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THÈSE

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Stabilité des équations différentielles impulsives et inclusion différentielles

Option :

Equations différentielles ordinaires

Présentée par :

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Publications

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Introduction

Impulsive diiferential equations, that is, differential equations involving impulse effect, appear as a natural description of observed evolution phenomena of several real world problems. Many processes studied in applied sciences are represented by differential equations. However, the situation is quite different in many physical phenomena that have a sudden change in their states such as mechanical systems with impact, biological systems such as heart beats, blood flows, population dynamics, theoretical physics, radiophysics, pharmacokinetics, mathematical economy, chemical technology, electric technology, metallurgy, ecology, industrial robotics, biotechnology processes, chemistry, engineering, control theory, medicine and so on. Adequate mathematical models of such processes are systems of differential equations with impulses.
The theory of impulsive differential equations is a new and important branch of differential equations. The first paper in this theory is related to

Milman and Myshkis [44] and then followed by a period of active research which culminated with the monograph by Halanay and Wexler [33].

A comprehensive introduction to the basic theory is well developed in the monographs by Benchohra *et al.* [11], Graef *et al.* [31], Laskshmikantham *et al.* [42], and Samoilenko and Perestyuk [61].

This thesis is devoted to the existence and stability via fixed point theorem of solutions for different classes of initial values problems for impulsive differential equation and inclusions with fixed and variable moments. This thesis is arranged as follows:

- In Chapters 1, 2, we introduce definitions, lemmas, notions of semigroup and fixed point theorem which are used throughout this thesis.

- In Chapter 3, we consider the following impulsive delay equations.

$$\begin{cases} x'(t) &= -a(t)x(t-r), t \in J := [0,\infty),\ t \neq t_k,\ k = 1,\dots, \\ \Delta x_{t=t_k} &= I_k(x(t_k^-)), k = 1,\dots, \\ x(t) &= \psi(t), t \in [-r,0] \end{cases} \tag{0.0.1}$$

  where $a : [0,\infty) \to \mathbb{R}$ be bounded and continuous, r be a positive constant, $0 = t_0 < t_1 < \dots < t_k < \dots,\ \lim\limits_{k\to\infty} t_k = \infty,\ \lim\limits_{h\to 0} x(t_k+h) = x(t_k^+), \lim\limits_{h\to 0} x(t_k-h) = x(t_k^-)$ and $\Delta x_{t=t_k} = x(t_k^+) - x(t_k^-)$.

For any function x defined on $[-r,+\infty)$ and any $t \in J$, we denote by x_t the element of $C([-r,0],\mathbb{R})$ defined by.

$$x_t(\theta) = x(t+\theta), \theta \in [-r,0]$$

Here $x_t(.)$ represents the history of the state from time t-r, up to the present time t.
By Krasnoselskii fixed point we studied the asymptotic stability as zero solution of problem 0.0.1 is provided in the first section of chapter 3. In the least section we investigate the stability of zero solution for some class of impulsive perturbation problem with delay.

- In Chapter 4, we study the existence, uniqueness, continuous dependance on initial condition and boundedness of solution for a system of impulsive differential equation using the fixed point approach in vector Banach space. Also the compactness and u.s.c of operator solution are investigated, we consider the following system

$$\begin{cases} x'(t) &= f(t,x,y), t \in J := [0,\infty), t \neq t_k,\ k = 1,\dots, \\ y'(t) &= g(t,x,y), t \in J, \quad t \neq t_k,\ k = 1,\dots, \\ x(t_k^+) - x(t_k^-) &= I_k(x(t_k), y(t_k)), \quad k = 1,\dots, \\ y(t_k^+) - y(t_k^-) &= \overline{I}_k(x(t_k), y(t_k)), \quad k = 1,\dots, \\ x(0) &= x_0, \\ y(0) &= y_0, \end{cases} \tag{0.0.2}$$

where $x_0, y_0 \in \mathbb{R}$, $f, g : J\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are a given functions, $I_k, \overline{I}_k \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$. The notations $x(t_k^+) = \lim\limits_{h\to 0^+} x(t_k + h)$ and $x(t_k^-) = \lim\limits_{h\to 0^+} x(t_k - h)$ stand for the right and the left limits of the functions y at $t = t_k$, respectively.

- In Chapter 5, we establish the measurable Filippov's theorem and relaxation problem for the following system of differential inclusions with impulse effects.

$$x^{'}(t) \in F_1(t, x(t), y(t)), y^{'}(t) \in F_2(t, x(t), y(t)), a.e. t \in [0,b] \tag{0.0.3}$$

$$x(0) = x_0, y(0) = y_0. \tag{0.0.4}$$

where $F_i : [0,b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ are multifunctions and $x_0, y_0 \in \mathbb{R}^n$.

- In Chapter 6 we prove existence result based on a nonlinear alternative of Leray-Schauder type theorem in generalized Banach spaces.

$$\begin{cases} x'(t) &\in Ax(t) + F^1(t, x(t), y(t)), \quad t \in [0,b], t \neq t_k, \\ y'(t) &\in Ay(t) + F^2(t, x(t), y(t)), \quad t \in [0,b], t \neq t_k, \\ \Delta x(t) &\in I_k(x(t_k)), \quad t = t_k \quad k = 1, 2, \dots, m \\ \Delta y(t) &\in \overline{I}_k(y(t_k)), \\ x(0) &= x_0 \in E, \\ y(0) &= y_0 \in E, \end{cases} \tag{0.0.5}$$

where $J := [0, b]$, E is a real separable Banach space with inner product $\langle \cdot, \cdot \rangle$ induced by norm $\| \cdot \|$, $A : D(A) \subset E \longrightarrow E$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $(S(t))_{t \geq 0}$ in X and $F^1, F^2 : [0, b] \times E \times E \longrightarrow \mathcal{P}(E)$ are given set-valued functions, where $\mathcal{P}(E)$ denotes the family of nonempty subsets of X, $I_k : E \to \mathcal{P}(E)$, $(k = 1, 2, \ldots, m)$.

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Chapter 1

Preliminaries

In this chapter, we introduce notations, definitions, and preliminary facts from multi-valued analysis, which are used throughout this thesis. We denote by
$\mathcal{P}(X) = \{Y \subset \mathcal{P}\colon\ Y \neq \emptyset\}$;
$\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X)\colon\ Y \text{ closed}\}$;
$\mathcal{P}_{b}(X) = \{Y \in \mathcal{P}(X)\colon\ Y \text{ bounded}\}$;
$\mathcal{P}_{cv}(X) = \{Y \in \mathcal{P}(X)\colon\ Y \text{ convex}\}$;
$\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X)\colon\ Y \text{ compact}\}$;
$\mathcal{P}_{cv,cp}(X) = \mathcal{P}_{cv}(X) \cap \mathcal{P}_{cp}(X)$.

1.1 Multivalued Analysis

Let (X, d) and (Y, ρ) be two metric spaces and $F : X \to \mathcal{P}(Y)$ be a multi-valued mapping. The map F is called *upper semi-continuous* (*u.s.c.*) on X if for each $x_0 \in X$ the set $F(x_0)$ is a nonempty, closed subset of X, and if for each open set N of Y containing $F(x_0)$, there exists an open neighborhood M of x_0 such that $F(M) \subseteq Y$. That is, if the set $F^{-1}(V)$ is closed for any closed set V in Y. Equivalently, F is *u.s.c.* if the set $F^{+1}(V)$ is open for any open set V in Y.

The mapping F is said to be *completely continuous* if it is *u.s.c.* and, for every bounded subset $A \subseteq X$, $F(A)$ is relatively compact, i.e., there exists a relatively compact set $K = K(A) \subset X$ such that

$$F(A) = \bigcup\{F(x) : x \in A\} \subset K.$$

Also, F is *compact* if $F(X)$ is relatively compact, and it is called *locally compact* if for each $x \in X$, there exists an open set U containing x such that $F(U)$ is relatively compact.

A multivalued map $F : X \to \mathcal{P}(X)$ has convex (closed) values if $F(x)$ is convex (closed) for all $x \in X$. We say that F is bounded on bounded sets if $F(B)$ is bounded in X for each bounded set B of X, that is, $\sup_{x\in B}\{\sup\{\| y \|: y \in F(x)\}\} < \infty$.

Finally, we say that F has a fixed point if there exists $x \in X$ such that $x \in F(x)$.

A multivalued map $F : J \to \mathcal{P}_{cl}(X)$ is said to be measurable if for each $x \in E$, the function $Y : J \to X$ defined by

$$Y(t) = dist(x, F(t)) = \inf\{\| x - z \|: z \in F(t)\} \tag{1.1.1}$$

is Lebesgue measurable

Theorem 1.1.1. *[24] Let $F : X \to \mathcal{P}_{cp}(Y)$ be a closed locally compact multifunction. Then F is u.s.c.*

In what follows, by E we will denote a Banach space over the field of real numbers $\mathbb{R}$ and by J a closed interval in $\mathbb{R}$. We let

$$C(J, E) = \{Y : J \to E \ y \ is \ continuous\}.$$

$C([0, b], \mathbb{R}^n)$ is the Banach space of all continuous functions from $[0, b]$ into $\mathbb{R}^n$ with the norm

$$\|y\|_\infty = sup\{\|y(t)\| : 0 \le t \le b\}.$$

$L^1([0, b], \mathbb{R}^n)$ denotes the Banach space of measurable functions $y : [0, b] \to \mathbb{R}^n$ which are Lebesgue integrable and normed by

$$\|y\|_{L^1} = \int_0^b | y(t) | \, dt \text{ for all } y \in L^1([0, b], \mathbb{R}^n).$$

Definition 1.1.1. *The map $f : [0, b] \times B \to \mathbb{R}^n$ is said to be L^1-Carathéodory if*

(i) $t \mapsto f(t, x)$ is measurable for each $x \in B$;

(ii) $x \mapsto f(t, x)$ is continuous for almost all $t \in [0, b]$;

(iii) For each $q > 0$, there exists $h_q \in L^1([0, b], \mathbb{R}_+)$ such that

$$\|f(t, x)\| \le h_q(t) \text{ for all } \|x\|_B \le q \text{ and for almost all } t \in [0, b].$$

Theorem 1.1.2. *(Kuratowki, Ryll and Nardzewski) [39] Let E be a separable Banach space and let $F : J \to \mathcal{P}_{cl}(E)$ be a measurable map, then there exists a measurable map $f : J \to E$ such that $f(t) \in F(t)$, for every $t \in J$.*

Let $\mathcal{A}$ be a subset of $J \times B, \mathcal{A}$ *is* $\mathcal{L} \otimes \mathcal{B}$ measurable if $\mathcal{A}$ belongs to the $\sigma-$ algebra generated by all sets of the form $N \times D$, where N is Lebesgue measurable in J and D is Borel measurable in B. A subset $\mathcal{A}$ of $L^1(J, E)$ is decomposable if for all $u, v \in \mathcal{A}$ and $N \subset J$ measurable, the function $u_{\chi N} + v_{\chi J-N} \in \mathcal{A}$, where χ stands for the characteristic function.
Let X be a nonempty closed subset of E and $G : X \to \mathcal{P}(E)$ a multivalued operator with nonempty closed values. G is lower semicontinuous (l.s.c.) if the set $\{x \in X : G(X) \cap B \neq \emptyset\}$ is open for any open set B in E. The following two results are easily deduced from the limit properties.

Lemma 1.1.1. *(See e.g. [8], Theorem 1.4.13) If* $G : X \longrightarrow \mathcal{P}_{cp}(Y)$ *is u.s.c., then for any* $x_0 \in X$,

$$\limsup_{x \to x_0} G(x) = G(x_0).$$

Lemma 1.1.2. *(See e.g. [8], Lemma 1.1.9) Let* $(K_n)_{n\in\mathbb{N}} \subset K \subset X$ *be a sequence of subsets where* K *is compact in the separable Banach space* X*. Then*

$$\overline{co}\,(\limsup_{n\to\infty} K_n) = \bigcap_{N>0} \overline{co}\,(\bigcup_{n\geq N} K_n),$$

where $\overline{co}\,A$ *refers to the closure of the convex hull of* A.

Lemma 1.1.3. *[41]. Given a Banach space* X*, let* $F : [a, b] \times X \longrightarrow \mathcal{P}_{cp,cv}(X)$ *be an* L^1*-Carathéodory multi-valued map such that for each* $y \in C([a, b], X)$, $S_{F,y} \neq \emptyset$ *and let* Γ *be a linear continuous mapping from* $L^1([a, b], X)$ *into* $C([a, b], X)$*. Then the operator*

$$\begin{array}{rl} \Gamma \circ S_F : C([a,b],X) & \longrightarrow \mathcal{P}_{cp,cv}(C([a,b],X)), \\ y & \longmapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y}) \end{array}$$

has a closed graph in $C([a, b], X) \times C([a, b], X)$.

1.2 C_0-Semigroups

Let E be a Banach space and $B(E)$ be the Banach space of linear bounded operators.

Definition 1.2.1. *A semigroup of class* C_0 *is a one parameter family* $\{S(t) \mid t \geq 0\} \subset B(E)$ *satisfying the conditions:*

(i) $S(t) \circ S(s) = S(t + s)$*, for* $t, s \geq 0$,

(ii) $S(0) = I$,

(iii) the map $t \to S(t)(x)$ *is strongly continuous, for each* $x \in E$*, i.e;*

$$\lim_{t\to 0} S(t)x = x, \ \forall x \in E.$$

A semigroup of bounded linear operators $S(t)$, is uniformly continuous if

$$\lim_{t \to 0} \|S(t) - I\| = 0.$$

Here I denotes the identity operator in E.

We note that if a semigroup $S(t)$ is class (C_0) then satisfies the growth condition.

Proposition 1.2.1. *Let $\{S(t)\}_{t \geq 0}$ be a semigroup of bounded linear operator. Then there exists some constant $M \geq 0$ and $\omega \in \mathbb{R}$ such that*

$$\|S(t)\|_{B(E)} \leq M e^{\omega t}, \quad \text{for } \; t \geq 0.$$

If, in particular $M = 1$ and $\beta = 0$, i.e; $\|S(t)\|_{B(E)} \leq 1$, for $t \geq 0$, then the semigroup $S(t)$ is called a *contraction semigroup* (C_0).

Definition 1.2.2. *Let $S(t)$ be a semigroup of class (C_0) defined on E. The infinitesimal generator A of $S(t)$ is the linear operator defined by*

$$A(x) = \lim_{h \to 0} \frac{S((h)(x) - x)}{h}, \quad \text{for } x \in D(A),$$

where $D(A) = \{x \in E \mid \lim_{h \to 0} \frac{S(h)(x) - x}{h}$ exists in $E\}$.

Let us recall the following property:

Proposition 1.2.2. *The infinitesimal generator A is closed linear and densely defined operator in E. If $x \in D(A)$, then $S(t)(x)$ is a C^1-map and*

$$\frac{d}{dt} S(t)(x) = A(S(t)(x)) = S(t)(A(x)) \quad \text{on } [0, \infty).$$

Theorem 1.2.1. (Hille and Yosida) *[54]. Let A be a densely defined linear operator with domain and range in a Banach space E. Then A is the infinitesimal generator of uniquely determined semigroup $S(t)$ of class (C_0) satisfying*

$$\|S(t)\|_{B(E)} \leq M \exp(\omega t), \quad t \geq 0,$$

where $M > 0$ and $\omega \in R$ if and only if $(\lambda I - A)^{-1} \in B(E)$ and $\|(\lambda I - A)^{-n}\| \leq M/(\lambda - \omega)^n$, $n = 1, 2, \ldots$, for all $\lambda \in R$.

For more details on strongly operators, we refer the reader to the books of Goldstein [32], Pazy [54].

1.3 Analytic semigroups

Definition 1.3.1. *Let* $\Delta = \{z : \varphi_1 < argz << \varphi_2, \varphi_1 < 0 << \varphi_2\}$ *and for* $z \in \Delta$ *let* $S(z)$ *be a bounded linear operator. The family* $S(z)$*,* $z \in \Delta$ *is an analytic semigroup in* Δ *if*

(i) $z \to S(z)$ *is analytic in* Δ.

(ii) $S(0) = I$ *and* $\lim_{z \longrightarrow 0} S(z)x = x$ *for every* $x \in E$.

(iii) $S(z_1 + z_2) = S(z_1)S(z_2)$ *for* $z_1, z_2 \in \Delta$.

A semigroup $S(t)$ *will be called analytic if it is analytic in some sector* Δ *containing the nonnegative real axis.*

Clearly, the restriction of an analytic semigroup to the real axis is a C_0 semigroup. We will be interested below in the possibility of extending a given C_0 semigroup to an analytic semigroup in some sector Δ around the nonnegative real axis.

Theorem 1.3.1. *[54] Let* $S(t)$ *be a uniformly bounded* C_0 *semigroup. Let* A *be the infinitesimal generator of* $S(t)$ *and assume* $0 \in \rho(A)$*. The following statements are equivalent:*

(a) $S(t)$ *can be extended to an analytic semigroup in a sector* $\Delta_\delta = \{z : |argz| < \delta\}$ *and* $\|S(z)\|$ *is uniformly bounded in every closed subsector* $\Delta_{\delta'}$, $\delta^{'} < \delta$*, of* Δ_δ.

(b) There exists a constant C *such that for every* $\sigma > 0$, $\tau \neq 0$

$$\|R(\sigma + it : A)\| \leq \frac{C}{\tau}.$$

(c) There exist $0 < \delta < \pi/2$ *and* $M > 0$ *such that*

$$\rho(A) \supset \Sigma = \{\lambda : |arg\lambda| < \frac{\pi}{2} + \delta\} \cup \{0\}$$

and

$$\|R(\lambda : A)\| \leq \frac{M}{|\lambda|} \text{ for } \lambda \in \Sigma, \lambda \neq 0.$$

(d) $S(t)$ *is differentiable for* $t > 0$ *and there is a constant* C *such that*

$$\|AS(t)\| \leq \frac{C}{t}, \quad t > 0.$$

1.4 Fractional Powers of Closed Operators

For our definition we will make the following assumption.

(DDC) Let A be a densely defined closed linear operator for which

$$\rho(A) \supset \Sigma^+ = \{\lambda : 0 < \omega < |arg\lambda| \leq \pi\} \cup V$$

where V is a neighborhood of zero, and

$$\|R(\lambda : A)\| \leq \frac{M}{1+|\lambda|} \text{ for } \lambda \in \Sigma^+.$$

If $M = 1$ and $w = \frac{\pi}{2}$ then $-A$ is the infinitesimal generator of a C_0 semigroup. If $w < \frac{\pi}{2}$ then, by Theorem 1.3.1, $-A$ is the infinitesimal generator of an analytic semigroup. The assumption that $0 \in \rho(A)$ and therefore a whole neighborhood V of zero is in $\rho(A)$ was made mainly for convenience. Most of the results on fractional powers that we will obtain in this section remain true even if $0 \in \rho(A)$.

