let F be a Carathéodory multivalued map and let Θ be a linear continuous map from $L^1(I) \to C(I)$. Then the operator*

$$\Theta \circ S_{F\circ u} : C(I) \to \mathcal{P}_{cv,cp}(C(I)), \quad u \mapsto (\Theta \circ S_{F\circ u})(u) = \Theta(S_{F\circ u})$$

is a closed graph operator in $C(I) \times C(I)$.

Definition 1.4.2 *Let E be Banach space. A multivalued mapping $T : E \to \mathcal{P}_{cl,b}(E)$ is called $k-$set- Lipschitz if there exists a constant $k > 0$, such that $\mu(T(X)) \leq k\mu(X)$ for all $X \in \mathcal{P}_{cl,b}(E)$ with $T(X) \in \mathcal{P}_{cl,b}(E)$. If $k < 1$, then T is called a $k-$set-contraction on E.*

Definition 1.4.3 *([52]) Let E be a Banach space and Ω_E the bounded subsets of E and B_1 the unit ball of E. The De Blasi measure of weak noncompactness is the map $\mu : \Omega_E \to [0, \infty)$ defined by*

$$\mu(B) = \inf\{\epsilon > 0 : \text{ there exists a weakly compact subset } \Omega \ \text{ of} E : X \subset \epsilon B_1 + \Omega\}.$$

The next result follows directly from the Hahn-Banach theorem.

Proposition 1.4.1 *If E is a normed space and $x_0 \in E\backslash\{0\}$, then there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\varphi(x_0) = \|\varphi\|$.*

The De Blasi measure of weak noncompactness satisfies the following properties.

Lemma 1.4.2 *([52]) Let A and B bounded sets.*

(1) $\mu(B) = 0 \Leftrightarrow \overline{B}$ *is compact (B is weakly relatively compact).*

(2) $\mu(cov(B)) = \mu(B)$.

(3) $\mu(B) = \alpha(\overline{B}^{\omega})$, *($\overline{B}^{\omega}$ denote the weak closure of B.)*

(4) $A \subset B \Rightarrow \mu(A) \leq \mu(B)$.

(5) $\mu(A + B) \leq \mu(A) + \mu(B)$, *where* $A + B = \{x + y : x \in A, \quad y \in B\}$.

(6) $\mu(\lambda B) = |\lambda|\mu(B)$; $\lambda \in \mathbb{R}$, *where* $\lambda B = \{\lambda x : x \in B\}$.

(7) $\mu(A \cup B) = max\{\mu(A), \mu(B)\}$.

(8) $\mu(B + x_0) = \mu(B)$ *for any* $x_0 \in E$.

Lemma 1.4.3 *([60]) Let $V \subset C(I, E)$ is a bounded and equicontinuous set, then*

(i) the function $t \longmapsto \mu(V(t))$ is continuous on I, and

$$\mu_C(V) = \max_{t \in I} \mu(V(t)),$$

(ii)

$$\mu\left(\int_a^T y(s)ds : y \in V\right) \leq \int_a^T \mu(V(s))ds,$$

where

$$V(t) = \{y(t) : y \in V\}, \ t \in I.$$

and μ_C is the De Blasi measure of weak noncompactness defined on the bounded sets of $C(I)$.

1.5 Random operators

Let $B_{\mathbb{R}^m}$ be the σ-algebra of Borel subsets of $\mathbb{R}^m$ and Ω is the sample space in a probability space (Ω, F). A mapping $v : \Omega \to \mathbb{R}^m$ is said to be measurable if for any $D \in B_{\mathbb{R}^m}$, one has

$$v^{-1}(D) = \{w \in \Omega : v(w) \in D\} \subset \mathcal{A}.$$

To define integrals of sample paths of a random process, it is necessary to define a jointly measurable map.

Definition 1.5.1 *A mapping $T : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ is called jointly measurable if for any $D \in B_{\mathbb{R}^m}$, one has*

$$T^{-1}(D) = \{(w, v) \in \Omega \times E : T(w, v) \in D\} \subset \mathcal{A} \times B_{\mathbb{R}^m},$$

where $\mathcal{A} \times B_{\mathbb{R}^m}$ is the direct product of the σ-algebras $\mathcal{A}$ and $B_{\mathbb{R}^m}$, those defined in Ω and $\mathbb{R}^m$, respectively.

Definition 1.5.2 *A function $T : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ is called jointly measurable if $T(\cdot, u)$ is measurable for all $u \in \mathbb{R}^m$ and $T(w, \cdot)$ is continuous for all $w \in \Omega$.*

A mapping $T : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ is called a random operator if $T(w, u)$ is measurable in w for all $u \in \mathbb{R}^m$, and it expressed as $T(w)u = T(w, u)$. In this case we also say that $T(w)$ is a random operator on $\mathbb{R}^m$. A random operator $T(w)$ on E is called continuous (resp. compact, totally bounded and completely continuous) if $T(w, u)$ is continuous (resp. compact, totally bounded and completely continuous) in u for all $w \in \Omega$. The details of completely continuous random operators in Banach spaces and their properties appear in Itoh [63].

Definition 1.5.3 *[56] Let $\mathcal{P}(Y)$ be the family of all nonempty subsets of Y and C be a mapping from Ω into $\mathcal{P}(Y)$. A mapping $T : \{(w, y) : w \in \Omega,\ y \in C(w)\} \to Y$ is called random operator with stochastic domain C, if C is measurable (i.e., for all closed $A \subset Y$, $\{w \in \Omega, C(w) \cap A \neq \emptyset\}$ is measurable) and for all open $D \subset Y$ and all $y \in Y$, $\{w \in \Omega : y \in C(w), T(w, y) \in D\}$ is measurable. T will be called continuous if every $T(w)$ is continuous. For a random operator T, a mapping $y : \Omega \to Y$ is called a random (stochastic) fixed point of T if for $P-$almost all $w \in \Omega$, $y(w) \in C(w)$ and $T(w)y(w) = y(w)$, and for all open $D \subset Y$, $\{w \in \Omega : y(w) \in D\}$ is measurable.*

Definition 1.5.4 *A function $f : I \times C([-r, \beta], \mathrm{I\!R}^n) \times C([-r, \beta], \mathrm{I\!R}^n) \times \mathrm{I\!R}^n \times \Omega \to \mathbb{R}^m$ is called random Carathéodory if the following conditions are satisfied:*

(i) The map $(t, w) \to f(t, u, v, x, w)$ is jointly measurable for all $(u, v, x) \in C([-r, \beta], \mathrm{I\!R}^n) \times C([-r, \beta], \mathrm{I\!R}^n) \times \mathrm{I\!R}^n$ and

(ii) The map $(u, v, x) \to f(t, u, v, x, w)$ is continuous for all $t \in I$ and $w \in \Omega$.

Let $x, y \in \mathbb{R}^m$ with $x = (x_1, x_2, \ldots, x_m)$, $y = (y_1, y_2, \ldots, y_m)$. By $x \leq y$ we mean $x_i \leq y_i,\ i = 1, \ldots, m$. Also $|x| = (|x_1|, |x_2|, \ldots, |x_m|)$,
$\max(x, y) = (\max(x_1, y_1),\ \ldots, \max(x_m, y_m))$, and $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x_i \in \mathbb{R}_+,\ i = 1, \ldots, m\}$. If $c \in \mathbb{R}$, then $x \leq c$ means $x_i \leq c,\ i = 1, \ldots, m$.

Definition 1.5.5 *Let X be a nonempty set. By a vector-valued metric on X we mean a map $d : X \times X \to \mathbb{R}^m$ with the following properties:*

(i) $d(x, y) \geq 0$ for all $x, y \in X$, and if $d(x, y) = 0$, then $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We call the pair (X, d) a generalized metric space with $d(x, y) := \begin{pmatrix} d_1(x, y) \\ d_2(x, y) \\ \cdot \\ \cdot \\ \cdot \\ d_m(x, y) \end{pmatrix}$.

Notice that d is a generalized metric space on X if and only if $d_i,\ i = 1, \ldots, m$, are metrics on X.

Definition 1.5.6 *[87] A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc, i.e. $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $det(M - \lambda I) = 0$, where I denotes the unit matrix of $M_{m \times m}(\mathbb{R})$.*

Example 1.5.1 *The matrix $A \in M_{2\times 2}(\mathbb{R})$ defined by*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

converges to zero in the following cases:

(1) $b = c = 0,\ a, d > 0$ and $\max\{a, d\} < 1$.

(2) $c = 0,\ a, d > 0,\ a + d < 1$ and $-1 < b < 0$.

(3) $a + b = c + d = 0,\ a > 1,\ c > 0$ and $|a - c| < 1$.

1.6 Some Fixed Point Theorems

Theorem 1.6.1 *[59, 79, 83] Let $(\Omega, \mathcal{F})$ be a measurable space, X be a real separable generalized Banach space and $F : \Omega \times X \to X$ be a continuous random operator, and let $M(w) \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ be a random variable matric such that for every $w \in \Omega$, the matrix $M(w)$ converges to 0 and*

$$d(F(w, x_1), F(w, x_2)) \leq M(w)d(x_1, x_2);\ \textit{for each } x_1, x_2 \in X \textit{ and } w \in \Omega.$$

Then there exists a random variable $x : \Omega \to X$ which is the unique random fixed point of F.

Theorem 1.6.2 *([58])**(Schauder's)**. Let X be a Banach space, $D \subset X$ a nonempty convex bounded closed set and let $N : D \longmapsto D$ be a completely continuous operator. Then N has at least one fixed point.*

Theorem 1.6.3 *([58])**(Schaefer's)**. Let X be a Banach space, and $N : X \longmapsto X$ be a completely continuous operator. If the set*

$$\xi = \{y \in X : y = \lambda N y,\ \textit{for some } \lambda \in (0, 1)\} \quad \textit{is bounded,}$$

then N has a fixed point.

Lemma 1.6.1 *(Darbo , [51]). Let D be a bounded, closed and convex subset of Banach space X. If the operator $N : D \to D$ is a strict set contraction, i.e there is a constant $0 \leq \lambda < 1$ such that $\alpha(N(S)) \leq \lambda\alpha(S)$ for any set $S \subset D$ then N has a fixed point in D.*

Theorem 1.6.4 *(Mönch , [76]). Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv}N(V) \quad \textit{or} \quad V = N(V) \cup 0 \Longrightarrow \alpha(V) = 0$$

holds for every subset V of D, then N has a fixed point.

Theorem 1.6.5 *([58]).Let D be a nonempty, closed, convex and equicontinuous subset of a metrizable locally convex vector space $C(I)$ such that $0 \in D$. Suppose $N : D \longrightarrow D$ is weakly-sequentially continuous. If the implication*

$$V = \overline{co}(N(V) \cup \{(0, 0)\}) \Longrightarrow \quad V \ \textit{ is relatively weakly compact,} \tag{1.5}$$

holds for every subset $V \subset D$ then the operator N has a fixed point.

Lemma 1.6.2 *([58]) (Nonlinear alternative for Kakutani maps) Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $N : U \longrightarrow \mathcal{P}_{cp,c}(C)$ is a upper semicontinuous compact map. Then either* *(i) N has a fixed point in U, or*

(ii) there is a $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda N(u)$.

Lemma 1.6.3 *(Covitz and Nadler ([50])) Let (X,d) be a complete metric space. If $N : X \longrightarrow \mathcal{P}_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

Theorem 1.6.6 *(Darbo fixed point theorem) [53] Let K be a bounded, closed and convex subset of a Banach space X and let $T : K \to \mathcal{P}_{cl,b}(K)$ be a closed and $k-$set-contraction. Then T has a fixed point.*

Theorem 1.6.7 *(Mönch fixed point theorem) [77] Let K be a closed and convex subset of a Banach space E, U be a relatively open subset of K, and $N : \overline{U} \mapsto \mathcal{P}(K)$. Assume that graph$N$ is closed, N maps compact sets into relatively compact sets, and for some $x_0 \in U$, the following two conditions are satisfied:*

(1) $M \subset \overline{U}$, $M \subset conv(x_0 \cup N(M))$, $\overline{M} = \overline{C} \Longrightarrow \overline{M}$ is compact

where C a countable subset of M,

(2)

$$x \notin (1-\lambda)x_0 + \lambda N(x) \quad \forall x \in \overline{U} \backslash U,\ \lambda \in (0,1). \tag{1.6}$$

Then there exists $x \in \overline{U}$ with $x \in N(x)$

Chapter 2

Nonlinear Implicit Fractional differential with Retarded and Advanced Arguments

2.1 Introduction

In this chapter, we establish in Section 2.2, the existence and uniqueness of solutions for implicit Caputo type modification of the Erdélyi-Kober fractional differential equations with retarded and advanced arguments see [39]. An extension of this problem is given in Section 2.4. More precisely, we shall present a result on the existence of solutions for nonlinear implicit of Caputo type modification of the Erdélyi-Kober fractional differential equations in Banach space with retarded and advanced arguments see [40].

2.2 Nonlinear IFDE with Retarded and Advanced Arguments

[1]

In this Section, we study the existence and uniqueness of solutions for a class of problem for nonlinear implicit fractional differential equations (NIFDE for short) involving both retarded and advanced arguments.

$$
\begin{aligned}
{}_c^\rho D_{a^+}^\alpha y(t) &= f(t, y^t, {}_c^\rho D_{a^+}^\alpha y(t)), \text{ for } && t \in I := [a, T],\ 1 < \alpha \leq 2, && (2.1)\\
y(t) &= \phi(t), && t \in [a-r, a],\ r > 0 && (2.2)\\
y(t) &= \psi(t), && t \in [T, T+\beta],\ \beta > 0, && (2.3)
\end{aligned}
$$

[1] M. Boumaaza, M. Benchohra, F. Berhoun, Nonlinear implicit Caputo type modification of the Erdélyi-Kober fractional differential equations with retarded and advanced Arguments, *Panam. Math. J*, **30** (2020), 21 - 36.

where ${}^{\rho}_{c}D^{\alpha}_{a^+}$ is the Caputo type modification of the Erdélyi-Kober fractional derivative, $f : I \times C([-r,\beta],\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ is a given function, $\phi \in C([a-r,a],\mathbb{R})$ with $\phi(a) = 0$ and $\psi \in C([T,T+\beta],\mathbb{R})$ with $\psi(T) = 0$. We denote by y^t the element of $C([-r,\beta])$ defined by:

$$y^t(s) = y(t+s) : s \in [-r,\beta]$$

here $y^t(\cdot)$ represents the history of the state from time $t-r$ up to time $t+\beta$.

Lemma 2.2.1 *Let $1 < \alpha \leq 2$, $\phi \in C([a-r,a],\mathbb{R})$ with $\phi(a) = 0$, $\psi \in C([T,T+\beta],\mathbb{R})$ with $\psi(T) = 0$ and $h : I \to \mathbb{R}$ be a continuous function. Then the linear problem*

$$ {}^{\rho}_{c}D^{\alpha}_{a^+}y(t) = h(t), \text{ for a.e. } \ t \in I := [a,T], \ 1 < \alpha \leq 2, \tag{2.4}$$

$$y(t) = \phi(t), \qquad t \in [a-r,a], \ r > 0 \tag{2.5}$$

$$y(t) = \psi(t), \qquad t \in [T,T+\beta], \ \beta > 0, \tag{2.6}$$

has a unique solution, which is given by

$$y(t) = \begin{cases} \phi(t), & \text{if} \quad t \in [a-r,a], \\ -\displaystyle\int_a^T G(t,s)h(s)ds, & \text{if} \quad t \in I \\ \psi(t), & \text{if} \quad t \in [T,T+\beta], \end{cases} \tag{2.7}$$

where

$$G(t,s) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \begin{cases} \dfrac{(t^\rho - a^\rho)(T^\rho - s^\rho)^{\alpha-1}s^{\rho-1}}{(T^\rho - a^\rho)} - s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1}, & a \leq s \leq t \leq T, \\ \dfrac{(t^\rho - a^\rho)(T^\rho - s^\rho)^{\alpha-1}s^{\rho-1}}{(T^\rho - a^\rho)}, & a \leq t \leq s \leq T. \end{cases} \tag{2.8}$$

Here $G(t,s)$ is called the Green function of the boundary value problem (2.4)-(2.6).
Proof. *From* (1.4), *we have*

$$y(t) = c_0 + c_1\left(\frac{t^\rho - a^\rho}{\rho}\right) +^{\rho} I^{\alpha}_{a^+}h(s), \quad c_0, c_1 \in \mathbb{R}, \tag{2.9}$$

therefore

$$y(a) = c_0 = 0,$$

$$y(T) = c_1\left(\frac{T^\rho - a^\rho}{\rho}\right) + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^T (T^\rho - s^\rho)^{\alpha-1}s^{\rho-1}h(s)ds,$$

and

$$c_1 = -\frac{\rho^{2-\alpha}}{(T^\rho - a^\rho)\Gamma(\alpha)}\int_a^T (T^\rho - s^\rho)^{\alpha-1}s^{\rho-1}h(s)ds.$$

Substitute the value of c_0 *and* c_1 *into equation (2.9), we get equation (2.7).*

$$y(t) = \begin{cases} \phi(t), & \text{if} \quad t \in [a-r, a], \\ -\displaystyle\int_a^T G(t,s)h(s)ds, & \text{if} \quad t \in I \\ \psi(t), & \text{if} \quad t \in [T, T+\beta], \end{cases}$$

where G *is defined by equation (2.4.1), the proof is complete.*

Lemma 2.2.2 *Let* $f : I \times C[-r,\beta] \times \mathbb{R} \longrightarrow \mathbb{R}$ *be a continuous function. A function* $y \in \Omega$ *is solution of problem* $(2.1)-(2.3)$ *if and only if* y *satisfies the following integral equation*

$$y(t) = \begin{cases} \phi(t), & \text{if} \quad t \in [a-r, a], \\ -\displaystyle\int_a^T G(t,s)h(s)ds, & \text{if} \quad t \in I \\ \psi(t), & \text{if} \quad t \in [T, T+\beta], \end{cases}$$

where $h \in C(I)$ *satisfies the functional equation*

$$h(t) = f(t, y_t, h(t)).$$

The following hypotheses will be used in the sequel:

(H_1) The function $f : I \times C[-r,\beta] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.

(H_2) There exist $K > 0$, $0 < \overline{K} < 1$ such that

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \leq K\|u - \bar{u}\|_{[-r,\beta]} + \overline{K}|v - \bar{v}|$$

for any $u, \bar{u} \in C([-r,\beta])$ and $v, \bar{v} \in \mathbb{R}$.

(H_3) There exists $p \in L^\infty([a,T],\mathbb{R}_+)$ such that

$$|f(t,u,v)| \leq p(t) \text{ for a.e. } t \in I, \quad \text{and each} \quad u \in C([-r,\beta]) \text{ and } v \in \mathbb{R}.$$

Set

$$p^* = ess\sup_{t\in I} p(t)$$

$$\widetilde{G} = \sup\left\{\int_a^T |G(t,s)|ds, t \in I\right\}.$$

Now, we state and prove our existence result for (2.1)-(2.3) based on the Banach contraction principle.

Theorem 2.2.1 *Assume (H_1) and (H_2) hold. If*

$$\frac{K\widetilde{G}}{(1-\overline{K})} < 1, \tag{2.10}$$

then the problem (2.1)-(2.3) has a unique solution.

Proof: Let the operator $N : \mathcal{C} \longmapsto \mathcal{C}$ defined by

$$(Ny)(t) = \begin{cases} \phi(t), & \text{if} \quad t \in [a-r,a], \\ \\ -\displaystyle\int_a^T G(t,s)h_y(s)ds, & \text{if} \quad t \in I \\ \\ \psi(t), & \text{if} \quad t \in [T,T+\beta]. \end{cases} \tag{2.11}$$

By Lemma 2.2.2 it is clear that the fixed points of N are solutions (2.1)-(2.3) .
Let $y_1, y_2 \in \mathcal{C}$. If $t \in [a-r,a]$ or $t \in [T,T+\beta]$ then

$$|(Ny_1)(t) - (Ny_2)(t)| = 0.$$

For $t \in I$, we have

$$|(Ny_1)(t) - (Ny_2)(t)| \leq \int_a^T |G(t,s)||h_{y_1}(s) - h_{y_2}(s)|ds, \tag{2.12}$$

and by (H_2) we have

$$\begin{aligned} |h_{y_1}(t) - h_{y_2}(t)| &= |f(t, y_1^t, {}^\rho_c D^\alpha_{a^+} y_1(t)) - f(t, y_2^t, {}^\rho_c D^\alpha_{a^+} y_2(t))| \\ &\leq K\|y_1 - y_2\|_{[-r,\beta]} + \overline{K}|h_{y_1}(t) - h_{y_2}(t)|. \end{aligned}$$

Then

$$|h_{y_1}(t) - h_{y_2}(t)| \le \frac{K}{(1-\overline{K})}\|y_1 - y_2\|_{[-r,\beta]}. \tag{2.13}$$

By replacing (2.13) in (2.12) we obtain,

$$\begin{aligned}|(Ny_1)(t) - (Ny_2)(t)| &\le \frac{K}{(1-\overline{K})}\int_a^T |G(t,s)|\|y_1 - y_2\|_{[-r,\beta]}ds \\ &\le \frac{K\widetilde{G}}{(1-\overline{K})}\|y_1 - y_2\|_{[-r,\beta]}.\end{aligned}$$

Therefore, for each $t \in I$, we have

$$|(Ny_1)(t) - (Ny_2)(t)| \le \frac{K\widetilde{G}}{(1-\overline{K})}\|y_1 - y_2\|_{\mathcal{C}}.$$

Thus

$$\|Ny_1 - Ny_2\|_{\mathcal{C}} \le \frac{K\widetilde{G}}{(1-\overline{K})}\|y_1 - y_2\|_{\mathcal{C}}.$$

Hence, by the Banach contraction principle, N has a unique fixed point which is a unique solution of the problem (2.1)-(2.3).

We now prove an existence result for (2.1)-(2.3) by using the Schauder's fixed point theorem.

Theorem 2.2.2 *Assume that the hypotheses (H_1) and (H_3) hold. Then problem (2.1)-(2.3) has at least one solution.*

Step 1. N is continuous. Let $\{y_n\}$ be a sequence such that $y_n \longrightarrow y$ in $\mathcal{C}$. If $t \in [a-r, a]$ or $t \in [T, T+\beta]$ then

$$|(Ny_n)(t) - (Ny)(t)| = 0.$$

For $t \in I$, we have

$$|(Ny_n)(t) - (Ny)(t)| \le \int_a^T |G(t,s)||h_n(s) - h(s)|ds, \tag{2.14}$$

where

$$h_n(t) = f(t, y_n^t, h_n(t)),$$

and

$$h(t) = f(t, y^t, h(t)).$$

Since $y_n \longrightarrow y$, and by (H_1) we get $h_n(t) \longrightarrow h(t)$ as $n \longrightarrow \infty$ for each $t \in I$.
By (H_3) we have for each $t \in I$,

$$|h_n(t)| \le p^*. \tag{2.15}$$

Then,

$$\begin{aligned} |G(t,s)||h_n(t)-h(t)| &\leq |G(t,s)|\left[|h_n(t)|+|h(t)|\right] \\ &\leq 2p^*|G(t,s)|. \end{aligned}$$

For each $t \in I$ the functions $s \longmapsto 2p^*|G(t,s)|$ are integrable on $[a,t]$, then by Lebesgue dominated convergence theorem, equation (2.14) implies

$$|(Ny_n)(t) - (Ny)(t)| \longrightarrow 0 \ as \ n \longrightarrow \infty,$$

and hence

$$\|N(y_n) - N(y)\|_{\mathcal{C}} \longrightarrow 0 \ as \ n \longrightarrow \infty.$$

Consequently, N is continuous.

Let the constant R be such that:

$$R \geq \max\left\{p^*\widetilde{G}, \|\phi\|_{[a-r,a]}, \|\psi\|_{[T,T+\beta]}\right\}, \tag{2.16}$$

and define

$$D_R = \{y \in \mathcal{C} : \|y\|_{\mathcal{C}} \leq R\}.$$

It is clear that D_R is a bounded, closed and convex subset of Ω.

Step 2. $N(D_R) \subset D_R$.

Let $y \in D_R$ we show that $Ny \in D_R$.
If $t \in [a-r,a]$, then

$$|N(y)(t)| \leq \|\phi\|_{[a-r,a]} \leq R,$$

and if $t \in [T, T+\beta]$, then

$$|N(y)(t)| \leq \|\psi\|_{[T,T+\beta]} \leq R.$$

For each $t \in I$, we have

$$|(Ny)(t)| \leq \int_a^T |G(t,s)||h(s)|ds.$$

By (H_3), we have

$$\begin{aligned} |(Ny)(t)| &\leq p^* \int_a^T |G(t,s)|ds \\ &\leq p^*\widetilde{G} \\ &\leq R, \end{aligned}$$

from which it follows that for each $t \in [a-r, T+\beta]$, we have $|Ny(t)| \leq R$, which implies that $\|Ny\|_{\Omega} \leq R$. Consequently,

$$N(D_R) \subset D_R.$$

Step 3: $N(D_R)$ is bounded and equicontinuous.
By Step 2 we have $N(D_R)$ is bounded.
Let $t_1, t_2 \in I = [a, T], t_1 < t_2,$ *and* $y \in D_R$ then

$$\begin{aligned} |(Ny)(t_2) - (Ny)(t_1)| &\leq \int_a^T |G(t_2, s) - G(t_1, s)||h(s)|ds \\ &\leq p^* \int_a^T |G(t_2, s) - G(t_1, s)|ds. \end{aligned}$$

As $t_1 \longrightarrow t_2$ the right hand side of the above inequality tends to zero. As consequence of Step 1 to Step 3, together withe the Arzela-Ascoli theorem, we can conclude that N is continuous and completely continuous. From Schauder's theorem, we conclude that N has a fixed point with is a solution of the problem (2.1)-(2.3).

We prove an existence result for the (2.1)-(2.3) problem by using the Schaefer's fixed point theorem.

Theorem 2.2.3 *Assume that* (H_1) *and*
(H_4) *There exist* $d, q, m \in C(I, \mathbb{R})$ *with* $m^* = \sup_{t \in I} m(t) < 1$ *such that*

$$|f(t, u, v)| \leq d(t) + q(t)\|u\|_{[-r,\beta]} + m(t)|v|$$

where $t \in I$, $u \in C([-r, \beta], \mathbb{R})$ *and* $v \in \mathbb{R}$.
If

$$\frac{q^* \widetilde{G}}{(1 - m^*)} < 1, \tag{2.17}$$

then problem (2.1)-(2.3) has at least one solution.

Proof. Consider the operator N defined in (2.11). We shall show that N satisfies the assumption of Schaefer's fixed point theorem. As shown in Theorem 2.2.2, we see that the operator N is continuous, and completely continuous.
Now it remains to show that the set

$$\xi = \{y \in \Omega : y = \lambda N y, \quad \text{for some } \lambda \in (0, 1)\} \quad \text{is bounded.}$$

Let $y \in \xi$, then $y = \lambda N y$ for some $0 < \lambda < 1$. Thus for each $t \in I$ we have

$$y(t) = -\lambda \int_a^T G(t, s) h_y(s) ds, \tag{2.18}$$

where

$$h_y(t) = f(t, y^t, h_y(t)).$$

By (H_4), we have for each $t \in I$

$$\begin{aligned} |h_y(t)| &\leq d(t) + q(t)\|y\|_{[-\alpha,\beta]} + m(t)|h_y(t)| \\ &\leq d^* + q^*\|y\|_{[-r,\beta]} + m^*|h_y(t)|. \end{aligned}$$

Thus

$$|h_y(t)| \leq \frac{1}{1-m^*}\left(d^* + q^*\|y\|_{[-\alpha,\beta]}\right).$$

This implies, by (2.18) that for each $t \in I$ we have

$$\begin{aligned} |y(t)| &\leq \int_a^T |G(t,s)|\frac{1}{1-m^*}\left(d^* + q^*\|y\|_{[-\alpha,\beta]}\right) ds \\ &\leq \frac{\left(d^* + q^*\|y\|_{[-r,\beta]}\right)\widetilde{G}}{(1-m^*)}. \end{aligned}$$

Then

$$\|y\|_{[-r,\beta]} \leq \frac{d^*\widetilde{G}}{(1-m^*)} + \frac{q^*\widetilde{G}\|y\|_{[-r,\beta]}}{(1-m^*)}.$$

Thus

$$\left[1 - \frac{q^*\widetilde{G}}{(1-m^*)}\right]\|y\|_{[-r,\beta]} \leq \frac{d^*\widetilde{G}}{(1-m^*)}.$$

Finally, by (2.17) we have

$$\|y\|_{[-r,\beta]} \leq \frac{\frac{d^*\widetilde{G}}{(1-m^*)}}{\left[1 - \frac{q^*\widetilde{G}}{(1-m^*)}\right]} = b_0.$$

If $t \in [a-r, a]$, then

$$|y(t)| \leq \|\phi\|_{[a-r,a]} \leq b_1,$$

and if $t \in [T, T+\beta]$, then

$$|y(t)| \leq \|\psi\|_{[T,T+\beta]} \leq b_2.$$

From which it follows that for each $t \in [a-r, T+\beta]$, we have $|y(t)| \leq \max\{b_2, b_1, b_0\}$, which implies that $\|y\|_{\mathcal{C}} \leq \max\{b_2, b_1, b_0\}$, this implies that ξ is bounded As a consequence of Schaefer's fixed point theorem, N admits a fixed point which is a solution of the problem (2.1)-(2.3).

2.3 Examples

Example 1: Consider the boundary value problem of implicit Caputo type modification of the Erdélyi-Kober fractional differential equation:

$$\begin{cases} y(t) = e^{t-2} - 1, & t \in [1,2], \\ {}_c^{\frac{1}{2}}D_{2^+}^{\frac{3}{2}} y(t) = \dfrac{1}{10e^{t+2}\left(1 + |y^t| + \left|{}_c^{\frac{1}{2}}D_{2^+}^{\frac{3}{2}} y(t)\right|\right)} + \dfrac{\sin(t)}{\ln(t^2+1)}, & t \in I = [2,4] \\ y(t) = t - 4, & t \in [4,6]. \end{cases} \tag{2.19}$$

Set

$$f(t,u,v)=\frac{1}{10e^{t+2}(1+|u|+|v|)}+\frac{\sin(t)}{\ln(t^2+1)},\quad t\in[2,4],u\in C([-r,\beta])$$

and $v\in \mathbb{R},\alpha=\frac{3}{2},\rho=\frac{1}{2},r=1,\beta=2$. For each $u,\bar{u}\in C([-r,\beta])$, $v,\bar{v}\in \mathbb{R}$ and $t\in[2,4]$, we have

$$\begin{aligned}|f(t,u,v)-f(t,\bar{u},\bar{v})| &\leq \left|\frac{1}{10e^{t+2}(1+|u|+|v|)}-\frac{1}{10e^{t+2}(1+|\bar{u}|+|\bar{v}|)}\right|\\ &\leq \frac{1}{10e^{t+2}}\left(|u-\bar{u}|+|v-\bar{v}|\right)\\ &\leq \frac{1}{10e^{t+2}}\left(\|u-\bar{u}\|_{[-r,\beta]}+|v-\bar{v}|\right).\end{aligned}$$

Therefore, (H_2) is verified with $K=\overline{K}=\frac{1}{10e^4}$.
For each $t\in I$ we have

$$\int_a^T|G(t,s)|ds\leq\frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho-a^\rho}{T^\rho-a^\rho}\right)\int_a^T\left|\left(\frac{T^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\right|ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_a^t\left|\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\right|ds.$$

Then

$$\int_a^T|G(t,s)|ds\leq\frac{2}{\Gamma(\alpha+1)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^\alpha.$$

Therefore

$$\widetilde{G}\leq\frac{2}{\Gamma(\alpha+1)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^\alpha.$$

The condition

$$\begin{aligned}\frac{K\widetilde{G}}{(1-\overline{K})} &\leq 2\frac{\frac{1}{10e^4}}{(1-\frac{1}{10e^4})\Gamma(\frac{5}{2})}\left(\frac{2-2^{\frac{1}{2}}}{\frac{1}{2}}\right)^{\frac{3}{2}}\approx 0.0035008\\ &< 1,\end{aligned}$$

is satisfied with $T=4,a=2$ and $\alpha=\frac{3}{2}$. Hence all conditions of Theorem 2.2.1 are satisfied, it follows that the problem (2.19) admit a unique solution defined on I.

Example 2: Consider the boundary value problem of implicit Caputo type modification of the Erdélyi-Kober fractional differential equation:

$$\begin{cases} y(t) = e^t - 1, & t \in [-1, 0], \\ {}^{\frac{1}{2}}_{c}D^{\frac{3}{2}}_{0^+} y(t) = \dfrac{\sin(2t)\left(2 + |y^t| + \left|{}^{\frac{1}{2}}_{c}D^{\frac{3}{2}}_{0^+} y(t)\right|\right)}{20e^{t+4}\left(1 + |y^t| + \left|{}^{\frac{1}{2}}_{c}D^{\frac{3}{2}}_{0^+} y(t)\right|\right)}, & t \in I = [0, e] \\ y(t) = \ln(t) - 1, & t \in [e, 4], \end{cases} \tag{2.20}$$

with

$$f(t,u,v) = \frac{\sin(2t)\,(2 + |u| + |v|)}{10e^{t+2}(1 + |u| + |v|)}, \quad t \in I = [0, e], u \in C([-r, \beta]) \text{ and } v \in \mathbb{R}$$

$$\alpha = \frac{3}{2}, \rho = \frac{1}{2}, r = 1, \beta = 4 - e.$$

Condition (H_4) is satisfied for each $u, \in C([-r, \beta])$, $v \in \mathbb{R}$ and $t \in [0, e]$:

$$\begin{aligned} |f(t,u,v)| &\leq \frac{2 + |u| + |v|}{20e^{t+4}} \\ &\leq \frac{1}{20e^{t+4}}\left(2 + |v| + \|u\|_{[-r,\beta]}\right). \end{aligned}$$

Therefore, (H_4) is verified with

$$d(t) = \frac{1}{10e^{t+4}}, \quad q(t) = m(t) = \frac{1}{20e^{t+4}} \quad \textit{and} \quad m^* = \frac{1}{20e^4} < 1.$$

Condition:

$$\begin{aligned} \frac{q^*\widetilde{G}}{(1 - m^*)} &\leq 2\frac{\frac{1}{20e^4}}{(1 - \frac{1}{20e^4})\Gamma(\frac{5}{2})}\left(\frac{e^{\frac{1}{2}}}{\frac{1}{2}}\right)^{\frac{3}{2}} \approx 0.0082575 \\ &< 1, \end{aligned}$$

is satisfied with $T = e, a = 0$ and $\alpha = \frac{3}{2}$. Hence all conditions of Theorem 2.2.3 are satisfied, it follows that the problem (2.20) has at least one solution on I.

2.4 Fractional implicit differential equations with retarded and advanced arguments in Banach Spaces [2]

[2] M. Benchohra, F. Berhoun, M. Boumaaza, S. Sivasundaram, Caputo type modification of the Erdelyi-Kober fractional implicit differential equations with retarded and advanced arguments in Banach Spaces, *Nonlinear Stud.* **27** (1) (2020), 285-296.

In this Section, we study the existence of solutions for the following problem of nonlinear implicit fractional differential equations (NIFDE for short), in Banach space with retarded and advanced arguments

$$ {}_c^\rho D_{a^+}^\nu y(t) = f(t, y^t, {}_c^\rho D_{a^+}^\nu y(t)), \ t \in I := [a, T], \ 1 < \nu \leq 2, \tag{2.21} $$

$$ y(t) = \phi(t), \qquad t \in [a-r, a], \ r > 0 \tag{2.22} $$

$$ y(t) = \psi(t), \qquad t \in [T, T+\beta], \ \beta > 0, \tag{2.23} $$

where ${}_c^\rho D_{a^+}^\nu$ is the Caputo type modification of the Erdélyi-Kober fractional derivative, $(E, \|\cdot\|)$ is a real Banach space and $f : I \times C([-r, \beta], E) \times E \to E$ is a given function, $\phi \in C([a-r, a], E)$ with $\phi(a) = 0$ and $\psi \in C([T, T+\beta], E)$ with $\psi(T) = 0$.
We denote by y^t the element of $C([-r, \beta])$ defined by:

$$ y^t(s) = y(t+s) : s \in [-r, \beta] $$

here $y^t(\cdot)$ represents the history of the state from time $t - r$ up to time $t + \beta$.

Definition 2.4.1 A function $y \in \mathcal{C}$, is said to be a solution of (2.21)-(2.23) if y satisfies the equation ${}_c^\rho D_{a^+}^\nu y(t) = f(t, y^t, {}_c^\rho D_{a^+}^\nu y(t))$ on I, and the conditions $y(t) = \phi(t), \phi(a) = 0$ on $[a-r, a]$ and $y(t) = \psi(t), \psi(T) = 0$ on $[T, T+\beta]$.

To prove the existence of solutions to (2.21)–(2.23), we need the following auxiliary Lemma.

Lemma 2.4.1 *Let $f : I \times C[-r, \beta] \times E \longrightarrow E$ be a continuous function. A function $y \in \Omega$ is solution of problem (2.21) – (2.23) if and only if y satisfies the following integral equation*

$$ y(t) = \begin{cases} \phi(t), & \text{if} \quad t \in [a-r, a], \\ -\displaystyle\int_a^T G(t,s) h(s) ds, & \text{if} \quad t \in I \\ \psi(t), & \text{if} \quad t \in [T, T+\beta], \end{cases} $$

where $h \in C(I)$ satisfies the functional equation

$$ h(t) = f(t, y^t, h(t)), $$

and

$$ G(t,s) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \begin{cases} \dfrac{(t^\rho - a^\rho)(T^\rho - s^\rho)^{\alpha-1} s^{\rho-1}}{(T^\rho - a^\rho)} - s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1}, & a \leq s \leq t \leq T, \\ \dfrac{(t^\rho - a^\rho)(T^\rho - s^\rho)^{\alpha-1} s^{\rho-1}}{(T^\rho - a^\rho)}, & a \leq t \leq s \leq T. \end{cases} $$

The following hypotheses will be used in the sequel:

(H_1) The function $f : I \times C[-r,\beta] \times E \longrightarrow E$ is continuous.

(H_2) There exist $d, q, m \in C(I, \mathrm{I\!R})$ with $m^* = \sup_{t\in I} m(t) < 1$ such that

$$\|f(t,u,v)\| \leq d(t) + q(t)\|u\|_{[-\alpha,\beta]} + m(t)\|v\|,\ u \in C([-r,\beta],E),\ v \in E,\ t \in I.$$

(H_3) for each bounded set $B \subset \mathcal{C}$, and for each $t \in I$, we have

$$\alpha(f(t,B_1,B_2)) \leq q(t) \sup_{t\in[-r,\beta]} \alpha(B_1) + m(t) \sup_{t\in[-r,\beta]} \alpha(B_2),$$

for any bounded sets, $B_1 \subset C([-r,\beta]), B_2 \subset E$.
Set

$$q^* = \sup_{t\in I} q(t),\ m^* = \sup_{t\in I} m(t),\ \widetilde{G} = \sup\left\{\int_a^T |G(t,s)|ds,\ t \in I\right\}.$$

We prove an existence result for the (2.21)-(2.23) problem, by using the Darbo fixed point theorem.

Theorem 2.4.1 *Assume that the hypotheses $(H_1) - (H_3)$ hold. If*

$$\frac{q^*\widetilde{G}}{1-m^*} < 1, \tag{2.24}$$

then problem (2.21)-(2.23) has at least one solution.

Proof. Let the operator $N : \mathcal{C} \longmapsto \mathcal{C}$ defined by

$$(Ny)(t) = \begin{cases} \phi(t), & \text{if} \quad t \in [a-r,a], \\ -\displaystyle\int_a^T G(t,s)h_y(s)ds, & \text{if} \quad t \in I \\ \psi(t), & \text{if} \quad t \in [T,T+\beta]. \end{cases} \tag{2.25}$$

By Lemma 2.4.1 it is clear that the fixed points of N are solutions (2.21)-(2.23).
Step 1: N is continuous. Let $\{y_n\}$ be a sequence such that $y_n \longrightarrow y$ in $\mathcal{C}$. If $t \in [a-r,a]$ or $t \in [T, T+\beta]$ then

$$\|(Ny_n)(t) - (Ny)(t)\| = 0.$$

For $t \in I$, we have

$$\|(Ny_n)(t) - (Ny)(t)\| \leq \int_a^T |G(t,s)|\|h_n(s) - h(s)\|ds, \tag{2.26}$$

where

$$h_n(t) = f(t, y_n^t, h_n(t)),$$

and

$$h(t) = f(t, y^t, h(t)).$$

Since $y_n \longrightarrow y$, by (H_1) we get $h_n(t) \longrightarrow h(t)$ as $n \longrightarrow \infty$ for each $t \in I$.
And let $\eta > 0$, such that, for each $t \in I$, we have $\|h_n(t)\| \leq \eta$ and $\|h(t)\| \leq \eta$.
Therefore

$$\begin{aligned} |G(t,s)|\|h_n(t) - h(t)\| &\leq |G(t,s)|\left[\|h_n(t)\| + \|h(t)\|\right] \\ &\leq 2\eta|G(t,s)|. \end{aligned}$$

For each $t \in I$ the function $s \longmapsto 2\eta|G(t,s)|$ is integrable on $[a,t]$, then by Lebesgue dominated convergence theorem, equation (2.26) implies

$$\|(Ny_n)(t) - (Ny)(t)\| \longrightarrow 0 \ as \ n \longrightarrow \infty,$$

and hence

$$\|N(y_n) - N(y)\|_{\mathcal{C}} \longrightarrow 0 \ as \ n \longrightarrow \infty.$$

Thus N is continuous.

Let the constant R be such that:

$$R \geq \max\left\{A\widetilde{G}, \|\phi\|_{[a-r,a]}, \|\psi\|_{[T,T+\beta]}\right\}, \tag{2.27}$$

and define

$$D_R = \{y \in \mathcal{C} : \|y\|_{\mathcal{C}} \leq R\}.$$

It is clear that D_R is a bounded, closed and convex subset of Ω.

Step 2: N maps D_R into itself.
Let $y \in D_R$ we show that $Ny \in D_R$.
If $t \in [a-r, a]$, then

$$\|N(y)(t)\| \leq \|\phi\|_{[a-r,a]} \leq R,$$

and if $t \in [T, T+\beta]$, then

$$\|N(y)(t)\| \leq \|\psi\|_{[T,T+\beta]} \leq R.$$

For each $t \in I$, we have

$$\|(Ny)(t)\| \leq \int_a^T |G(t,s)|\|h(s)\|ds.$$

By (H_2) we have for each $t \in I$

$$\begin{aligned} \|h(t)\| &\leq d(t) + q(t)\|y\|_{[-\alpha,\beta]} + m(t)\|h(t)\| \\ &\leq d^* + q^*\|y\|_{[-\alpha,\beta]} + r^*\|h(t)\| \\ &\leq d^* + q^*R_1 + r^*\|h(t)\|, \end{aligned}$$

where

$$d^* = \sup_{t\in I} d(t), \quad q^* = \sup_{t\in I} q(t) \quad and \quad m^* = \sup_{t\in I} m(t).$$

Then

$$\|h(t)\| \leq \frac{d^* + q^*R}{1-m^*} = A. \tag{2.28}$$

By (2.28), for $t \in I$, we have

$$\begin{aligned}\|(Ny)(t)\| &\leq A\int_a^T |G(t,s)|ds \\ &\leq A\widetilde{G} \\ &\leq R,\end{aligned}$$

from which it follows that for each $t \in [a-r, T+\beta]$, we have $\|Ny(t)\| \leq R$, which implies that $\|Ny\|_\Omega \leq R$. This proves that N transforms the set D_R into itself.
Step 3: $N(D_R)$ is bounded and equicontinuous.
Since $N(D_R) = \{N(y) : y \in D_R\} \subset D_R$ and D_R is bounded, then $N(D_R)$ is bounded.
Now, let $t_1, t_2 \in I = [a, T], t_1 < t_2,$ *and* $y \in D_R$ then

$$\begin{aligned}\|(Ny)(t_2) - (Ny)(t_1)\| &\leq \int_a^T |G(t_2,s) - G(t_1,s)|\|h(s)\|ds \\ &\leq A\int_a^T |G(t_2,s) - G(t_1,s)|ds.\end{aligned}$$

As $t_1 \longrightarrow t_2$ the right hand side of the above inequality tends to zero.
Step 4: The operator $N : D_R \longmapsto D_R$ is a strict set contraction.
Let $V \subset D_R$ if t $[a-r, a]$, then

$$\begin{aligned}\alpha(N(V)(t)) &= \alpha(N(y)(t), y \in V) \\ &= \alpha(\phi(t)) \\ &= 0,\end{aligned}$$

also if t $[T, T+\beta]$, then

$$\begin{aligned}\alpha(N(V)(t)) &= \alpha(N(y)(t), y \in V) \\ &= \alpha(\psi(t)) \\ &= 0.\end{aligned}$$

And if $t \in I$, we have

$$\begin{aligned}\alpha(N(V)(t)) &= \alpha(N(y)(t), y \in V) \\ &\leq \left\{\int_a^T |G(t,s)|\alpha(h(s))ds, y \in V\right\}.\end{aligned}$$

By (H_3) we have

$$\begin{aligned}\alpha(h(s), y \in V) &= \alpha(\{f(s, y(s), h(s)), y \in V\}) \\ &\leq q(t)\alpha(\{y(s), y \in V\}) + m(t)\alpha(\{h(s), y \in V\}) \\ &\leq q^*\alpha(\{y(s), y \in V\}) + m^*\alpha(\{h(s), y \in V\}).\end{aligned}$$

Then

$$\alpha(\{h(s), y \in V\}) \leq \frac{q^*}{1-m^*}\alpha(\{y(s), y \in V\}). \tag{2.29}$$

Thus

$$\begin{aligned}\alpha(N(V)(t)) &\leq \frac{q^*}{1-m^*}\int_a^T |G(t,s)|\alpha(\{y(s), y \in V\})ds \\ &\leq \frac{q^*\widetilde{G}}{1-m^*}\alpha_c(V).\end{aligned}$$

Therefore

$$\alpha_c(NV) \leq \frac{q^*\widetilde{G}}{1-m^*}\alpha_c(V).$$

So by (2.24) the operator N is a set contraction. And thus, by Theorem 1.6.1, N has a fixed point, which is solution to problem $(2.21)-(2.23)$.

We prove an existence result for the (2.21)-(2.23) problem, by using the Mönch's fixed point theorem.

Theorem 2.4.2 *Assume that* $(H_1)-(H_3)$ *hold. If*

$$\frac{q^*\widetilde{G}}{1-m^*} < 1, \tag{2.30}$$

then problem $(2.21)-(2.23)$ *has at least one solution.*

Proof: Consider the operator N defined in (2.25). According to Theorem 2.4.1, the operator N is bounded into itself, and equicontinuous.
Now let V be a subset of D_R such that $V \subset conv(N(V) \cup \{0\})$. Since V is bounded and equicontinuous, the function $t \longmapsto v(t) = \alpha(V(t))$ is continuous on $[a-r, T+\beta]$. By $(H_1)-(H_3)$, Lemma 1.4.3, and the properties of measure α, for each $t \in I$, we have

$$\begin{aligned}v(t) &\leq \alpha(N(V)(t) \cup \{0\}) \\ &\leq \alpha\left(\{(Ny)(t), y \in V\}\right) \\ &\leq \int_a^T |G(t,s)|\frac{q^*}{1-m^*}\alpha(V(s))ds \\ &\leq \frac{q^*\widetilde{G}}{1-m^*}\|v\|_c.\end{aligned}$$

Thus

$$\|v\|_c \leq \frac{q^*\widetilde{G}}{1-m^*}\|v\|_c.$$

From (2.30), we get $\|v\|_c = 0$, that is $\alpha(V(t)) = 0$ for each $t \in I$.
For $t \in [a-r, a]$, we have

$$\begin{array}{rcl} v(t) & = & \alpha(\phi(t)) \\ & = & 0. \end{array}$$

Also for $t \in [T, T+\beta]$ we have

$$\begin{array}{rcl} v(t) & = & \alpha(\psi(t)) \\ & = & 0, \end{array}$$

then $V(t)$ is relatively compact in E. In view of Ascoli-Arzela theorem, V is relatively compact in D_R. Applying Theorem 1.6.5, we conclude that N has a fixed point which is a solution of the problem (2.21) – (2.23).