Definition 1.4.1. *Let A satisfy Assumption (DDC) with $w < \frac{\pi}{2}$. For every $\alpha > 0$ we define*

$$A^\alpha = (A^{-\alpha})^{-1}.$$

For $\alpha = 0$, $A^\alpha = I$.

Theorem 1.4.1. *[54] Let A^α be defined by Definition 1.4.1 then,*

(a) A^α is a dosed operator with domain $D(A^\alpha) = R(A^{-\alpha})$ = the range of $A^{-\alpha}$.

(b) $\alpha \geq \beta > 0$ implies $D(A^\alpha) \subset D(A^\beta)$.

(c) $\overline{D(A^\alpha)} = E$ for every $\alpha \geq 0$.

(d) If α, β are real then

$$A^{\alpha+\beta}x = A^\alpha \cdot A^\beta x$$

for every $x \in D(A^\gamma)$ where $\gamma = \max(\alpha, \beta, \alpha+\beta)$.

Theorem 1.4.2. *[54] Let $-A$ be the infinitesimal generator of an analytic semigroup $S(t)$. if $0 \in \rho(A)$ then,*

(a) $S(t) : E \to D(A^\alpha)$ for every $t > 0$ and $\alpha \geq 0$.

(b) For every $x \in D(A^\alpha)$ we have $S(t)A^\alpha x = A^\alpha S(t)x$.

(c) For every $t > 0$ the operator $A^\alpha S(t)$ is bounded and

$$\|A^\alpha S(t)\| \leq M_\alpha t^{-\alpha} e^{-\delta t}.$$

(d) Let $0 < \alpha \leq 1$ and $x \in D(A^\alpha)$ then

$$\|S(t)x - x\| \leq C_\alpha t^\alpha \|A^\alpha x\|.$$

1.5 Fixed point theorems

In this section we present some classical fixed point theorems.

Theorem 1.5.1. *(Schaefer's fixed point theorem) (see also [62], page 29). Let X be a Banach space and let $N : X \to X$ be a completely continuous map. If the set*

$$\Phi = \{x \in X : \lambda x = N(x) \text{ for some } \lambda > 1\}$$

is bounded, then N has a fixed point.

Theorem 1.5.2. *(nonlinear alternative [30]). let X be a a Banach space with $C \subset X$ closed and convex. Assume U is a relatively open subset of C with $0 \in U$ and $G : \overline{U} \to C$ is a compact map. Then either,*

(i) *G has a fixed point in $\overline{U}$; or*

(ii) *there is a point $u \in \partial U$ and $\lambda \in (0, 1)$ with $u = \lambda G(u)$.*

Before stating our next fixed point theorem, we need some preliminaries.
Let (X, d) be a metric space induced from the normed space$(X, | \, . \, |)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$, given by

$$H_d(\mathcal{A}, \mathcal{B}) = \max\{\sup_{a\in\mathcal{A}} d(a, \mathcal{B}), \sup_{b\in\mathcal{B}} d(\mathcal{A}, b)\}, \tag{1.5.1}$$

where $d(\mathcal{A}, b) = \inf_{a\in\mathcal{A}} d(a, b), d(a, \mathcal{B}) = \inf_{b\in\mathcal{B}} d(a, b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized (complete) metric space (see [39]).

Definition 1.5.1. *A multivalued operator $G : X \to \mathcal{P}_{cl}(X)$ is called*

(a) *$\gamma-$ Lipschitz if there exists $\gamma > 0$ such that*

$$H_d(G(x), G(y)) \le \gamma d(x, y), \text{ for each } x, y \in X; \tag{1.5.2}$$

(b) *a contraction if it is $\gamma-$ Lipschitz with $\gamma < 1$.*

The following is due to Schauder.

Theorem 1.5.3. *[50] Let B is a closed, convex and nonempty subset of a Banach space E. Let $N : B \to B$ be a continuous mapping such that $N(B)$ is a relatively compact subset of E. Then N has at least one fixed point in B. That is, there exists $y \in B$ such that $Ny = y$.*

Theorem 1.5.4. *(Krasnoselskii) [43] Let X be a Banach space. Suppose that A and B map X into X such that*

(i) A is completely continuous operator,

(ii) B is a contraction with constant $0 < \alpha < 1$.

(iii) the set

$$\mathcal{M} = \{x \in X : \ x = \lambda B\left(\frac{x}{\lambda}\right) + \lambda A(x), \lambda \in (0,1)\}$$

is bounded. Then there exists $x \in X$ *with* $Ax + Bx = x$.

For further readings and details on multi-valued analysis and fixed point theory, we refer the reader to the books by Aubin and Celina [6], Aubin and Frankowska [8], Deimling [24], Djebali *et al* [25], Dugundji and Granas, [27], Górniewicz [29], Hu and Papageorgiou [53], Kamenskii [38], Kisielewicz [39], Smirnov [63], and Tolstonogov [64].

Chapter 2

Vector metric spaces

The classical Banach contraction principle was extended for contractive maps on spaces endowed with vector-valued metric space by Perov in 1964 [57] and Perov and Kibenko [56]. Till now, there have been a number of attempts to generalize the Perov fixed point theorem in several directions and also have been a number of applications in various fields of nonlinear analysis, system of ordinary differential and semilinear differential equations.

2.1 Generalized metric space

In this section we define generalized metric space (or vector metric spaces) a prove some properties. If, $x, y \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n)$, by $x \leq y$ we mean $x_i \leq y_i$ for all $i = 1, \ldots, n$. Also $|x| = (|x_1|, \ldots, |x_n|)$ and $\max(x, y) = \max(\max(x_1, y_1), \ldots, \max(x_n, y_n))$. If $c \in \mathbb{R}$, then $x \leq c$ means $x_i \leq c$ for each $i = 1, \ldots, n$. For $x \in \mathbb{R}^n,\ (x)_i = x_i,\ i = 1, \ldots, n$.

Definition 2.1.1. *Let X be a nonempty set. By a generalized metric on X (or vector-valued metric) we mean a map $d : X \times X \to \mathbb{R}^n$ with he following properties:*

(i) $d(u, v) \geq 0$ *for all* $u, v \in X$*; if* $d(u, v) = 0$ *then* $u = v$

(ii) $d(u, v) = d(v, u)$ *for all* $u, v \in X$

(iii) $d(u, v) \leq d(u, w) + d(w, v)$ *for all* $u, v, w \in X$.

Note that for any $i \in \{1, \ldots, n\}$ $(d(u, v))_i = d_i(u, v)$ *is a metric space in* X.

We call the pair (X, d) generalized metric space. For $r = (r_1, r_2, \ldots, r_n) \in \mathbb{R}^n_+$,we will denote by

$$B(x_0, r) = \{x \in X : \ d(x_0, x) < r\}$$

the open ball centrad in x_0 with radius r and

$$\overline{B(x_0, r)} = \{x \in X : d(x_0, x) \leq r\}$$

the closed ball centered in x_0 with radius $r = (r_1, \dots, r_n) > 0,\ r_i > 0,\ i = 1; \dots, n$.

Definition 2.1.2. *Let (X, d) be a generalized metric space. A subset $A \subseteq X$ is called open if, for any $x_0 \in A$, there exists $r \in \mathbb{R}^n_+$ with $r > 0$ such that*

$$B(x_0, r) \subseteq A.$$

Any open ball is an open set and the collection of all open balls of X generates the generalized metric topology on X.

Definition 2.1.3. *Let (X, d) be a generalized metric space*

(a) A sequence (x_p) in X converge (or $\mathbb{R}^n_+$-converges) to some $x \in X$, if for every $\epsilon \in \mathbb{R}^n_+,\ \epsilon > 0$ there exists $p_0(\epsilon) \in \mathbb{N}$ such that for each

$$d(x_p, x) \leq \epsilon \quad \text{for all} \quad p \geq p_0(\epsilon).$$

(b) A sequence (x_p) is called Cauchy sequence if for every $\epsilon \in \mathbb{R}^n_+,\ \epsilon > 0$ there exists $p_0(\epsilon) \in \mathbb{N}$ such that for each

$$d(x_p, x_q) \leq \epsilon \quad \text{for all} \quad p, q \geq p_0(\epsilon).$$

(c) A generalized metric space X is called complete if each Cauchy sequence in X converges to a limit in$X$.

(d) A subset Y of a generalized metric space X is said to be closed whenever $(x_p) \subseteq Y$ and $x_p \to x$, as $p \to \infty$ imply $x \in Y$.

Using the above definitions, we have the following properties: If $x_p \to x$ as $p \to \infty$, then

(i) The limit x is unique.

(ii) Every subsequence of (x_p) converges to x.

(iii) If also $x_p \to x$ as $p \to \infty$, then

$$d(x_p, y_p) \to d(x, y) \text{ as } p \to \infty.$$

Theorem 2.1.1. *For the generalized metric space (X, d) the followings hold:*

(a) Every convergent sequence is an Cauchy sequence,

(b) Every Cauchy sequence is bounded,

(c) If an Cauchy sequence (x_p) *has a subsequence* (x_{p_k}) *such that*

$$x_{p_k} \to x \ \ as \ \ p_k \to \infty,$$

then

$$x_p \to x \ \ as \ \ p \to \infty.$$

Proof. (a) Let $(x_p)_{p\in\mathbb{N}}$ be a convergent sequence in X. The for every $\epsilon \in \mathbb{R}^n_+$ there exists $p_0(\epsilon) \in \mathbb{N}$ such that

$$d(x_p, x) \leq \frac{\epsilon}{2} \qquad \text{for all } p \geq p_0(\epsilon).$$

Then for every $p, q \geq p_0(\epsilon)$ we have

$$d(x_p, x_q) \leq d(x_p, x) + d(x_q, x) \Rightarrow d(x_p, x_q) \leq \epsilon.$$

Hence $(x_p)_{p\in\mathbb{N}}$ is an Cauchy sequence in X.

(b) Let $(x_p)_{p\in\mathbb{N}}$ be an Cauchy sequence. Fixe $\epsilon \in \mathbb{R}^n_+$ there exists $p_0(\epsilon) \in \mathbb{N}$ such that

$$d(x_p, x_q) \leq \epsilon, \quad \text{for all } p, q \geq p_0(\epsilon).$$

Hence for each $p \in \mathbb{N}$, we get

$$x_p \in B(x_{p_0(\epsilon)}, \epsilon + r), \quad r = \max_{1\leq i,j\leq p_0(\epsilon)-1} d(x_i, x_j),$$

this implies that $(x_p)_{p\in\mathbb{N}}$ bounded in X.
(c) Let $(x_p)_{p\in\mathbb{N}}$ be an Cauchy sequence and let $(x_{p_k})_{p_k\in\mathbb{N}}$ be a subsequence of $(x_p)_{p\in\mathbb{N}}$ such that $\lim_{p_k\to\infty} x_{p_k} = x$. The for every $\epsilon \in \mathbb{R}^n_+$ there exist $p_*(\epsilon), q_*(\epsilon) \in \mathbb{N}$ such that

$$d(x_p, x_q) \leq \frac{\epsilon}{2} \quad \text{for all} p, q \geq p_*(\epsilon)$$

and

$$d(x_{p_k}, x) \leq \frac{\epsilon}{2} \quad \text{for all} p_k \geq q_*(\epsilon)$$

Then

$$d(x_p, x) \leq d(x_p, x_{p_k}) + d(x_{p_k}, x) \leq \epsilon \quad \text{for all } \ p \geq \max(q_*(\epsilon), p_*(\epsilon)).$$

Hence

$$x_p \to x \text{ as } \ p \to \infty.$$

□

Definition 2.1.4. *Let* (X, d) *and* (Y, ρ) *be generalized metric spaces, and let* $x \in X$.

(a) A function $f : X \to Y$ is said to be continuous (or topologically continuous) at x if for every $\epsilon \in \mathbb{R}^n_+, \epsilon > 0$ there exists some $\delta(\epsilon) \in \mathbb{R}^n_+, \delta(\epsilon) > 0$ such that

$$\rho(f(x), f(y)) < \epsilon$$

whenever $x, y \in X$ and $d(x, y) < \delta(\epsilon)$.

The function f is said to be topologically continuous if it is topologically continuous at each point of X.

Definition 2.1.5. *Let (X, d) be an generalized metric space. We say that a subset $Y \subset X$ is closed if, $(x_p) \subset Y$ and $x_p \to x$ as $p \to \infty$ imply $x \in Y$.*

Definition 2.1.6. *Let (X, d) be a generalized metric space. A subset C of X is called compact if, every open cover of C has a finite subcover. A subset C of X is sequentially compact if, every sequence in C contains a convergent subsequence with limit in C.*

Definition 2.1.7. *A subset C of X is totally bounded if, for each $\epsilon\mathbb{R}^n_+$ with $\epsilon > 0$, there exists a finite number of elements $x_1, x_2, \ldots, x_p \in X$ such that*

$$C \subseteq \cup_{i=1}^{p} B(x_i, \epsilon).$$

The set $x_1, \ldots, x_p$is called a finite ϵ-net.

Theorem 2.1.2. *If C is a subset of X, then the following affirmations hold:*

i) C is compact if and only if, C is sequentially compact if and only if , C is closed and totally bounded;

ii) C relatively compact, if and only if, C sequentially relatively compact, if and only if, C totally bounded.

Definition 2.1.8. *Let (X, d) be an generalized metric space. If $A \subset X$ is a nonempty set, then the function*

$$\delta(A) = \sup\{d(x, y) : \ x; y \in A\}$$

is called the diameter of A. If $\delta(A) < \infty$, then A is called an bounded set.

Let (X, d) be a generalized metric space we define the following metric spaces: Let $X_i = X, \ i = 1, \ldots, n$. Consider $\prod_{i=1}^{n} X_i$ with $\bar{d}$:

$$\bar{d}((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sum_{i=1}^{n} d_i(x_i, y_i).$$

The diagonal space of $\prod_{i=1}^{n} X_i$ defined by

$$\widetilde{X} = \{(x, \ldots, x) \in \prod_{i=1}^{n} X_i : x \in X, \ i = 1, \ldots, n\}.$$

Thus is a metric space with the following distance

$$d_*((x,\dots,x),(y,\dots,y)) = \sum_{i=1}^{n} d_i(x,y), \text{ for each } x,y \in X.$$

It is clear that $\widetilde{X}$ is closed set in $\prod_{i=1}^{n} X_i$.

Intuitively, X and $\widetilde{X}$. This is showed in the following result.

Lemma 2.1.1. *Let (X,d) be a generalized metric space. Then there exists $h : X \to \widetilde{X}$ an homeomorphism map.*

Proof. Consider $h : X \to \widetilde{X}$ defined by

$$h(x) = (x,\dots,x) \quad \text{for all } x \in X.$$

Obviously h is bijective.

- To prove that h is a continuous map.

Let $x, y \in X$. Thus

$$d_*(h(x),h(y)) \leq \sum_{i=1}^{n} d_i(x,y).$$

For $\epsilon > 0$ we take $\delta = (\frac{\epsilon}{n},\dots,\frac{\epsilon}{n})$, let fixed $x_0 \in X$ and $B(x_0,\delta) = \{x \in X : d(x_0,x) < \delta\}$, then for every $x \in B(x_0,\delta)$ we have

$$d_*(h(x_0),h(x)) \leq \epsilon.$$

- Now, $h^{-1} : \widetilde{X} \to X$ is a continuous map defined by

$$h^{-1}(x,\dots,x) = x, \qquad (x,\dots,x) \in \widetilde{X}.$$

To show that h^{-1} is continuous. Let $(x,\dots,x),(y,\dots,y) \in \widetilde{X}$, then

$$d(h^{-1}(x,\dots,x),h^{-1}(y,\dots,y)) = d(x,y).$$

Let $\epsilon = (\epsilon_1,\dots,\epsilon_n) > 0$ we take $\delta = \frac{\min\limits_{1\leq i\leq n} \epsilon_i}{n}$ and we fix $(x_0,\dots,x_0) \in \widetilde{X}$. Set

$$B((x_0,\dots,x_0),\delta) = \{(x,\dots,x) \in \widetilde{X} : d_*((x_0,\dots,x_0),(x,\dots,x)) < \delta\}.$$

For $(x,\dots,x) \in B((x_0,\dots,x_0),\delta)$ we have

$$d_*((x_0,\dots,x_0),(x,\dots,x)) < \delta \Rightarrow \sum_{i=1}^{n} d_i(x_0,x) < \frac{\min\limits_{1\leq i\leq n} \epsilon_i}{n}.$$

Then

$$d_i(x_0,x) < \frac{\min\limits_{1\leq i\leq n} \epsilon_i}{n},\ i = 1,\dots,n \Rightarrow d(x_0,x) < \epsilon.$$

Hence h^{-1} is continuous. □

Theorem 2.1.3. *Every generalized metric space is paracompact.*

Proof. Let X be a generalized metric space, by Lemma 2.1.1 there exists $\widetilde{X}$ metric space homeomorphic to X. Since every metric space is paracompact hence X is paracompact. □

2.2 Matrix convergent

Definition 2.2.1. *A square matrix $M \in \mathcal{M}_{n\times n}(\mathbb{R})$ of real numbers is said to be convergent to zero if*

$$M^k \to 0, \qquad as\ k \to \infty.$$

Lemma 2.2.1. *[60] Let M be a square matrix of nonnegative numbers. The following assertions are equivalent:*

(i) M is convergent towards zero;

(ii) the matrix $I - M$ is non-singular and

$$(I - M)^{-1} = I + M + M^2 + \ldots + M^k + \ldots;$$

(iii) $|\lambda| < 1$ for every $\lambda \in \mathbb{C}$ with $det(M - \lambda I) = 0$

(iv) $(I - M)$ is non-singular and $(I - M)^{-1}$ has nonnegative elements;

Proof. Assume that M is converge to zero. We show that $I - M$ is non-singular it suffices to prove that the linear system

$$(I - M)x = 0 \tag{2.2.1}$$

has only the null solution. Let $x \in \mathbb{C}$ be a solution of the system (2.2.1), then

$$x = Mx = M^2x = \ldots M^kx = \ldots$$

and letting $k \to \infty$ we deduce $x = 0$. Hence $I - M$ is non-singular. Furthermore, we have

$$I - (I - M)(I + M + M^2 + \ldots M^k) = M^{k+1} \to 0 \quad \text{as} \quad k \to \infty.$$

This implies that

$$(I - M)^{-1} = I + M + M^2 + \ldots M^k \ldots.$$

□

Lemma 2.2.2. *A square matrix $M \in \mathcal{M}_{n\times n}(\mathbb{R})$ of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words ,this means that all the eigenvalues of M are in the open unit disc.*

Lemma 2.2.3. *Let* $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ *is convergent towards zero, then*

$$z \le (I-M)^{-1}z \text{ for every } z \in \mathbb{R}^n_+.$$

Proof. Since $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ is convergent towards zero, then from lemma 2.2.1, $(I-M)^{-1} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ and

$$(I-M)^{-1} = I + M + M^2 + \dots$$

Thus for every $z \in \mathbb{R}^n_+$ we have

$$(I-M)^{-1}z = \sum_{i=0}^{\infty} M^i z \Rightarrow z \le (I-M)^{-1}z.$$

□

Lemma 2.2.4. *Let* $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ *is convergent towards zero, then*

$$P_M = \{z \in \mathbb{R}^n_+ : (I-M)z > 0\},$$

is nonempty and coincides with the set

$$\{(I-M)^{-1}z_0 : z_0 \in \mathbb{R}^n,\ z_0 > 0\}.$$

Proof. Clear that $I - M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ and singular matrix, then for every $z \in \mathbb{R}^n_+, z = (z_1, \dots, z_n)$ with $z_i > 0,\ i = 1, \dots, n$, we get $(I-M)z > 0$. This implies that $P_m \neq \emptyset$. Now we show that

$$P_M = \{(I-M)^{-1}z_0 : z_0 \in \mathbb{R}^n,\ z_0 > 0\}.$$

Indeed, if $z_0 \in \mathbb{R}^n$ and $z_0 > 0$, then

$$z := (I-M)^{-1}z_0 \ge z_0 \Rightarrow z > 0.$$

Hence $(I-M)z > 0$ and so $z \in P_M$. Conversely, if $z \in P_M$, then $z_0 := (I-M)z > 0$ and $z = (I-M)^{-1}z$. □

Definition 2.2.2. *We say that a non-singular matrix* $A = (a_{ij})_{1\le i,j\le n} \in \mathcal{M}_{n\times n}(\mathbb{R})$ *has the absolute value property if*

$$A^{-1}|A| \le I,$$

where

$$|A| = (|a_{ij}|)_{1\le i,j\le n} \in \mathcal{M}_{n\times n}(\mathbb{R}_+).$$

Some examples of matrices convergent to zero are the following:

1) $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ and $\max(a, b) < 1$

2) $A = \begin{pmatrix} a & -c \\ 0 & b \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ and $a + b < 1,\ c < 1$

3) $A = \begin{pmatrix} a & -a \\ b & -b \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ and $|a - b| < 1,\ a > 1, b > 0$.

Lemma 2.2.5. *Let $M = (a_{ij})_{1 \le i,j \le n} \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$ be a triangular matrix with*

$$\max\{|a_{ii}| i = 1, \dots, n\} < \frac{1}{2}.$$

Then the matrix $A = (I - M)^{-1}M$ is convergent to zero.

Proof. Suppose $M := \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ & \dots & \\ 0 & \dots & a_{nn} \end{pmatrix} \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$. Then the eigenvalues of M are $\lambda_i = \frac{a_{ii}}{1 - a_{ii}}$, for all $i = 1, \dots, n$. Since all of the eigenvalues of M are in the open unit disc, the conclusion follows from Theorem 2.2.1. □

Example 2.2.1. *Some examples of matrices convergent to zero are:*

1. $M = \begin{pmatrix} a & a \\ b & b \end{pmatrix}$, *where $a, b \in \mathbb{R}_+$ and $a + b < 1$;*

2. $M = \begin{pmatrix} a & b \\ a & b \end{pmatrix}$, *where $a, b \in \mathbb{R}_+$ and $a + b < 1$;*

3. $M = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, *where $a, b, c \in \mathbb{R}_+$ and $\max\{a, c\} < 1$.*

2.3 Fixed point results in generalized metric spaces

Now, we recall how to define the contraction and other known helpful results for the proof of Krasnoselskii's theorem for single valued operators in generalized Banach spaces.

Definition 2.3.1. *Let (X, d) be a generalized metric space and let $f : X \to X$ be a single valued operator. Then, f is called a single valued M-contraction if and only if, $M \in \mathcal{M}_{m \times m}(\mathbb{R}_+)$ is a matrix convergent to zero and*

$$d(f(x), f(y)) \le M d(x, y),\ \textit{for any } x, y \in X.$$

Theorem 2.3.1. *[57]Let* (X, d) *be a complete generalized metric space with* $d : X \times X \longrightarrow \mathbb{R}^n$ *and let* $N : X \longrightarrow X$ *be such that*

$$d(N(x), N(y)) \leq Md(x, y)$$

for all $x, y \in X$ *and some square matrix* M *of nonnegative numbers. If the matrix* M *is convergent to zero, that is* $M^k \longrightarrow 0$ *as* $k \longrightarrow \infty$*, then* N *has a unique fixed point* $x_* \in X$

$$d(N^k(x_0), x_*) \leq M^k(I - M)^{-1}d(N(x_0), x_0)$$

for every $x_0 \in X$ *and* $k \geq 1$*.*

Proof. Let $x \in X$ and define the sequence $x_n = N^n(x)$, where $N^n = N \circ \ldots \circ N$. Using the fact N is $M-$contraction, we get

$$d(x_{k+1}, x_k) \leq M^k d(N(x), x)$$

and, as a consequence,

$$d(x_k, x_{k+m}) \leq (M^k + M^{k+1} + \ldots + M^{k+m-1})d(N(x), x)$$

From lemma 2.2.1 we deduce that

$$d(x_k, x_{k+m}) \leq M^k(I - M)^{-1}d(N(x), x).$$

Hence (x_k) is a Cauchy sequence with respect to d and thus converges to some limit $x_* \in X$. The continuity of N guarantees that

$$x_* = N(x_*).$$

For uniqueness let y_1, y_2 be two fixed points of N, then

$$d(y_1, y_2) = d(N^k(y_1), N^k(y_2)) \leq M^k d(N(y_1), N(y_2)).$$

Since $M^k \to 0$ as $k \to \infty$, this implies $d(y_1, y_2) = 0$, so $y_l = y_2$. □

In [65], the following version of the Krasnoselskii's fixed point theorem in generalized Banach space was obtained.

Theorem 2.3.2. *(Krasnoselskii type) [65] Let* X *be a generalized Banach space. Suppose that* A *and* B *map* X *into* X *such that*

(i) A *is a completely continuous operator.*

(ii) B *is a contraction with constant* $\alpha < 1$*.*

(iii) the set $\mathcal{M} = \{x \in X : \ x = \lambda B(\frac{x}{\lambda}) + \lambda A(x), \lambda \in (0, 1)\}$ *is bounded.*

Then there exists $x \in \mathcal{M}$ with $A(x) + B(x) = x$.

Theorem 2.3.3. *Let X be a generalized Banach space, C be a nonempty compact convex subset of X, $G : C \to \mathcal{P}_{cp,cv}(C)$ be an u.s.c. multivalued map, then the operator inclusion G has at leat one fixed point, that is there exists $x \in C$ such that $x \in G(x)$.*

As a consequence of the above result we present the multivalued version of Schaefer's fixed point theorem and nonlinear alternative Leray-Schauder type theorem in generalized Banach spaces.

Theorem 2.3.4. *Let $(X, \|\cdot\|)$ be a generalized Banach space and $F : X \to \mathcal{P}_{cp,cv}(X)$ be a completely continuous multivalued mapping and u.s.c. Moreover assume that the set*

$$\mathcal{A} = \{x \in X : x \in \lambda F(x) \quad \text{for some } \lambda \in (0,1)\}$$

is bounded. Then F has a fixed point.

Chapter 3

Impulsive differential equations with delay

The fixed point theory has been proven to be a powerful tool for dealing with the stability of functional differential equations was studied in [13–20].

In this chapter we consider the following impulsive delay equations

$$\left\{\begin{array}{rcl} x'(t) &=& -a(t)x(t-r), t \in J := [0,\infty),\ t \neq t_k,\ k=1,\ldots, \\ \Delta x_{t=t_k} &=& I_k(x(t_k^-)), k=1,\ldots, \\ x(t) &=& \psi(t), t\in[-r,0] \end{array}\right. \tag{3.0.1}$$

where $a : [0,\infty) \to \mathbb{R}$ be bounded and continuous, r be a positive constant, $0 = t_0 < t_1 < \ldots < t_k < \ldots$, $\lim\limits_{k\to\infty} t_k = \infty$. $\lim\limits_{h\to 0} x(t_k+h) = x(t_k^+), \lim\limits_{h\to 0} x(t_k-h) = x(t_k^-)$, $\Delta x_{t=t_k} = x(t_k^+) - x(t_k^-)$ and $I_k \in C(\mathbb{R},\mathbb{R})$
For any function x defined on $[-r,+\infty)$ and any $t\in J$, we denote by x_t the element of $C([-r,0],\mathbb{R})$ defined by.

$$x_t(\theta) = x(t+\theta), \theta\in[-r,0]$$

Here $x_t(.)$ represents the history of the state from time t-r, up to the present time t.

3.1 Stability via Banach fixed point

Consider the Banach space

$$PC_b = \{y \in PC([-r,\infty),\mathbb{R}) : y \text{ is bounded}\},$$

where

$$PC([-r,\infty),\mathbb{R}) = \{y\colon [-r,\infty)\to\mathbb{R},\ x_k\in C((t_k,t_{k+1}],\mathbb{R}), x(t_k^-) \text{ and } x(t_k^+) \text{ exist and satisfy } x(t_k) = x(t_k^-) \text{ for } k=1,\ldots\}$$

and $x_k := y_{|(t_k,t_{k+1}]}$. Endowed with the norm

$$\|x\|_b = \sup\{|x(t)| : t \in [-r, \infty)\},$$

PC_b is a Banach space. Next we define what we mean by a solution to problem (3.0.1).

Lemma 3.1.1. *([10]) Let $C \subset PC_b$. Then C is relatively compact if it satisfies the following conditions:*

(a) C is uniformly bounded in $PC_b(\mathbb{R}^+, \mathbb{R})$.

(b) The functions belonging to C are almost equicontinuous on $\mathbb{R}_+$, i.e. equicontinuous on every compact interval of $\mathbb{R}_+$.

(c) The functions from C are equiconvergent, that is, given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that $|x(\tau_1) - x(\tau_2)| < \varepsilon$ for any $\tau_1, \tau_2 \geq T(\varepsilon)$ and $x \in C$.

Definition 3.1.1. *A function $x \in PC(J, \mathbb{R})$ is said to be a solution of (3.0.1) if $x'(t) = -a(t)x(t-r), t \in \mathbb{R}_+\ t \neq t_k,\ k = 1, \ldots, \Delta x_{t=t_k} = I_k(x(t_k^-)), k = 1, \ldots,$ and $x(t) = \psi(t), t \in [-r, 0]$.*

Lemma 3.1.2. *The solution of above problem can be expressed by the formula.*

$$\begin{aligned} x(t) &= x(0)e^{-\int_0^t a(s+r)ds} + \int_{t-r}^t a(u+r)x(u)du - e^{-\int_0^t a(u+r)du} \int_{-r}^0 a(u+r)x(u)du \\ &\quad - \int_0^t a(s+r)e^{-\int_s^t a(u+r)du} \int_{s-r}^s a(u+r)x(u)duds \\ &\quad + \sum_{0<t_k<t} I_k(x(t_k))e^{-\int_{t_k}^t a(s+r)ds}. \end{aligned}$$

and $x(t) = \psi(t),\ t \in [-r, 0]$.

Proof. Let $x \in PC$ be solution of problem (3.0.1), then for $t \in [0, t_1]$ we have

$$\int_0^t \left(x(s)e^{\int_0^{s+r}} a(u)du\right)' ds = \int_0^t \left(e^{\int_0^{s+r} a(u)du}\left(\frac{d}{ds}\int_{s-r}^s a(u+r)x(u)du\right)\right) ds,$$

then

$$\begin{aligned} \int_0^t \left(x(s)e^{\int_0^{s+r}} a(u)du\right)' ds &= \int_0^t \left(e^{\int_0^{s+r} a(u)du}(\frac{d}{ds}\int_{s-r}^s a(u+r)x(u)du\right) ds \\ x(t)e^{\int_0^{t+r} a(u)du} - x(0)e^{\int_0^r a(u)du} &= e^{\int_0^{t+r} a(u)du}\int_{t-r}^t a(u+r)x(u)du \\ &\quad -e^{\int_0^r a(u)du}\int_{-r}^0 a(u+r)x(u)du \\ &\quad -\int_0^t \left(\frac{d}{ds}(e^{\int_0^{s+r} a(u)du})\int_{s-r}^s a(u+r)x(u)du\right) ds. \end{aligned}$$

Thus

$$
\begin{aligned}
x(t) &= x(0)e^{-\int_0^t}a(s+r)ds + \int_{t-r}^{t} a(u+r)x(u)du \\
&-e^{-\int_0^t a(u+r)du}\int_{-r}^{0} a(u+r)x(u)du \\
&-\int_0^t a(s+r)e^{-\int_s^t a(u+r)du)}\int_{s-r}^{s} a(u+r)x(u)duds.
\end{aligned}
$$

For $t \in (t_1, t_2]$, we get

$$
\begin{aligned}
\int_{t_1}^{t}\left(x(s)e^{\int_{t_1}^{s+r}}a(u)du\right)' ds &= \int_{t_1}^{t}\left(e^{\int_{t_1}^{s+r}a(u)du}(\frac{d}{ds}\int_{s-r}^{s} a(u+r)x(u)du\right) ds \\
x(t)e^{\int_{t_1}^{t+r}a(u)du} - x(t_1^+)e^{\int_{t_1}^{t_1+r}a(u)du} &= e^{\int_{t_1}^{t+r}a(u)du}\int_{t-r}^{t} a(u+r)x(u)du \\
&-e^{\int_{t_1}^{t_1+r}a(u)du}\int_{t_1-r}^{t_1} a(u+r)x(u)du \\
&-\int_{t_1}^{t} a(s+r)e^{\int_{t_1}^{s+r}a(u)du)}\int_{s-r}^{s} a(u+r)x(u)duds \\
x(t) &= x(t_1^+)e^{-\int_{t_1}^{t}a(s+r)ds} + \int_{t-r}^{t} a(u+r)x(u)du \\
&-e^{-\int_{t_1}^{t}a(u+r)du}\int_{t_1-r}^{t_1} a(u+r)x(u)du \\
&-\int_{t_1}^{t} a(s+r)e^{-\int_s^t a(u+r)du}\int_{s-r}^{s} a(u+r)x(u)duds.
\end{aligned}
$$

Then

$$
\begin{aligned}
x(t) &= x(0)e^{-\int_0^t a(s+r)ds} + \int_{t-r}^{t} a(u+r)x(u)du \\
&-e^{-\int_0^t a(u+r)du}\int_{-r}^{0} a(u+r)x(u)du \\
&-\int_0^t a(s+r)e^{-\int_s^t a(u+r)du}\int_{s-r}^{s} a(u+r)x(u)duds \\
&+I_1(x(t_1))e^{-\int_{t_1}^{t}a(s+r)ds}.
\end{aligned}
$$

We continue this process we obtain for $t \in [0,b]$, we concluded

$$\begin{aligned} x(t) &= x(0)e^{-\int_0^t a(s+r)ds} + \int_{t-r}^t a(u+r)x(u)du - e^{-\int_0^t a(u+r)du} \int_{-r}^0 a(u+r)x(u)du \\ &\quad - \int_0^t a(s+r)e^{-\int_s^t a(u+r)du} \int_{s-r}^s a(u+r)x(u)duds \\ &\quad + \sum_{0<t_k<t} I_k(x(t_k))e^{-\int_{t_k}^t a(s+r)ds}. \end{aligned}$$

□

We will use the Banach fixed point theorem to prove that under the Lipschitz conditions of the jumps functions $I_k,\ k = 1,\dots,$ and for each small initial condition ψ a solution of problem (3.0.1) bounded and tends to zero as $t \to \infty$.