2.5 An Example

Let

$$E = l^1 = \left\{ y = (y_1, y_2, \ldots, y_n, \ldots), \sum_{k=1}^{\infty} |y_n| < \infty \right\},$$

be the Banach space with the norm

$$\|y\|_E = \sum_{k=1}^{\infty} |y_n|.$$

Consider the boundary value problem of implicit Caputo type modification of the Erdélyi-Kober fractional differential equation

$$\begin{cases} y(t) = \ln(t) - 1, & t \in [e, 4], \\ {}^3_cD^{\frac{3}{2}}_{2^+} y_n(t) = f(t, y_n^t, {}^3_c D^{\frac{3}{2}}_{2^+} y_n(t)), & t \in I = [2, e] \\ y(t) = \frac{1}{2}t - 1, & t \in [-1, 2], \end{cases} \tag{2.31}$$

here $T = e, \quad a = 2, \quad \nu = \dfrac{3}{2}, \quad \rho = 3.$
Set

$$y = (y_1, y_2, \ldots, y_n, \ldots), \quad f = (f_1, f_2, \ldots, f_n, \ldots)$$

$$f(t, y^t, {}^3_c D^{\frac{3}{2}}_{2^+} y(t)) = \frac{cos(t) + \|y_t\|_{C([-3,4-e])} + \|{}^3_c D^{\frac{3}{2}}_{2^+} y(t)\|}{2e^{t-2}\left(1 + \|y_t\|_{C([-3,4-e])} + \|{}^3_c D^{\frac{3}{2}}_{2^+} y\|_E\right)}.$$

For each $y \in E$ and $t \in [2, e]$, we have

$$\|f(t, y(t), {}^3_c D^{\frac{3}{2}}_{2^+} y(t))\| \leq \frac{1}{2e^{t-2}}\left(cos(t) + \|y_t\|_{C([-3,4-e])} + \|{}^3_c D^{\frac{3}{2}}_{2^+} y(t)\|\right),$$

hence. (H_2) is satisfied with $m^* = q^* = \frac{1}{2}$.
For each $t \in I$ we have

$$\int_a^T |G(t,s)|ds \leq \frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{T^\rho - a^\rho}\right)\int_a^T \left|\left(\frac{T^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\right| ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_a^t \left|\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\right| ds,$$

then

$$\int_a^T |G(t,s)|ds \leq \frac{2}{\Gamma(\nu+1)}\left(\frac{T^\rho - a^\rho}{\rho}\right)^\nu.$$

Therefore

$$\widetilde{G} \leq \frac{2}{\Gamma(\nu+1)}\left(\frac{T^\rho - a^\rho}{\rho}\right)^\nu.$$

Condition (2.24) holds, indeed,

$$\begin{aligned} \frac{q^*\widetilde{G}}{1-m^*} &\leq \frac{2}{\Gamma(\frac{3}{2}+1)}\left(\frac{e^{\frac{3}{2}} - 2^{\frac{3}{2}}}{3}\right)^{\frac{3}{2}} \approx 0.61549 \\ &< 1. \end{aligned}$$

Hence all conditions of Theorem 2.4.1 are satisfied. It follows that the problem (2.31) has at least one solution.

Chapter 3

Coupled Implicit Fractional Differential Systems with Retarded and Advanced Arguments

3.1 Introduction

In Section 3.2, we study the existence and uniqueness of solutions to the following coupled system nonlinear implicit of Caputo type modification of the Erdélyi-Kober fractional differential equations involving both retarded and advanced arguments see [42]. An extension of this problem is given in Section 3.4. More precisely, we shall present a result of existence of weak solutions to coupled system nonlinear implicit of Caputo type modification of the Erdélyi-Kober fractional differential equations involving both retarded and advanced arguments see [44]. This chapter generalizes the previous one.

3.2 Existence Results for the Coupled Implicit Fractional Differential Systems with Retarded and Advanced Arguments

[1]. In this Section,, we study the existence and uniqueness of solutions to the following Coupled system nonlinear implicit fractional differential equations (CSIFD for short) involving both retarded and advanced arguments

$$\begin{cases} {}^{\rho}_{c}D^{\alpha}_{a^+}u(t) = f_1(t, u^t, v^t, {}^{\rho}_{c}D^{\alpha}_{a^+}u(t), {}^{\rho}_{c}D^{\alpha}_{a^+}v(t)) \\ {}^{\rho}_{c}D^{\alpha}_{a^+}v(t) = f_2(t, u^t, v^t, {}^{\rho}_{c}D^{\alpha}_{a^+}u(t), {}^{\rho}_{c}D^{\alpha}_{a^+}v(t)) \end{cases} \quad t \in I := [a, T], \tag{3.1}$$

[1]M. Boumaaza, M. Benchohra and Juan J. Nieto, Caputo type modification of the Erdélyi-Kober coupled implicit fractional differential systems with retardation and anticipation, *Differ. Equ. Appl.* (accepted)

$$\begin{cases} (u(t), v(t)) = (\phi_1(t), \phi_2(t)), \ t \in [a-r, a], \ r > 0 \\ (u(t), v(t)) = (\psi_1(t), \psi_2(t)), \ t \in [T, T+\beta], \ \beta > 0, \end{cases} \tag{3.2}$$

where ${}^{\rho}_{c}D^{\alpha}_{a+}$ is the Caputo type modification of the Erdélyi-Kober fractional derivative and $f_i : I \times C([-r, \beta], \mathbb{R})^2 \times \mathbb{R}^2 \to \mathbb{R}$ is a given function, $\phi_i \in C([a-r, a], \mathbb{R})$ with $\phi_i(a) = 0$ and $\psi_i \in C([T, T+\beta], \mathbb{R})$ with $\psi_i(T) = 0$,$i = 1, 2$
We denote by u^t the element of $C([-r, \beta])$ defined by:

$$u^t(s) = u(t+s) : s \in [-r, \beta]$$

here $u^t(\cdot)$ represents the history of the state from time $t - r$ up to time $t + \beta$.

Lemma 3.2.1 *Let $1 < \alpha \leq 2$, $\phi \in C([a-r, a], \mathbb{R})$ with $\phi(a) = 0$, $\psi \in C([T, T+\beta], \mathbb{R})$ with $\psi(T) = 0$ and $h : I \to \mathbb{R}$ be a continuous function. Then the linear problem*

$$ {}^{\rho}_{c}D^{\alpha}_{a+}u(t) = h(t), \ \textit{for a.e } t \in I := [a, T], \ 1 < \alpha \leq 2, \tag{3.3}$$

$$u(t) = \phi(t), \qquad t \in [a-r, a], \ r > 0 \tag{3.4}$$

$$u(t) = \psi(t), \qquad t \in [T, T+\beta], \ \beta > 0, \tag{3.5}$$

has a unique solution, which is given by

$$u(t) = \begin{cases} \phi(t), & \textit{if} \quad t \in [a-r, a], \\ -\displaystyle\int_a^T G(t,s)h(s)ds, & \textit{if} \quad t \in I \\ \psi(t), & \textit{if} \quad t \in [T, T+\beta], \end{cases} \tag{3.6}$$

where

$$G(t,s) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \begin{cases} \dfrac{(t^\rho - a^\rho)(T^\rho - s^\rho)^{\alpha-1}s^{\rho-1}}{(T^\rho - a^\rho)} - s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1}, & a \leq s \leq t \leq T. \\ \dfrac{(t^\rho - a^\rho)(T^\rho - s^\rho)^{\alpha-1}s^{\rho-1}}{(T^\rho - a^\rho)}, & a \leq t \leq s \leq T. \end{cases} \tag{3.7}$$

Here $G(t, s)$ is called the Green function of the boundary value problem (3.3)-(3.5).
Proof. *From* (1.4), *we have*

$$u(t) = c_0 + c_1\left(\frac{t^\rho - a^\rho}{\rho}\right) +^{\rho} I^{\alpha}_{a+}h(s), \quad c_0, c_1 \in \mathbb{R}, \tag{3.8}$$

therefore

$$u(a) = c_0 = 0,$$

$$u(T) = c_1\left(\frac{T^\rho - a^\rho}{\rho}\right) + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^T (T^\rho - s^\rho)^{\alpha-1} s^{\rho-1} h(s)ds,$$

and

$$c_1 = -\frac{\rho^{2-\alpha}}{(T^\rho - a^\rho)\Gamma(\alpha)}\int_a^T (T^\rho - s^\rho)^{\alpha-1} s^{\rho-1} h(s)ds.$$

Substitute the value of c_0 and c_1 into equation (3.8), we get equation (3.6).

$$u(t) = \begin{cases} \phi(t), & \text{if} \quad t \in [a-r, a], \\ \\ -\displaystyle\int_a^T G(t,s)h(s)ds, & \text{if} \quad t \in I \\ \\ \psi(t), & \text{if} \quad t \in [T, T+\beta], \end{cases}$$

where G is defined by equation (3.7), the proof is complete.

Lemma 3.2.2 *Let $f_i : I \times C[-r,\beta]^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ i=1,2, be continuous functions. A function $(u,v) \in \mathcal{C}^2$ is solution of system $(3.1)-(3.2)$ if and only if (u,v) satisfies the following coupled system of integral equations*

$$u(t) = \begin{cases} \phi_1(t), & \text{if} \quad t \in [a-r, a], \\ \\ -\displaystyle\int_a^T G(t,s)h_1(s)ds, & \text{if} \quad t \in I \\ \\ \psi_1(t), & \text{if} \quad t \in [T, T+\beta], \end{cases}$$

$$v(t) = \begin{cases} \phi_2(t), & \text{if} \quad t \in [a-r, a], \\ \\ -\displaystyle\int_a^T G(t,s)h_2(s)ds, & \text{if} \quad t \in I \\ \\ \psi_2(t), & \text{if} \quad t \in [T, T+\beta], \end{cases}$$

where $h_i \in C(I)$ satisfies the system of functional equations

$$\begin{cases} h_1(t) = f_1(t, u^t, v^t, h_1(t), h_2(t)), \\ h_2(t) = f_2(t, u^t, v^t, h_1(t), h_2(t)). \end{cases}$$

The following hypotheses will be used in the sequel:

(H_1) The functions $f_i : I \times C[-r,\beta]^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are continuous.

(H_2) There exist $K_i, \overline{K_i}, C_i, \overline{C_i} > 0$, $0 < \overline{C_2} < 1$, $0 < C_1 < 1$ such that

$$|f_i(t,u,v,w,z) - f_i(t,\bar{u},\bar{v},\bar{w},\bar{z})| \leq K_i\|u-\bar{u}\|_{[-r,\beta]} + \overline{K_i}\|v-\bar{v}\|_{[-r,\beta]} + C_i|w-\bar{w}| + \overline{C_i}|z-\bar{z}|$$

for any $u,\bar{u} \in C([-r,\beta])$ and $v,\bar{v} \in \mathbb{R}, i = 1,2$.

(H_3) There exist $p_i, q_i \in L^\infty([a,T],\mathbb{R}_+)$ such that

$$|f_i(t,u,v,\bar{u},\bar{v})| \leq \frac{p_i(t)\|u\|_{[-r,\beta]} + q_i(t)\|v\|_{[-r,\beta]}}{1 + \|u\|_{[-r,\beta]} + \|v\|_{[-r,\beta]} + |\bar{u}| + |\bar{v}|}$$

for a.e. $t \in I$, and each $u,v \in C([-r,\beta])$ and $\bar{u},\bar{v} \in \mathbb{R}$.
Set

$$p_i^* = ess\sup_{t\in I} p_i(t), \qquad q_i^* = ess\sup_{t\in I} q_i(t), \quad i = 1,2$$

$$\widetilde{G} = \sup\left\{\int_a^T |G(t,s)|ds, t \in I\right\}.$$

Now, we state and prove our existence result for (3.1)-(3.2) based on the Banach fixed point theorem.

Theorem 3.2.1 *Assume (H_1) and (H_2) hold. If*

$$\frac{C_2\overline{C_1}}{(1-C_1)(1-\overline{C_2})} < 1, \tag{3.9}$$

and

$$G_1^* + G_2^* < 1, \tag{3.10}$$

then the problem (3.1)-(3.2) has a unique solution.

Proof: Let the operator $N : \mathcal{C} \times \mathcal{C} \longmapsto \mathcal{C} \times \mathcal{C}$ defined by

$$N(u,v)(t) = (N_1(u,v), N_2(u,v))$$

$$= \begin{cases} (\phi_1(t), \phi_2(t)), & \text{if} \quad t \in [a-r,a], \\ -\left(\int_a^T G(t,s)h_1(s)ds, \int_a^T G(t,s)h_2(s)ds\right), & \text{if} \quad t \in I \\ (\psi_1(t), \psi_2(t)), & \text{if} \quad t \in [T, T+\beta]. \end{cases} \tag{3.11}$$

By Lemma 3.2.2 it is clear that the fixed points of N are solutions (3.1)-(3.2) .
Let $(u_2,v_2),(u_1,v_1)\in\mathcal{C}^2$. If $t\in[a-r,a]$ or $t\in[T,T+\beta]$ then

$$|N(u_2,v_2)(t)-N(u_1,v_1)(t)|=0.$$

For $t\in I$, we have

$$|N_1(u_2,v_2)(t)-N_1(u_1,v_1)(t)|\leq\int_a^T|G(t,s)||{}_c^\rho D^\alpha u_2(t)-{}_c^\rho D^\alpha u_1(t)|ds, \tag{3.12}$$

and by (H_2) we have

$$\begin{aligned}
|{}_c^\rho D_{a^+}^\alpha u_2(t)-{}_c^\rho D_{a^+}^\alpha u_1(t)| &= |f_1(t,u_2^t,v_2^t,{}_c^\rho D_{a^+}^\alpha u_2(t),{}_c^\rho D_{a^+}^\alpha v_2(t))\\
&- f_1(t,u_1^t,v_1^t,{}_c^\rho D_{a^+}^\alpha u_1(t),{}_c^\rho D_{a^+}^\alpha v_1(t))|\\
&\leq K_1\|u_2-u_1\|_{[-r,\beta]}+\overline{K_1}\|v_2-v_1\|_{[-r,\beta]}\\
&+ C_1|{}_c^\rho D_{a^+}^\alpha u_2(t)-{}_c^\rho D_{a^+}^\alpha u_1(t)|+\overline{C_1}|{}_c^\rho D_{a^+}^\alpha v_2(t)-{}_c^\rho D_{a^+}^\alpha v_1(t)|.
\end{aligned}$$

Then

$$\begin{aligned}
|{}_c^\rho D_{a^+}^\alpha u_2(t)-{}_c^\rho D_{a^+}^\alpha u_1(t)| &\leq \frac{K_1}{(1-C_1)}\|u_2-u_1\|_{[-r,\beta]}+\frac{\overline{K_1}}{(1-C_1)}\|v_2-v_1\|_{[-r,\beta]}\\
&+ \frac{\overline{C_1}}{(1-C_1)}|{}_c^\rho D_{a^+}^\alpha v_2(t)-{}_c^\rho D_{a^+}^\alpha v_1(t)|.
\end{aligned}$$

Similarly, one can find that

$$\begin{aligned}
|{}_c^\rho D_{a^+}^\alpha v_2(t)-{}_c^\rho D_{a^+}^\alpha v_1(t)| &\leq \frac{K_2}{(1-\overline{C_2})}\|u_2-u_1\|_{[-r,\beta]}+\frac{\overline{K_2}}{(1-\overline{C_2})}\|v_2-v_1\|_{[-r,\beta]}\\
&+ \frac{C_2}{(1-\overline{C_2})}|{}_c^\rho D_{a^+}^\alpha u_2(t)-{}_c^\rho D_{a^+}^\alpha u_1(t)|.
\end{aligned}$$

Therefore

$$\begin{aligned}
|{}_c^\rho D_{a^+}^\alpha u_2(t)-{}_c^\rho D_{a^+}^\alpha u_1(t)| &\leq \frac{K_1}{(1-C_1)}\|u_2-u_1\|_{[-r,\beta]}+\frac{\overline{K_1}}{(1-C_1)}\|v_2-v_1\|_{[-r,\beta]}\\
&+ \frac{\overline{C_1}}{(1-C_1)}\left[\frac{K_2}{(1-\overline{C_2})}\|u_2-u_1\|_{[-r,\beta]}+\frac{\overline{K_2}}{(1-\overline{C_2})}\|v_2-v_1\|_{[-r,\beta]}\right]\\
&+ \frac{C_2\overline{C_1}}{(1-C_1)(1-\overline{C_2})}|{}_c^\rho D_{a^+}^\alpha u_2(t)-{}_c^\rho D_{a^+}^\alpha u_1(t)|,
\end{aligned}$$

then

$$\begin{aligned}
|{}^{\rho}_{c}D^{\alpha}_{a^+}u_2(t) - {}^{\rho}_{c}D^{\alpha}_{a^+}u_1(t)| &\leq \frac{K_1(1-\overline{C_2})+\overline{C_1}K_2}{(1-C_1)(1-\overline{C_2})-C_2\overline{C_1}}\|u_2-u_1\|_{[-r,\beta]} \\
&+ \frac{\overline{K_1}(1-\overline{C_2})+\overline{C_1}\overline{K_2}}{(1-C_1)(1-\overline{C_2})-C_2\overline{C_1}}\|v_2-v_1\|_{[-r,\beta]},
\end{aligned}$$

and

$$\begin{aligned}
|{}^{\rho}_{c}D^{\alpha}_{a^+}v_2(t) - {}^{\rho}_{c}D^{\alpha}_{a^+}v_1(t)| &\leq \frac{K_2(1-C_1)+C_2K_1}{(1-C_1)(1-\overline{C_2})-C_2\overline{C_1}}\|u_2-u_1\|_{[-r,\beta]} \\
&+ \frac{\overline{K_2}(1-C_1)+\overline{K_1}C_2}{(1-C_1)(1-\overline{C_2})-C_2\overline{C_1}}\|v_2-v_1\|_{[-r,\beta]}.
\end{aligned}$$

From it we get

$$\begin{aligned}
|N_1(u_2,v_2)(t)-N_1(u_1,v_1)(t)| &\leq \int_a^T |G(t,s)|(\frac{K_1(1-\overline{C_2})+\overline{C_1}K_2}{(1-C_1)(1-\overline{C_2})-C_2\overline{C_1}}\|u_2-u_1\|_{[-r,\beta]} \\
&+ \frac{\overline{K_1}(1-\overline{C_2})+\overline{C_1}\overline{K_2}}{(1-C_1)(1-\overline{C_2})-C_2\overline{C_1}}\|v_2-v_1\|_{[-r,\beta]}ds) \\
&\leq \frac{\widetilde{G}\left(K_1(1-\overline{C_2})+\overline{C_1}K_2\right)}{(1-C_1)(1-\overline{C_2})-C_2\overline{C_1}}\|u_2-u_1\|_{[-r,\beta]} \\
&+ \frac{\widetilde{G}\left(\overline{K_1}(1-\overline{C_2})+\overline{C_1}\overline{K_2}\right)}{(1-C_1)(1-\overline{C_2})-C_2\overline{C_1}}\|v_2-v_1\|_{[-r,\beta]}.
\end{aligned}$$

Therefore, for each $t \in I$, we have

$$\begin{aligned}
|N_1(u_2,v_2)(t)-N_1(u_1,v_1)(t)| &\leq \frac{\widetilde{G}\left(K_1(1-\overline{C_2})+\overline{C_1}K_2\right)}{(1-C_1)(1-\overline{C_2})-C_2\overline{C_1}}\|u_2-u_1\|_{\mathcal{C}} \\
&+ \frac{\widetilde{G}\left(\overline{K_1}(1-\overline{C_2})+\overline{C_1}\overline{K_2}\right)}{(1-C_1)(1-\overline{C_2})-C_2\overline{C_1}}\|v_2-v_1\|_{\mathcal{C}}.
\end{aligned}$$

Thus

$$\|N_1(u_2,v_2)-N_1(u_1,v_1)\|_{\mathcal{C}} \leq G_1^*\left[\|u_2-u_1\|_{\mathcal{C}}+\|v_2-v_1\|_{\mathcal{C}}\right], \tag{3.13}$$

with

$$G_1^* = \frac{\widetilde{G}\left((\overline{K_1}+K_1)(1-\overline{C_2})+\overline{C_1}(K_2+\overline{K_2})\right)}{(1-C_1)(1-\overline{C_2})-C_2\overline{C_1}}.$$

likewise, we get

$$\|N_2(u_2,v_2) - N_2(u_1,v_1)\|_{\mathcal{C}} \le G_2^* \left[\|u_2-u_1\|_{\mathcal{C}} + \|v_2-v_1\|_{\mathcal{C}}\right], \tag{3.14}$$

with

$$G_2^* = \frac{\widetilde{G}\left((\overline{K_2}+K_2)(1-C_1)+C_2(K_1+\overline{K_1})\right)}{(1-C_1)(1-\overline{C_2})-C_2\overline{C_1}}.$$

Thus it follows from (3.13) and (3.14), that

$$\|N(u_2,v_2)-N(u_1,v_1)\|_{\overline{\mathcal{C}}} \;\le\; (G_1^*+G_2^*)\left[\|u_2-u_1\|_{\overline{\mathcal{C}}} + \|v_2-v_1\|_{\overline{\mathcal{C}}}\right],$$

with

$$G_1^*+G_2^* = \widetilde{G}\left(\frac{(\overline{K_1}+K_1)(1-\overline{C_2}+C_2)+(1-C_1+\overline{C_1})(K_2+\overline{K_2})}{(1-C_1)(1-\overline{C_2})-C_2\overline{C_1}}\right).$$

So by (3.10) the operator N is a contraction. By the Banach contraction principle, N has a fixed point, which is solution to problem (3.1)-(3.2).

We now prove an existence result for (3.1)-(3.2) by using the Schauder's fixed point theorem.

Theorem 3.2.2 *Suppose that (H_1) and (H_3) hold. Then problem (3.1)-(3.2) has at least one solution.*

Step 1. N is continuous. Let $\{(u_n,v_n)\}$ be a sequence such that $(u_n,v_n) \longrightarrow (u,v)$ in $\mathcal{C}\times\mathcal{C}$. If $t\in[a-r,a]$ or $t\in[T,T+\beta]$ then

$$|(N(u_n,v_n))(t)-(N(u,v))(t)| = 0.$$

For $t \in I$, we have

$$|(N_i(u_n,v_n)(t)) - (N_i(u,v))(t)| \le \int_a^T |G(t,s)||h_{i,n}(s)-h_i(s)|ds, i=1,2, \tag{3.15}$$

where

$$h_{i,n}(t) = f_i(t,u_n^t,v_n^t,h_{1,n}(t),h_{2,n}(t)),$$

and

$$h_i(t) = f_i(t,u^t,v^t,h_1(t),h_2(t)).$$

Since $(u_n,v_n) \longrightarrow (u,v)$, and by (H_1) we get $h_{i,n}(t) \longrightarrow h(t), i=1,2$ as $n \longrightarrow \infty$ for each $t\in I$. By (H_3) we have for each $t \in I, i=1,2,$

$$|h_{i,n}(t)| \le p_i^* + q_i^*. \tag{3.16}$$

Then,

$$\begin{aligned}|G(t,s)||h_{i,n}(t)-h_i(t)| &\leq |G(t,s)|\left[|h_{i,n}(t)|+|h_i(t)|\right]\\ &\leq 2(p_i^*+q_i^*)|G(t,s)|.\end{aligned}$$

For each $t \in I$ the functions $s \longmapsto 2(p_i^*+q_i^*)|G(t,s)|$ are integrable on $[a,t]$, then by Lebesgue dominated convergence theorem, equation (3.15) implies

$$|(N_i(u_n,v_n))(t)-(N_i(u,v))(t)| \longrightarrow 0 \ as \ n \longrightarrow \infty,$$

and hence

$$\|N(u_n,v_n)-N(u,v)\|_{\overline{\mathcal{C}}} \longrightarrow 0 \ as \ n \longrightarrow \infty.$$

Consequently, N is continuous.

Let the constant R be such that:

$$R \geq \max\left\{L_1+L_2, \|\phi_1\|_{[a-r,a]}+\|\phi_2\|_{[a-r,a]}, \|\psi_1\|_{[T,T+\beta]}+\|\psi_2\|_{[T,T+\beta]}\right\}, \tag{3.17}$$

and define

$$D_R=\{(u,v)\in \mathcal{C}\times\mathcal{C} : \|(u,v)\|_{\overline{\mathcal{C}}} \leq R\}.$$

It is clear that D_R is a bounded, closed and convex subset of $\overline{\mathcal{C}}$.

Step 2. $N(D_R) \subset D_R$.

Let $(u,v) \in D_R$ we show that $N(u,v)=(N_1(u,v),N_2(u,v)) \in D_R$.
If $t \in [a-r,a]$, then

$$|N(u,v)(t)| \leq \|\phi_1\|_{[a-r,a]}+\|\phi_2\|_{[a-r,a]} \leq R,$$

and if $t \in [T,T+\beta]$, then

$$|N(u,v)(t)| \leq \|\psi_1\|_{[T,T+\beta]}+\|\psi_2\|_{[T,T+\beta]} \leq R.$$

For each $t \in I$, we have

$$|(N_i(u,v))(t)| \leq \int_a^T |G(t,s)||h_i(s)|ds, i=1,2.$$

By (H_3), we have

$$\begin{aligned}|(N_i(u,v))(t)| &\leq (p_i^*+q_i^*)\int_a^T |G(t,s)|ds\\ &\leq (p_i^*+q_i^*)\widetilde{G}=L_i\end{aligned}$$

from which it follows that for each $t \in [a-r,T+\beta]$, we have

$$|N_i(u,v)(t)| \leq L_i,$$

which implies that $\|N_i(u,v)\|_{\mathcal{C}} \leq L_i$,
hence we get

$$\begin{aligned} \|N(u,v)\|_{\overline{\mathcal{C}}} &\leq L_1 + L_2 \\ &\leq R. \end{aligned}$$

Consequently,

$$N(D_R) \subset D_R.$$

Step 3: $N(D_R)$ is bounded and equicontinuous.
By Step 2 we have $N(D_R)$ is bounded.
Let $t_1, t_2 \in I = [a,T], t_1 < t_2,\ and\ (u,v) \in D_R$ then

$$\begin{aligned} |(N_i(u,v))(t_2) - (N_i(u,v))(t_1)| &\leq \int_a^T |G(t_2,s) - G(t_1,s)||h_i(s)|ds \\ &\leq (p_i^* + q_i^*) \int_a^T |G(t_2,s) - G(t_1,s)|ds. \end{aligned}$$

As $t_1 \longrightarrow t_2$ the right hand side of the above inequality tends to zero.Therefore, the operator $N(u,v)$ is equicontinuous. As consequence of Step 1 to Step 3, together withe the Arzela-Ascoli theorem, we can conclude that N is continuous and completely continuous and satisfies the assumptions of Schauder's fixed point theorem. Then N has a fixed point, which is a solution of the problem (3.1)-(3.2).