Theorem 3.1.1. *Assume that:*

(H_1) $\int_{t-r}^t |a(u+r)|du + \int_0^t |a(s+r)|e^{-\int_s^t a(u+r)du} \int_{s-r}^s |a(u+r)|duds \leq \alpha$

(H_2) $\int_0^t a(s+r)ds \to \infty$ *as* $t \to \infty$

(H_3) *There exist* $c_k \geq 0,\ k = 1,\dots$ *such that*

$$I_k(0) = 0,\ |I_k(x) - I_k(y)| \leq c_k|x-y|,\ \textit{for all } x, y \in \mathbb{R}.$$

hold. If $\alpha + \sum_{k=1}^{\infty} c_k < 1$. *Then the problem (3.0.1) has unique bounded solution and tends to zero as* $t \to \infty$

Proof. Consider $N : PC_b \to PC_b$ by

$$(Nx)(t) = \begin{cases} \psi(0)e^{-\int_0^t a(s+r)ds} + \int_{t-r}^t a(u+r)x(u)du \\ -e^{-\int_0^t a(u+r)du} \int_{-r}^0 a(u+r)x(u)du \\ -\int_0^t a(s+r)e^{-\int_s^t a(u+r)du} \int_{s-r}^s a(u+r)x(u)duds \\ +\sum_{0<t_k<t} I_k(x(t_k))e^{-\int_{t_k}^t a(s+r)ds} & \text{if } t \in [0,\infty), \\ \\ \psi(t) & \text{if } t \in [-r, 0]. \end{cases}$$

From (H_1)-(H_3), we can easily prove that $N(PC_b) \subset PC_b$ and N is contraction operator, then by Banach fixed point there exists unique $x \in PC_b$ such that $x = N(x)$ which is solution of problem (3.0.1). Now we show that x is bounded and tends to zero at

$t \to \infty$. Let $t \in [0, \infty)$ then, we get

$$\begin{aligned} |x(t)| &\le \int_{t-r}^{t} |a(u+r)||x(u)|du + e^{-\int_0^t a(u+r)du} \int_{-r}^{0} |a(u+r)||x(u)|du \\ &+ \int_0^t |a(s+r)|e^{-\int_s^t a(u+r)du} \int_{s-r}^{s} |a(u+r)||x(u)|duds \\ &+ \sum_{0<t_k<t} |I_k(x(t_k))|e^{-\int_{t_k}^t a(s+r)ds}. \end{aligned}$$

Thus

$$\begin{aligned} |x(t)| &\le \int_{t-r}^{t} |a(u+r)| \sup_{u\in[0,t]} |x(u)|du + e^{-\int_0^t a(u+r)du} \int_{-r}^{0} |a(u+r)| \sup_{u\in[0,t]} |x(u)|du \\ &+ \int_0^t |a(s+r)|e^{-\int_s^t a(u+r)du} \int_{s-r}^{s} |a(u+r)| \sup_{u\in[0,t]} |x(u)|duds \\ &+ \sum_{0<t_k<t} c_k \sup_{u\in[0,t]} |x(u)|. \end{aligned}$$

Hence

$$\begin{aligned} |x(t)| &\le \alpha \sup_{s\in(0,t)} |x(s)| + e^{-\int_0^t a(u+r)du} \int_{-r}^{0} |a(u+r)| \sup_{s\in(0,t)} |x(s)| \\ &+ \sum_{0\le t_k\le t} c_k \sup_{s\in(0,t)} |x(s)|. \end{aligned}$$

This implies that

$$\sup_{t\in\in[-r,\infty)} |x(t)| \le \frac{\epsilon}{1 - \sum_{k=1}^{\infty} c_k}.$$

□

3.2 Stability via Krasnoselskii fixed point theorem

In this section we present the stability result of problem (3.0.1) via the following theorem

Theorem 3.2.1. *Let* $(H_1) - (H_2)$ *and the following condition*

(H_4) *there exist* $\alpha_k, \beta_k \ge 0,\ k = 1, \ldots$ *such that*

$$|I_k(x)| \le \alpha_k |x| + \beta_k, \text{ for all } x \in \mathbb{R}.$$

are satisfied. If $\alpha+\sum_{k=1}^{\infty}\alpha_k<1$ *and* $\sum_{k=1}^{\infty}\beta_k<\infty$*. Then the problem (3.0.1) has unique bounded solution. If* $\beta_k=0,\ k=1,\dots$ *the solution of problem (3.0.1) tends to zero as* $t\to\infty$*.*

Proof. Let $N:PC_b\to PC_b$ be operator defined in theorem 3.1.1. $N=A+B$ where $A,B:PC_b\to PC_b$ by

$$B\phi(t)=\begin{cases}\psi(0)e^{-\int_0^t a(s+r)ds}+\int_{t-r}^t a(u+r)x(u)du\\ -e^{-\int_0^t a(u+r)du}\int_{-r}^0 a(u+r)x(u)du\\ -\int_0^t a(s+r)e^{-\int_s^t a(u+r)du}\int_{s-r}^s a(u+r)x(u)duds & \text{if } t\in[0,\infty),\\ \\ \psi(t) & \text{if } t\in[-r,0].\end{cases}$$

and

$$A\phi(t)=\begin{cases}\displaystyle\sum_{0<t_k<t} I_k(\phi(t_k))e^{-\int_{t_k}^t a(s+r)ds} & ,t\in[0,\infty)\\ \\ 0 & \text{if } t\in[-r,0].\end{cases}$$

Step 1 B is a contraction. Let $\phi,\eta\in PC_b$ then

$$\begin{aligned}|(B\phi)(t)-(B\eta)(t)| &\leq \left|\int_{t-r}^t a(u+r)(\phi(u)-\eta(u))du\right|\\ &\quad+\left|\int_0^t a(s+r)e^{-\int_s^t a(u+r)du}\int_{s-r}^s a(u+r)(\phi(u)-\eta(u))duds\right|\\ &\leq \int_{t-r}^t |a(u+r)||\phi(u)-\eta(u)|du\\ &\quad+\int_0^t |a(s+r)|e^{-\int_s^t a(u+r)du}\int_{s-r}^s |a(u+r)||\phi(u)-\eta(u)|duds\end{aligned}$$

Hence

$$\|B(\phi)-B(\eta)\|_b \leq \alpha\|\phi-\eta\|_b, \quad \text{for all } \phi,\eta\in PC_b.$$

Step 2. A is continuous

Given $\phi_n\to\phi$ in PC_b, then there exists $M>0$ such that

$$\|\phi_n\|_b\leq M \text{ for every } n\in\mathbb{N},$$

and

$$|(A\phi_n)(t)-(A\phi)(t)| \leq \sum_{0<t_k<t} |I_k(\phi_n(t_k)) - I_k(\phi(t_k))|.$$

Since

$$\sum \alpha_k < \infty, \quad \text{and} \sum \beta_k < \infty,$$

hence for every $\epsilon > 0, \exists k_0, k_0 \in \mathbb{N}$ such that

$$\sum_{k=k_0}^{\infty} \alpha_k < \frac{\epsilon}{6M}, \ \sum_{k=k_0}^{\infty} \beta_k < \frac{\epsilon}{6},$$

Using the fact $\lim_{k\to\infty} t_k = \infty, \exists n_0 \in \mathbb{N}$, such that for each $k \geq n_0 \Rightarrow t_k \geq k_0$. From (H_4), we get

$$\begin{aligned} \| A\phi_n - A\phi \|_b &\leq \sum_{0<t_k\leq t_{n_0-1}} |I_k(\phi_n(t_k)) - I_k(\phi(t_k))| \\ &\quad + \sum_{k=n_0}^{\infty} (2M\alpha_k + 2\beta_k) \\ &\leq \sum_{k=1}^{n_0-1} |I_k(\phi_n(t_k)) - I_k(\phi(t_k))| \\ &\quad + \frac{2\epsilon}{3}. \end{aligned}$$

Used the fact that I_k are continuous functions, then we have

$$\sum_{k=0}^{n_0-1} |I_k(\phi_n(t_k)) - I_k(\phi(t_k))| \to 0 \text{ as } n \to \infty$$

Hence

$$\|A\phi_n - A\phi\|_b \to 0 \text{ as } n \to \infty.$$

Step 3 From, (H_4), we can easily prove that A sends bounded sets into bounded sets in PC_b. We will now show that $A(M)$ is contained in a compact set

Step 4. *A sends bounded sets in PC_b into almost equicontinuous sets of PC_b.* Let $r > 0$, $B_r := \{y \in PC_b : \|y\|_\infty \leq r\}$ be a bounded set in PC_b, $\tau_1, \tau_2 \in [0, \infty)$, $\tau_1 < \tau_2$, and $\phi \in B_r$ we have

$$A\phi(\tau_1) = \sum_{0\leq t_k\leq \tau_1} I_k(\phi(t_k)) e^{-\int_{t_k}^{\tau_1} a(s+r)ds}$$

and

$$\begin{aligned} A\phi(\tau_2) &= \sum_{0\le t_k\le \tau_2} I_k(\phi(t_k))e^{-\int_{t_k}^{\tau_2} a(s+r)ds} \\ &= \sum_{0\le t_k\le \tau_2} I_k(\phi(t_k))e^{-\left(\int_{t_k}^{\tau_1} a(s+r)ds+\int_{\tau_1}^{\tau_2} a(s+r)ds\right)}. \end{aligned}$$

Then

$$|A\phi(\tau_1) - A\phi(\tau_2)| \le \left(1 - e^{-\int_{\tau_1}^{\tau_2} a(s+r)ds}\right)\sum_{k=1}^{\infty}(\alpha_k r + \beta_k).$$

Thus

$$|A\phi(\tau_1) - A\phi(\tau_2)| \to 0 \text{ as } \tau_1 \to \tau_2.$$

Step 5. We now show that the set $A(\overline{B}(0,r))$ is equiconvergent, i.e. for every $\varepsilon > 0$, there exists $T(\varepsilon) > 0$ such that $|A\phi(t) - A\phi(s)| \le \varepsilon$ for every $t, s \ge T(\epsilon)$ and each $\phi \in \overline{B}(0,r)$. Letting $\phi \in \overline{B}(0,r)$. Then

$$|A\phi(t) - A\phi(s)| \ \le \ \sum_{s\le t_k<t} |I_k(\phi(t_k))|$$

Since $\sum_{k=1}^{\infty} c_k < \infty$, $\sum_{k=1}^{\infty} d_k < \infty$, then there exist $k_0(\epsilon) \in \mathbb{N}$ such that

$$\sum_{k=k_0(\epsilon)}^{\infty} (\alpha_k r + \beta_k) \le \varepsilon.$$

Hence

$$|A\phi(t) - A\phi(s)| \le \varepsilon, \ \forall\, t \ge k_0(\epsilon).$$

Then $A(B(0,r))$ is equiconvergent. With Lemma 3.1.1 and Steps 2-4, we conclude that A is completely continuous.

Step 6. Now, we show that the set

$$\mathcal{M} = \{x \in PC(J,\mathbb{R}) : \ x = \lambda B\left(\frac{x}{\lambda}\right) + \lambda A(x), \lambda \in (0,1)\}$$

is bounded. Let $x \in \mathcal{M}$ then

Let $t \in [0,\infty)$ then, we get

$$\begin{aligned} |x(t)| \ \le \ & \int_{t-r}^{t} |a(u+r)||x(u)|du + e^{-\int_0^t a(u+r)du}\int_{-r}^{0} |a(u+r)||x(u)|du \\ & + \int_0^t |a(s+r)|e^{-\int_s^t a(u+r)du}\int_{s-r}^{s} |a(u+r)||x(u)|duds \\ & + \sum_{0<t_k<t} |I_k(x(t_k))|e^{-\int_{t_k}^t a(s+r)ds}. \end{aligned}$$

Thus

$$\begin{aligned}|x(t)| &\leq \int_{t-r}^{t}|a(u+r)|\sup_{u\in[0,t]}|x(u)|du+e^{-\int_0^t a(u+r)du}\int_{-r}^{0}|a(u+r)|\sup_{u\in[0,t]}|x(u)|du\\ &+\int_0^t|a(s+r)|e^{-\int_s^t a(u+r)du}\int_{s-r}^{s}|a(u+r)|\sup_{u\in[0,t]}|x(u)|duds\\ &+\sum_{0<t_k<t}\alpha_k\sup_{u\in[0,t]}|x(u)|+\sum_{0\leq t_k\leq t}\beta_k.\end{aligned}$$

Therefore

$$\begin{aligned}|x(t)| &\leq \alpha\sup_{s\in(0,t)}|x(s)|+e^{-\int_0^t a(u+r)du}\int_{-r}^{0}|a(u+r)|\sup_{s\in(0,t)}|x(s)|\\ &+\sum_{0<t_k<t}\alpha_k\sup_{s\in(0,t)}|x(s)|+\sum_{0<t_k<t}\beta_k.\end{aligned}$$

Hence

$$|x(t)| \leq \frac{\sum_{k=1}^{\infty}\beta_k}{1-\alpha-\sum_{k=1}^{\infty}\alpha_k}.$$

By theorem 1.5.4 the problem (3.0.1) has a bounded solution.

Let ψ be an initial condition and $\epsilon>0$ such that $\|\psi\|_\infty\leq\epsilon$, then

$$|x(t)| \leq \epsilon+\sum_{0\leq t_k\leq t}\alpha_k\sup_{s\in(0,t)}|x(s)|.$$

So,

$$|x(t)| \leq \frac{\epsilon}{1-\sum_{k=1}^{\infty}\alpha_k}.$$

□

3.3 Perturbated problem

In this section, we will show prove the bounded of solution and zero asymptotically stability of the following problem:

$$\begin{cases} x'(t) = -a(t)x(t-r)+f(t,x_t), t\in J:=[0,\infty),\ t\neq t_k,\ k=1,\ldots,\\ \Delta x_{t=t_k} = I_k(x(t_k^-)), k=1,\ldots,\\ x(t) = \psi(t), t\in[-r,0]\end{cases} \quad (3.3.1)$$

where $f : [0,\infty) \times C([-r,0],\mathbb{R}) \to \mathbb{R}$ be a continuous function and $0 = t_0 < t_1 < \ldots < t_k < \ldots, \lim_{k\to\infty} t_k = \infty$.

Our main results are based on Krasnoslkii's fixed point theorem 1.5.4.

Theorem 3.3.1. *Suppose that (H_1) and (H_2) and the following condition*

(H_5) there exist $p : [0,\infty) \to \mathbb{R}_+$ measurable, integrable function and positive number $M \geq 0$ such that

$$|f(t,x)| \leq p(t)\|x\|_\infty \quad \textit{for all } x \in C([-r,0],\mathbb{R})$$

where

$$\int_0^t p(s)e^{-\int_u^s a(l)dl}ds \leq M, \ \textit{for all } t \in [0,\infty).$$

hold. Then problem (3.3.1) has at least one solution and all this solution are bounded. If in (H_4) we have $\sum_{k=1}^{\infty} \beta_k = 0$, then for every small initial condition, the solution of problem (3.3.1) tends to zero as $t \to \infty$.

Proof. Consider $N_* : PC_b \to PC_b$ by

$$(N_*x)(t) = \begin{cases} \psi(0)e^{-\int_0^t a(s+r)ds} + \int_{t-r}^t a(u+r)x(u)du \\ -e^{-\int_0^t a(u+r)du} \int_{-r}^0 a(u+r)x(u)du \\ -\int_0^t a(s+r)e^{-\int_s^t a(u+r)du} \int_{s-r}^s a(u+r)x(u)duds \\ + \sum\limits_{0<t_k<t} I_k(x(t_k))e^{-\int_{t_k}^t a(s+r)ds} + \displaystyle\int_0^t f(s,x_s)e^{-\int_s^t a(u+r)du}ds & \text{if } t \in [0,\infty), \\ \\ \psi(t) & \text{if } t \in [-r,0]. \end{cases}$$

Let $A_* : PC_b \to PC_b$ be operator defined by

$$(A_*x)(t) = \begin{cases} \sum\limits_{0<t_k<t} I_k(x(t_k))e^{-\int_{t_k}^t a(s+r)ds} + \displaystyle\int_0^t f(s,x_s)e^{-\int_s^t a(u+r)du}ds, & t \in [0,\infty) \\ \\ 0 & \text{if } t \in [-r,0]. \end{cases}$$

Then

$$N_*(x) = A_*(x) + B(x), \ \text{for each } x \in PC_b,$$

where B is defined in theorem 3.2.1. As in theorem 3.2.1, we can prove that A_* completely continuous and from (H_1) the operator A is contractive. Also we can easily prove that the set

$$\mathcal{M} = \{x \in PC(J,\mathbb{R}) : \ x = \lambda B_*\left(\frac{x}{\lambda}\right) + \lambda A(x), \lambda \in (0,1)\}$$

is bounded. Hence by theorem 1.5.4 then problem (3.3.1) has at least one solution and all this solutions are bounded. Now we show that for every small initial condition the solution x of problem (3.3.1) tends to zero as $t \to \infty$.

Let $t \in [0, \infty)$ then, we get

$$\begin{aligned} |x(t)| &\leq \int_{t-r}^{t} |a(u+r)||x(u)|du + e^{-\int_0^t a(u+r)du} \int_{-r}^{0} |a(u+r)||x(u)|du \\ &+ \int_0^t |a(s+r)|e^{-\int_s^t a(u+r)du} \int_{s-r}^{s} |a(u+r)||x(u)|duds \\ &+ \sum_{0<t_k<t} |I_k(x(t_k))|e^{-\int_{t_k}^t a(s+r)ds} + \int_0^t |f(s,x_s)|e^{-\int_s^t a(u+r)du}ds. \end{aligned}$$

Thus

$$\begin{aligned} |x(t)| &\leq \int_{t-r}^{t} |a(u+r)| \sup_{u\in[0,t]} |x(u)|du + e^{-\int_0^t a(u+r)du} \int_{-r}^{0} |a(u+r)| \sup_{u\in[0,t]} |x(u)|du \\ &+ \int_0^t |a(s+r)|e^{-\int_s^t a(u+r)du} \int_{s-r}^{s} |a(u+r)| \sup_{u\in[0,t]} |x(u)|duds \\ &+ \sum_{0<t_k<t} \alpha_k \sup_{u\in[0,t]} |x(u)| + \int_0^t \|x_s\|_\infty p(s)e^{-\int_s^t a(u+r)du}ds. \end{aligned}$$

Therefore

$$\mu(t) \leq \frac{1}{1-\sum_{k=1}^{\infty}\alpha_k}\left(\epsilon + \int_0^t \mu(s)p(s)e^{-\int_s^t a(u+r)du}ds\right),$$

where

$$\mu(t) = \sup\{|x(s)| : s \in [-r,t]\}.$$

By Gronwall lemma, we obtain

$$|x(t)| \leq \frac{\epsilon}{1-\sum_{k=1}^{\infty}\alpha_k}\left(1 + \frac{Me^M\epsilon}{1-\sum_{k=1}^{\infty}\alpha_k}\right).$$

□

Chapter 4

Impulsive differential equations on the half-line

In [58], Precup, established the role of vector-valued metric convergence in the study of semilinear operator systems. In recent years, many authors studied the existence of solutions for system of differential equations by using the vector version fixed point theorem; see [12, 47–49, 59] and in the references therein.

In this paper we consider the following system of impulsive differential equations

$$\begin{cases} x'(t) &= f(t,x,y), t \in J := [0,\infty), t \neq t_k, \ k = 1, \ldots, \\ y'(t) &= g(t,x,y), t \in J, \quad t \neq t_k, \ k = 1, \ldots, \\ x(t_k^+) - x(t_k^-) &= I_k(x(t_k), y(t_k)), \quad k = 1, \ldots, \\ y(t_k^+) - y(t_k^-) &= \overline{I}_k(x(t_k), y(t_k)), \quad k = 1, \ldots, \\ x(0) &= x_0, \\ y(0) &= y_0, \end{cases} \tag{4.0.1}$$

where $x_0, y_0 \in \mathbb{R}$, $f, g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are a given functions, $I_k, \overline{I}_k \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. The notations $x(t_k^+) = \lim_{h \to 0^+} x(t_k + h)$ and $x(t_k^-) = \lim_{h \to 0^+} x(t_k - h)$ stand for the right and the left limits of the functions y at $t = t_k$, respectively.

By using Perov's and Krasnoselskii fixed point type theorems in generalized Banach spaces, we prove the existence, uniqueness and compactness of solution sets of above problem .

4.1 Uniqueness and continuous dependence on initial data

In order to define a solution for problem (4.0.1), consider the space of piecewise continuous functions

$$PC_b = \{y \in PC([0,\infty),\mathbb{R}) :\ y \text{ is bounded}\}$$

where $PC([0,\infty),\mathbb{R}) = \{y : [0,\infty) \to \mathbb{R}, y_k \in C((t_k,t_{k+1}],\mathbb{R}),\ k = 0,\ldots, y(t_k^-)$ and $y(t_k^+)$ exist and satisfy $y(t_k) = y(t_k^-)$ for $k = 1,\ldots\}$.

PC_b is a Banach space with norm

$$\|y\|_b = \sup\{|y(t)| :\ t \in [0,\infty)\}.$$

Definition 4.1.1. *A function $(x,y) \in PC_b(J,\mathbb{R}) \times PC_b(J,\mathbb{R})$ is said to be a solution of (4.0.1) if and only if*

$$\begin{cases} x(t) &= x_0 + \int_0^t f(s,x(s),y(s))ds + \displaystyle\sum_{0<t_k<t} I_k(x(t_k),y(t_k)), t \in J, \\ y(t) &= y_0 + \int_0^t g(s,x(s),y(s))ds + \displaystyle\sum_{0<t_k<t} \overline{I}_k(x(t_k),y(t_k)), t \in J. \end{cases}$$

In this section we assume the following conditions:

(H_1) There exist functions $l_i \in L^1(J,\mathbb{R}^+), i = 1,\ldots,4$, such that

$$|f(t,x,y) - f(s,\overline{x},\overline{y})| \leq l_1(t)|x-\overline{x}| + l_2(t)|y-\overline{y}|, \text{ for all } x,\overline{x},y,\overline{y} \in \mathbb{R}$$

and

$$|g(t,x,y) - g(s,\overline{x},\overline{y})| \leq l_3(t)|x-\overline{x}| + l_4(t)|y-\overline{y}|, \text{ for all } x,\overline{x},y,\overline{y} \in \mathbb{R}.$$

(H_2) There exist constants $a_{1k}, a_{2k} \geq 0,\ k = 1,\ldots,$ such that

$$|I_k(x,y) - I_k(\overline{x},\overline{y})| \leq a_{1k}|x-\overline{x}| + a_{2k}|y-\overline{y}|, \text{ for all } x,\overline{x},y,\overline{y} \in \mathbb{R}$$

and

$$\sum_{k=1}^{\infty} |I_k(0,0)| < \infty.$$

(H_3) There exist constants $b_{1k}, b_{2k} \geq 0,\ k = 1,\ldots,$ such that

$$|\overline{I}_k(x,y) - \overline{I}_k(\overline{x},\overline{y})| \leq b_{1k}|x-\overline{x}| + b_{2k}|y-\overline{y}|, \text{ for all } x,\overline{x},y,\overline{y} \in \mathbb{R}$$

and

$$\sum_{k=1}^{\infty} |\bar{I}_k(0,0)| < \infty.$$

We will use the Perov fixed point theorem to prove that a solution of problem (4.0.1) is bounded and tends to zero as $t \to \infty$.

Theorem 4.1.1. *Assume that* $(H_1) - (H_3)$ *are satisfied and the matrix*

$$M = \begin{pmatrix} \|l_1\|_{L^1} + \sum_{k=1}^{\infty} a_{1k} & \|l_2\|_{L^1} + \sum_{k=1}^{\infty} a_{2k} \\ \|l_3\|_{L^1} + \sum_{k=1}^{\infty} b_{1k} & \|l_4\|_{L^1} + \sum_{k=1}^{\infty} b_{2k} \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R}^+) \qquad (4.1.1)$$

where

$$\sum_{k=1}^{\infty} a_{ik} < \infty \ \text{ and } \sum_{k=1}^{\infty} b_{ik} < \infty, i = 1, 2,$$

converges to zero and $f(\cdot, 0, 0), g(\cdot, 0, 0) \in L^1(J, \mathbb{R})$. *Then the problem (4.0.1) has unique solution. If we add that and*

$$\sum_{k=1}^{\infty} a_{1k} + \sum_{k=1}^{\infty} a_{2k} + \sum_{k=1}^{\infty} b_{1k} + \sum_{k=1}^{\infty} b_{2k} < 1,$$

the unique solution of (4.0.1) is bounded.

Proof. Consider the operator $N : PC \times PC \to PC \times PC$ defined by

$$N(x, y) = (N_1(x, y), N_2(x, y))$$

where

$$N_1(x, y)(t) = x_0 + \int_0^t f(s, x(s), y(s))ds + \sum_{0<t_k<t} I_k(x(t_k), y(t_k)), t \in [0, \infty)$$

and

$$N_2(x, y)(t) = y_0 + \int_0^t g(s, x(s), y(s))ds + \sum_{0<t_k<t} \bar{I}_k(x(t_k), y(t_k)), t \in [0, \infty).$$

We show that the operator N was well defined. Given $(x, y) \in PC_b \times PC_b, t \in [0, \infty)$, then

$$\begin{aligned} \|N_1(x, y)\|_b &\leq |x_0| + \int_0^t |f(s, x(s), y(s))|ds + \sum_{0<t_k<t} |I_k(x(t_k), y(t_k))| \\ &\leq \|l_1\|_{L^1}\|x\|_b + \|l_2\|_{L^1}\|y\|_b + \sum_{0<t_k<t} (a_{1k}\|x\|_b + a_{2k}\|y\|_b) \\ &\quad +\|f(\cdot, 0, 0)\|_{L^1} + \sum_{0<t_k<t} (|I_k(0, 0)| + |\bar{I}_k(0, 0)|). \end{aligned}$$

Similarly we have

$$\begin{aligned}\|N_2(x,y)\|_b &\le \|l_3\|_{L^1}\|x\|_b + \|l_4\|_{L^1}\|y\|_b + \sum_{0<t_k<t}(b_{1k}\|x\|_b + b_{2k}\|y\|_b)\\ &\quad +\|g(\cdot,0,0)\|_{L^1} + \sum_{0<t_k<t}(|I_k(0,0)| + |\bar{I}_k(0,0)|).\end{aligned}$$

Thus

$$\begin{aligned}\begin{pmatrix}\|N_1(x,y)\|_b\\ \|N_1(x,y)\|_b\end{pmatrix} &\le \begin{pmatrix}\|l_1\|_{L^1} + \sum_{k=1}^{\infty} a_{1k} & \|l_2\|_{L^1} + \sum_{k=1}^{\infty} a_{2k}\\ \|l_3\|_{L^1} + \sum_{k=1}^{\infty} b_{1k} & \|l_4\|_{L^1} + \sum_{k=1}^{\infty} b_{1k}\|x\| + b_{2k}\end{pmatrix}\begin{pmatrix}\|x\|_b\\ \|y\|_b\end{pmatrix}\\ &+\begin{pmatrix}\|f(.,0,0)\|_{L^1} + \sum_{k=1}^{\infty}(\|I_k(0,0)\|_b + \|\bar{I}_k(0,0)\|_b)\\ \\ \|g(.,0,0)\|_{L^1} + \sum_{k=1}^{\infty}(|I_k(0,0)| + |\bar{I}_k(0,0)|)\end{pmatrix}.\end{aligned}$$

This implies that the operator N is well defined. Clearly, fixed points of the operator N are solutions of problem (4.0.1). We show that N is a contraction. Let $(x,y),(\overline{x},\overline{y}) \in PC_b \times PC_b$. Then (H_1) and (H_2) imply

$$\begin{aligned}|N_1(x,y)(t) - N_1(\overline{x},\overline{y})(t)| &\le \int_0^t |f(s,x(s),y(s)) - f(s,\overline{x}(s),\overline{y}(s))|ds\\ &+ \sum_{0<t_k<t}^{\infty} |I_k(x(t_k),y(t_k)) - I_k(\overline{x}(t_k),\overline{y}(t_k))|\\ &\le \int_0^t (l_1(s)|x(s)-\overline{x}(s)| + l_2(s)|y(s)-\overline{y}(s)|)\,ds\\ &+ \sum_{0<t_k<t}(a_{1k}|x(t_k)-\overline{x}(t_k)| + a_{2k}|y(t_k)-\overline{y}(t_k)|).\end{aligned}$$

Thus

$$\begin{aligned}\|N_1(x,y) - N_1(\overline{x},\overline{y})\|_b &\le (\|l_1\|_{L^1} + \sum_{k=1}^{\infty} a_{1k})\|x-\overline{x}\|_b\\ &+ (\|l_2\|_{L^1} + \sum_{k=1}^{\infty} a_{2k})\|y-\overline{y}\|_b.\end{aligned}$$

Similarly we have

$$\begin{aligned}\|N_2(x,y)-N_2(\overline{x},\overline{y})\|_b &\leq (\|l_3\|_{L^1}+\sum_{k=1}^{\infty}b_{1k})\|x-\overline{x}\|_b\\ &+ (\|l_4\|_{L^1}+\sum_{k=1}^{\infty}b_{2k})\|y-\overline{y}\|_b.\end{aligned}$$

Therefore

$$\|N(x,y)-N(\overline{x},\overline{y})\|_b \leq M\begin{pmatrix}\|x-\overline{x}\|_b\\ \|y-\overline{y}\|_b\end{pmatrix},\ \text{for all } (x,y),(\overline{x},\overline{y})\in PC_b\times PC_b.$$

Hence, by Theorem 2.3.1, the operator N has at least one fixed point which is a solution of problem (4.0.1).

Now we show that the solution (x,y) is bounded. Let $t\in[0,\infty)$. Then, we get

$$\begin{aligned}|x(t)| &\leq |x_0|+\int_0^t|f(s,x(s),y(s))|ds+\sum_{0<t_k<t}|I_k(x(t_k),y(t_k))|\\ &\leq |x_0|+\int_0^t(l_1(s)|x|+l_2(s)|y|)ds+\sum_{k=1}^{\infty}a_{1k}|x(t_k)|+\sum_{k=1}^{\infty}a_{2k}|y(t_k)|\\ &\quad +\|f(.,0,0)\|_{L^1}+\|g(.,0,0)\|_{L^1}+\sum_{k=1}^{\infty}|I_k(0,0)+\sum_{k=1}^{\infty}|\bar{I}_k(0,0)|\end{aligned}$$

and

$$\begin{aligned}|y(t)| &\leq |y_0|+\int_0^t(l_3(s)|x(s)|+l_4(s)|y(s)|)ds+\sum_{k=1}^{\infty}b_{1k}|x(t_k)|+\sum_{k=1}^{\infty}b_{2k}|y(t_k)|\\ &\quad +\|f(.,0,0)\|_{L^1}+\|g(.,0,0)\|_{L^1}+\sum_{k=1}^{\infty}|I_k(0,0)+\sum_{k=1}^{\infty}|\bar{I}_k(0,0)|.\end{aligned}$$

Thus

$$\begin{aligned}|x(t)|+|y(t)| &\leq |x_0|+|y_0|+\int_0^t((l_1(s)+l_3(s))|x(s)|+(l_2(s)+l_4(s))|y(s)|)ds\\ &+ (\sum_{k=1}^{\infty}a_{1k}+\sum_{k=1}^{\infty}a_{2k}+\sum_{k=1}^{\infty}b_{1k}+\sum_{k=1}^{\infty}b_{2k})(|x(t_k)|+|y(t_k)|)\\ &\quad +2\|f(.,0,0)\|_{L^1}+2\|g(.,0,0)\|_{L^1}+2\sum_{k=1}^{\infty}|I_k(0,0)+2\sum_{k=1}^{\infty}|\bar{I}_k(0,0)|.\end{aligned}$$

Hence

$$\begin{aligned}\sup_{s\in(0,t)}(|x(s)|+|y(s)|) &\le |x_0|+|y_0|+\int_0^t (l_1(s)+l_3(s)+l_2(s)+l_4(s))\times \\ &\quad \sup_{s\in[0,t]}(|x(s)|+|y(s)|)ds \\ &+ \left(\sum_{k=1}^{\infty}a_{1k}+\sum_{k=1}^{\infty}a_{2k}+\sum_{k=1}^{\infty}b_{1k}+\sum_{k=1}^{\infty}b_{2k}\right)\sup_{s\in[0,t]}(|x(t_k)|+|y(t_k)|) \\ &+ 2\|f(.,0,0)\|_{L^1}+2\|g(.,0,0)\|_{L^1}+2\sum_{k=1}^{\infty}|I_k(0,0)+2\sum_{k=1}^{\infty}|\bar{I}_k(0,0)|.\end{aligned}$$

This implies that

$$\sup_{s\in(0,t)}(|x(s)|+|y(s)|)\le \alpha+\int_0^t l(s)\sup_{s\in[0,t]}(|x(s)|+|y(s)|)ds$$

where

$$\alpha=\frac{|x_0|+|y_0|+2\|f(.,0,0)\|_{L^1}+2\|g(.,0,0)\|_{L^1}+2\sum_{k=1}^{\infty}|I_k(0,0)+2\sum_{k=1}^{\infty}|\bar{I}_k(0,0)|}{1-\left(\sum_{k=1}^{\infty}a_{1k}+\sum_{k=1}^{\infty}a_{2k}+\sum_{k=1}^{\infty}b_{1k}+\sum_{k=1}^{\infty}b_{2k}\right)}$$

and

$$l(s)=\frac{l_1(s)+l_2(s)+l_3(s)+l_4(s)}{1-\left(\sum_{k=1}^{\infty}a_{1k}+\sum_{k=1}^{\infty}a_{2k}+\sum_{k=1}^{\infty}b_{1k}+\sum_{k=1}^{\infty}b_{2k}\right)}.$$

Using the Gronwall Inequality we obtain

$$\sup_{s\in[0,t]}(|x(s)|+|y(s)|)\le \alpha\exp\left(\int_0^t l(s)ds\right).$$

Then

$$\|x\|_b+\|y\|_b\le \alpha\exp\left(\int_0^{\infty} l(s)ds\right).$$

This implies that the solution (x,y) is bounded. □

For the next result we prove the continuous dependence of solutions on initial conditions.

Theorem 4.1.2. *Assume the conditions $(H_1)-(H_3)$ hold, that the matrix M defined in (4.1.1) converges to zero, that $I_k(0,0)=\bar{I}_k(0,0),\ k=1,\dots$ and $f(t,0,0)=g(t,0,0)=0,\ t\in J$. For every $(x_0,y_0)\in\mathbb{R}\times\mathbb{R}$ we denote by $(x(t,x_0),y(t,y_0))$ the solution of* (4.0.1). *Then the map $(x_0,y_0)\to(x(\cdot,x_0),y(\cdot,y_0))$ is continuous.*

Proof. Let $(x_0, y_0), (\bar{x}_0, \bar{y}_0) \in \mathbb{R}\times\mathbb{R}$. Then from Theorem 4.1.1, there exist $(x(\cdot, x_0), y(\cdot, y_0))$, $(\bar{x}(\cdot, \bar{x}_0), \bar{y}(\cdot, \bar{y}_0)) \in PC_b \times PC_b$ such that

$$x(t, x_0) = x_0 + \int_0^t f(s, x(s, x_0), y(s, y_0))ds + \sum_{0<t_k<t} I_k(x(t_k, x_0), y(t_k, y_0)), t \in [0, \infty),$$

$$y(t, y_0) = y_0 + \int_0^t g(s, x(s, x_0), y(s, y_0))ds + \sum_{0<t_k<t} \overline{I}_k(x(t_k, x_0), y(t_k, y_0)), t \in [0, \infty),$$

$$x(t, \bar{x}_0) = \bar{x}_0 + \int_0^t f(s, x(s, \bar{x}_0), y(s, \bar{y}_0))ds + \sum_{0<t_k<t} I_k(x(t_k, \bar{x}_0), y(t_k, \bar{y}_0)), t \in [0, \infty),$$

and

$$\bar{y}(t, \bar{y}_0) = \bar{y}_0 + \int_0^t g(s, x(s, \bar{x}_0), y(s, \bar{y}_0))ds + \sum_{0<t_k<t} \overline{I}_k(x(t_k, \bar{x}_0), y(t_k, \bar{y}_0)), t \in [0, \infty).$$

Hence from the proof of Theorem 4.1.1 we deduce that

$$\|x(\cdot, x_0) - \bar{x}(\cdot, \bar{x}_0)\|_b + \|y(\cdot, y_0) - \bar{y}(\cdot, \bar{y}_0)\|_b \leq \frac{|x_0 - \bar{x}_0| + |y_0 - \bar{y}_0|}{1 - \left(\sum_{k=1}^{\infty} a_{1k} + \sum_{k=1}^{\infty} a_{2k} + \sum_{k=1}^{\infty} b_{1k} + \sum_{k=1}^{\infty} b_{2k}\right)} \times \exp\left(\int_0^{\infty} l(s)ds\right).$$

Then

$$\|x(\cdot, x_0) - \bar{x}(\cdot, \bar{x}_0)\|_b + \|y(\cdot, y_0) - \bar{y}(\cdot, \bar{y}_0)\|_b \to 0, \text{ as } (x_0, y_0) \to (\bar{x}_0, \bar{y}_0).$$

□

4.2 Existence and compactness of solution sets

In this section we present an application of e Krasnoselskii's type fixed point theorem to problem (4.0.1).