3.3 An Example

Consider the boundary value problem of implicit Caputo type modification of the Erdélyi-Kober fractional differential equation:

$$\begin{cases} (u(t),v(t)) = (e^{t-2} - 1, 2t-4), \quad t \in [1,2], \\ {}^{\frac{1}{2}}_{c}D^{\frac{3}{2}}_{2^+}u(t) = \dfrac{\ln(t)}{200e^{t+2}\left(1 + |u^t| + |v^t| + \left|{}^{\frac{1}{2}}_{c}D^{\frac{3}{2}}_{2^+}u(t)\right| + \left|{}^{\frac{1}{2}}_{c}D^{\frac{3}{2}}_{2^+}v(t)\right|\right)}, \quad t \in I = [2,e] \\ {}^{\frac{1}{2}}_{c}D^{\frac{3}{2}}_{2^+}v(t) = \dfrac{\arctan(t)}{100e^{t+2}\left(1 + |u^t| + |v^t| + \left|{}^{\frac{1}{2}}_{c}D^{\frac{3}{2}}_{2^+}u(t)\right| + \left|{}^{\frac{1}{2}}_{c}D^{\frac{3}{2}}_{2^+}v(t)\right|\right)}, \quad t \in I = [2,e] \\ (u(t),v(t)) = (\ln(t) - 1, t-e), \quad t \in [e,6]. \end{cases} \tag{3.18}$$

Set

$$f_1(t,u,v,\bar{u},\bar{v}) = \frac{\ln(t)}{200e^{t+2}\left(1+|u^t|+|v^t|+|\bar{u}|+|\bar{v}|\right)},\ \ t\in[2,4], u,v\in C([-r,\beta]), \bar{u},\bar{v}\in \mathbb{R},$$

$$f_2(t,u,v,\bar{u},\bar{v}) = \frac{\arctan(t)}{100e^{t+2}\left(1+|u^t|+|v^t|+|\bar{u}|+|\bar{v}|\right)},\ \ t\in[2,4], u,v\in C([-r,\beta]), \bar{u},\bar{v}\in \mathbb{R},$$

$$v\in\mathbb{R}, \alpha=\frac{3}{2}, \rho=\frac{5}{2}, r=1, \beta=6-e.$$

Condition (H_2) is satisfied, indeed, for each $u,v\in C([-r,\beta])$, $\bar{u},\bar{v}\in\mathbb{R}$ and $t\in[2,e]$, we have

$$\begin{aligned} |f_1(t,u_2,v_2,\bar{u}_2,\bar{v}_2)-f_1(t,u_1,v_1,\bar{u}_1,\bar{v}_1)| &\le \frac{1}{200e^{t+2}}(\|u_2-u_1\|_{[-r,\beta]}+\|u_2-v_1\|_{[-r,\beta]} \\ &+ |\bar{u}_2-\bar{u}_1|+|\bar{v}_2-\bar{v}_1|), \end{aligned}$$

and

$$\begin{aligned} |f_2(t,u_2,v_2,\bar{u}_2,\bar{v}_2)-f_2(t,u_1,v_1,\bar{u}_1,\bar{v}_1)| &\le \frac{\pi}{200e^{t+2}}(\|u_2-u_1\|_{[-r,\beta]}+\|u_2-v_1\|_{[-r,\beta]} \\ &+ |\bar{u}_2-\bar{u}_1|+|\bar{v}_2-\bar{v}_1|). \end{aligned}$$

Therefore, (H_2) is verified with

$$K_i=\overline{K}_i=C_i=\overline{C}_i=\begin{cases}\frac{\pi}{200e^4} & for\ \ i=2,\\ \frac{1}{200e^4} & for\ \ i=1.\end{cases}$$

For each $t\in I$ we have

$$\begin{aligned} \int_a^T|G(t,s)|ds &\le \frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho-a^\rho}{T^\rho-a^\rho}\right)\int_a^T\left|\left(\frac{T^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\right|ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_a^t\left|\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\right|ds \\ &\le \frac{2}{\Gamma(\alpha+1)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^\alpha. \end{aligned}$$

Therefore

$$\widetilde{G}\le\frac{2}{\Gamma(\alpha+1)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^\alpha.$$

We have

$$\begin{aligned} G_1^*+G_2^* &\le \frac{\frac{1}{100e^4}+\frac{\pi}{100e^4}}{((1-\frac{1}{200e^4})(1-\frac{\pi}{200e^4})-\frac{\pi}{(200e^4)^2})}\frac{2}{\Gamma(\frac{5}{2})}\left(\frac{e^{\frac{5}{2}}-2^{\frac{5}{2}}}{\frac{5}{2}}\right)^{\frac{3}{2}} \\ &\approx 6.689246337.10^{-7} \\ &< 1. \end{aligned}$$

Hence (3.10) is satisfied with $T = e, a = 2$ and $\alpha = \frac{3}{2}$. Hence all conditions of Theorem 3.2.1 are satisfied, it follows that the problem (3.18) admit a unique solution defined on I.

3.4 Existence Results for the Weak Solutions Of Coupled Implicit Fractional Differential Systems with Retarded and Advanced Arguments

[2]

In this Section, we study the existence of weak solutions to the following coupled system nonlinear implicit fractional differential equations (CSIFD for short) involving both retarded and advanced arguments:

$$\begin{cases} {}^{\rho}_{c}D^{\alpha}_{a^+}u(t) = f_1(t, u^t, v^t, {}^{\rho}_{c}D^{\alpha}_{a^+}u(t), {}^{\rho}_{c}D^{\alpha}_{a^+}v(t)) \\ {}^{\rho}_{c}D^{\alpha}_{a^+}v(t) = f_2(t, u^t, v^t, {}^{\rho}_{c}D^{\alpha}_{a^+}u(t), {}^{\rho}_{c}D^{\alpha}_{a^+}v(t)) \end{cases} \quad t \in I := [a, T], \tag{3.19}$$

$$\begin{cases} (u(t), v(t)) = (\phi_1(t), \phi_2(t)), \ t \in [a - r, a], \ r > 0 \\ (u(t), v(t)) = (\psi_1(t), \psi_2(t)), \ t \in [T, T + \beta], \ \beta > 0, \end{cases} \tag{3.20}$$

where ${}^{\rho}_{c}D^{\alpha}_{a^+}$ is the Caputo type modification of the Erdélyi-Kober fractional derivative and E is a real (or complex) Banach space with norm $\|\cdot\|_E$ and dual E^*, such that E is the dual of a weakly compactly generated Banach space X, $f_i : I \times C([-r, \beta], E)^2 \times E^2 \to E$ is a given function, $\phi_i \in C([a - r, a], E)$ with $\phi_i(a) = 0$ and $\psi_i \in C([T, T + \beta], E)$ with $\psi_i(T) = 0, \ i = 1, 2$.
We denote by u^t the element of $C([-r, \beta])$ defined by:

$$u^t(s) = u(t + s) : s \in [-r, \beta].$$

Definition 3.4.1 A function $(u, v) \in \mathcal{C} \times \mathcal{C}$, is said to be a solution of (3.19)-(3.20), if (u, v) satisfies the system equation

$$\begin{cases} {}^{\rho}_{c}D^{\alpha}_{a^+}u(t) = f_1(t, u^t, v^t, {}^{\rho}_{c}D^{\alpha}_{a^+}u(t), {}^{\rho}_{c}D^{\alpha}_{a^+}v(t)) \\ {}^{\rho}_{c}D^{\alpha}_{a^+}v(t) = f_2(t, u^t, v^t, {}^{\rho}_{c}D^{\alpha}_{a^+}u(t), {}^{\rho}_{c}D^{\alpha}_{a^+}v(t)) \end{cases}$$

on I, and the conditions $(u(t), v(t)) = (\phi_1(t), \phi_2(t)), (\phi_1(a), \phi_2(a)) = (0, 0)$ on $[a - r, a]$ and $(u(t), v(t)) = (\psi_1(t), \psi_2(t)), (\psi_1(T), \psi_2(T)) = (0, 0)$ on $[T, T + \beta]$.

To prove the existence of solutions to (3.19)-(3.20), we need the following auxiliary Lemma.

[2]M. Boumaaza, M. Benchohra and Juan J. Trujillo, Weak solutions of coupled Caputo-type modification of the Erdélyi-Kober implicit fractional differential systems with retarded and advanced arguments, *Jordan J. Math. Stat.* (accepted).

Lemma 3.4.1 *Let $f_i : I\times C[-r,\beta]\times C[-r,\beta]\times E^2 \longrightarrow E$ i=1,2, be continuous functions. A function $(u,v)\in \mathcal{C}^2$ is solution of system $(3.19)-(3.20)$ if and only if (u,v) satisfies the following coupled system of integral equations*

$$u(t)=\begin{cases}\phi_1(t), & \text{if} \quad t\in[a-r,a],\\ \\ -\displaystyle\int_a^T G(t,s)h_1(s)ds, & \text{if} \quad t\in I\\ \\ \psi_1(t), & \text{if} \quad t\in[T,T+\beta],\end{cases}$$

$$v(t)=\begin{cases}\phi_2(t), & \text{if} \quad t\in[a-r,a],\\ \\ -\displaystyle\int_a^T G(t,s)h_2(s)ds, & \text{if} \quad t\in I\\ \\ \psi_2(t), & \text{if} \quad t\in[T,T+\beta],\end{cases}$$

where $h_i\in C(I)$ satisfies the system of functional equations

$$\begin{cases}h_1(t)=f_1(t,u^t,v^t,h_1(t),h_2(t)),\\ h_2(t)=f_2(t,u^t,v^t,h_1(t),h_2(t)).\end{cases}$$

and

$$G(t,s)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\begin{cases}\dfrac{(t^\rho-a^\rho)(T^\rho-s^\rho)^{\alpha-1}s^{\rho-1}}{(T^\rho-a^\rho)}-s^{\rho-1}(t^\rho-s^\rho)^{\alpha-1}, & a\le s\le t\le T.\\ \\ \dfrac{(t^\rho-a^\rho)(T^\rho-s^\rho)^{\alpha-1}s^{\rho-1}}{(T^\rho-a^\rho)}, & a\le t\le s\le T.\end{cases} \tag{3.21}$$

The following hypotheses will be used in the sequel:

(H_1) The functions $(u,v,\bar{u},\bar{v})\longrightarrow f_i(t,u,v,\bar{u},\bar{v})$ are weakly sequentially continuous for *a.e.* $t\in I$.

(H_2) For all $u,v\in C([-r,\beta])$, $\bar{u},\bar{v}\in E$ the functions $t\longrightarrow f_i(t,u,v,\bar{u},\bar{v}), i=1,2,$ are Pettis integrable.

(H_3) There exist $p_i, q_i \in C([a,T], \mathbb{R}_+)$ such that, for all $\varphi \in E^*$,

$$|\varphi(f_i(t,u,v,\bar{u},\bar{v}))| \leq \frac{p_i(t)\|u\|_{[-r,\beta]} + q_i(t)\|v\|_{[-r,\beta]}}{1 + \|\varphi\| + \|u\|_{[-r,\beta]} + \|v\|_{[-r,\beta]} + \|\bar{u}\|_E + \|\bar{v}\|_E}$$

for a.e. $t \in I$, and each $u, v \in C([-r,\beta])$ and $\bar{u}, \bar{v} \in E$.

(H_4) For each bounded measurable sets $B_i \subset C[-r,\beta], i = 1,2$, and each $t \in I$, we have

$$\mu(f_1(t, B_1, B_2, {}^{\rho}_{c}D^{\alpha}_{a^+}(B_1), {}^{\rho}_{c}D^{\alpha}_{a^+}(B_2)), 0) \leq p_1(t) \sup_{s\in[-r,\beta]} \mu(B_1(s)) + q_1(t) \sup_{s\in[-r,\beta]} \mu(B_2(s))$$

and

$$\mu(0, f_2(t, B_1, B_2, {}^{\rho}_{c}D^{\alpha}_{a^+}(B_1), {}^{\rho}_{c}D^{\alpha}_{a^+}(B_2))) \leq p_2(t) \sup_{s\in[-r,\beta]} \mu(B_1(s)) + q_2(t) \sup_{s\in[-r,\beta]} \mu(B_2(s)),$$

where

$${}^{\rho}_{c}D^{\alpha}_{a^+}(B_i) = \{{}^{\rho}_{c}D^{\alpha}_{a^+}(w) : w \in B_i\}, i = 1,2.$$

Set

$$p_i^* = \sup_{t\in I} p_i(t), \quad q_i^* = \sup_{t\in I} q_i(t), \quad i = 1,2, \quad \widetilde{G} = \sup\left\{\int_a^T |G(t,s)|ds, t \in I\right\}.$$

We now prove an existence result for (3.19)-(3.20) by using the Mönch fixed point theorem.

Theorem 3.4.1 *Suppose that* (H_1) - (H_4) *hold. If*

$$\widetilde{G}(p_1^* + q_1^* + p_2^* + q_2^*) < 1, \tag{3.22}$$

then the coupled system (3.19)-(3.20) has at least one weak solution defined on I.
Proof: *Let the operator* $N : \mathcal{C} \times \mathcal{C} \longmapsto \mathcal{C} \times \mathcal{C}$ *defined by*

$$N(u,v)(t) = (N_1(u,v), N_2(u,v))$$

$$= \begin{cases} (\phi_1(t), \phi_2(t)), & \text{if} \quad t \in [a-r, a], \\ -\left(\int_a^T G(t,s)h_1(s)ds, \int_a^T G(t,s)h_2(s)ds\right), & \text{if} \quad t \in I \\ (\psi_1(t), \psi_2(t)), & \text{if} \quad t \in [T, T+\beta]. \end{cases} \tag{3.23}$$

First, notice that the hypotheses imply that, for each $h_i \in C(I)$, $i = 1,2$, *the function* $t \longrightarrow G(t,s)h_i(t)$ *are Pettis integrable over* I.

Let the constant R be such that:

$$R \geq \max\left\{L_1 + L_2, \|\phi_1\|_{[a-r,a]} + \|\phi_2\|_{[a-r,a]}, \|\psi_1\|_{[T,T+\beta]} + \|\psi_2\|_{[T,T+\beta]}\right\}, \tag{3.24}$$

and define

$$D = \left\{ (u,v) \in \mathcal{C} \times \mathcal{C} : \left\{ \begin{array}{c} \|(u,v)\|_{\overline{\mathcal{C}}} \leq R, \\ \\ \|u(t_2) - u(t_1)\|_E \leq (p_1^* + q_1^*) \int_a^T |G(t_2,s) - G(t_1,s)|ds, \\ \\ \|v(t_2) - v(t_1)\|_E \leq (p_2^* + q_2^*) \int_a^T |G(t_2,s) - G(t_1,s)|ds. \end{array} \right. \right\}$$

Clearly, the subset D is closed, convex end equicontinuous. We shall show that the operator N satisfies all the assumptions of Theorem 1.6.5. The proof will be given in several steps.

Step 1. *N maps D into itself.*

Let $(u,v) \in D,\ t \in I$ and assume that $(N(u,v))(t) \neq (0,0)$. Then there exists $\varphi \in E^*$ such that $\|N_i(u,v)(t)\|_E = \varphi\left(N_i(u,v)(t)\right)$. Thus, for any $i \in \{1,2\}$ we have

$$\|N_i(u,v)(t)\|_E = \varphi\left(\int_a^T G(t,s)h_i(s)ds\right),$$

where $h_i \in C(I)$, with

$$h_i(t) = f_i(t, u^t, v^t, h_1(t), h_2(t)).$$

If $t \in [a-r,a]$, then

$$\|N(u,v)(t)\|_E \leq \|\phi_1\|_{[a-r,a]} + \|\phi_2\|_{[a-r,a]} \leq R,$$

and if $t \in [T, T+\beta]$, then

$$\|N(u,v)(t)\|_E \leq \|\psi_1\|_{[T,T+\beta]} + \|\psi_2\|_{[T,T+\beta]} \leq R.$$

For each $t \in I$, we have

$$\|(N_i(u,v))(t)\|_E \leq \int_a^T |G(t,s)||\varphi(h_i(s))|ds, i = 1,2.$$

By (H_3), we get

$$|\varphi(h_i(t))| \leq p_i^* + q_i^*.$$

Therefore

$$\begin{aligned} \|(N_i(u,v))(t)\|_E &\leq (p_i^* + q_i^*) \int_a^T |G(t,s)|ds \\ &\leq (p_i^* + q_i^*)\widetilde{G} = L_i \end{aligned}$$

from which it follows that for each $t \in [a-r, T+\beta]$, we have

$$\|N_i(u,v)(t)\|_E \leq L_i,$$

which implies that $\|N_i(u,v)\|_{\mathcal{C}} \leq L_i$,
hence we get

$$\begin{aligned}\|N(u,v)\|_{\overline{\mathcal{C}}} &\leq L_1 + L_2 \\ &\leq R.\end{aligned}$$

Next, Let $t_1, t_2 \in I = [a,T], t_1 < t_2,$ *and* $(u,v) \in D$ be such that

$$(N_i(u,v))(t_2) - (N_i(u,v))(t_1) \neq 0.$$

Then there exists $\varphi \in E^*$ such that

$$\|(N_i(u,v))(t_2) - (N_i(u,v))(t_1)\|_E = \varphi((N_i(u,v))(t_2) - (N_i(u,v))(t_1)),$$

and $\|\varphi\| = 1$. Then, for any $i \in \{1,2\}$, we get

$$\begin{aligned}\|(N_i(u,v))(t_2) - (N_i(u,v))(t_1)\|_E &= \varphi((N_i(u,v))(t_2) - (N_i(u,v))(t_1)) \\ &\leq \varphi\left(\int_a^T |G(t_2,s) - G(t_1,s)| h_i(s)\right)\end{aligned}$$

where $h_i \in C(I)$, with

$$h_i(t) = f_i(t, u^t, v^t, h_1(t), h_2(t)).$$

Thus, we have

$$\begin{aligned}\|(N_i(u,v))(t_2) - (N_i(u,v))(t_1)\|_E &\leq \int_a^T |G(t_2,s) - G(t_1,s)||\varphi(h_i(s))| ds \\ &\leq (p_i^* + q_i^*) \int_a^T |G(t_2,s) - G(t_1,s)| ds.\end{aligned}$$

Consequently,

$$N(D) \subset D.$$

Step 2. N is weakly sequentially continuous.
Let $\{(u_n, v_n)\}_n$ be a sequence in $D \times D$, and let $(u_n(t), v_n(t)) \longrightarrow (u(t), v(t))$ in $(E,\omega) \times (E,\omega)$ for each $t \in [a-r, T+\beta]$. Fix $t \in [a-r, T+\beta]$. Since for any $i \in 1,2$, the function $f_i(t, u_n^t, v_n^t, {}^\rho_c D^\alpha_{a^+} u_n(t), {}^\rho_c D^\alpha_{a^+} v_n(t))$ satisfies assumption (H_1), we have that it converges

weakly uniformly to $f_i(t, u^t, v^t, {}^\rho_c D^\alpha_{a^+} u(t), {}^\rho_c D^\alpha_{a^+} v(t))$. Hence the Lebesgue dominated convergence theorem for Pettis integral implies that $(N(u_n, v_n))(t)$ converges weakly uniformly to $(N(u, v))(t)$ in (E, ω). We do it for each $t \in I$, so $N(u_n, v_n) \longrightarrow N(u, v)$. Then $N : D \longrightarrow D$ is weakly sequentially continuous.

Step 3. Now let V be a subset of D such that $V = conv(N(V) \cup \{(0, 0)\})$. Obviously

$$V(t) \subset conv(N(V)(t) \cup \{(0, 0)\}).$$

Since V is bounded and equicontinuous, the function $t \longmapsto v(t) = \mu(V(t))$ is continuous on $[a-r, T+\beta]$. By $(H_1)-(H_3)$, Lemma 1.4.3, and the properties of measure μ, for each $t \in I$, we have

$$\begin{aligned}
v(t) &\leq \mu(N(V)(t) \cup \{(0, 0)\}) \\
&\leq \mu((NV)(t)) \\
&\leq \mu\left(\{((N_1 u)(t), (N_2 v)(t)) : (u, v) \in V\}\right) \\
&\leq \int_a^T |G(t, s)| \mu(\{(f_1(t, u^t, v^t, {}^\rho_c D^\alpha_{a^+} u(t), {}^\rho_c D^\alpha_{a^+} v(t)), 0)\}) ds \\
&+ \int_a^T |G(t, s)| \mu(\{(0, f_2(t, u^t, v^t, {}^\rho_c D^\alpha_{a^+} u(t), {}^\rho_c D^\alpha_{a^+} v(t)))\}) ds \\
&\leq \int_a^T |G(t, s)| (p_1(s) \mu(\{(u(s), 0); (u, 0) \in V\}) \\
&+ q_1(s) \mu(\{(v(s), 0); (v, 0) \in V\}) ds \\
&+ \int_a^T |G(t, s)| (p_2(s) \mu(\{(0, u(s)); (0, u) \in V\}) \\
&+ q_2(s) \mu(\{(0, v(s)); (0, v) \in V\})) ds \\
&\leq \int_a^T |G(t, s)| (p_1(s) + q_1(s) + p_2(s) + q_2(s)) \mu(V(s)) ds \\
&\leq \widetilde{G}(p_1^* + q_1^* + p_2^* + q_2^*) \|v\|_c.
\end{aligned}$$

Thus

$$\|v\|_c \leq \widetilde{G}(p_1^* + q_1^* + p_2^* + q_2^*) \|v\|_c.$$

From (3.22), we get $\|v\|_c = 0$, that is $\mu(V(t)) = 0$ for each $t \in I$.
For $t \in [a - r, a]$, we have

$$\begin{aligned}
v(t) &= \mu((\phi_1(t), \phi_2(t))) \\
&= 0.
\end{aligned}$$

Also for $t \in [T, T + \beta]$ we have

$$\begin{aligned}
v(t) &= \mu(\psi_1(t), \psi_2(t)) \\
&= 0,
\end{aligned}$$

then $V(t)$ is relatively compact in E. In view of Ascoli-Arzela theorem V is weakly relatively compact in $\overline{\mathcal{C}}$. Applying Theorem 1.6.5, we conclude that N has a fixed point which is a solution of the problem (3.19) − (3.20).

3.5 An Example

Let

$$E = l^1 = \left\{ u = (u_1, u_2, \ldots, u_n, \ldots), \sum_{k=1}^{\infty} |u_n| < \infty \right\},$$

be the Banach space with the norm

$$\|u\|_E = \sum_{k=1}^{\infty} |u_n|.$$

Consider the boundary value problem of implicit Caputo type modification of the Erdélyi-Kober fractional differential equation

$$\begin{cases} (u(t), v(t)) = (\frac{1}{2}t, t^2 + t), \quad t \in [-1, 0], \\ \\ {}_c^1D_{0^+}^{\frac{3}{2}} u_n(t) = f(t, u_n^t, v_n^t, {}_c^1D_{0^+}^{\frac{3}{2}} u_n(t), {}_c^1D_{0^+}^{\frac{3}{2}} u_n(t)), \quad t \in I = [0, 1] \\ \\ {}_c^1D_{0^+}^{\frac{3}{2}} v_n(t) = g(t, u_n^t, v_n^t, {}_c^1D_{0^+}^{\frac{3}{2}} u_n(t), {}_c^1D_{0^+}^{\frac{3}{2}} u_n(t)), \quad t \in I = [0, 1] \\ \\ (u(t), v(t)) = (t - 1, t^2 - t), \quad t \in [1, 2], \end{cases} \tag{3.25}$$

here $T = 1, \quad a = 0, \quad \alpha = \dfrac{3}{2}, \quad \rho = 1.$

Set

$$y = (y_1, y_2, \ldots, y_n, \ldots), \quad f = (f_1, f_2, \ldots, f_n, \ldots)$$

$$f(t, u^t, v^t, {}_c^1D_{0^+}^{\frac{3}{2}} u(t), {}_c^1D_{0^+}^{\frac{3}{2}} v(t))$$

$$= \frac{sin(t)(\|u^t\|_{C([-1,1])} + \|v^t\|_{C([-1,1])})}{8(t+1)\left(1 + \|u^t\|_{C([-1,1])} + \|v^t\|_{C([-1,1])} + \|{}_c^3D_{0^+}^{\frac{3}{2}} u\|_E + \|{}_c^3D_{2^+}^{\frac{3}{2}} v\|_E\right)}.$$

and

$$g(t, u^t, v^t, {}_c^1D_{0^+}^{\frac{3}{2}} u(t), {}_c^1D_{0^+}^{\frac{3}{2}} v(t))$$

$$= \frac{cos(t)(\|u^t\|_{C([-1,1])} + \|v^t\|_{C([-1,1])})}{8(t^2+1)\left(1 + \|u^t\|_{C([-1,1])} + \|v^t\|_{C([-1,1])} + \|{}_c^1D_{0^+}^{\frac{3}{2}} u\|_E + \|{}_c^1D_{0^+}^{\frac{3}{2}} v\|_E\right)}.$$

For each $y \in E$ and $t \in [0, 1]$, we have

$$\|f(t,u^t,v^t,{}_c^1D^{\frac{3}{2}}_{0^+}u(t),{}_c^1D^{\frac{3}{2}}_{0^+}v(t))\|_E \leq \frac{sin(t)}{8(t+1)}$$

and

$$\|g(t,u^t,v^t,{}_c^1D^{\frac{3}{2}}_{0^+}u(t),{}_c^1D^{\frac{3}{2}}_{0^+}v(t))\|_E \leq \frac{cos(t)}{8(t^2+1)}.$$

Hence (H_2) is satisfied with $P_i^* = q_i^* = \frac{1}{8}, i = 1,2$.
For each $t \in I$ we have

$$\begin{aligned}
\int_a^T |G(t,s)|ds &\leq \frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{T^\rho - a^\rho}\right)\int_a^T \left|\left(\frac{T^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\right| ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_a^t \left|\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\right| ds \\
&\leq \frac{2}{\Gamma(\alpha+1)}\left(\frac{T^\rho - a^\rho}{\rho}\right)^{\alpha}.
\end{aligned}$$

Therefore

$$\widetilde{G} \leq \frac{2}{\Gamma(\alpha+1)}\left(\frac{T^\rho - a^\rho}{\rho}\right)^{\alpha}.$$

Condition (3.22) holds, indeed,

$$\begin{aligned}
\widetilde{G}(p_1^* + q_1^* + p_2^* + q_2^*) &\leq \frac{1}{\Gamma(\frac{3}{2}+1)} \\
&\approx 0.7522527778 \\
&< 1.
\end{aligned}$$

Hence all conditions of Theorem 3.4.1 are satisfied. It follows that the problem (3.25) has at least one solution.