Theorem 4.2.1. *Let* (H_1) *be satisfied and the following conditions:*

(H_4) *There exist* $\alpha_k, \beta_k \geq 0, \ k = 1, \dots,$ *such that*

$$|I_k(x, y)| \leq \alpha_k|x| + \beta_k|y| + c_k, \text{ for all } (x, y) \in \mathbb{R} \times \mathbb{R}.$$

(H_5) *There exist* $\overline{\alpha_k}, \overline{\beta_k} \geq 0,\ k = 1, \ldots,$ *such that*

$$|\overline{I_k}(x,y)| \leq \overline{\alpha_k}|x| + \overline{\beta_k}|y| + \overline{c_k}, \text{ for all } (x,y) \in \mathbb{R} \times \mathbb{R}.$$

If

$$M_* = \begin{pmatrix} \|l_1\|_{L^1} & \|l_2\|_{L^1} \\ \|l_3\|_{L^1} & \|l_4\|_{L^1} \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R}^+) \tag{4.2.1}$$

converges to zero and $\sum_{k=1}^{\infty} \alpha_k + \sum_{k=1}^{\infty} \overline{\alpha}_k + \sum_{k=1}^{\infty} \beta_k + \sum_{k=1}^{\infty} \overline{\beta}_k < 1, \sum_{k=1}^{\infty} c_k < \infty$ *and* $\sum_{k=1}^{\infty} \overline{c}_k < \infty,$ *then the problem (4.0.1) has at least one bounded solution.*

Proof. Let $N : PC_b \times PC_b \to PC_b \times PC_b$ be the operator defined in Theorem 4.1.1. $N = A + B$ where $A, B : PC_b \times PC_b \to PC_b \times PC_b$ are defined by

$$B(x(t), y(t)) = (B_1(x(t), y(t)),\ B_2(x(t), y(t)), \quad t \in J,$$

where

$$\begin{cases} B_1(x,y) &= x_0 + \int_0^t f(s, x(s), y(s))ds \\ B_2(x,y)) &= y_0 + \int_0^t g(s, x(s), y(s))ds \end{cases}$$

and

$$A(x(t), y(t)) = (A_1(x(t), y(t)),\ A_2(x(t), y(t)), \quad t \in J$$

where

$$\begin{cases} A_1(x,y) &= \sum\limits_{0<t_k<t}^{\infty} I_k(x(t_k), y(t_k)) \\ A_2(x,y)) &= \sum\limits_{0<t_k<t}^{\infty} \overline{I}_k(x(t_k), y(t_k)). \end{cases}$$

Step 1 B is a contraction. Let $(x,y), (\overline{x}, \overline{y}) \in PC_b \times PC_b$. Then

$$\begin{aligned} |B_1(x(t), y(t)) - B_1(\overline{x}t), \overline{y}(t))| &\leq \int_0^t |f(s, x(s), y(s)) - f(s, \overline{x}(s), \overline{y}(s))| \\ &\leq \int_0^t (l_1(s)|x(s) - \overline{x}(s)| + l_2(s)|y(s) - \overline{y}(s)|)\, ds. \end{aligned}$$

Hence

$$\|B_1(x,y) - B_1(\overline{x}, \overline{y})\|_b \leq \|l_1\|_{L^1}\|x - \overline{x}\|_b + \|l_2\|_{L^1}\|y - \overline{y}\|_b.$$

Similarly we have

$$\|B_2(x,y) - B_2(\overline{x}, \overline{y})\|_b \leq \|l_3\|_{L^1}\|x - \overline{x}\|_b + \|l_4\|_{L^1}\|y - \overline{y}\|_b.$$

Therefore

$$\|B(x,y) - B(\overline{x}, \overline{y})\|_b \leq \begin{pmatrix} \|l_1\|_{L^1} & \|l_2\|_{L^1} \\ \|l_3\|_{L^1} & \|l_4\|_{L^1} \end{pmatrix} \begin{pmatrix} \|x - \overline{x}\|_b \\ \|y - \overline{y}\|_b \end{pmatrix}.$$

Step 2. A is continuous

Given $(x_n, y_n) \to (x, y)$ in $PC_b \times PC_b$, then there exists $M, M^{'} > 0$ such that

$$\|x_n\|_b \leq M \text{ and } \|y_n\|_b \leq M' \text{ for every } n \in \mathbb{N},$$

and

$$|(A_1 x_n)(t) - (A_1 x)(t)| \leq \sum_{0 \leq t_k < t} |I_k(x_n, y_n) - I_k(x, y)|.$$

Since

$$\sum \alpha_k < \infty, \quad \text{ and } \sum \beta_k < \infty,$$

hence for every $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{k=k_0}^{\infty} \alpha_k < \frac{\epsilon}{6M}, \ \sum_{k=k_0}^{\infty} \beta_k < \frac{\epsilon}{6M'}.$$

Using the fact $\lim_{k\to\infty} t_k = \infty$, there exists $n_0 \in \mathbb{N}$, such that for each $k \geq n_0 \Rightarrow t_k \geq k_0$.

From (H_3), we get

$$\begin{aligned} \| \ A_1(x_n, y_n) - A_1(x, y) \ \|_{PC_b} \ &\leq \ \sum_{0 \leq t_k \leq t_{n_0 - 1}} |I_k(x_n, y_n) - I_k(x, y)| \\ &\quad + \sum_{k=k_0}^{\infty} (2M\alpha_k + 2M'\beta_k) \\ &\leq \ \sum_{k=1}^{k_0 - 1} |I_k(x_n, y_n) - I_k(x, y)| \\ &\quad + \frac{2\epsilon}{3}. \end{aligned}$$

Using the fact that I_k are continuous functions, then we have

$$\sum_{k=0}^{n_0 - 1} |I_k(x_n, y_n) - I_k(x, y)| \to 0 \text{ as } n \to \infty.$$

Hence

$$\|A_1(x_n, y_n) - A_1(x, y)\|_b \to 0 \text{ as } n \to \infty.$$

Similarly we have

$$\|A_2(x_n, y_n) - A_2(x, y)\|_b \to 0 \text{ as } n \to \infty.$$

Thus

$$\|A(x_n, y_n) - A(x, y)\|_b \to 0 \text{ as } n \to \infty.$$

Step 3 From (H_3), we can easily prove that A maps bounded sets into bounded sets in $PC \times PC$.

We will now show that $A(M)$ is contained in a compact set.

Step 4. *A maps bounded sets in $PC_b \times PC_b$ into almost equicontinuous sets of $PC_b \times PC_b$.* Let $r = (r_1, r_2) > 0$, $B_r := \{(x, y) \in PC_b \times PC_b : \|(x, y)\|_\infty \leq r\}$ be a bounded set in $PC \times PC$, $\tau_1, \tau_2 \in [0, \infty)$, $\tau_1 < \tau_2$, and $\phi \in B_r$. We have

$$A\phi(\tau_1) = (A_1\phi(\tau_1), A_2\phi(\tau_1)) \text{ where } \begin{cases} A_1\phi(\tau_1) &= \sum_{0 \leq t_k \leq \tau_1} I_k(\phi_1(t_k), \phi_2(t_k)) \\ A_2\phi(\tau_1). &= \sum_{0 \leq t_k \leq \tau_2} I_k(\phi_1(t_k), \phi_2(t_k)) \end{cases}.$$

Then

$$|A_1\phi(\tau_2) - A_1\phi(\tau_1)| \leq \sum_{\tau_1 \leq t_k \leq \tau_2} I_k(\phi_1(t_k), \phi_2(t_k)).$$

Thus

$$|A_1\phi(\tau_2) - A_1\phi(\tau_1)| \to 0 \text{ as } \tau_1 \to \tau_2.$$

Similarly we have

$$|A_2\phi(\tau_2) - A_2\phi(\tau_1)| \to 0 \text{ as } \tau_1 \to \tau_2.$$

Thus

$$|A\phi(\tau_1) - A\phi(\tau_2)| \to 0 \text{ as } \tau_1 \to \tau_2.$$

Step 5. We now show that the set $A(\overline{B}(0, r))$ is equiconvergent, i.e. for every $\varepsilon > 0$, there exists $T(\varepsilon) > 0$ such that $\|A(\phi(t)) - A(\phi(s))\| \leq \varepsilon$ for every $t, s \geq T(\epsilon)$ and each $\phi \in \overline{B}(0, r)$. Letting $\phi \in \overline{B}(0, r)$, then for every $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{k=k_0}^{\infty} \alpha_k < \frac{\epsilon}{2r_1}, \sum_{k=k_0}^{\infty} \beta_k < \frac{\epsilon}{2r_2},$$

$$\begin{aligned} |A_1\phi(t) - A_1\phi(s)| &\leq \sum_{s \leq t_k \leq t} I_k(\phi_1(t_k), \phi_2(t_k)) \\ &\leq \sum_{s \leq t_k \leq t} (\alpha_k r_1 + \beta_k r_2). \end{aligned}$$

Then, for every $s, t > k_0$, we get

$$|A_1\phi(t) - A_1\phi(s)| \leq r_1 \sum_{k=k_0}^{\infty} \alpha_k + r_2 \sum_{k=k_0}^{\infty} \beta_k.$$

Therefore for all $\phi \in B(0,r)$ and $s,t > k_0$ we have

$$|A_1\phi(t) - A_1\phi(s)| \;\; \leq \;\; \epsilon.$$

Similarly we can prove that there exists $\bar{k}_0 > 0$ such that for all $\phi \in B(0,r)$ and $s,t > k_0$ we have

$$|A_2\phi(t) - A_2\phi(s)| \;\; \leq \;\; \epsilon.$$

Thus, for every $(\epsilon,\epsilon) > 0$ there exists $(k_0,\bar{k}_0) > 0$ such that for all $s,t > k_0$ and $s,t > \bar{k}_0$ we have

$$|A\phi(\tau_1) - A\phi(\tau_2)| \leq (\epsilon,\epsilon), \text{ for every } \phi \in B(0,r).$$

Step 6 Now, we show that the set

$$\mathcal{M} = \{(x,y) \in PC \times PC; (x,y) = \lambda B(\frac{x}{\lambda},\frac{y}{\lambda}) + \lambda A(x,y), \lambda \in (0,1)\}$$

is bounded. Let $(x,y) \in \mathcal{M}$ then

$$\begin{aligned} |x(t)| &\leq |x_0| + \int_0^t \lambda \left| f\left(s, \frac{x(s)}{\lambda}, \frac{y(s)}{\lambda}\right)\right| ds + \sum_{0<t_k<t} |I_k(x(t_k),y(t_k))| \\ &\leq |x_0| + \int_0^t (l_1(s)|x(s)| + l_2(s)|y(s)|)ds + \sum_{k=1}^{\infty} \alpha_k |x(t_k)| + \sum_{k=1}^{\infty} \beta_k |y(t_k)| + \sum_{k=1}^{\infty} c_k \end{aligned}$$

$$|y(t)| \leq |y_0| + \int_0^t (l_3(s)|x(s)| + l_4(s)|y(s)|)ds + \sum_{k=1}^{\infty} \overline{\alpha}_k |x(t_k)| + \sum_{k=1}^{\infty} \overline{\beta}_k |y(t_k)| + \sum_{k=1}^{\infty} \overline{c}_k.$$

Thus

$$\begin{aligned} |x(t)| + |y(t)| \leq\ & |x_0| + |y_0| + \int_0^t ((l_1(s) + l_3(s))|x(s)| + (l(s)_2 + l(s)_4)|y(s)|)ds \\ +\ & (\sum_{k=1}^{\infty} \alpha_k + \sum_{k=1}^{\infty} \overline{\alpha}_k)|x(t_k| + (\sum_{k=1}^{\infty} \beta_k + \sum_{k=1}^{\infty} \overline{\beta}_k)|y(t_k| \\ +\ & \sum_{k=1}^{\infty} c_k + \sum_{k=1}^{\infty} \overline{c}_k. \end{aligned}$$

Hence

$$\begin{aligned}\sup_{s\in[0,t]}(|x(s)|+|y(s)|) &\leq |x_0|+|y_0|+\int_0^t (l_1(s)+l_3(s)+l_2(s)+l_4(s)) \\ &\quad \times \sup_{s\in[0,t]}(|x(s)|+|y(s)|)ds \\ &+ (\sum_{k=1}^{\infty}\alpha_k+\sum_{k=1}^{\infty}\overline{\alpha}_k)\sup_{s\in(0,t)}|x(t_k|+(\sum_{k=1}^{\infty}\beta_k+\sum_{k=1}^{\infty}\overline{\beta}_k)\sup_{s\in(0,t)}|y(t_k| \\ &+ \sum_{k=1}^{\infty}c_k+\sum_{k=1}^{\infty}\overline{c}_k.\end{aligned}$$

This implies that

$$\sup_{s\in(0,t)}(|x(s)|+|y(s)|) \leq \beta+\int_0^t l_*(s)\sup_{s\in[0,t]}(|x(s)|+|y(s)|)ds$$

where

$$\beta=\frac{|x_0|+|y_0|+\sum_{k=1}^{\infty}c_k+\sum_{k=1}^{\infty}\overline{c}_k}{1-(\sum_{k=1}^{\infty}\alpha_k+\sum_{k=1}^{\infty}\overline{\alpha}_k+\sum_{k=1}^{\infty}\beta_k+\sum_{k=1}^{\infty}\overline{\beta}_k)}$$

and

$$l_*(s)=\frac{l_1(s)+l_3(s)+l_2(s)+l_4(s)}{1-(\sum_{k=1}^{\infty}\alpha_k+\sum_{k=1}^{\infty}\overline{\alpha}_k+\sum_{k=1}^{\infty}\beta_k+\sum_{k=1}^{\infty}\overline{\beta}_k)}.$$

By the Gronwall Inequality, we have

$$\sup_{s\in(0,t)}(|x(s)|+|y(s)|)\leq \beta e^{\|l_*\|_{L^1}}.$$

Then

$$\|x\|_b\leq \beta e^{\|l_*\|_{L^1}}, \quad \text{and} \quad \|y\|_b\leq \beta e^{\|l_*\|_{L^1}}.$$

Hence from Theorem 2.3.2 the problem (4.0.1) has at least one solution.

□

By simple modification in the prove we can obtain the following result.

Theorem 4.2.2. *Let (H_2) be satisfied and the following condition:*

(H_6) *There exists* $p \in L^1(J, \mathbb{R}_+)$, *and let* $\psi : \mathbb{R}_+ \to (0, \infty)$ *be a continuous nondecreasing function such that*

$$|f(t,x,y)| \leq p(t)\psi(|x|+|y|), \ \textit{for all } (x,y) \in \mathbb{R} \times \mathbb{R}$$

and

$$|g(t,x,y)| \leq p(t)\psi(|x|+|y|), \ \textit{for all } (x,y) \in \mathbb{R} \times \mathbb{R}.$$

If

$$\bar{M}_* = \begin{pmatrix} \sum_{k=1}^{\infty} a_{1k} & \sum_{k=1}^{\infty} a_{2k} \\ \sum_{k=1}^{\infty} b_{1k} & \sum_{k=1}^{\infty} b_{2k} \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R}^+) \tag{4.2.2}$$

converges to zero, then the problem (4.0.1) has unique bounded solution.

By the nonlinear alternative in generalized Banach space we can also prove the following result.

Theorem 4.2.3. *Assume that* $(H_4) - (H_6)$ *hold. If*

$$\sum_{k=1}^{\infty} \alpha_k + \sum_{k=1}^{\infty} \overline{\alpha}_k + \sum_{k=1}^{\infty} \beta_k + \sum_{k=1}^{\infty} \overline{\beta}_k < \infty, \sum_{k=1}^{\infty} c_k < \infty \ \textit{and} \ \sum_{k=1}^{\infty} \overline{c}_k < \infty,$$

then the problem (4.0.1) has at least one solution. Moreover, the solution set

$$S(x_0, y_0) = \{(x,y) \in PC_b \times PC_b : (x,y) \ \textit{is solution of} (4.0.1)\}$$

is compact and the multivalued map $S : (x_0, y_0) \to S(x_0, y_0)$ *is u.s.c.*

Chapter 5

Differential Inclusions

It is well known that in $C([0,\infty),\mathbb{R}^n)$, the distance between two trajectories of an ordinary differential equation $y' = f(x,y)$ (with a Lipschitz continuous vector field) is majorized by the distance between two initial points multiplied by an exponential function of time. The object of this chapter is to prove a Filippov type theorem and a Filippov-Wazewski type theorem for impulsive differential inclusions. More precisely, we consider the problem

$$y' \in F(t,y(t)), \quad a.e.\ t \in J := [0,b],\ t \neq t_k, \quad k = 1,\dots,m, \tag{5.0.1}$$

$$y(t_k^+) - y(t_k) = I_k(y(t_k^-)), \quad k = 1,\dots,m, \tag{5.0.2}$$

$$y(0) = y_0, \tag{5.0.3}$$

where $F : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a multivalued map, $y_0 \in \mathbb{R}^n$, and $I_k \in C(\mathbb{R}^n,\mathbb{R}^n)$, $k = 1,2,\dots,m$. In the case where the impulses are absent (i.e., $I_k \equiv 0$, $k = 1,2,\dots,m$) and the problem (5.0.1)–(5.0.3) reduces to an autonomous control system, that is, $F(t,y) := f(t,U)$, where f is single map and U is a control set, a property of Filippov's theorem. Some generalizations of Filippov's theorem were considered by Frankowska [28] and Zhu [68].

5.1 Filippov's Theorem

In order to define a solution of (5.0.1)–(5.0.3), we consider the space

$$PC(J,\mathbb{R}^n) = \{y : J \to \mathbb{R}^n \mid y(t) \text{ is continuous everywhere except for some } t_k \text{ at which } y(t_k^-) \text{ and } y(t_k^+) \text{ exist and } y(t_k^-) = y(t_k),\ k = 1,2,\dots,m\ \}.$$

Clearly, $PC(J,\mathbb{R}^n)$ is a Banach space with the norm

$$\|y\|_{PC} = \sup\{|y(t)| : t \in J\}.$$

We begin by defining what is meant by a solution of the problem (5.0.1)–(5.0.3).

Definition 5.1.1. *A function $y \in PC \cap AC(t_k, t_{k+1})$, $k = 0, \ldots, m$, is said to be a solution of the impulsive differential inclusion (5.0.1)–(5.0.3) if y satisfies the differential inclusion $y'(t) \in F(t, y(t))$ a.e. on $J \setminus \{t_1, \ldots, t_m\}$, and the conditions $y(t_k^+) - y(t_k) = I_k(y(t_k^-))$, $k = 1, 2, \ldots, m$, and $y(0) = a$.*

We will consider a map generated by (5.0.1)–(5.0.3) by associating with each initial point $y_0 \in \mathbb{R}^n$ the set

$$S_{[0,b]}(y_0) = \{y \,|\, y \text{ is a solution to } (5.0.1) - (5.0.3) \text{ on } [0, b] \text{ and } y(0) = y_0\}.$$

The following two lemma are needed in this chapter.

Lemma 5.1.1. *([68], Lemma 3.2) Let $F : [a, b] \to \mathcal{P}(Y)$ be a measurable multi-valued map and $u : [a, b] \to Y$ a measurable function. Then for any measurable $v : [a, b] \to (0, +\infty)$, there exists a measurable selection f_v of F such that for a.e. $t \in [a, b]$,*

$$|u(t) - f_v(t)| \le d(u(t), F(t)) + v(t).$$

Lemma 5.1.2. *(Mazur's Lemma, [69], Theorem 21.4) Let E be a normed space and $\{x_k\}_{k\in\mathbb{N}} \subset E$ be a sequence weakly converging to a limit $x \in E$. Then there exists a sequence of convex combinations $y_m = \sum_{k=1}^{m} \alpha_{mk} x_k$ with $\alpha_{mk} > 0$ for $k = 1, 2, \ldots, m$ and $\sum_{k=1}^{m} \alpha_{mk} = 1$, which converges strongly to x.*

We now present a Filippov type result for the problem (5.0.1)–(5.0.3).

Theorem 5.1.1. *Consider a function $x \in PC \cap AC(t_k, t_{k+1})$, $k = 1, 2, \ldots, m$, and assume that the following conditions hold.*

(5.1.1.A) The function $F : J \times \mathbb{R}^n \to \mathcal{P}_{cl}(\mathbb{R}^n)$ satisfies:
a) for all $u \in \mathbb{R}^n$, $t \mapsto F(t, u)$ is measurable,
b) the map $\gamma : t \to d(x'(t), F(t, x(t))$ is integrable.

(5.1.1.B) There exists a function $p \in L^1(J, \mathbb{R}_+)$ such that

$$H_d(F(t, u), F(t, z)) \le p(t)|u - z| \quad \text{for all } x,\ \overline{x} \in \mathbb{R}^n.$$

(5.1.1.C) There exist constants $c_k \ge 0$, $k = 1, 2, \ldots, m$, such that

$$|I_k(u) - I_k(z)| \le c_k|u - z| \quad \text{for each } u, z \in \mathbb{R}^n.$$

If $\sum_{k=1}^{m} c_k < 1$ *then for every* $y_0 \in \mathbb{R}^n$*, there exists* $y \in S(y_0)$ *with* $|y(0) - x(0)| < \delta$ *such that*

$$|x(t) - y(t)| \leq \eta(t) + \frac{\sum_{k=1}^{m} c_k \delta}{1 - \sum_{k=1}^{m} c_k} + \frac{\sum_{k=1}^{m} c_k |x(t_k)|}{1 - \sum_{k=1}^{m} c_k}$$

and

$$|x'(t) - y'(t)| \leq p(t)\eta(t) + \gamma(t)$$

for all $t \in [0, b]$*, where*

$$\eta(t) = \frac{\delta}{1 - \sum_{k=1}^{m} c_k} \exp\left(\frac{m(t)}{1 - \sum_{k=1}^{m} c_k}\right) + \int_0^t \gamma(s) \exp\left(\frac{m(t) - m(s)}{1 - \sum_{k=1}^{m} c_k}\right) ds$$

and

$$m(t) = \int_0^t p(s)ds.$$

Proof. Let $f_0 = x'$, $y_0(t) = x(0) + \int_0^t f_0(s)ds + \sum_{0<t_k<t} I_k(y_0(t_k))$, and $y_0(t_k) = x(t_k)$, $k = 1, 2, \ldots, m$. Let $U_1 : [0, b] \to \mathcal{P}(\mathbb{R}^n)$ be given by $U_0(t) = F(t, y_0(t)) \cap (f_0(t) + (\gamma(t)))B(0, 1)$. Since g and γ are measurable, Theorem III.4.1 in [22] tells us that the ball $(f_0(t) + \gamma(t))B(0, 1))$ is measurable. Moreover $F(t, y_0(t))$ is measurable and U_1 is nonempty. From Lemma 5.1.1, there exists a function u which is a measurable selection of $F(t, y_t^0)$ and such that

$$|u(t) - \bar{g}(t)| \leq d(\bar{g}(t), F(t, y_0(t)) = \gamma(t).$$

Then $u \in U_1(t)$, proving our claim. We deduce that the intersection multivalued operator $U_1(t)$ is measurable. By Lemma 1.1.2 (Kuratowski-Ryll-Nardzewski selection theorem), there exists a function $t \to f_1(t)$ which is a measurable selection for U_1. Since the multivalued operator $U_1(t)$ is measurable (see Proposition III.4 in [22]), there exists a function $t \to f_1(t)$, which is a measurable selection for U_1, and so

$$y_1(t) = y_0 + \int_0^t f_1(s)ds + \sum_{0<t_k<t} I_k(y_1(t_k)), \; y_1(t_k) = y_0(t_k), \; k = 1, 2, \ldots, m.$$

Then, we have

$$\begin{aligned}
|y_1(t)-y_0(t)| &\leq |y(0)-x(0)|+\int_0^t |f_0(s)-f_1(s)|\,ds \\
&\quad + \sum_{0<t_k<t} |I_k(y_1(t_k^-))-I_k(y_0(t_k^-))| \\
&\leq |y(0)-x(0)|+\int_0^t \gamma(s)ds \\
&\quad + \sum_{0<t_k<t} c_k|y_1(t_k)-y_0(t_k)|,
\end{aligned}$$

and so

$$\begin{aligned}
|y_1(t)-y_0(t)| &\leq \frac{1}{1-\sum_{k=1}^m c_k}\left[|x(0)-y(0)|+\int_0^t \gamma(s)ds\right] \\
&\leq \frac{1}{1-\sum_{k=1}^m c_k}\left[\delta+\int_0^t \gamma(s)ds\right].
\end{aligned}$$

Define the set valued map $U_2(t)=F(t,y_1(t))\cap B(f_1(t),p(t)|y_1(t)-y_0(t)|)$. It follows that there exists a measurable selection $f_2(t)\in U_2(t)$ so that

$$y_2(t)=y_0+\int_0^t f_2(s)ds+\sum_{0<t_k<t} I_k(y_2(t_k)),\ y_2(t_k)=y_1(t_k),\ k=1,\dots,m.$$

Hence,

$$\begin{aligned}
|y_2(t)-y_1(t)| &\leq \frac{1}{1-\sum_{k=1}^m c_k}\int_0^t |f_2(s)-f_1(s)|\,ds \\
&\leq \frac{\delta}{(1-\sum_{k=1}^m c_k)^2}\int_0^t p(s)ds+\frac{1}{(1-\sum_{k=1}^m c_k)^2}\int_0^t p(s)\int_0^s \gamma(u)duds
\end{aligned}$$

Then

$$|y_2(t)-y_1(t)| \leq \frac{\delta}{(1-\sum_{k=1}^m c_k)^2}m(t)+\frac{1}{(1-\sum_{k=1}^m c_k)^2}\int_0^t \gamma(s)[m(t)-m(s)]ds.$$

We consider $U_3(t) = F(t, y_2(t)) \cap B(f_2(t), p(t)|y_2(t) - y_1(t)|)$. By Proposition III.4 in [22], there exists a measurable selection $f_3(t) \in U_3(t)$ so that

$$y_3(t) = y_0 + \int_0^t f_3(s)ds + \sum_{0<t_k<t} I_k(y_3(t_k)),\ y_3(t_k) = y_2(t_k),\ k = 1, \dots, m.$$

Then,

$$\begin{aligned}
|y_3(t) - y_2(t)| &\leq \frac{1}{1 - \sum_{k=1}^m c_k} \int_0^t |f_2(s) - f_3(s)|\, ds \\
&\leq \frac{\delta}{(1 - \sum_{k=1}^m c_k)^3} \int_0^t p(s) e^{2\int_0^s p(\tau)d\tau} ds \\
&\quad + \frac{1}{(1 - \sum_{k=1}^m c_k)^3} \int_0^t p(s) e^{2\int_0^s p(\tau)d\tau} \int_0^s \gamma(u)[m(s) - m(u)]duds \\
&\leq \frac{\delta}{2(1 - \sum_{k=1}^m c_k)^3} m^2(t) + \frac{1}{2(1 - \sum_{k=1}^m c_k)^3} \int_0^t \gamma(s)[m(t) - m(s)]^2 ds.
\end{aligned}$$

Thus, we have

$$|y_3(t) - y_2(t)| \leq \frac{\delta}{2(1 - \sum_{k=1}^m c_k)^3} m^2(t) + \frac{1}{2(1 - \sum_{k=1}^m c_k)^3} \int_0^t \gamma(s)[m(t) - m(s)]^2 ds.$$

Proceeding by induction, we have

$$|y_{n+1}(t) - y_n(t)| \leq \frac{\delta m^n(b)}{(n+1)!\left(1 - \sum_{k=1}^m c_k\right)^{n+1}} \tag{5.1.1}$$

$$+ \frac{\int_0^b \gamma(s)[m(b) - m(s)]^n ds}{(n+1)!(1 - \sum_{k=1}^m c_k)^{n+1}}. \tag{5.1.2}$$

We deduce that $\{y_n\}$ is a Cauchy sequence in PC, converging uniformly to a function $y \in PC$. From the definition of $U_n,\ n \in \mathbb{N}$,

$$|f_{n+1}(t) - f_n(t)| \leq p(t)|y_n(t) - y_{n-1}(t)|, \text{ for } n \in \mathbb{N}, \text{a.e. } t \in [0, b].$$

Hence, for almost every $t \in [0,b]$, the sequence $\{f_n(t)\}$ is Cauchy in $\mathbb{R}^n$, and so $\{f_n(t)\}$ converges almost everywhere to a measurable function $\{f(\cdot)\}$ in $\mathbb{R}^n$. Moreover, since $f_0 = x'$, and by the last inequality, we obtain

$$\begin{aligned} |f_n(t) - f_0(t) + f_0(t)| &\leq |f_n(t) - f_0(t)| + |f_0(t)| \\ &\leq \sum_{k=1}^{n} p(t)|y_k(t) - y_{k-1}(t)| + |f_0(t)| \\ &\leq p(t)\sum_{k=1}^{\infty} |y_k(t) - y_{k-1}(t)| + |f_0(t)| \\ &\leq p(t)A + \gamma(t), \end{aligned}$$

where

$$A = \delta \frac{1}{1 - \sum_{k=1}^{m} c_k} \exp\left(\frac{m(b)}{1 - \sum_{k=1}^{m} c_k}\right) + \int_0^b \gamma(s) \exp\left(\frac{m(b) - m(s)}{1 - \sum_{k=1}^{m} c_k}\right) ds.$$

Then,

$$\text{for all } n \in \mathbb{N}, \quad |y_n(t)| \leq Ap(t) + \gamma(t). \tag{5.1.3}$$

From (5.1.3), we conclude that f_n converges to f in $L^1([0,b], \mathbb{R}^n)$. Consequently,

$$y(t) = y_0 + \int_0^t f(s)ds + \sum_{0<t_k<t} I_k(y(t_k)), \ t \in [0,b],$$

is a solution of the problem (5.0.1)–(5.0.2) with the condition $y(0) = y_0$. Thus,

$$y \in S_{[0,b]}(y_0).$$

Next, we prove that

$$|x(t) - y(t)| \leq \eta(t) + \frac{\sum_{k=1}^{m} c_k\delta}{1 - \sum_{k=1}^{m} c_k} + \frac{\sum_{k=1}^{m} c_k|x(t_k)|}{1 - \sum_{k=1}^{m} c_k}, \text{ for } t \in [0,b].$$

Hence,

$$|x(t)-y(t)| \leq \frac{1}{1-\sum_{k=1}^{m} c_k}\Bigg[\delta+\int_0^t |f(s)-f_n(s)|ds$$
$$+\int_0^t p(s)\delta \exp\left(\frac{m(s)}{1-\sum_{k=1}^{m} c_k}\right) ds$$
$$+\int_0^t p(s)\int_0^s \gamma(\tau)d\tau \exp\left(\frac{m(t)-m(s)}{1-\sum_{k=1}^{m} c_k}\right) ds+\sum_{0<t_k<t} c_k|x(t_k)|\Bigg].$$

Therefore

$$|x(t)-y(t)| \leq \frac{1}{1-\sum_{k=1}^{m} c_k}\delta+\frac{1}{1-\sum_{k=1}^{m} c_k}\int_0^t |f(s)-f_n(s)|ds+\exp\left(\frac{m(t)}{1-\sum_{k=1}^{m} c_k}\right)\delta$$
$$-\delta-\int_0^t [\exp\left(\frac{m(t)-m(s)}{1-\sum_{k=1}^{m} c_k}\right)]'\int_0^s \gamma(\tau)d\tau ds+\sum_{0<t_k<t} c_k|x(t_k)|\Bigg].$$

So

$$|x(t)-y(t)| \leq \frac{\delta}{1-\sum_{k=1}^{m} c_k}+\frac{1}{1-\sum_{k=1}^{m} c_k}\int_0^b |f(s)-f_n(s)|ds+\exp\left(\frac{m(t)}{1-\sum_{k=1}^{m} c_k}\right)\delta$$
$$-\delta+\int_0^t \exp\left(\frac{m(t)-m(s)}{1-\sum_{k=1}^{m} c_k}\right)\gamma(s)ds+\frac{1}{1-\sum_{k=1}^{m} c_k}\sum_{k=1}^{m} c_k|x(t_k)|.$$

Then

$$\begin{aligned}|x(t)-y(t)| &\leq \frac{\sum_{k=1}^m c_k}{1-\sum_{k=1}^m c_k}\delta + \frac{1}{1-\sum\limits_{k=1}^m c_k}\int_0^b |f(s)-f_n(s)|ds \\ &+\eta(t) + \frac{\sum_{k=1}^m c_k}{1-\sum_{k=1}^m c_k}|x(t_k)|.\end{aligned}$$

Letting $n \to \infty$ yields

$$|x(t)-y(t)| \leq \eta(t) + \frac{\sum_{k=1}^m c_k\delta}{1-\sum_{k=1}^m c_k} + \frac{\sum_{k=1}^m c_k|x(t_k)|}{1-\sum_{k=1}^m c_k}.$$

Finally, we prove that

$$|x'(t)-y'(t)| \leq p(t)\eta(t)+\gamma(t), \quad \text{a.e. } t\in[0,b].$$

$$\begin{aligned}|x'(t)-y'(t)| &\leq |f_0(s)-f_n(s)|+|f_n(t)-f(t)| \\ &\leq \sum_{k=1}^n p(t)|y_k(t)-y_{k-1}(t)|+|f_n(t)-f(t)| \\ &\leq p(t)\sum_{k=1}^\infty |y_k(t)-y_{k-1}(t)|+|f_n(t)-f(t)| \\ &\leq |f_n(t)-f(t)|+p(t)\sum_{n=0}^\infty \frac{\delta m^n(t)}{n!(1-\sum\limits_{k=1}^m c_k)^{n+1}} \\ &+p(t)\sum_{n=0}^\infty \frac{\int_0^t \gamma(s)[m(s)-m(u)]^n ds}{n!(1-\sum\limits_{k=1}^m c_k)^{n+1}} + |f_1(t)-f_0(t)| \\ &\leq p(t)\eta(t)+\gamma(t)+|f_n(t)-f(t)|.\end{aligned}$$

Thus,

$$|x'(t)-y'(t)| \leq p(t)\eta(t)+\gamma(t).$$

Set

$$PCA([0,b],\mathbb{R}^n)=\{y:[0,b]\to\mathbb{R}^n : y\in AC(J_k,\mathbb{R}^n)\},$$

where $J_k=[t_k,t_{k+1}],\ k=0,\dots,m$, which is a Banach space with the norm

$$\|y\|_{APC}=\sup_{t\in[0,b]}|y(t)|+\int_0^b |y'(t)|dt.$$

□

Corollary 5.1.1. *Assume that the conditions (5.1.1.B), (5.1.1.C) and the first part of (5.1.1.A) hold. In addition, assume that there exists* $f \in L^1(J, \mathbb{R}_+)$ *such that*

$$H_d(0, F(t,0)) \le f(t) \quad \text{for a.e.} \quad t \in J.$$

Then the set-valued map $S_{[0,b]} : \mathbb{R}^n \to \mathcal{P}(APC)$ *is Lipschitz with Lipschitz constant*

$$L = \frac{1 - \left(1 - \sum_{k=1}^{m} c_k\right)^2}{1 - \sum_{k=1}^{m} c_k} + \left(2 - \sum_{k=1}^{m} c_k\right) \exp\left(\frac{m(b)}{1 - \sum_{k=1}^{m} c_k}\right).$$

Moreover, if F *has compact and convex values, then* $S_{[0,b]}(\cdot) \in \mathcal{P}_{cp}(PC)$*, that is,* $S_{[0,b]}$ *is compact in* PC.

Proof. We show that

$$H_d(S_{[0,b]}(x_0), S_{[0,b]}(y_0)) \le L|x_0 - y_0|, \text{ for all } x_0, y_0 \in \mathbb{R}^n.$$

Let $x_0, y_0 \in \mathbb{R}^n$ and $y \in S_{[0,b]}(y_0)$. Then there exists $f \in L^1([0,b], \mathbb{R}^n)$ such that $f(t) \in F(t, y(t))$, and

$$y(t) = y_0 + \int_0^t f(s)ds + \sum_{0<t_k<t} I_k(y(t_k)).$$

Then $\gamma(t) = d(y'(t), F(t, y(t)) = 0$. From Theorem 5.1.1, there exist $x \in S_{[0,b]}(x_0)$ and $f_1(t) \in F(t, x(t))$, such that

$$x(t) = x_0 + \int_0^t f_1(s) + \sum_{0<t_k<t} I_k(x(t_k))$$

and

$$\begin{aligned} |x(t) - y(t)| &\le \frac{|x_0 - y_0|}{1 - \sum_{k=1}^{m} c_k} + |x_0 - y_0| \exp\left(\frac{m(t)}{1 - \sum_{k=1}^{m} c_k}\right) \\ &\le \left(\frac{1}{1 - \sum_{k=1}^{m} c_k} + \exp\left(\frac{m(t)}{1 - \sum_{k=1}^{m} c_k}\right)\right) |x_0 - y_0|. \end{aligned}$$

Also from Theorem 5.1.1, we have

$$\begin{aligned}\int_0^t |x'(s)-y'(s)|ds &\leq \int_0^t p(s)\exp\left(\frac{m(s)}{1-\sum_{k=1}^{m}c_k}\right)ds\\ &= \left(1-\sum_{k=1}^{m}c_k\right)\int_0^t \frac{1}{m(s)}\left[\exp\left(\frac{m(s)}{1-\sum_{k=1}^{m}c_k}\right)\right]'|x_0-y_0|ds\\ &\leq (1-\sum_{k=1}^{m}c_k)\left(\exp\left(\frac{m(t)}{1-\sum_{k=1}^{m}c_k}\right)-1\right)|x_0-y_0|.\end{aligned}$$

By an analogous relation, obtained by interchanging the roles of x_0 and y_0, it follows that

$$\begin{aligned}H_d(S_{[0,b]}(x_0),S_{[0,b]}(y_0)) &\leq \left[\frac{1-\left(1-\sum_{k=1}^{m}c_k\right)^2}{1-\sum_{k=1}^{m}c_k}\right.\\ &\quad\left.+\left(2-\sum_{k=1}^{m}c_k\right)\exp\left(\frac{m(b)}{1-\sum_{k=1}^{m}c_k}\right)\right]|x_0-y_0|.\end{aligned}$$

Now, we shall prove that for every $x_0\in\mathbb{R}^n$, $S_{[0,b]}(x_0)$ is compact. As in we can see that $S_{[0,b]}(x_0)$. Let $y_n\in S_{[0,b]}(x_0)$; then there exist $g_n\in S_{F,y_n}$ such that, for each $t\in J$,

$$y_n(t) = x_0+\int_0^t g_n(s)ds+\sum_{0<t_k<t}I_k(y_n(t_k^-)),\quad t\in[0,b].$$