Chapter 4

Random Coupled Fractional Differential Systems in Generalized Banach Spaces

[1]

4.1 Introduction

This chapter generalizes the previous chapter so that we study the existence and uniqueness of random solutions to the following coupled system of nonlinear implicit fractional differential equations (CSIFD for short) involving both retarded and advanced arguments see [43]:

$$\begin{cases} ({}_c^{\rho}D_{a^+}^{\alpha_1}u)(t,w) = f_1(t,u^t(w),v^t(w),({}_c^{\rho}D_{a^+}^{\alpha_1}u)(t,w),w) \\ ({}_c^{\rho}D_{a^+}^{\alpha_2}v)(t,w) = f_2(t,u^t(w),v^t(w),({}_c^{\rho}D_{a^+}^{\alpha_2}v)(t,w),w) \end{cases} \quad t \in I := [a,T], w \in \Omega, \tag{4.1}$$

$$\begin{cases} (u(t,w),v(t,w)) = (\phi_1(t,w),\phi_2(t,w)),\ t \in [a-r,a],\ r > 0 \\ (u(t,w),v(t,w)) = (\psi_1(t,w),\psi_2(t,w)),\ t \in [T,T+\beta],\ \beta > 0, \end{cases} \quad ; w \in \Omega \tag{4.2}$$

where $\alpha_i \in (1,2]$, ${}_c^{\rho}D_{a^+}^{\alpha_i}, i = 1,2$ is the Caputo type modification of the Erdélyi-Kober fractional derivative and $f_i : I \times C([-r,\beta],\mathbb{R}^n)^2 \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ is a given function, $\phi_i \in C([a-r,a],\mathbb{R}^n)$ with $\phi_i(a,w) = 0$ and $\psi_i \in C([T,T+\beta],\mathbb{R}^n)$ with $\psi_i(T,w) = 0$, $i = 1,2$.
We denote by u_t the element of $C([-r,\beta])$ defined by:

$$u_t(s) = u(t+s) : s \in [-r,\beta].$$

[1]M. Boumaaza, M. Benchohra and J. Henderson, Random coupled Caputo-type modification of Erdélyi-Kober fractional differential systems in generalized Banach spaces with retarded and advanced arguments, *Commun. Optim. Theory*, **2021**(2021), 1-14.

4.2 Existence Results

Lemma 4.2.1 *Let* $1 < \alpha \leq 2$, $\phi \in C([a-r,a],\mathbb{R})$ *with* $\phi(a) = 0$, $\psi \in C([T,T+\beta],\mathbb{R})$ *with* $\psi(T) = 0$ *and* $h : I \to \mathbb{R}$ *be a continuous function. Then the linear problem*

$$ {}^{\rho}_{c}D^{\alpha}_{a+}u(t) = h(t), \text{ for a.e. } t \in I := [a,T], \ 1 < \alpha \leq 2, \tag{4.3}$$

$$u(t) = \phi(t), \qquad t \in [a-r,a], \ r > 0, \tag{4.4}$$

$$u(t) = \psi(t), \qquad t \in [T,T+\beta], \ \beta > 0, \tag{4.5}$$

has a unique solution, which is given by

$$u(t) = \begin{cases} \phi(t), & \text{if} \quad t \in [a-r,a], \\ -\displaystyle\int_a^T G(t,s)h(s)ds, & \text{if} \quad t \in I, \\ \psi(t), & \text{if} \quad t \in [T,T+\beta], \end{cases} \tag{4.6}$$

where

$$G(t,s) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \begin{cases} \dfrac{(t^\rho - a^\rho)(T^\rho - s^\rho)^{\alpha-1}s^{\rho-1}}{(T^\rho - a^\rho)} - s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1}, & a \leq s \leq t \leq T, \\ \dfrac{(t^\rho - a^\rho)(T^\rho - s^\rho)^{\alpha-1}s^{\rho-1}}{(T^\rho - a^\rho)}, & a \leq t \leq s \leq T. \end{cases} \tag{4.7}$$

Here $G(t,s)$ *is called the Green function of the boundary value problem (4.3)-(4.5).*

Proof. From (1.4), we have

$$u(t) = c_0 + c_1 \left(\frac{t^\rho - a^\rho}{\rho} \right) + {}^{\rho} I^{\alpha}_{a+}h(s), \quad c_0, c_1 \in \mathbb{R}. \tag{4.8}$$

Therefore,

$$u(a) = c_0 = 0,$$

$$u(T) = c_1 \left(\frac{T^\rho - a^\rho}{\rho} \right) + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^T (T^\rho - s^\rho)^{\alpha-1} s^{\rho-1} h(s) ds,$$

and

$$c_1 = -\frac{\rho^{2-\alpha}}{(T^\rho - a^\rho)\Gamma(\alpha)} \int_a^T (T^\rho - s^\rho)^{\alpha-1} s^{\rho-1} h(s) ds.$$

Substitute the values of c_0 and c_1 into equation (4.8), and we get equation (4.6), that is,

$$u(t) = \begin{cases} \phi(t), & \text{if} \quad t \in [a-r, a], \\ -\displaystyle\int_a^T G(t,s)h(s)ds, & \text{if} \quad t \in I, \\ \psi(t), & \text{if} \quad t \in [T, T+\beta], \end{cases}$$

where G is defined by equation (4.7), and the proof is complete. □

Lemma 4.2.2 *Let $f_i : I \times C([-r,\beta], \mathbb{R}^n) \times C([-r,\beta], \mathbb{R}^n) \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n, \ i = 1,2$ be continuous functions. A functions $(u,v) \in \mathcal{C}^2$ is a random solution of system (4.1)-(4.2) if and only if (u,v) satisfies the following random coupled system integral equations,*

$$u(t,w) = \begin{cases} \phi_1(t,w), & \text{if} \quad t \in [a-r, a], \\ -\displaystyle\int_a^T G_{\alpha_1}(t,s)h_1(s,w)ds, & \text{if} \quad t \in I, \\ \psi_1(t,w), & \text{if} \quad t \in [T, T+\beta], \end{cases} \qquad w \in \Omega,$$

$$v(t,w) = \begin{cases} \phi_2(t,w), & \text{if} \quad t \in [a-r, a], \\ -\displaystyle\int_a^T G_{\alpha_2}(t,s)h_2(s,w)ds, & \text{if} \quad t \in I, \\ \psi_2(t,w), & \text{if} \quad t \in [T, T+\beta], \end{cases} \qquad w \in \Omega,$$

where $h_i(\cdot,w) \in C(I), w \in \Omega$, satisfies the system of functional equations,

$$\begin{cases} h_1(t,w) = f_1(t, u^t(w), v^t(w), h_1(t,w)), \\ h_2(t,w) = f_2(t, u^t(w), v^t(w), h_2(t,w)), \end{cases} \qquad w \in \Omega,$$

and the Green function $G_{\alpha_i}, \ i = 1,2$ is given by

$$G_{\alpha_i}(t,s) = \frac{\rho^{1-\alpha_i}}{\Gamma(\alpha_i)} \begin{cases} \dfrac{(t^\rho - a^\rho)(T^\rho - s^\rho)^{\alpha_i - 1} s^{\rho-1}}{(T^\rho - a^\rho)} - s^{\rho-1}(t^\rho - s^\rho)^{\alpha_i - 1}, & a \le s \le t \le T, \\ \dfrac{(t^\rho - a^\rho)(T^\rho - s^\rho)^{\alpha_i - 1} s^{\rho-1}}{(T^\rho - a^\rho)}, & a \le t \le s \le T. \end{cases} \tag{4.9}$$

The following hypotheses will be used in the sequel:

(H_1) The functions $f_i, i = 1,2$, are random Carathéodory.

(H_2) There exist continuous functions $p_i, q_i, r_i : I \longrightarrow L^\infty(\Omega, \mathbb{R}_+)$, with $\|r_i(\cdot,w)\|_{[a,T]} < 1$ such that

$$\begin{aligned} \|f_i(t,u,v,x,w) - f_i(t,\bar{u},\bar{v},\bar{x},w)\| &\le p_i(t,w)\|u-\bar{u}\|_{[-r,\beta]} + q_i(t,w)\|v-\bar{v}\|_{[-r,\beta]} \\ &+ r_i(t,w)\|x-\bar{x}\| \end{aligned}$$

for any $u, \bar{u}, v, \bar{v} \in C([-r,\beta])$ and $x, \bar{x} \in \mathbb{R}^n, i = 1,2$.

(H_3) There exist measurable functions $a_i, b_i, c_i, d_i : I \longrightarrow L^\infty(\Omega, \mathbb{R}_+)$; $i = 1,2$, with $d_i^*(\cdot, w) < 1$, such that

$$\|f_i(t,u,v,x,w)\| \leq a_i(t,w) + b_i(t,w)\|u\|_{[-r,\beta]} + c_i(t,w)\|v\|_{[-r,\beta]} + d_i(t,w)\|x\|,$$

for a.e. $t \in I$, $w \in \Omega$, *and each* $u, v \in C_{[-r,\beta]}, x \in \mathbb{R}^n$.

Set

$$a_i^*(\cdot,w) = ess\sup_{t\in I} a_i(t,w), \quad b_i^*(\cdot,w) = ess\sup_{t\in I} b_i(t,w),$$

$$c_i^*(\cdot,w) = ess\sup_{t\in I} c_i(t,w), \quad d_i^*(\cdot,w) = ess\sup_{t\in I} d_i(t,w), \quad i = 1,2$$

$$\widetilde{G_{\alpha_i}} = \sup\left\{ \int_a^T |G_{\alpha_i}(t,s)|ds, t \in I \right\}.$$

Now, we state and prove our existence and uniqueness of random solutions result for of the problem (4.1)-(4.2)

Theorem 4.2.1 *Assume* (H_1) *and* (H_2) *hold. If for every* $w \in \Omega$, *the matrix*

$$M(w) := \begin{pmatrix} \frac{\widetilde{G_{\alpha_1}}\|p_1(\cdot,w)\|_{[a,T]}}{1-\|r_1(\cdot,w)\|_{[a,T]}} & \frac{\widetilde{G_{\alpha_1}}\|q_1(\cdot,w)\|_{[a,T]}}{1-\|r_1(\cdot,w)\|_{[a,T]}} \\ \\ \frac{\widetilde{G_{\alpha_2}}\|p_2(\cdot,w)\|_{[a,T]}}{1-\|r_2(\cdot,w)\|_{[a,T]}} & \frac{\widetilde{G_{\alpha_2}}\|q_2(\cdot,w)\|_{[a,T]}}{1-\|r_2(\cdot,w)\|_{[a,T]}} \end{pmatrix}$$

converges to 0, *then the problem (4.1)-(4.2) has a unique solution.*

Proof: Let the operator $N : \mathcal{C}^2 \times \Omega \longmapsto \mathcal{C}^2$ be defined by

$$N(u,v)(t,w) = (N_1(u,v)(t,w), N_2(u,v)(t,w))$$

$$= \begin{cases} (\phi_1(t,w), \phi_2(t,w)), & \text{if} \quad t \in [a-r,a], \\ -\left(\int_a^T G_1(t,s)h_1(s,w)ds, \int_a^T G_2(t,s)h_2(s,w)ds \right), & \text{if} \quad t \in I, \\ (\psi_1(t,w), \psi_2(t,w)), & \text{if} \quad t \in [T, T+\beta]. \end{cases} \tag{4.10}$$

By Lemma 4.2.2 it is clear that the fixed points of N are solutions (4.1)-(4.2).

Let $(u_1,v_1), (u_2,v_2) \in \mathcal{C}^2$ *and* $w \in \Omega$. If $t \in [a-r,a]$ or $t \in [T, T+\beta]$. Then

$$\|N(u_2,v_2)(t,w) - N(u_1,v_1)(t,w)\| = 0.$$

For $t \in I$, we have

$$\|N_1(u_2,v_2)(t,w) - N_1(u_1,v_1)(t,w)\| \leq \int_a^T |G(t,s)| \|h_1(t,w) - \bar{h_1}(t,w)\| ds, \tag{4.11}$$

where $h_i(\cdot,w), \bar{h_i}(\cdot,w) \in C(I)$ for $w \in \Omega$ are given by

$$h_i(t,w) = f_i(t, u^t(w), v^t(w), h_i(t,w)), i = 1,2,$$
$$\bar{h_i}(t,w) = f_i(t, \bar{u^t}(w), \bar{v^t}(w), \bar{h_i}(t,w)), i = 1,2,$$

and by (H_2) we have

$$\begin{aligned}
\|h_1(t,w) - \bar{h_1}(t,w)\| &\leq p_i(t,w)\|u-\bar{u}\|_{[-r,\beta]} + q_i(t,w)\|v-\bar{v}\|_{[-r,\beta]} \\
&+ r_i(t,w)\|h_1(t,w) - \bar{h_1}(t,w)\| \\
&\leq \|p_i(\cdot,w)\|_{[a,T]}\|u-\bar{u}\|_{[-r,\beta]} + \|q_i(\cdot,w)\|_{[a,T]}\|v-\bar{v}\|_{[-r,\beta]} \\
&+ \|r_i(\cdot,w)\|_{[a,T]}\|h_1(t,w) - \bar{h_1}(t,w)\|.
\end{aligned}$$

Then

$$\begin{aligned}
\|h_1(t,w) - \bar{h_1}(t,w)\| &\leq \frac{\|p_i(\cdot,w)\|_{[a,T]}}{1-\|r_i(\cdot,w)\|_{[a,T]}}\|u-\bar{u}\|_{[-r,\beta]} \\
&+ \frac{\|q_i(\cdot,w)\|_{[a,T]}}{1-\|r_i(\cdot,w)\|_{[a,T]}}\|v-\bar{v}\|_{[-r,\beta]},
\end{aligned}$$

from which we conclude,

$$\begin{aligned}
\|N_1(u,v)(t,w) - N_1(\bar{u},\bar{v})(t,w)\| &\leq \frac{\|p_1(\cdot,w)\|_{[a,T]}}{1-\|r_1(\cdot,w)\|_{[a,T]}}\int_a^T |G_1(t,s)|\|u-\bar{u}\|_{[-r,\beta]}ds \\
&+ \frac{\|q_1(\cdot,w)\|_{[a,T]}}{1-\|r_1(\cdot,w)\|_{[a,T]}}\int_a^T |G_1(t,s)|\|v-\bar{v}\|_{[-r,\beta]}ds \\
&\leq \frac{\widetilde{G_1}\|p_1(\cdot,w)\|_{[a,T]}}{1-\|r_1(\cdot,w)\|_{[a,T]}}\|u-\bar{u}\|_{[-r,\beta]} \\
&+ \frac{\widetilde{G_1}\|q_1(\cdot,w)\|_{[a,T]}}{1-\|r_1(\cdot,w)\|_{[a,T]}}\|v-\bar{v}\|_{[-r,\beta]}.
\end{aligned}$$

Therefore, for each $t \in I$, and $w \in \Omega$

$$\begin{aligned}
\|N_1(u,v)(\cdot,w) - N_1(\bar{u},\bar{v})(\cdot,w)\|_{\mathcal{C}} &\leq \frac{\widetilde{G_1}\|p_1(\cdot,w)\|_{[a,T]}}{1-\|r_1(\cdot,w)\|_{[a,T]}}\|u-\bar{u}\|_{\mathcal{C}} \\
&+ \frac{\widetilde{G_1}\|q_1(\cdot,w)\|_{[a,T]}}{1-\|r_1(\cdot,w)\|_{[a,T]}}\|v-\bar{v}\|_{\mathcal{C}}.
\end{aligned}$$

Also, for any $w \in \Omega$ and each $(u,v), (\bar{u},\bar{v}) \in \mathcal{C}^2$ and $t \in I$, we get

$$\begin{aligned}
\|N_2(u,v)(\cdot,w) - N_2(\bar{u},\bar{v})(\cdot,w)\|_{\mathcal{C}} &\leq \frac{\widetilde{G_2}\|p_2(\cdot,w)\|_{[a,T]}}{1-\|r_2(\cdot,w)\|_{[a,T]}}\|u-\bar{u}\|_{\mathcal{C}} \\
&+ \frac{\widetilde{G_2}\|q_2(\cdot,w)\|_{[a,T]}}{1-\|r_2(\cdot,w)\|_{[a,T]}}\|v-\bar{v}\|_{\mathcal{C}}.
\end{aligned}$$

Thus,

$$d(N(u,v)(\cdot,w),N(\bar{u},\bar{v})(\cdot,w)) \leq M(w)d((u(\cdot,w),v(\cdot,w)),(\bar{u}(\cdot,w),\bar{v}(\cdot,w)),$$

where

$$d((u(\cdot,w),v(\cdot,w)),(\bar{u}(\cdot,w),\bar{v}(\cdot,w))) = \begin{pmatrix} \|u(\cdot,w)-v(\cdot,w)\|_{\mathcal{C}} \\ \\ \|\bar{u}(\cdot,w)-\bar{v}(\cdot,w)\|_{\mathcal{C}} \end{pmatrix}.$$

Since for every $w \in \Omega$, the matrix $M(w)$ converges to zero, then by Theorem 1.6.1, N has a unique random fixed point which is a solution to problem (4.1)-(4.2). □

Theorem 4.2.2 *Suppose that* (H_1) *and* (H_3) *hold. Then problem (4.1)-(4.2) has at least one solution.*

Proof: The proof will be established in steps.

Step 1: $N(\cdot,\cdot,w)$ is continuous. Let $\{(u_n,v_n)\}$ be a sequence such that $(u_n,v_n) \to (u,v)$ in $\mathcal{C}\times\mathcal{C}$, for any $w \in \Omega$. If $t \in [a-r,a]$ or $t \in [T,T+\beta]$ then

$$\|(N(u_n,v_n))(t,w) - (N(u,v))(t,w)\| = 0.$$

For $t \in I$, we have

$$\begin{aligned} \|(N_i(u_n,v_n)(t,w)) - (N_i(u,v))(t,w)\| \quad &\leq \int_a^T |G_{\alpha_i}(t,s)| \|f_i(t,u_{t,n}(w),v_{t,n}(w),h_{i,n}(t,w)) \\ &\quad -f_i(t,u_t(w),v_t(w),h_i(t,w))\|ds \\ &\leq \widetilde{G_{\alpha_i}} \|f_i(\cdot,u_n(\cdot,w),v_n(\cdot,w),h_{i,n}(\cdot,w)) \\ &\quad -f_i(\cdot,u_t(w),v(\cdot,w),h_i(\cdot,w))\|_{\mathcal{C}}, i=1,2, \end{aligned}$$

where

$$h_i(t,w) = f_i(t,u^t(w),v^t(w),h_i(t,w)).$$

Since f_i is Carathéodory, we have:

$$\|(N_i(u_n,v_n))(\cdot,w) - (N_i(u,v))(\cdot,w)\|_{\mathcal{C}} \longrightarrow 0 \ as \ n \longrightarrow \infty, i=1,2,$$

and hence

$$\|N(u_n,v_n)(\cdot,w) - N(u,v)(\cdot,w)\|_{\overline{\mathcal{C}}} \longrightarrow 0 \ as \ n \longrightarrow \infty.$$

Consequently, N is continuous.

Let the constant $R(w)$ be such that,

$$R(w) \geq \max\{L_1(w)+L_2(w), \|\phi_1(\cdot,w)\|_{[a-r,a]} + \|\phi_2(\cdot,w)\|_{[a-r,a]}, \|\psi_1(\cdot,w)\|_{[T,T+\beta]}$$

$$+\|\psi_2(\cdot,w)\|_{[T,T+\beta]}\},$$

and define

$$D_{R(w)} = \{(u,v) \in \mathcal{C}\times\mathcal{C} : \|u\|_{\mathcal{C}} \leq R(w) \ \ and \ \ \|v\|_{\mathcal{C}} \leq R(w)\}.$$

It is clear that D_R is a bounded, closed and convex subset of $\overline{\mathcal{C}}$.

Step 2: $N(D_R(w)) \subset D_R(w)$.

Let $(u,v) \in D_R(w)$. We show that $N(u,v) = (N_1(u,v), N_2(u,v)) \in D_R(w)$. For any $w \in \Omega$, if $t \in [a-r,a]$, then

$$\|N(u,v)(t,w)\| \leq \|\phi_1(\cdot,w)\|_{[a-r,a]} + \|\phi_2(\cdot,w)\|_{[a-r,a]} \leq R(w),$$

and if $t \in [T, T+\beta]$, then

$$\|N(u,v)(t,w)\| \leq \|\psi_1(\cdot,w)\|_{[T,T+\beta]} + \|\psi_2(\cdot,w)\|_{[T,T+\beta]} \leq R(w).$$

For any $w \in \Omega$ and each $t \in I$, we have

$$\|(N_i(u,v))(t,w)\| \leq \int_a^T |G_{\alpha_i}(t,s)|\|h_{\alpha_i}(s,w)\|ds, i = 1,2.$$

By (H_3) we have for any $w \in \Omega$ and each $t \in I$

$$\begin{aligned}\|h_i(t,w)\| &\leq a_i(t,w) + b_i(t,w)\|u\|_{[-r,\beta]} + c_i(t,w)\|v\|_{[-r,\beta]} + d_i(t,w)\|h_i(t,w)\| \\ &\leq a_i^*(\cdot,w) + b_i^*(\cdot,w)\|u\|_{[-r,\beta]} + c_i^*(\cdot,w)\|v\|_{[-r,\beta]} + d_i^*(\cdot,w)\|h_i(t,w)\|,\end{aligned}$$

where

$$h_i(t,w) = f_i(t, u^t(w), v^t(w), h_i(t,w)), i = 1,2.$$

Then

$$\|h_i(t,w)\| \leq \frac{a_i^*(\cdot,w) + (b_i^*(\cdot,w) + c_i^*(\cdot,w))R}{1 - d_i^*(\cdot,w)} = A(w). \tag{4.12}$$

By (4.12), for any $w \in \Omega$ and $t \in I$, we have

$$\begin{aligned}\|(N_i(u,v))(t,w)\| &\leq A(w)\int_a^T |G_{\alpha_i}(t,s)|ds \\ &\leq A(w)\widetilde{G_{\alpha_i}} \\ &= L_i(w),\end{aligned}$$

from which it follows that for each $t \in [a-r, T+\beta]$, we have

$$\|N_i(u,v)(t,w)\| \leq L_i(w),$$

which implies that $\|N_i(u,v)(\cdot,w)\|_{\mathcal{C}} \leq L_i(w)$, hence we get

$$\begin{aligned} \|N(u,v)(\cdot,w)\|_{\overline{\mathcal{C}}} &\leq L_1(w) + L_2(w) \\ &\leq R(w). \end{aligned}$$

Consequently,

$$N(D_R(w)) \subset D_R(w).$$

Step 3: $N(D_R(w))$ is bounded and equicontinuous.

By Step 2 we have $N(D_R(w))$ is bounded. For $t_1, t_2 \in I = [a,T],\ t_1 < t_2$, $(u,v) \in D_R(w),\ w \in \Omega$, we have

$$\begin{aligned} \|(N_i(u,v))(t_2,w) - (N_i(u,v))(t_1,w)\| &\leq \int_a^T |G_{\alpha_i}(t_2,s) - G_{\alpha_i}(t_1,s)|\|h_i(s,w)\|ds \\ &\leq A(w)\int_a^T |G_{\alpha_i}(t_2,s) - G_{\alpha_i}(t_1,s)|ds. \end{aligned}$$

As $t_1 \longrightarrow t_2$ the right hand side of the above inequality tends to zero.Therefore, the operator $N(u,v)(\cdot,w)$ is equicontinuous. As consequence of Step 1 to Step 3, together withe the Arzela-Ascoli theorem, we can conclude that N is continuous and completely continuous. Theorem 4.2.2 implies that the operator equation $N(u,v)(\cdot,w) = (u,v)$ has a random solution. This shows that the random system (4.1)-(4.2) has a random solution. □

4.3 An Example

We equip the space $\mathbb{R}^*_- := (-\infty, 0)$ with the usual σ-algebra consisting of Lebesgue measurable subsets of $\mathbb{R}^*_-$. Consider the boundary value problem of implicit Caputo type modification of the Erdélyi-Kober fractional differential equation:

$$\begin{cases} (u(t,w),v(t,w)) = (e^{t-1}-1, 2t-2), & t \in [0,1], \\ {}^1_cD^{\frac{3}{2}}_{1^+}u(t,w) = \dfrac{\frac{1}{2}t}{(w^2+10)\left(1+\|u^t(\cdot,w)\|+\|v^t(\cdot,w)\|+|{}^1_cD^{\frac{3}{2}}_{1^+}u(t,w)|\right)}, & t \in I = [1,2] \\ {}^1_cD^{\frac{3}{2}}_{1^+}v(t,w) = \dfrac{(t-1)\cos(t)}{(w^2+10)\left(1+\|u^t(\cdot,w)\|+\|v^t(\cdot,w)\|+|{}^1_cD^{\frac{3}{2}}_{1^+}v(t,w)|\right)}, & t \in I = [1,2] \\ (u(t,w),v(t,w)) = (\ln(t-1), t-2), & t \in [2,3]. \end{cases} \tag{4.13}$$

Set

$$f_1(t,u,v,\bar{u},w) = \frac{\frac{1}{2}t}{(w^2+10)(1+|u^t|+|v^t|+|\bar{u}|)},\ t \in [1,2], u,v \in C([-1,1]), \bar{u} \in \mathbb{R}^n$$

$$f_2(t,u,v,\bar{v},w) = \frac{(t-1)\cos(t)}{(w^2+10)(1+|u^t|+|v^t|+|\bar{v}|)},\ t \in [1,2], u,v \in C([-1,1]), \bar{v} \in \mathbb{R}^n$$

and $\alpha_1 = \alpha_2 = \frac{3}{2}, \rho = 1, r = 1, \beta = 1$. Indeed for each $u, v, \bar{u}, \bar{v} \in C([-1,1]), x, \bar{x} \in \mathbb{R}^n$ and $t \in [1,2]$, we have

$$\begin{aligned} \|f_1(t,u,v,x,w) - f_1(t,\bar{u},\bar{v},\bar{x},w)\| &\leq \frac{1}{(w^2+10)}(\|u-\bar{u}\|_{[-r,\beta]} + \|v-\bar{v}\|_{[-r,\beta]} \\ &+ \|x-\bar{x}\|), \end{aligned}$$

and

$$\begin{aligned} \|f_2(t,u,v,x,w) - f_2(t,\bar{u},\bar{v},\bar{x},w)\| &\leq \frac{1}{(w^2+10)}(\|u-\bar{u}\|_{[-r,\beta]} + \|v-\bar{v}\|_{[-r,\beta]} \\ &+ \|x-\bar{x}\|). \end{aligned}$$

Therefore, (H_2) is verified with

$$\|p_i(\cdot,w)\|_{[-r,\beta]} = \|q_i(\cdot,w)\|_{[-r,\beta]} = \|r_i(\cdot,w)\|_{[-r,\beta]} = \frac{1}{w^2+10}.$$

For each $t \in I,\ i = 1,2$ we have

$$\begin{aligned} \int_a^T |G_{\alpha_i}(t,s)|ds &\leq \frac{1}{\Gamma(\alpha_i)}\left(\frac{t^\rho - a^\rho}{T^\rho - a^\rho}\right)\int_a^T \left|\left(\frac{T^\rho - s^\rho}{\rho}\right)^{\alpha_i - 1} s^{\rho-1}\right| ds \\ &+ \frac{1}{\Gamma(\alpha_i)}\int_a^t \left|\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_i - 1} s^{\rho-1}\right| ds \\ &\leq \frac{2}{\Gamma(\alpha_i+1)}\left(\frac{T^\rho - a^\rho}{\rho}\right)^{\alpha_i}. \end{aligned}$$

Therefore

$$\widetilde{G_{\alpha_i}} \leq \frac{2}{\Gamma(\alpha_i+1)}\left(\frac{T^\rho - a^\rho}{\rho}\right)^{\alpha_i}.$$

Furthermore, for every $w \in \Omega$, the matrix

$$\frac{1}{(w^2+9)\frac{3}{4}\sqrt{\pi}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

converges to 0. Hence, Theorem 4.2.1 implies that the system (4.13) has a unique random solution defined on $[1,2]$.

Chapter 5

Caputo type modification of the Erdélyi-Kober Fractional Differential Inclusions

5.1 Introduction

This chapter generalizes the second chapter, we establish, in Section 5.2, the existence of solutions for a class of problem for nonlinear Caputo type modification of the Erdélyi-Kober fractional differential inclusions (FDI for short) involving both retarded and advanced arguments see [45]. An extension of this problem is given in Section 5.5. More precisely, we shall present a result of existence the existence of solutions of Caputo type modification of the Erdélyi-Kober fractional differential inclusions in Banach spaces. with retarded and advanced arguments see [41].

5.2 Fractional Differential Inclusions with Retarded and Advanced Arguments

[1]

This section is concerned with the existence of solutions for a class of problem for nonlinear fractional differential inclusions (FDI for short) involving both retarded and advanced arguments given by:

$$ {}^{\rho}_{c}D^{\alpha}_{a^+}y(t) \in F(t, y^t), \text{ for } t \in I := [a, T], \ 1 < \alpha \leq 2, \tag{5.1} $$

$$ y(t) = \phi(t), \qquad t \in [a - r, a], \ r > 0 \tag{5.2} $$

$$ y(t) = \psi(t), \qquad t \in [T, T + \beta], \ \beta > 0, \tag{5.3} $$

[1]M. Boumaaza, M. Benchohra, Caputo type modification of the Erdélyi-Kober fractional differential inclusions with retarded and advanced arguments, *Adv. Dyn. Syst. Appl.* **15** (2) (2020), 63-78.

where ${}^{\rho}_{c}D^{\alpha}_{a^+}$ is the Caputo type modification of the Erdélyi-Kober fractional derivative, $F : I \times C([-r,\beta],\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ is a given function, $\phi \in C([a-r,a],\mathbb{R})$ with $\phi(a) = 0$ and $\psi \in C([T,T+\beta],\mathbb{R})$ with $\psi(T) = 0$. We denote by y^t the element of $C([-r,\beta])$ defined by:

$$y^t(s) = y(t+s) : s \in [-r,\beta].$$

5.3 Existence Results

Lemma 5.3.1 *Let $1 < \alpha \le 2$, $\phi \in C([a-r,a],\mathbb{R})$ with $\phi(a) = 0$, $\psi \in C([T,T+\beta],\mathbb{R})$ with $\psi(T) = 0$ and $h : I \to \mathbb{R}$ be a integrable function. Then the linear problem*

$$ {}^{\rho}_{c}D^{\alpha}_{a^+}y(t) = h(t), \text{ for a.e. } \ t \in I := [a,T], \ 1 < \alpha \le 2, \tag{5.4}$$

$$y(t) = \phi(t), \ t \in [a-r,a], \ r > 0 \tag{5.5}$$

$$y(t) = \psi(t), \ t \in [T,T+\beta], \ \beta > 0, \tag{5.6}$$

has a unique solution, which is given by

$$y(t) = \begin{cases} \phi(t), & \text{if} \quad t \in [a-r,a], \\ \\ -\displaystyle\int_a^T G(t,s)h(s)ds, & \text{if} \quad t \in I \\ \\ \psi(t), & \text{if} \quad t \in [T,T+\beta], \end{cases} \tag{5.7}$$

where

$$G(t,s) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \begin{cases} \dfrac{(t^\rho - a^\rho)(T^\rho - s^\rho)^{\alpha-1}s^{\rho-1}}{(T^\rho - a^\rho)} - s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1}, & a \le s \le t \le T, \\ \\ \dfrac{(t^\rho - a^\rho)(T^\rho - s^\rho)^{\alpha-1}s^{\rho-1}}{(T^\rho - a^\rho)}, & a \le t \le s \le T. \end{cases} \tag{5.8}$$

Here $G(t,s)$ is called the Green function of the boundary value problem (5.4)-(5.6).
Proof. *From* (1.4)*, we have*

$$y(t) = c_0 + c_1\left(\frac{t^\rho - a^\rho}{\rho}\right) +^{\rho} I^{\alpha}_{a^+}h(s), \quad c_0, c_1 \in \mathbb{R}, \tag{5.9}$$

therefore

$$y(a) = c_0 = 0,$$

$$y(T) = c_1\left(\frac{T^\rho - a^\rho}{\rho}\right) + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^T (T^\rho - s^\rho)^{\alpha-1}s^{\rho-1}h(s)ds,$$

and

$$c_1 = -\frac{\rho^{2-\alpha}}{(T^\rho - a^\rho)\Gamma(\alpha)}\int_a^T (T^\rho - s^\rho)^{\alpha-1}s^{\rho-1}h(s)ds.$$

Substitute the value of c_0 and c_1 into equation (5.9), we get equation (5.7).

$$y(t) = \begin{cases} \phi(t), & \text{if} \quad t \in [a-r, a], \\ -\displaystyle\int_a^T G(t,s)h(s)ds, & \text{if} \quad t \in I \\ \psi(t), & \text{if} \quad t \in [T, T+\beta], \end{cases}$$

where G is defined by equation (5.8), the proof is complete.

Lemma 5.3.2 *Let $F : I \times C[-r,\beta] \longrightarrow \mathcal{P}(\mathbb{R})$ be a Carathéodory multivalued map. A function $y \in \mathcal{C}$ is a solution for the inclusion problem $(5.1)-(5.3)$ if and only if y satisfies the following integral equation*

$$y(t) = \begin{cases} \phi(t), & \text{if} \quad t \in [a-r, a], \\ -\displaystyle\int_a^T G(t,s)h(s)ds, & \text{if} \quad t \in I \\ \psi(t), & \text{if} \quad t \in [T, T+\beta], \end{cases}$$

where $h \in L^1(I)$ with

$$h(t) \in F(t, y^t) \quad a.e. \quad t \in I,$$

$$\widetilde{G} = \sup\left\{\int_a^T |G(t,s)|ds, t \in I\right\}.$$

The following hypotheses will be used in the sequel:

(H_1) The multivalued map $F : I \times C([-r,\beta]) \to \mathcal{P}_{cp,c}(\mathbb{R})$ is Carathéodory,

(H_2) There exist $p \in L^\infty(I, \mathbb{R}_+)$, and $\Omega : [0, \infty) \longrightarrow (0, \infty)$ continuous and nondecreasing such that

$$\|F(t,u)\|_{\mathcal{P}} = \sup\{\|v\|_C : v(t) \in F(t,u)\} \leq p(t)\Omega(\|v\|_{[-r,\beta]}),$$

for a.e. $t \in I$, and each $u \in C([-r, \beta])$,

(H_3) there exists $l \in L^1(I, \mathbb{R})$, such that

$H_d(F(t,u^t), F(t,\bar{u^t})) \leq l(t)\|u - \bar{u}\|_{[-r,\beta]}$ for every $u, \bar{u} \in C([-r, \beta])$, and

$$d(0, F(0,u^t)) \leq l(t) \quad a.e. \quad t \in I.$$

(H_4) There exists a number $K_1 > 0$ such that

$$\frac{K_1}{\widetilde{G}p^*\Omega(K_1)} > 1, \tag{5.10}$$

where

$$p^* = ess\sup_{t\in I} p(t)$$

Now, we state and prove our existence result for problem (5.1)-(5.3) based on a nonlinear alternative for Kakutani maps.

Theorem 5.3.1 *Assume that (H_1), (H_2) and (H_4) hold. Then the problem (5.1)-(5.3) has at least one solution.*

Proof: Let the operator $N : \mathcal{C} \longmapsto \mathcal{P}(\mathcal{C})$ defined by

$$(Ny)(t) = \left\{ \begin{array}{c} h : I \longrightarrow \mathcal{C} : \\ \\ h(t) = \begin{cases} \phi(t), & \text{if} \quad t \in [a-r, a], \\ \\ -\displaystyle\int_a^T G(t,s)v(s)ds, & \text{if} \quad t \in I \\ \\ \psi(t), & \text{if} \quad t \in [T, T+\beta], \end{cases} \end{array} \right\} \tag{5.11}$$

where

$$v \in S_{F\circ y} = \{v : \Omega \longrightarrow L^1(I) : v(t) \in F(t, y^t) \text{ a.e. } t \in I\}.$$

By Lemma 5.3.2 it is clear that the fixed points of N are solutions (5.1)-(5.3) .

Step 1. *$N(u)$ is convex for each $u \in C(I)$.*
Indeed, if h_1, h_2 belong to $N(u)$, then there exist $v_1, v_2 \in S_{F\circ u}$ such that for each $t \in I$ we have

$$h_i(t) = \int_a^T G(t,s)v_i(s)ds;\ i = 1,2.$$

Let $0 \le \lambda \le 1$. Then, for each $t \in I$, we have

$$(\lambda h_1 + (1-\lambda)h_2)(t) = \int_a^T G(t,s)(\lambda v_1(s) + (1-\lambda)v_2(s))ds.$$

Since $S_{F\circ u}$ is convex (because F has convex values), we have $\lambda h_1 + (1-\lambda)h_2 \in N(u)$.

Let the constant R be such that:

$$R \ge \max\left\{p^*\Omega(R))\widetilde{G}, \|\phi\|_{[a-r,a]}, \|\psi\|_{[T,T+\beta]}\right\}, \tag{5.12}$$

and define

$$D_R = \{u \in \mathcal{C} : \|u\|_{\mathcal{C}} \le R\}.$$

It is clear that D_R is a bounded, closed and convex subset of $\mathcal{C}$.

Step 2. *$N(D_R) \subset D_R$.*

Let $u \in D_R$ Then for each $h \in N(u)$, there exists $v \in S_{F\circ u}$ such that

$$h(t) = \int_a^T G(t,s)v(s)ds;\ i = 1,2.$$

If $t \in [a-r,a]$, then

$$|h(t)| \le \|\phi\|_{[a-r,a]} \le R,$$

and if $t \in [T,T+\beta]$, then

$$|h(t)| \le \|\psi\|_{[T,T+\beta]} \le R.$$

For each $t \in I$, we have

$$|h(t)| \le \int_a^T |G(t,s)||v(s)|ds.$$

By (H_2), we have

$$\begin{aligned} |h(t)| &\le \int_a^T |G(t,s)|p(t)\Omega(\|u\|_{[-r,\beta]})ds \\ &\le p^*\Omega(R)\int_a^T |G(t,s)|ds \\ &\le p^*\Omega(R))\widetilde{G} \\ &\le R, \end{aligned}$$

from which it follows that for each $t \in [a-r, T+\beta]$, we have $|h(t)| \leq R$, which implies that $\|h\|_{\mathcal{C}} \leq R$, and so $N(D_R) \subset D_R$.

Step 3: N maps bounded sets in $\mathcal{C}$ into equicontinuous sets.
We consider D_R is bounded set in $\mathcal{C}$. By Step 2 we have $N(D_R) \subset D_R$. Now let $t_1, t_2 \in I = [a, T], t_1 < t_2,\ and\ \ let\ u \in D_R,\ h \in N(u)$. Then, there exists $v \in S_{F\circ u}$ such that

$$\begin{aligned} |h(t_2) - h(t_1)| &\leq \int_a^T |G(t_2, s) - G(t_1, s)||v(s)|ds \\ &\leq p^*\Omega(R) \int_a^T |G(t_2, s) - G(t_1, s)|ds. \end{aligned}$$

As $t_1 \longrightarrow t_2$ the right hand side of the above inequality tends to zero. As consequence of Step 1 to Step 3, together withe the Arzela-Ascoli theorem, we can conclude that N is completely continuous multi-valued operator.

Step 4. *The graph of N is closed.*
Let $(u_n, h_n) \in \text{graph}(N)$, $n \geq 1$, with $(\|u_n - u\|, \|h_n - h\|) \to (0, 0)$, as $n \to \infty$. We have to show that $(u, h) \in \text{graph}(N)$. $(u_n, h_n) \in \text{graph}(N)$ means that $h_n \in N(u_n)$, which implies that there exists $v_n \in S_{F\circ u_n}$, such that for each $t \in I$,

$$h_n(t) = \int_a^T G(t, s)v_n(s)ds.$$

Consider the continuous linear operator $\Theta : L^1(I) \to \mathcal{C}$,

$$\Theta(v)(t) \mapsto h_n(t) = \int_a^T G(t, s)v_n(s)ds.$$

Clearly, $\|h_n(t) - h(t)\| \to 0$ as $n \to \infty$. From Lemma 1.4.1 it follows that $\Theta \circ S_F$ is a closed graph operator. Moreover, $h_n(t) \in \Theta(S_{F\circ u_n})$. Since $u_n \to u$, then

$$h(t) = \int_a^T G(t, s)v(s)ds,$$

for some $v \in S_{F\circ u}$.

Step 5: A priori bounds on solutions.
Let $u \in \mathcal{C}$ be such that $u \in \lambda N(u)$ for all $\lambda \in (0, 1)$. Then, there exists $v \in S_{F\circ u}$ such that for each $t \in I$, we have

$$u(t) = -\lambda \int_a^T G(t, s)v(s)ds, \tag{5.13}$$

This implies, by (5.13) that for each $t \in I$ we have

$$\begin{aligned}|u(t)| &\leq \int_a^T |G(t,s)|p(t)\Omega(\|u\|_{[-\alpha,\beta]})ds \\ &\leq \widetilde{G}p^*\Omega(\|u\|_{[-\alpha,\beta]}).\end{aligned}$$

Thus

$$\frac{\|u\|_{\mathcal{C}}}{\widetilde{G}p^*\Omega(\|u\|_{\mathcal{C}})} \leq 1$$

By (H_4), we have $\|u\|_{\mathcal{C}} \neq K_1$. Set

$$U = \{u \in \mathcal{C} : \|u\|_{\mathcal{C}} < K_1 + 1\}$$

From the choice of U there is no $u \in \partial U$ such that $u \in \lambda N(u)$ for some $\lambda \in (0,1)$. As a consequence of Lemma 1.6.2, we deduce that N has a fixed point u in U which is a solution of $(5.1) - (5.3)$.

We now prove an existence result for (5.1)-(5.3) with non-convex valued right hand side. Our considerations are based on the fixed point theorem for contraction multivalued maps given by Covitz and Nadler.

Theorem 5.3.2 *Assume that* (H_3) *and*
(H_5) $F : [a,T] \times C([-r,\beta], \mathbb{R}) \longrightarrow \mathcal{P}_{cp}(\mathbb{R})$ *such that*
$F(\cdot, u) : [a,T] \longrightarrow \mathcal{P}_{cp}(\mathbb{R})$ *is measurable for each* $u \in C([-r,\beta], \mathbb{R})$ *and,*
If

$$\widetilde{G}\|l\|_{[a,T]} < 1, \tag{5.14}$$

then problem (5.1)-(5.3) has at least one solution.

Proof. We shall show that N, as defined in the proof of Theorem 5.3.1, satisfies the assumptions of Lemma 1.6.3. The proof will be given in two steps.
Step 1. $N(\cdot)$ is closed valued.
$N(u) \in \mathcal{P}_{cl}(D_R)$ *for each* $u \in D_R$. Let $\{u_n\}_{n\geq 0} \in N(u)$ such that $u_n \longrightarrow \tilde{u}$ in $\mathcal{C}$. Then, $\tilde{u} \in D_R$ and there exists $g_n \in S_{F\circ u}$ be such that, for each $t \in I$, we have

$$u_n(t) = \int_a^T G(t,s)g_n(s)ds.$$

Using (H_5) together with the fact that F has compact values, we may pass to a subsequence to see that g_n converges to g in $L^1(I)$, and hence $g \in S_{F\circ u}$. Then, for each $t \in I$, we get

$$u_n(t) \longrightarrow \tilde{u}(t) = \int_a^T G(t,s)g(s)ds.$$

So, $\tilde{u} \in N(u)$.

Step 2. There exist $\gamma < 1$ such that $H_d(N(u), N(\bar{u})) \leq \gamma \|u - \bar{u}\|_{\mathcal{C}}$ for each $u, \bar{u} \in \mathcal{C}$. Let $u, \bar{u} \in \mathcal{C}$ and $h_1 \in N(u)$. Then, there exists $v_1 \in F(t, u^t)$ such that for each $t \in I$

$$u_1 = \int_a^T G(t,s)v_1(s)ds.$$

From (H_3) it follows that

$$H_d(F(t,u^t), F(t,\bar{u^t})) \leq l(t)\|u^t - \bar{u^t}\|$$

Hence, there exists $w \in F(t, \bar{u^t})$ such that

$$|v_1 - w| \leq l(t)\|u^t - \bar{u^t}\| \quad t \in I.$$

Consider $U : I \longrightarrow \mathcal{P}(\mathrm{I\!R})$ given by

$$U(t) = \{w \in \mathrm{I\!R} : |v_1 - w| \leq l(t)\|u^t - \bar{u^t}\|\}$$

Since the multivalued operator $V(t) = U(t) \bigcap F(t, \bar{u}^t)$ is measurable, there exists a function $v_2(t)$ which is measurable selection for V. So, $v_2 \in F(t, \bar{u}^t)$, and for each $t \in I$

$$|v_1 - v_2| \leq l(t)\|u^t - \bar{u^t}\|.$$

Let us define for each $t \in I$

$$u_2 = \int_a^T G(t,s)v_2(s)ds.$$

For $t \in I$, we have

$$\begin{aligned}
|h_1(t) - h_2(t)| &\leq \int_a^T |G(t,s)||v_1(s) - v_2(s)|ds \\
&\leq \int_a^T |G(t,s)|l(t)\|u^s - \bar{u^s}\|ds \\
&\leq \int_a^T |G(t,s)|l(t)\|u - \bar{u}\|_{[-r,\beta]}ds \\
&\leq \widetilde{G}\|l\|_{[a,T]}\|u - \bar{u}\|_{\mathcal{C}}.
\end{aligned}$$

Thus

$$\|h_1 - h_2\|_{\mathcal{C}} \leq \widetilde{G}\|l\|_{[a,T]}\|u - \bar{u}\|_{\mathcal{C}}.$$

Analogously, interchanging the roles of u and $\bar{u}$, we obtain

$$H_d(N(u), N(\bar{u})) \leq \widetilde{G}\|l\|_{[a,T]}\|u - \bar{u}\|_{\mathcal{C}}.$$

Since N is a contraction, it follows by Lemma 1.6.3 that N has a fixed point u which is a solution of (5.1)-(5.3). This completes the proof.

5.4 An Example

Consider the boundary value problem of Caputo type modification of the Erdélyi-Kober fractional differential inclusion:

$$\begin{cases} y(t) = e^{t-2} - 1, & t \in [1,2], \\ {}^{\frac{1}{2}}_{c}D^{\frac{3}{2}}_{2^+} y(t) \in F(t, u^t), & t \in [2,4] \\ y(t) = t - 4, & t \in [4,6]. \end{cases} \tag{5.15}$$

Set

$$F(t, u^t) = \{v \in \mathbb{R} : 0 \leq v \leq \frac{1}{t+1}(\|u\|_{[-r,\beta]} + 1)\}, \quad t \in [2,4], \ u \in C([-r,\beta]),$$

and

$$\alpha = \frac{3}{2}, \ \rho = \frac{1}{2}, \ r = 1, \ \beta = 2.$$

For each $u \in C([-r,\beta])$, $t \in [2,4]$, we have

$$\|F(t,u^t)\| \leq \frac{1}{t+1}(\|u\|_{[-r,\beta]} + 1).$$

Therefore, (H_2) is verified with $p^* = \frac{1}{3}$
For each $t \in I$ we have

$$\begin{aligned} \int_a^T |G(t,s)|ds &\leq \frac{1}{\Gamma(\alpha)} \left(\frac{t^\rho - a^\rho}{T^\rho - a^\rho} \right) \int_a^T \left| \left(\frac{T^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} \right| ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_a^t \left| \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} \right| ds. \end{aligned}$$

Then

$$\int_a^T |G(t,s)|ds \leq \frac{2}{\Gamma(\alpha+1)} \left(\frac{T^\rho - a^\rho}{\rho} \right)^\alpha.$$

Therefore

$$\widetilde{G} \leq \frac{2}{\Gamma(\alpha+1)} \left(\frac{T^\rho - a^\rho}{\rho} \right)^\alpha.$$

The condition (5.10)is satisfied. Indeed, we have

$$\begin{aligned} \widetilde{G}p^* &\leq \frac{2}{3\Gamma(\frac{3}{2}+1)} \left(\frac{4^{\frac{1}{2}} - 2^{\frac{1}{2}}}{\frac{1}{2}} \right)^{\frac{3}{2}} \\ &\approx 0.6359551731 \\ &< 1, \end{aligned}$$

with $T = 4, a = 2$ and $\alpha = \frac{3}{2}$. Hence all conditions of Theorem 5.3.1 are satisfied, and F is compact, convex valued, and upper semi-continuous. It follows that the problem (5.15) admit a unique solution defined on I.

5.5 Existence of solutions in Banach Space

[2]

In this section we discuss the existence of solutions for a class of problem for nonlinear fractional differential inclusions involving both retarded and advanced arguments in Banach space given by:

$$ {}_c^{\rho}D_{a+}^{\alpha} y(t) \in F(t, y^t), \text{ for } t \in I := [a, T], \ 1 < \alpha \leq 2, \tag{5.16}$$

$$ y(t) = \phi(t), \qquad t \in [a-r, a], \ r > 0 \tag{5.17}$$

$$ y(t) = \psi(t), \qquad t \in [T, T+\beta], \ \beta > 0, \tag{5.18}$$

where ${}_c^{\rho}D_{a+}^{\alpha}$ is the Caputo type modification of the Erdélyi-Kober fractional derivative, $F : I \times C([-r, \beta], E) \to \mathcal{P}(E)$ is a given function, $\phi \in C([a-r, a], E)$ with $\phi(a) = 0$ and $\psi \in C([T, T+\beta], E)$ with $\psi(T) = 0$. We denote by y^t the element of $C([-r, \beta])$ defined by:

$$ y^t(s) = y(t+s) : s \in [-r, \beta]. $$

Definition 5.5.1 A function $y \in \mathcal{C}$, is said to be a solution of (5.16)-(5.18) if there exist a function $v \in L^1(I)$ with $v(t) \in F(t, y^t)$, for a.e. $t \in I$, such that ${}_c^{\rho}D_{a+}^{\nu} y(t) = v(t)$ and the conditions $y(t) = \phi(t), \phi(a) = 0$ on $[a-r, a]$ and $y(t) = \psi(t), \psi(T) = 0$ on $[T, T+\beta]$. is satisfied.

To prove the existence of solutions to (5.16)–(5.18), we need the following auxiliary Lemma.

Lemma 5.5.1 *Let $F : I \times C[-r, \beta] \longrightarrow \mathcal{P}(E)$ be a Carathéodory. A function $y \in \mathcal{C}$ is a solution for the inclusion problem* $(5.16) - (5.18)$ *if and only if y satisfies the following integral equation*

$$ y(t) = \begin{cases} \phi(t), & \textit{if} \quad t \in [a-r, a], \\ \\ -\displaystyle\int_a^T G(t,s)h(s)ds, & \textit{if} \quad t \in I \\ \\ \psi(t), & \textit{if} \quad t \in [T, T+\beta], \end{cases} $$

[2]M. Boumaaza, M. Benchohra, and C. Tunç, Erdélyi-Kober fractional differential inclusions in Banach spaces with retarded and advanced arguments, *Discus. Mathem. Differ. Incl., Contr. Optim.* **40** (2020), 75-92.

where $h \in L^1(I)$ *with*

$$h(t) \in F(t, y^t) \quad a.e. \quad t \in I.$$

and

$$G(t,s) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \begin{cases} \dfrac{(t^\rho - a^\rho)(T^\rho - s^\rho)^{\alpha-1} s^{\rho-1}}{(T^\rho - a^\rho)} - s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1}, & a \le s \le t \le T, \\ \dfrac{(t^\rho - a^\rho)(T^\rho - s^\rho)^{\alpha-1} s^{\rho-1}}{(T^\rho - a^\rho)}, & a \le t \le s \le T. \end{cases} \tag{5.19}$$

Set

$$\widetilde{G} = \sup\left\{ \int_a^T |G(t,s)| ds, t \in I \right\}.$$

The following hypotheses will be used in the sequel:

(H_1) The multivalued map $F : I \times C([-r,\beta]) \to \mathcal{P}_{cp,c}(E)$ is Carathéodory,

(H_2) There exist $p \in L^\infty(I, \mathbb{R}_+)$, such that

$$\|F(t,u)\|_{\mathcal{P}} = \sup\{\|v\|_C : v(t) \in F(t,u)\} \le p(t)(\|v\|_{[-r,\beta]} + 1),$$

for a.e. $t \in I$, and each $u \in C([-r,\beta])$,

(H_3) For each bounded set $B \subset C([-r,\beta])$ and for each $t \in I$, we have

$$\mu(F(t, B(t)) \le p(t) \sup_{s \in [-r,\beta]} \mu(B(s)),$$

where $B(t) = \{u(t) : u \in B\}$,

(H_4) The function $\Phi \equiv 0$ is the unique solution in $C(I)$ of the inequality

$$\Phi(t) \le 2p^* \int_a^T G(t,s)\Phi(s) ds,$$

where

$$p^* = ess \sup_{t \in I} p(t)$$

Theorem 5.5.1 *Assume that* $(H_1) - (H_3)$ *hold. If*

$$L := 2p^* \widetilde{G} < 1, \tag{5.20}$$

then the problem (5.16)-(5.18) has at least one solution.

Proof: Let the operator $N : \mathcal{C} \longmapsto \mathcal{P}(\mathcal{C})$ defined by

$$(Nu)(t) = \left\{ \begin{array}{c} h : I \longrightarrow \mathcal{C} : \\ \\ h(t) = \begin{cases} \phi(t), & \text{if} \quad t \in [a-r, a], \\ \\ -\displaystyle\int_a^T G(t,s)v(s)ds, & \text{if} \quad t \in I \\ \\ \psi(t), & \text{if} \quad t \in [T, T+\beta], \end{cases} \end{array} \right\} \tag{5.21}$$

where

$$v \in S_{F\circ u} = \{v : \Omega \longrightarrow L^1(I) : v(t) \in F(t, u^t) \text{ a.e. } t \in I\}.$$

Let the constant R be such that:

$$R \geq \max\left\{p^*(R+1))\widetilde{G}, \|\phi\|_{[a-r,a]}, \|\psi\|_{[T,T+\beta]}\right\}, \tag{5.22}$$

and define

$$D_R = \{u \in \mathcal{C} : \|u\|_{\mathcal{C}} \leq R\}.$$

It is clear that D_R is a bounded, closed and convex subset of $\mathcal{C}$.

Step 1. $N(D_R) \subset D_R$.

Let $u \in D_R$ Then for each $h \in N(u)$, there exists $v \in S_{F\circ u}$ such that

$$h(t) = \int_a^T G(t,s)v(s)ds; \ i = 1, 2.$$

If $t \in [a-r, a]$, then

$$\|h(t)\| \leq \|\phi\|_{[a-r,a]} \leq R,$$

and if $t \in [T, T+\beta]$, then

$$\|h(t)\| \leq \|\psi\|_{[T,T+\beta]} \leq R.$$

For each $t \in I$, we have

$$\|h(t)\| \ \leq \ \int_a^T |G(t,s)||v(s)|ds.$$

By (H_2), we have

$$\begin{aligned} \|h(t)\| \ &\leq \ \int_a^T |G(t,s)|p(t)(\|u\|_{[-r,\beta]} + 1)ds \\ &\leq \ p^*(R+1)\int_a^T |G(t,s)|ds \\ &\leq \ p^*(R+1))\widetilde{G} \\ &\leq \ R, \end{aligned}$$

from which it follows that for each $t \in [a-r, T+\beta]$, we have $\|h(t)\| \leq R$, which implies that $\|h\|_{\mathcal{C}} \leq R$, and so $N(D_R) \subset D_R$.

Step 2. $N(\cdot)$ is upper semicontinuous.
$N(u) \in \mathcal{P}_{cl}(D_R)$ *for each* $u \in D_R$. Let $\{u_n\}_{n\geq 0} \in N(u)$ such that $u_n \longrightarrow \tilde{u}$ in $\mathcal{C}$. Then, $\tilde{u} \in D_R$ and there exists $g_n \in S_{F\circ u}$ be such that, for each $t \in I$, we have

$$u_n(t) = \int_a^T G(t,s)g_n(s)ds.$$

Using (H_1) together with the fact that F has compact values, we may pass to a subsequence to see that g_n converges to g in $L^1(I)$, and hence $g \in S_{F\circ u}$. Then, for each $t \in I$, we get

$$u_n(t) \longrightarrow \tilde{u}(t) = \int_a^T G(t,s)g(s)ds.$$

So, $\tilde{u} \in N(u)$.
Step 3. *N satisfies the Darbo condition.*
The operator $N : D_R \longmapsto D_R$ is a L- set contraction.
Let $U \subset D_R$ if t $[a-r, a]$, then

$$\begin{aligned}\mu(N(U)(t)) &= \mu(N(y)(t), y \in U)\\ &= \alpha(\phi(t))\\ &= 0,\end{aligned}$$

also if t $[T, T+\beta]$, then

$$\begin{aligned}\mu(N(U)(t)) &= \mu(N(y)(t), y \in U)\\ &= \mu(\psi(t))\\ &= 0.\end{aligned}$$

If $t \in I$, we have

$$\mu((NU)(t)) = \mu(\{(Nu)(t) : u \in U\}).$$

For each $h \in N(u)$, there exists $f \in S_{F\circ u}$ such that

$$h(t) = \int_a^T G(t,s)f(s)ds.$$

By Theorem 1.4.1 and the fact that $U \subset D_R \subset \mathcal{C}$, we obtain

$$\mu((NU)(t)) \leq 2\int_a^T \mu\left(\{G(t,s)f(s) : u \in U\}\right)ds.$$

Now, since $f \in S_{F\circ u}$ and $u(s) \in U(s)$, we have

$$\mu(\{G(t,s)f(s)\}) = |G(t,s)|p(s)\mu(U(s)).$$

It follows that

$$\mu((NU)(t)) \leq 2 \int_a^T \mu\left(\{G(t,s)f(s)\}\right) ds.$$

Thus

$$\mu((NU)(t)) \leq 2p^* \int_a^T G(t,s)\mu(U(s))ds.$$

Hence

$$\mu((NU)(t)) \leq 2p^* \widetilde{G} \mu(U).$$

Therefore,

$$\mu(N(U)) \leq L\mu(U),$$

So by (5.20) the operator N is a $L-$ set contraction. and thus, by Theorem 1.6.6 , N has a fixed point, which is solution to problem $(5.16)-(5.18)$.

We now prove an another existence result for (5.16)-(5.18) by using Mönch's fixed point theorem.

Theorem 5.5.2 *Assume that (H_1)-(H_4) hold. If*

$$p^* \widetilde{G} < 1, \tag{5.23}$$

then problem (5.16)-(5.18) has at least one solution.

Proof. We shall show in five steps that the multivalued operator N satisfies all assumptions of Theorem 1.6.7.

Step 1. *$N(u)$ is convex for each $u \in \mathcal{C}$.*
If h_1, h_2 belong to $N(u)$, then there exist $v_1, v_2 \in S_{F\circ u}$ such that for each $t \in I$ we have

$$h_i(t) = \int_a^T G(t,s)v_i(s)ds;\ i = 1, 2.$$

Let $0 \leq \lambda \leq 1$. For each $t \in I$, we have

$$(\lambda h_1 + (1-\lambda)h_2)(t) = \int_a^T G(t,s)(\lambda v_1(s) + (1-\lambda)v_2(s))ds.$$

Since $S_{F\circ u}$ is convex (because F has convex values), we have $\lambda h_1 + (1-\lambda)h_2 \in N(u)$.

Step 2. *For each compact $M \subset \mathcal{C}$, $N(M)$ is relatively compact.*
let $M \subset \mathcal{C}$ be a compact set and let (h_n) be any sequence of elements of $N(M)$. We show that (h_n) has a convergent subsequence by using the Arzéla-Ascoli criterion of non-compactness in $\mathcal{C}$. Since $(h_n) \in N(M)$ there exist $(u_n) \in M$ and $v_n \in S_{F\circ u_n}$, such that

$$h_n(t) = \int_a^T G(t,s)v_n(s)ds.$$

Using Theorem 1.4.1 and the properties of the Kuratowski measure of noncompactness, we have

$$\mu(\{h_n(t)\}) \leq 2\int_a^T \mu\left(\{G(t,s)v_n(s)\}\right) ds. \tag{5.24}$$

On the other hand, since M is compact, the set $\{v_n(s) : n \geq 1\}$ is compact. Consequently, $\mu(\{v_n(s) : n \geq 1\}) = 0$ for a.e. $s \in I$. Furthermore

$$\mu(\{G(t,s)v_n(s)\}) = |G(t,s)|\mu(\{v_n(s) : n \geq 1\}) = 0.$$

for a.e. $t, s \in I$. Now (5.24) implies that $\{h_n(t) : n \geq 1\}$ is relatively compact for each $t \in I$. In addition, for each t_1 and t_2 from I, with $t_1 < t_2$, we have

$$\begin{aligned} \|h(t_2) - h(t_1)\| &\leq \int_a^T |G(t_2,s) - G(t_1,s)|\|v(s)\|ds \\ &\leq p^*(R+1)\int_a^T |G(t_2,s) - G(t_1,s)|ds. \end{aligned} \tag{5.25}$$

As $t_1 \longrightarrow t_2$ the right hand side of the above inequality tends to zero. This shows that $\{h_n : n \geq 1\}$ is equicontinuous. Consequently, $\{h_n : n \geq 1\}$ is relatively compact in $\mathcal{C}$.

Step 3. *The graph of N is closed.*

Let $(u_n, h_n) \in \text{graph}(N)$, $n \geq 1$, with $(\|u_n - u\|, \|h_n - h\|) \to (0.0)$, as $n \to \infty$. We have to show that $(u, h) \in \text{graph}(N)$. $(u_n, h_n) \in \text{graph}(N)$ means that $h_n \in N(u_n)$, which implies that there exists $v_n \in S_{F\circ u_n}$, such that for each $t \in I$,

$$h_n(t) = \int_a^T G(t,s)v_n(s)ds.$$

Consider the continuous linear operator $\Theta : L^1(I) \to \mathcal{C}$,

$$\Theta(v)(t) \mapsto h_n(t) = \int_a^T G(t,s)v_n(s)ds.$$

Clearly, $\|h_n(t) - h(t)\| \to 0$ as $n \to \infty$. From Lemma 1.4.1 it follows that $\Theta \circ S_F$ is a closed graph operator. Moreover, $h_n(t) \in \Theta(S_{F\circ u_n})$. Since $u_n \to u$, Lemma 1.4.1 implies

$$h(t) = \int_a^T G(t,s)v(s)ds,$$

for some $v \in S_{F\circ u}$.

Step 4. *M is relatively compact in $\mathcal{C}$.*

Suppose $M \subset \overline{U}$, $M \subset \text{conv}(\{0\} \cup N(M))$, and $\overline{M} = \overline{C}$ for some countable set $C \subset M$. Using an estimation of type (5.25), we see that $N(M)$ is equicontinuous. Then from $M \subset \text{conv}(\{0\} \cup N(M))$, we deduce that M is equicontinuous, too. To apply the

Arzéla-Ascoli theorem, it remains to show that $M(t)$ is relatively compact in E for each $t \in I$. Since $C \subset M \subset \operatorname{conv}(\{0\} \cup N(M))$, and C is countable, we can find a countable set $H = \{h_n : n \geq 1\} \subset N(M)$ with $C \subset \operatorname{conv}(\{0\} \cup H)$. Then, there exist $u_n \in M$ and $v_n \in S_{F \circ u_n}$ such that

$$h_n(t) = \int_a^T G(t,s) v_n(s) ds.$$

Taking into account Theorem 1.4.1 and the fact that $M \subset \overline{C} \subset \overline{conv}(\{0\} \cup H))$, we obtain

$$\mu(M(t)) \leq \mu(\overline{C}(t)) \leq \mu(H(t)) = \mu(\{h_n(t)\ : n \geq 1\}).$$

Using (5.24), we obtain

$$\mu(M(t)) \leq 2 \int_a^T \mu\left(\{G(t,s)v_n(s)\}\right) ds.$$

Now, since $v_n \in S_{F \circ u_n}$ and $u_n(s) \in M(s)$, we have

$$\mu(M(t)) \leq 2 \int_a^T \mu\left(\{G(t,s)v_n(s) : n \geq 1\}\right) ds.$$

Also, since $v_n \in S_{F \circ u_n}$ and $u_n(s) \in M(s)$, then from (H_3) we have

$$\mu(\{G(t,s)v_n(s);\ n \geq 1\}) = G(t,s)p(s)\mu(M(s)).$$

It follows that

$$\mu(M(t)) \leq 2p^* \int_a^T G(t,s)\mu(M(s))ds.$$

Consequently, by (H_4), the function Φ given by $\Phi(t) = \mu(M(t))$ satisfies $\Phi \equiv 0$, that is, $\mu(M(t)) = 0$ for all $t \in I$. Now, by the Arzéla-Ascoli theorem, M is relatively compact in $\mathcal{C}$.

Step 5: A priori bounds on solutions.
Let $u \in \mathcal{C}$ be such that $u \in \lambda N(u)$ for all $\lambda \in (0,1)$. Then, there exists $v \in S_{F \circ u}$ such that for each $t \in I$, we have

$$u(t) = -\lambda \int_a^T G(t,s)v(s)ds. \tag{5.26}$$

This implies, by (5.26) that for each $t \in I$ we have

$$\begin{aligned} \|u(t)\| &\leq \int_a^T |G(t,s)| p(t)(\|u\|_{[-\alpha,\beta]} + 1) ds \\ &\leq \widetilde{G} p^*(\|u\|_{[-\alpha,\beta]} + 1), \end{aligned}$$

then

$$\|u(t)\|_{[-\alpha,\beta]} \ \leq \ \widetilde{G}p^*(\|u\|_{[-\alpha,\beta]}+1).$$

Thus by (5.23), we have

$$\|u\|_{\mathcal{C}} \ \leq \ \frac{p^*\widetilde{G}}{1-p^*\widetilde{G}} = d.$$

Set

$$U = \{u \in \mathcal{C} : \|u\|_{\mathcal{C}} < 1+d\}.$$

Condition (1.6) is satisfied by our choice of the open set U. From the above steps and Theorem 1.6.7, we deduce that N has a fixed point u in U which is a solution of (5.16)−(5.18).

5.6 An Example

Let

$$E = l^1 = \left\{ u = (u_1, u_2, \ldots, u_n, \ldots), \sum_{n=1}^{\infty} |u_n| < \infty \right\}$$

be the Banach space with the norm

$$\|u\|_E = \sum_{n=1}^{\infty} |u_n|.$$

We consider the problem

$$\begin{cases} u(t) = e^{t-2} - 1, & t \in [1,2], \\ \frac{\frac{1}{2}}{c} D_{2^+}^{\frac{3}{2}} u(t) \in F(t, u^t), & t \in [2,4] \\ u(t) = t - 4, & t \in [4,6]. \end{cases} \tag{5.27}$$

Set

$$F_n(t,u^t) = \left\{ v_n \in E : 0 \leq v_n \leq \frac{1}{e^t+1}(\|u_n\|_{[-r,\beta]}+1) \right\}, \quad t \in [2,4], u \in C([-r,\beta])$$

Set

$$u = (u_1, u_2, \ldots, u_n, \ldots), \quad F = (F_1, F_2, \ldots, F_n, \ldots).$$

and

$$\alpha = \frac{3}{2},\ \rho = \frac{1}{2},\ r = 1,\ \beta = 2.$$

For each $u \in C([-r,\beta])$, $t \in [2,4]$, we have

$$\|F(t,u^t)\|_{\mathcal{P}} \ \leq \ \frac{1}{e^t+1}\left(\|u\|_{[-r,\beta]}+1\right).$$

Therefore, (H_2) is verified with $p^* = \frac{1}{e^2+1}$.
For each $t \in I$ we have

$$\int_a^T |G(t,s)|ds \leq \frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{T^\rho - a^\rho}\right)\int_a^T \left|\left(\frac{T^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\right| ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_a^t \left|\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\right| ds.$$

Then

$$\int_a^T |G(t,s)|ds \leq \frac{2}{\Gamma(\alpha+1)}\left(\frac{T^\rho - a^\rho}{\rho}\right)^\alpha.$$

Therefore

$$\widetilde{G} \leq \frac{2}{\Gamma(\alpha+1)}\left(\frac{T^\rho - a^\rho}{\rho}\right)^\alpha.$$

The condition (5.23)is satisfied. Indeed, we have

$$\begin{aligned}\widetilde{G}p^* \ &\leq \ \frac{2}{(e^2+1)\Gamma(\frac{3}{2}+1)}\left(\frac{4^{\frac{1}{2}} - 2^{\frac{1}{2}}}{\frac{1}{2}}\right)^{\frac{3}{2}}\\ &\approx \ 0.2274231447\\ &< \ 1,\end{aligned}$$

with $T = 4, a = 2$ and $\alpha = \frac{3}{2}$. Hence from Theorem 5.5.2 the problem (5.27) admits at least one solution.

Chapter 6

Weak Solutions of Coupled Systems Fractional Differential Inclusions

[1]

6.1 Introduction

This chapter generalizes the previous chapter, so that we study the existence of weak solutions to the coupled system fractional differential inclusions (CSFDI for short) involving both retarded and advanced arguments in Banach space see [46].

$$\begin{cases} ({}^{\rho}_{c}D^{\alpha_1}_{a^+}u)(t) \in F_1(t, u^t, v^t) \\ ({}^{\rho}_{c}D^{\alpha_2}_{a^+}v)(t) \in F_2(t, u^t, v^t) \end{cases} ;\ t \in I := [a, T], \tag{6.1}$$

$$\begin{cases} (u(t), v(t)) = (\phi_1(t), \phi_2(t)),\ t \in [a - r, a],\ r > 0 \\ (u(t), v(t)) = (\psi_1(t), \psi_2(t)),\ t \in [T, T + \beta],\ \beta > 0, \end{cases} \tag{6.2}$$

where $\beta > 0, \alpha_i \in (1, 2]$ $(E, \|\cdot\|)$ is a real Banach space and ${}^{\rho}_{c}D^{\alpha}_{a^+}$ is the Caputo type modification of the Erdélyi-Kober fractional derivative, $F : I \times C([-r, \beta], E) \times C([-r, \beta], E) \to \mathcal{P}(E)$ is a given function, $\phi_i \in C([a - r, a], E)$ with $\phi_i(a) = 0$ and $\psi_i \in C([T, T + \beta], E)$ with $\psi_i(T) = 0, i = 1, 2$. We denote by y^t the element of $C([-r, \beta])$ defined by:

$$y^t(s) = y(t + s) : s \in [-r, \beta].$$

[1] M. Boumaaza, S. Abbas, M. Benchohra, Weak solutions of coupled systems Caputo type modification of the Erdélyi-Kober fractional differential inclusions with retarded and advanced arguments, *Abst. Anal. Optim*, (accepted).

6.2 Existence of Weak Solutions

Lemma 6.2.1 *Let $1 < \alpha \leq 2$, $\phi \in C([a-r,a],\mathbb{R})$ with $\phi(a) = 0$, $\psi \in C([T,T+\beta],\mathbb{R})$ with $\psi(T) = 0$ and $h : I \to \mathbb{R}$ be a integrable function. Then the linear problem*

$$ {}^{\rho}_{c}D^{\alpha}_{a+}y(t) = h(t), \text{ for a.e. } t \in I := [a,T], \ 1 < \alpha \leq 2, \tag{6.3}$$

$$ y(t) = \phi(t), \qquad t \in [a-r,a], \ r > 0 \tag{6.4}$$

$$ y(t) = \psi(t), \qquad t \in [T,T+\beta], \ \beta > 0, \tag{6.5}$$

has a unique solution, which is given by

$$ y(t) = \begin{cases} \phi(t), & \text{if} \quad t \in [a-r,a], \\ \\ -\displaystyle\int_a^T G(t,s)h(s)ds, & \text{if} \quad t \in I \\ \\ \psi(t), & \text{if} \quad t \in [T,T+\beta], \end{cases} \tag{6.6}$$

where

$$ G(t,s) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \begin{cases} \dfrac{(t^\rho - a^\rho)(T^\rho - s^\rho)^{\alpha-1}s^{\rho-1}}{(T^\rho - a^\rho)} - s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1}, & a \leq s \leq t \leq T, \\ \\ \dfrac{(t^\rho - a^\rho)(T^\rho - s^\rho)^{\alpha-1}s^{\rho-1}}{(T^\rho - a^\rho)}, & a \leq t \leq s \leq T. \end{cases} \tag{6.7}$$

Here $G(t,s)$ is called the Green function of the boundary value problem (6.3)-(6.5).
Proof. *From* (1.4)*, we have*

$$ y(t) = c_0 + c_1\left(\frac{t^\rho - a^\rho}{\rho}\right) +^{\rho} I^{\alpha}_{a+}h(s), \quad c_0, c_1 \in \mathbb{R}, \tag{6.8}$$

therefore

$$ y(a) = c_0 = 0,$$

$$ y(T) = c_1\left(\frac{T^\rho - a^\rho}{\rho}\right) + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^T (T^\rho - s^\rho)^{\alpha-1}s^{\rho-1}h(s)ds,$$

and

$$ c_1 = -\frac{\rho^{2-\alpha}}{(T^\rho - a^\rho)\Gamma(\alpha)}\int_a^T (T^\rho - s^\rho)^{\alpha-1}s^{\rho-1}h(s)ds.$$

Substitute the value of c_0 and c_1 into equation (6.8), we get equation (6.6).

$$y(t) = \begin{cases} \phi(t), & \text{if} \quad t \in [a-r, a], \\ -\int_a^T G(t,s)h(s)ds, & \text{if} \quad t \in I \\ \psi(t), & \text{if} \quad t \in [T, T+\beta], \end{cases}$$

where G is defined by equation (6.7), the proof is complete.

Lemma 6.2.2 *Let $F_i : I \times C[-r, \beta] \times C[-r, \beta] \longrightarrow \mathcal{P}(E), i = 1, 2$ be such that $S_{F \circ u} \subset \mathcal{C}$ for any $u \in \mathcal{C}$ and $S_{F \circ v} \subset \mathcal{C}$ for any $v \in \mathcal{C}$ Then solving the system* (6.1) − (6.2) *is equivalent to the finding the solutions of the system of integral equations*

$$u(t) = \begin{cases} \phi_1(t), & \text{if} \quad t \in [a-r, a], \\ -\int_a^T G_{\alpha_1}(t,s)w_1(s)ds, & \text{if} \quad t \in I \\ \psi_1(t), & \text{if} \quad t \in [T, T+\beta], \end{cases}$$

and

$$v(t) = \begin{cases} \phi_2(t), & \text{if} \quad t \in [a-r, a], \\ -\int_a^T G_{\alpha_2}(t,s)w_2(s)ds, & \text{if} \quad t \in I \\ \psi_2(t), & \text{if} \quad t \in [T, T+\beta], \end{cases}$$

where

$$w_1 \in S_{F_1 \circ u}, w_2 \in S_{F_2 \circ v},$$

and

$$\widetilde{G_{\alpha_i}} = \sup \left\{ \int_a^T |G_{\alpha_i}(t,s)| ds, t \in I \right\}, i = 1, 2.$$

The following hypotheses will be used in the sequel:

(H_1) $F_1, F_2 : I \times C([-r, \beta]) \times C([-r, \beta]) \to \mathcal{P}_{cp,cl,cv}(E)$ have weakly sequentially closed graph;

(H_2) For all continuous functions $u, v : [-r, \beta] \to E$, there exist measurable functions $w \in S_{F_1 \circ u}$, $z \in S_{F_2 \circ v}$, a.e. on I and w, z are Pettis integrable on I;

(H_3) There exist $p_1, p_2 \in L^\infty(I, \mathbb{R}_+)$ such that for all $\varphi \in E^*$, we have

$$\|F_1(t,u,v)\|_{\mathcal{P}} \leq p_1(t), \text{ for a.e. } t \in I, \text{ and each } u, v \in C([-r,\beta]),$$
$$\|F_2(t,u,v)\|_{\mathcal{P}} \leq p_2(t), \text{ for a.e. } t \in I, \text{ and each } u, v \in C([-r,\beta]);$$

(H_4) For each bounded set $B_i \subset C([-r,\beta]), i = 1,2$ and for each $t \in I$, we have

$$\mu(F_1(t, B_1, B_2) \leq p_1(t) \sup_{s\in[-r,\beta]} \mu(B_1(s)),$$

and

$$\mu(F_2(t, B_1, B_2) \leq p_2(t) \sup_{s\in[-r,\beta]} \mu(B_2(s)).$$

where

$$B_i(t) = \{u(t) : u \in B_i\}, i = 1,2.$$

Set

$$p_i^* = ess \sup_{t\in I} p_i(t), i = 1,2.$$

Now, we state and prove our existence result for Equations (6.1)-(6.2) based on the Mönch's fixed point.

Theorem 6.2.1 *Assume that* (H_1)-(H_4) *hold. If*

$$\widetilde{G_{\alpha_1}} p_1^* + \widetilde{G_{\alpha_2}} p_2^* < 1, \tag{6.9}$$

then problem (6.1)-(6.2) has at least one solution.

Proof: Let the operator $N_i : \mathcal{C} \longmapsto \mathcal{P}(\mathcal{C}), i = 1,2$ defined by

$$(N_1u)(t) = \left\{ \begin{array}{c} h_1 : I \longrightarrow \mathcal{C} : \\ \\ h_1(t) = \begin{cases} \phi_1(t), & \text{if} \quad t \in [a-r, a], \\ \\ -\displaystyle\int_a^T G_{\alpha_1}(t,s) w_1(s) ds, & \text{if} \quad t \in I \\ \\ \psi_1(t), & \text{if} \quad t \in [T, T+\beta]. \end{cases} \end{array} \right\}, \tag{6.10}$$

and

$$(N_2v)(t) = \left\{ \begin{array}{c} h_2 : I \longrightarrow \mathcal{C} : \\ \\ h_2(t) = \begin{cases} \phi_2(t), & \text{if} \quad t \in [a-r, a], \\ \\ -\displaystyle\int_a^T G_{\alpha_2}(t,s) w_2(s) ds, & \text{if} \quad t \in I \\ \\ \psi_2(t), & \text{if} \quad t \in [T, T+\beta]. \end{cases} \end{array} \right\} \tag{6.11}$$

where

$$w_1 \in S_{F_1 \circ u} = \{u : \Omega \longrightarrow L^1(I) : w_1(t) \in F_1(t, u^t, v^t) \text{ a.e. } t \in I\},$$

and

$$w_2 \in S_{F_2 \circ v} = \{v : \Omega \longrightarrow L^1(I) : w_2(t) \in F_2(t, u^t, v^t) \text{ a.e. } t \in I\}.$$

Consider the multi-valued map $N : \overline{\mathcal{C}} \to \mathcal{P}(\overline{\mathcal{C}})$ defined by:

$$(N(u, v))(t) = ((N_1 u)(t), (N_2 v)(t)),$$

By Lemma 6.2.2 it is clear that the fixed points of N are solutions (6.1)-(6.2) .

Let the constant R be such that:

$$R \geq \max\left\{L_1 + L_2, \|\phi_1\|_{[a-r,a]} + \|\phi_2\|_{[a-r,a]}, \|\psi_1\|_{[T,T+\beta]} + \|\psi_2\|_{[T,T+\beta]}\right\}, \tag{6.12}$$

and define

$$Q = \left\{ (u, v) \in \mathcal{C} \times \mathcal{C} : \left\{ \begin{array}{c} \|(u, v)\|_{\overline{\mathcal{C}}} \leq R, \\ \\ \|u(t_2) - u(t_1)\|_E \leq p_1^* \int_a^T |G_{\alpha_1}(t_2, s) - G_{\alpha_1}(t_1, s)| ds, \\ \\ \|v(t_2) - v(t_1)\|_E \leq p_2^* \int_a^T |G_{\alpha_2}(t_2, s) - G_{\alpha_2}(t_1, s)| ds, \ t_1, t_2 \in I, \end{array} \right. \right\}$$

It is clear that Q is a bounded, closed and convex subset of $\mathcal{C}$.

Step 1. *$N(u, v)$ is convex for each $(u, v) \in \mathcal{C}$.*
If $(h_1, d_1), (h_2, d_2)$ belong to $N(u, v)$, then there exist $v_1, v_2 \in S_{F \circ u}$ and $z_1, z_2 \in S_{F \circ v}$ such that for each $t \in I$ we have

$$h_i(t) = \int_a^T G_{\alpha_1}(t, s) v_i(s) ds; \ i = 1, 2,$$

and

$$d_i(t) = \int_a^T G_{\alpha_2}(t, s) z_i(s) ds; \ i = 1, 2.$$

Let $0 \leq \lambda \leq 1$. For each $t \in I$, we have

$$(\lambda h_1 + (1 - \lambda) h_2)(t) = \int_a^T G_{\alpha_1}(t, s)(\lambda v_1(s) + (1 - \lambda) v_2(s)) ds.$$

Since $S_{F \circ u}$ is convex (because F has convex values), we have $\lambda h_1 + (1 - \lambda) h_2 \in N_1(u)$. Also, for each $t \in I$, we have

$$(\lambda d_1 + (1 - \lambda) d_2)(t) = \int_a^T G_{\alpha_2}(t, s)(\lambda z_1(s) + (1 - \lambda) z_2(s)) ds.$$

Since $S_{F\circ v}$ is convex (because F has convex values), we have $\lambda d_1 + (1-\lambda)d_2 \in N_2(v)$. Hence $\lambda(h_1,d_1) + (1-\lambda)(h_2,d_2) \in N(u,v)$.

Step 2. *N maps Q into itself.*

Let $h_i \in N_i(Q), i = 1,2$ then there exists $u, v \in Q$, such that $h_1 \in N_1(u), h_2 \in N_2(v)$ and there exists a Pettis integrable function $w_1 \in F_1 \circ u$ and $w_2 \in F_2 \circ v$, assume that $h_i(t) \neq 0$ Then there exists $\varphi \in E^*$ such that $\|h_i(t)\|_E = |\varphi(h_i(t))|$ Thus, for any $i \in \{1,2\}$ we have

$$\|h_i(t)\|_E = \varphi\left(\int_a^T G_{\alpha_i}(t,s)w_i(s)ds\right).$$

If $t \in [a-r, a]$, then

$$\|h(t)\|_E = \|(h_1(t), h_2(t))\|_E \leq \|\phi_1\|_{[a-r,a]} + \|\phi_2\|_{[a-r,a]} \leq R,$$

and if $t \in [T, T+\beta]$, then

$$\|h(t)\|_E = \|(h_1(t), h_2(t))\|_E \leq \|\psi_1\|_{[T,T+\beta]} + \|\psi_2\|_{[T,T+\beta]} \leq R.$$

For each $t \in I$, we have

$$\|h_i(t)\|_E \leq \int_a^T |G_{\alpha_i}(t,s)||\varphi(w_i(s))|ds, i = 1,2.$$

By (H_3), we get

$$|\varphi(h_i(t))| \leq p_i^*$$

Therefore

$$\begin{aligned}\|h_i(t)\|_E &\leq p_i^* \int_a^T |G_{\alpha_i}(t,s)|ds\\ &\leq p_i^* \widetilde{G_{\alpha_i}} = L_i\end{aligned}$$

which implies that $\|h_i(t)\|_E \leq L_i$,
hence we get

$$\begin{aligned}\|h(t)\|_E &\leq L_1 + L_2\\ &\leq R.\end{aligned}$$

Now, suppose that $h_1 \in N_1(u), h_2 \in N_2(v)$ and $t_1, t_2 \in I = [a,T]$ with $t_1 < t_2$ so that $(h_i(t_2) - (h_i(t_1) \neq 0, i = 1,2$ then, there exists $\varphi \in E^*$ such that

$$\|h_1(t_2) - h_1(t_1)\|_E = \varphi(h_1(t_2) - h_1(t_1)),$$

and $\|\varphi\| = 1$. Then, for any $i \in \{1, 2\}$, we get

$$\begin{aligned}
\|h_1(t_2) - h_1(t_1)\|_E &= \varphi(h_1(t_2) - h_1(t_1)) \\
&\leq \varphi\left(\int_a^T |G_{\alpha_1}(t_2, s) - G_{\alpha_1}(t_1, s)| w_1(s)\right).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\|h_1(t_2) - h_1(t_1)\|_E &\leq \int_a^T |G_{\alpha_1}(t_2, s) - G_{\alpha_1}(t_1, s)||\varphi(w_1(s))| ds \\
&\leq p_1^* \int_a^T |G_{\alpha_1}(t_2, s) - G_{\alpha_1}(t_1, s)| ds.
\end{aligned}$$

Similarly,

$$\|h_2(t_2) - h_2(t_1)\|_E \leq p_2^* \int_a^T |G_{\alpha_2}(t_2, s) - G_{\alpha_2}(t_1, s)| ds.$$

Consequently,

$$N(Q) \subset Q.$$

Step 3. *N has weakly-sequentially closed graph.*
Let $(u_n, w_n), (x_n, y_n)$ be a sequence in $Q \times Q$, with

$$\begin{cases} u_n(t) \to u(t) & \text{in } (E, \omega), \\ x_n(t) \to x(t) & \text{in } (E, \omega), \end{cases} \quad \text{for each } t \in I,$$

and

$$\begin{cases} w_n \in N_1(u_n), \\ y_n \in N_2(x_n). \end{cases} \quad \text{for } n \in \{1, 2, 3, \ldots\}, \tag{6.13}$$

We shall show that

$$\begin{cases} w \in N_1(u), \\ y \in N_2(x). \end{cases}$$

By (6.13) there exist $f_n \in S_{F_1 \circ u_n}$ and $g_n \in S_{F_2 \circ x_n}$ such that

$$\begin{cases} w_n = \int_a^T G_{\alpha_1}(t, s) f_n(s) ds, \\ y_n = \int_a^T G_{\alpha_2}(t, s) g_n(s) ds. \end{cases}$$

We must show that there exist $f \in S_{F_1 \circ u}$ and $g \in S_{F_2 \circ x}$ such that for each $t \in I$

$$\begin{cases} w = \int_a^T G_{\alpha_1}(t,s)f(s)ds, \\ y = \int_a^T G_{\alpha_2}(t,s)g(s)ds. \end{cases}$$

Since $F_i, i = 1,2$ has compact values (so weakly compact), there exist a Pettis integrable subsequence f_{n_m}, g_{n_m} such that

$$f_{n_m}(t) \in F_1(t, u_n^t, x_n^t) \text{ a.e. } t \in I,$$

$$f_{n_m}(\cdot) \to f(\cdot) \text{ in } (E,\omega) \text{ as } m \to \infty.$$

and

$$g_{n_m}(t) \in F_2(t, u_n^t, x_n^t) \text{ a.e. } t \in I,$$

$$g_{n_m}(\cdot) \to g(\cdot) \text{ in } (E,\omega) \text{ as } m \to \infty.$$

As $F_i(t,\cdot,\cdot), i = 1,2$ has weakly sequentially closed graph, $f(t) \in F_1(t, u^t, x^t)$ *and* $g(t) \in F_2(t, u^t, x^t)$. Then by the Lebesgue Dominated Convergence Theorem for the Pettis integral, we obtain

$$\varphi(w_n(t)) \to \varphi\left(\int_a^T G_{\alpha_1}(t,s)f_n(s)ds\right),$$

i.e., $w_n(t) \to (N_1 u)(t)$ in (E,ω) for each $t \in I$, which implies that $w \in N_1(u)$,
and

$$\varphi(y_n(t)) \to \varphi\left(\int_a^T G_{\alpha_2}(t,s)g_n(s)ds\right),$$

i.e., $y_n(t) \to (N_2 u)(t)$ in (E,ω) for each $t \in I$, which implies that $y \in N_2(x)$.

Step 4. Now let $V = V_1 \times V_2$ be a subset of Q such that $V = conv(N(V) \cup \{(0,0)\})$. Obviously

$$V(t) \subset conv(N(V)(t) \cup \{(0,0)\})$$

. Since V is bounded and equicontinuous, the function $t \longmapsto v(t) = \mu(V(t))$ is continuous on $[a-r, T+\beta]$. By $(H_1)-(H_4)$, Lemma 1.3.1, and the properties of measure μ, for each

$t \in I$, we have

$$\begin{aligned}
v(t) &\leq \mu(N(V)(t) \cup \{(0,0)\}) \\
&\leq \mu((NV)(t)) \\
&\leq \mu\left(\{((N_1u)(t),(N_2v)(t)) : (u,v) \in V\}\right) \\
&\leq \mu\Big\{\int_a^T |G_{\alpha_1}(t,s)|(d(s),0)ds \\
&+ \int_a^T |G_{\alpha_2}(t,s)|(0,z(s))ds \;\; d(t) \in F_1(t,u^t,v^t), z(t) \in F_2(t,u^t,v^t), (u,v) \in V\Big\} \\
&\leq \int_a^T |G_{\alpha_1}(t,s)| \left(p_1(s)\mu(\{(d(s),0), d(t) \in F_1(t,u^t,v^t), (u,v) \in V\}ds\right) \\
&+ \int_a^T |G_{\alpha_2}(t,s)|(p_2(s)\mu(\{(0,z(s)); z(t) \in F_2(t,u^t,v^t), (u,v) \in V\}ds) \\
&\leq \int_a^T |G_{\alpha_1}(t,s)|p_1(s)\mu(V(s))ds \\
&+ \int_a^T |G_{\alpha_2}(t,s)|p_2(s)\mu(V(s))ds \\
&\leq \left(\widetilde{G_{\alpha_1}}p_1^* + \widetilde{G_{\alpha_2}}p_2^*\right)\|v\|_c.
\end{aligned}$$

Thus

$$\|v\|_c \leq \left(\widetilde{G_{\alpha_1}}p_1^* + \widetilde{G_{\alpha_2}}p_2^*\right)\|v\|_c.$$

From (6.9), we get $\|v\|_c = 0$, that is $\mu(V(t)) = 0$ for each $t \in I$.
For $t \in [a-r,a]$, we have

$$\begin{aligned}
v(t) &= \mu((\phi_1(t),\phi_2(t))) \\
&= 0.
\end{aligned}$$

Also for $t \in [T, T+\beta]$ we have

$$\begin{aligned}
v(t) &= \mu(\psi_1(t),\psi_2(t)) \\
&= 0,
\end{aligned}$$

then $V(t)$ is weakly relatively compact in E. In view of Ascoli-Arzela theorem , V is weakly relatively compact in $\overline{\mathcal{C}}$. Applying Theorem 1.6.5, we conclude that N has a fixed point that is a weak solution of the problem (6.1) $-$ (6.2).

6.3 An Example

Let

$$E = l^1 = \left\{u = (u_1,u_2,\ldots,u_n,\ldots), \sum_{n=1}^{\infty}|u_n| < \infty\right\}$$

be the Banach space with the norm

$$\|u\|_E = \sum_{n=1}^{\infty} |u_n|.$$

As an application of the main results, we consider the Coupled System Caputo type modification of the Erdélyi-Kober fractional differential inclusions with retarded and advanced arguments.

$$\begin{cases} (u(t), v(t)) = (e^t - 1, t^2) \quad t \in [-1, 0], \\ \\ {}_c^2 D_{0^+}^{\frac{3}{2}} u(t) \in F_n(t, u^t, v^t), \quad t \in I = [0, 1] \\ \\ {}_c^2 D_{0^+}^{\frac{4}{3}} v(t) \in G_n(t, u^t, v^t), \quad t \in I = [0, 1] \\ \\ (u(t), v(t)) = (t - 1, e^t - e) \quad t \in [1, 2]. \end{cases} \tag{6.14}$$

where

$$F_n(t, u^t, v^t) = \frac{e^{-3}}{1 + \|u\|_{C([-1,1])} + \|v\|_{C([-1,1])}} \left[u_n^t - 1; u_n^t\right] \quad t \in [0, 1], u, v \in C([-r, \beta]),$$

and

$$G_n(t, u^t, v^t) = \frac{e^{-t-6}}{1 + \|u\|_{C([-1,1])} + \|v\|_{C([-1,1])}} \left[v_n^t; v_n^t + 1\right] \quad t \in [0, 1], u, v \in C([-r, \beta]).$$

Set

$$u = (u_1, u_2, \ldots, u_n, \ldots), \quad F = (F_1, F_2, \ldots, F_n, \ldots)$$

and

$$v = (v_1, v_2, \ldots, v_n, \ldots), \quad G = (G_1, G_2, \ldots, G_n, \ldots),$$

with

$$\alpha_1 = \frac{3}{2}, \alpha_2 = \frac{4}{3}, \ \rho = 2, \ r = 1, \ \beta = 1.$$

For each $u, v \in C([-1, 1])$, $t \in [2, 4]$, we have

$$\|F(t, u^t, v^t)\|_{\mathcal{P}} \leq e^{-3},$$

and

$$\|G(t, u^t, v^t)\|_{\mathcal{P}} \leq e^{-t-6},$$

hence, (H_2) is verified with $p_1^* = e^{-3}, p_2^* = e^{-6}$.
For each $t \in I, i = 1, 2$ we have

$$\int_a^T |G_{\alpha_i}(t,s)|ds \leq \frac{1}{\Gamma(\alpha_i)} \left(\frac{t^\rho - a^\rho}{T^\rho - a^\rho}\right) \int_a^T \left| \left(\frac{T^\rho - s^\rho}{\rho}\right)^{\alpha_i - 1} s^{\rho-1} \right| ds$$

$$+ \frac{1}{\Gamma(\alpha_i)} \int_a^t \left| \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_i - 1} s^{\rho-1} \right| ds.$$

Then

$$\int_a^T |G_{\alpha_i}(t,s)|ds \leq \frac{2}{\Gamma(\alpha_i + 1)} \left(\frac{T^\rho - a^\rho}{\rho}\right)^{\alpha_i}.$$

Therefore

$$\widetilde{G_{\alpha_i}} \leq \frac{2}{\Gamma(\alpha_i + 1)} \left(\frac{T^\rho - a^\rho}{\rho}\right)^{\alpha_i}, i = 1, 2.$$

The condition (5.23)is satisfied. Indeed, we have

$$\begin{aligned} \widetilde{G_{\alpha_1}} p_1^* + \widetilde{G_{\alpha_2}} p_2^* &\leq \frac{2e^{-3}}{2^{\frac{3}{2}} \Gamma(\frac{3}{2} + 1)} + \frac{2e^{-6}}{2^{\frac{4}{3}} \Gamma(\frac{4}{3} + 1)} \\ &\approx 0.02813526732 \\ &< 1. \end{aligned}$$

with $T = 1, a = 0$ and $\alpha_1 = \frac{3}{2}, \alpha_2 = \frac{4}{3}$. Since all the conditions of Theorem 6.2.1 are satisfied, problem (6.14) has at least one solution defined on I.

Conclusion and Perspective

In this thesis, we have considered the following set of nonlinear fractional differential equations and inclusions with retarded and advanced arguments

(1) Fractional differential equations and inclusion

$$ {}^{\rho}_{c}D^{\alpha}_{a^+}u(t) = f(t, u^t, {}^{\rho}_{c}D^{\alpha}_{a^+}u(t)), \text{ for each } t \in I := [a, T], \ 1 < \alpha \leq 2, $$

$$ {}^{\rho}_{c}D^{\alpha}_{a^+}u(t) \in F(t, u^t), \qquad \text{for each } t \in I := [a, T], \ 1 < \alpha \leq 2. $$

(2) Coupled system nonlinear implicit and random coupled fractional differential systems problem

$$ \begin{cases} {}^{\rho}_{c}D^{\alpha}_{a^+}u(t) = f_1(t, u^t, v^t, {}^{\rho}_{c}D^{\alpha}_{a^+}u(t), {}^{\rho}_{c}D^{\alpha}_{a^+}v(t)) \\ {}^{\rho}_{c}D^{\alpha}_{a^+}v(t) = f_2(t, u^t, v^t, {}^{\rho}_{c}D^{\alpha}_{a^+}u(t), {}^{\rho}_{c}D^{\alpha}_{a^+}v(t)) \end{cases} \quad t \in I, \ 1 < \alpha \leq 2, $$

$$ \begin{cases} {}^{\rho}_{c}D^{\alpha_1}_{a^+}u(t, w) = f_1(t, u^t(w), v^t(w), {}^{\rho}_{c}D^{\alpha_1}_{a^+}u(t, w)) \\ {}^{\rho}_{c}D^{\alpha_2}_{a^+}v(t, w) = f_2(t, u^t(w), v^t(w), {}^{\rho}_{c}D^{\alpha_2}_{a^+}v(t, w)) \end{cases} \quad t \in I, \ 1 < \alpha \leq 2. $$

(3) Coupled systems fractional differential inclusions

$$ \begin{cases} {}^{\rho}_{c}D^{\alpha_1}_{a^+}u(t) \in F_1(t, u^t, v^t) \\ {}^{\rho}_{c}D^{\alpha_2}_{a^+}v(t) \in F_2(t, u^t, v^t) \end{cases} \qquad t \in I := [a, T], \ 1 < \alpha \leq 2, $$

Here ${}^{\rho}_{c}D^{\alpha}_{a^+}$ is the Caputo type modification of the Erdélyi-Kober fractional derivative. We established the existence, the uniqueness of the solution for fractional differential equations and inclusions, coupled system of differential equations and inclusions, weak solution for coupled systems of differential equations and inclusions and random solutions for coupled systems of differential equations. The results are based on the measure of noncompactness, suitable deterministe and radom fixed point theorems; in particular we have used the theorems of Schauder, Schaefer, the nonlinear alternative for Kakutani maps, Covitz-Nadler, Darbo, and Mönch.

In the futur, we plan to study the qualitative aspect of the solutions for the above mentioned problems, in particular, we will look for the stability and controllability of the above cited problems.

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