Since there exists $f\in L^1([0,b],\mathbb{R}^n)$ such that $f(t)\in F(t,0)$ a.e. $t\in[0,b]$, by (5.1.1.B), we have

$$\begin{aligned}|g_n(t)| &\leq H_d(F(t,0),F(t,y_n(t)))+|f(t)|\\ &\leq p(t)|y_n(t)|+|f(t)|\\ &\leq Mp(t)+|f(t)|,\ \text{a.e. } t\in[0,b],\end{aligned}\tag{5.1.4}$$

where $|y_n(t)| \leq M$ for all $n \in \mathbb{N}$ for some $M > 0$. For $t \in [0, t_1]$, we have

$$y_n(t) = x_0 + \int_0^t g_n(s)ds,$$

and hence

$$|y_n(t)| \leq |x_0| + \int_0^t |g_n(s)|ds. \tag{5.1.5}$$

By the above inequality and (5.1.4), we obtain

$$|y_n(t)| \leq |x_0| + M\int_0^t p(s)ds + \int_0^t |f(s)|ds, \quad t \in [0, t_1]. \tag{5.1.6}$$

Hence,

$$\sup\{|y_n(t)| : t \in [0, t_1]\} \leq |x_0| + M\|p\|_{L^1} + \|f\|_{L^1} := K_0, \text{ for all } n \geq 1.$$

Let $t \in (t_1, t_2]$; then

$$y_n(t) = \int_{t_1}^t g_n(s)ds + y_n(t_1) + I_1(y(t_1)).$$

It is clear that

$$y_n(t_1^+) = y_n(t_1) + I_1(y_n(t_1)),$$

Thus,

$$\begin{aligned} |y_n(t_1^+)| &\leq |y_n(t_1)| + |I_1(y_n(t_1))| \\ &\leq K_1 + \sup\{|x| : x \in B(0, K_0)\}, \end{aligned}$$

where

$$B(0, K_0) = \{x \in \mathbb{R}^n : |x| \leq K_0\}.$$

Analogous to what we did above, we can show that there exists $\widetilde{K}_2 > 0$ such that

$$\sup\{|y(t)| : t \in [t_1, t_2]\} \leq K_1.$$

We continue this process and also take into account that

$$y_n(t) = \int_{t_m}^t g_n(s)ds + y_n(t_m) + I_m(y(t_m)), \ t \in (t_m, b].$$

We obtain that there exists a constant K_m such that

$$\sup\{|y_n(t)| : t \in [t_m, b]\} \leq K_m.$$

Consequently, for each $n \geq 1$ we have

$$\|y_n\|_{PC} \leq \max\{K_i : i = 1, \ldots, m\} := \overline{K}.$$

By the Arzela-Ascoli theorem, we can conclude that $\{y_n : n \geq 1\}$ is compact in PC, so there exists y such that $y_n \to y$. We shall prove that $y \in S_{[0,b]}(x_0)$. We need to show that there exists $g \in S_{F,y}$ such that for each $t \in J$,

$$y(t) \;=\; x_0 + \int_0^t g(s)ds + \sum_{0<t_k<t} I_k(y(t_k^-)).$$

Since $g_n(t) \in (Mp(t) + f(t))B(0,1)$ then there exists subsequence of g_n converge to g.

It remains to prove that $g(t) \in F(t, y(t)),$ for a.e. $t \in J$. Lemma 5.1.2 yields the existence of $\alpha_i^n \geq 0,\ i = n, \ldots, k(n)$ such that $\sum_{i=1}^{k(n)} \alpha_i^n = 1$ and the sequence of convex combinaisons $v_n(\cdot) = \sum_{i=1}^{k(n)} \alpha_i^n g_i(\cdot)$ converges strongly to v in L^1. Since F takes convex values, using Lemma 1.1.2, we obtain that

$$\begin{aligned} v(t) &\in \bigcap_{n\geq 1} \overline{\{v_n(t)\}}, \quad \text{a.e. } t \in J \\ &\subset \bigcap_{n\geq 1} \overline{co}\{g_k(t),\ k \geq n\} \\ &\subset \bigcap_{n\geq 1} \overline{co}\{\bigcup_{k\geq n} F(t, y_k(t))\} \\ &= \overline{co}(\limsup_{k\to\infty} F(t, y_k(t))). \end{aligned} \tag{5.1.7}$$

Since F is H_d-u.s.c. with compact values, then by Lemma 1.1.1, we have

$$\limsup_{n\to\infty} F(t, y_n(t)) = F(t, y(t), \quad \text{for a.e. } t \in J.$$

This with (5.1.7) imply that $v(t) \in \overline{co}\, F(t, y(t))$. By Lebesgue dominated convergence theorem and since I_k, $k = 1, 2, \ldots, m$, are continuous, we have

$$\begin{aligned} &\Big\| \Big(y_n(t) - x_0 - \sum_{0<t_k<t} I_k(y_n(t_k^-)) - \int_0^t g_n(s)ds \Big) \\ &- \Big(y(t) - x_0 - \sum_{0<t_k<t} I_k(y(t_k^-)) - \int_0^t g(s)ds \Big) \Big\|_{PC} \to 0, \text{ as } n \to \infty. \end{aligned}$$

Therefore,

$$y(t) = x_0 + \sum_{0<t_k<t} I_k(y(t_k^-)) + \int_0^t g(s)ds, \quad t \in [0, b].$$

□

5.2 Relaxation Theorem

In this subsection, we examine to what extent the convexification of the right-hand side of the inclusion introduces new solutions. More precisely, we want to find out if the solutions of the nonconvex problem are dense in those of the convex one. Such a result is known in the literature as a relaxation theorem and has important implications in optimal control theory. It is well-known that in order to have optimal state-control pairs, the system has to satisfy certain convexity requirements. If these conditions are not present, then in order to guarantee existence of optimal solutions we need to pass to an augmented system with convex structure by introducing the so-called relaxed (generalized, chattering) controls. The resulting relaxed problem has a solution. The relaxation theorems tell us that the relaxed optimal state can be approximated by the original states, which are generated by a more economical set of controls that are much simpler to build. In particular, "strong relaxation" theorems imply that this approximation can be achieved using states generated by bang-bang controls. More precisely, we compare trajectories of (5.0.1)–(5.0.3) to those of the relaxation impulsive differential inclusion,

$$y'(t) \in \overline{co}F(t, y(t)), \quad a.e.\ t \in J =: [0, b]\backslash\{t_1, \ldots, t_m\}, \tag{5.2.1}$$

$$y(t_k^+) - y(t_k) = I_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{5.2.2}$$

$$y(0) = y_0. \tag{5.2.3}$$

The following result is known.

Theorem 5.2.1. ([37]) *Let $U : [0, b] \to \mathcal{P}_{cl}(\mathbb{R}^n)$ be a measurable, integrable bounded set-valued map and let $t \to d(0, U(t))$ be an integrable map. Then the integral $\int_0^b U(t)dt$ is convex, the mapping $t \to \overline{co}U(t)$ is measurable, and for every $\epsilon > 0$, and every measurable selection of u of $\overline{co}U(t)$, there exists a measurable selection $\overline{u}$ of U such that*

$$\sup_{t\in[0,b]} \Big| \int_0^t u(s)ds - \int_0^t \overline{u}(s)ds \Big| \leq \epsilon$$

and

$$\overline{\int_0^t \overline{co}U(t)dt} = \overline{\int_0^t U(t)dt} = \int_0^t \overline{co}U(t)dt.$$

Our relaxation result in this subsection is the following.

Theorem 5.2.2. *Assume that the conditions (5.1.1.A)–(5.1.1.C) are satisfied. If $\sum_{k=1}^m c_k < 1$ then for every $\delta > 0$, there exists a solution y to the problem (5.0.1)–(5.0.3) on $[0, b]$ satisfying*

$$x(0) = y(0) \text{ and } \|x - y\|_{PC} \leq \delta.$$

This implies that $S^{co}_{[0,b]}(y_0) = \overline{S_{[0,b]}(y_0)}$, *where*

$$S^{co}_{[0,b]} = \{y \mid y \text{ is a solution to (5.2.1)–(5.2.3) on } [0,b],\ y(0) = y_0\}.$$

Proof. Let y be a solution of the problem (5.2.1)–(5.2.3). Let $\epsilon > 0$ be given and let y' be an integral solution of $t \to \overline{co}F(t, y(t))$. Let $\delta = \dfrac{\epsilon}{\exp\Big(\frac{m(b)}{1-\sum_{k=1}^{m} c_k}\Big)}$, From Theorem 5.2.1, there exists a measurable selection f of $t \to F(t, y(t))$ such that

$$\sup_{t\in[0,b]} \Big| \int_0^t f(s)ds - \int_0^t y'(s)ds \Big| \leq \delta.$$

Set

$$x(t) = x(0) + \int_0^t f(s)ds + \sum_{0<t_k<t} I_k(x(t_k)),\ x(t_k) = y(t_k),\ k = 1, 2, \ldots, m.$$

Then,

$$d(x'(t), F(t, x(t)) \leq p(t)|x(t) - y(t)| \leq p(t)\delta,\ a.e\ t \in [0,b].$$

From Theorem 5.1.1, there exists a solution y_* of the problem (5.0.1)–(5.0.3) such that

$$\begin{aligned} |y_*(t) - y(t)| &\leq \frac{1}{1 - \sum_{k=1}^{m} c_k} \int_0^t \delta p(s) \exp\Big(\frac{m(t) - m(s)}{1 - \sum_{k=1}^{m} c_k}\Big) ds \\ &\leq \delta \exp\Big(\frac{m(b)}{1 - \sum_{k=1}^{m} c_k}\Big) \\ &\leq \epsilon \end{aligned}$$

for all $t \in [0,b]$. Thus, $S^{co}_{[0,b]}(y_0) = \overline{S_{[0,b]}(y_0)}$. □

Chapter 6

Impulsive Semilinear Differential Inclusions

In this chapter we consider the system with impulse effects

$$x'(t) - A_1 x(t) \in F_1(t, x(t), y(t)), \quad y'(t) - A_2 y(t) \in F_2(t, x(t), y(t)), \quad t \in [0, b], \quad (6.0.1)$$

$$x(t_k^+) - x(t_k^-) \in I_k(x(t_k)), \quad y(t_k^+) - y(t_k^-) \in \overline{I}_k(y(t_k)), \quad k = 1, \ldots, m \qquad (6.0.2)$$

$$x(0) = x_0, \quad y(0) = y_0, \qquad (6.0.3)$$

where $J := [0, b]$, E is a Banach space, $F_1, F_2 \colon J \times E \times E \to \mathcal{P}(E)$ are a multifunctions, $x_0, y_0 \in E$. $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = b$. The operators $A_i,\ i = 1, 2$ are infinitesimal generator of a C_0-semigroup $\{T_i(t)\}_{t\geq 0}$ on a Banach space $(E, |\cdot|)$ respectively, $I_k, \overline{I}_k : E \to \mathcal{P}(E)\,(k = 1, \ldots, m)$, and $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$. The notations $y(t_k^+) = \lim\limits_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim\limits_{h\to 0^+} y(t_k - h)$ stand for the right and the left limits of the function y at $t = t_k$, respectively.

In the absence of impulses and multimap framework the above system was used to study the initial value problems and boundary value problems for nonlinear competitive or cooperative differential systems from mathematical biology [46] and mathematical economics [51] where the model can be set in the operator form (6.0.1)-(6.0.3).

In [47, 59] the authors present existence and uniqueness results for system of semilinear differential equations without impulses. Recently Precup [58] proved the role of matrix convergence and vector metric in the study of semilinear operator systems.

6.1 Mild Solutions

In order to define mild solutions for problem (6.0.1) − (6.0.3), we consider the space

$$PC = \{z\colon [0,b] \to E,\ z_k \in C(J_k, E),\ k = 0,\dots,m, \text{ such that } z(t_k^-) \text{ and } z(t_k^+) \text{ exist and satisfy } z(t_k^-) = z(t_k) \text{ for } k = 1,\dots,m\}.$$

Endowed with the norm

$$\|z\|_{PC} = \max\{\|z_k\|_\infty, \quad k = 0,\dots,m\},$$

PC is a Banach space. Throughout this chapter, A is an infinitesimal generator of a C_0 − semigroup $\{T_i(t)\}_{t\geq 0}, i = 1,2$ and there exists $M > 0$ such that

$$\|T_i(t)\| \leq M \qquad t \in \mathbb{R}_+.$$

Definition 6.1.1. *A function $(x,y) \in PC\times PC$ is said to be a mild solution of problem (6.0.1)-(6.0.3) if there exists $v_1, v_2 \in L^1(J,E)$ such that $v_i(t) \in F_i(t,x(t),y(t))$ $i = 1,2$ a.e. on J, and $\mathcal{I}_k(x(t_k)) \in I_k(x(t_k))$, $\overline{\mathcal{I}}_k(x(t_k)) \in \overline{I}_k(x(t_k))$, $k = 1,\dots,m$*

$$x(t) = T_1(t)x_0 + \int_0^t T_1(t-s)v_i(s)ds + \sum_{0<t_k<t} T(t-t_k)\mathcal{I}_k(x(t_k^-)).$$

and

$$y(t) = T_2(t)y_0 + \int_0^t T_2(t-s)v_i(s)ds + \sum_{0<t_k<t} T_2(t-t_k)\overline{\mathcal{I}}_k(x(t_k^-)).$$

6.2 Existences result

Let $(E, |\cdot|)$ be a separable Banach space and $F_i\colon J\times E\times E \to \mathcal{P}_{cl,b,cv}(E), i = 1,2$ are Carathéodory multimaps which satisfies some of the following assumptions:

($\mathcal{G}_1$) There exist a functions $p_i \in L^1(J,\mathbb{R}^+)$ and a continuous nondecreasing functions $\psi_i\colon [0,\infty) \to [0,\infty), i = 1,2$ such that

$$\|F_i(t,x,y)\|_{\mathcal{P}} \leq p(t)\psi_i(|x|+|y|) \text{ for a.e. } t \in J \text{ and each } x,y \in E,$$

with

$$\int_0^b p_i(s)ds < \int_1^\infty \frac{du}{\psi(u)}.$$

($\mathcal{G}_2$) $I_k,\ \overline{I}_k\colon E \to \mathcal{P}_{cv}(E), k = 1,\dots,m$ are closed and there exist constants $c_k, \overline{c}_k > 0$ and continuous functions $\phi_k, \overline{\phi}_k : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|I_k(z)\|_{\mathcal{P}} \leq c_k\phi_k(|z|) \text{ for each } z \in E,\ k = 1,\dots,m$$

and

$$\|\overline{I}_k(z)\|_{\mathcal{P}} \leq \overline{c}_k\overline{\phi}_k(|z|) \text{ for each } z \in E,\ k = 1,\dots,m.$$

$(\mathcal{G}_3)$ The semigroup $T_i(\cdot),\ i = 1, 2$ is compact for $t > 0$.

The following result is known as Gronwal-Bihari Theorem.

Lemma 6.2.1. *[21] Let $u,\ \bar{g}\colon\ [a, b] \longrightarrow \mathbb{R}$ be positive real continuous functions. Assume there exist $c > 0$ and a continuous nondecreasing function $\phi : \mathbb{R}_+ \longrightarrow (0, +\infty)$ such that*

$$u(t) \leq c + \int_a^t \bar{g}(s)\phi(u(s))\, ds, \quad \forall\, t \in J.$$

Then

$$u(t) \leq H^{-1}\left(\int_a^t \bar{g}(s)\, ds\right), \quad \forall\, t \in J$$

provided

$$\int_c^{+\infty} \frac{dy}{\phi(y)} > \int_a^b \bar{g}(s)\, ds.$$

Here H^{-1} refers to the inverse of the function $H(u) = \int_c^u \frac{dy}{\phi(y)}$ for $u \geq c$.

Theorem 6.2.1. *Assume that F satisfies either $(\mathcal{G}_1)$, $(\mathcal{G}_2)$ and $(\mathcal{G}_3)$. Then the set of solutions for problem (6.0.1)-(6.0.3) is nonempty. compact.*

Proof. Consider the operator $N\colon\ PC \times PC \to \mathcal{P}(PC \times PC)$ defined for $(x, y) \in PC \times PC$ by

$$N(x,y) = \left\{ (h_1, h_2) \in PC \times PC :\ (h_1(t), h_2(t)) = \left\{ \begin{array}{r} T_1(t)x_0 + \displaystyle\int_0^t T_1(t-s)v_1(s)ds \\ + \displaystyle\sum_{0<t_k<t} T_1(t-t_k)\mathcal{I}_k(y(t_k)),\ t \in J \\ \\ T_2(t)y_0 + \displaystyle\int_0^t T_2(t-s)v_2(s)ds \\ + \displaystyle\sum_{0<t_k<t} T_2(t-t_k)\overline{\mathcal{I}}_k(y(t_k)),\ t \in J \end{array} \right. \right\},$$

where $v_i \in S_{F_i,x,y} = \{v \in L^1(J, E)\colon\ f(t) \in F_i(t, x(t), y(t)),\ a.e.\ t \in J\}$ and $\mathcal{I}_k(x(t_k)) \in I_k(x(t_k)), \overline{\mathcal{I}}_k(y(t_k)) \in I_k(y(t_k)),\ k = 1, \ldots, m$. Clearly, fixed points of the operator N are solutions of Problem (6.0.1)-(6.0.3). Let

$$N_1(x,y) = \left\{ h_1 \in PC :\ h_1(t) = \left\{ \begin{array}{r} T_1(t)x_0 + \displaystyle\int_0^t T_1(t-s)v_1(s)ds \\ + \displaystyle\sum_{0<t_k<t} T_1(t-t_k)\mathcal{I}_k(y(t_k)),\ t \in J \end{array} \right. \right\}$$

and

$$N_2(x,y)=\left\{h_2\in PC:\ h_2(t)=\left\{\begin{array}{c} T_2(t)y_0+\displaystyle\int_0^t T_2(t-s)v_2(s)ds \\ +\displaystyle\sum_{0<t_k<t} T_2(t-t_k)\overline{\mathcal{I}}_k(y(t_k)),\ t\in J \end{array}\right.\right\}.$$

Hence

$$N(x,y)=(N_1(x,y),N_2(x,y))\quad \text{for every } (x,y)\in PC\times PC.$$

Since, for each $(x,y)\in PC\times PC$, the nonlinearity F takes convex values, the selection set $S_{F,x,y}$ is convex and by $(\mathcal{G}_2)$, then N has convex values. From $(\mathcal{G}_1)$ and $(\mathcal{G}_2)$, we can prove that N maps bounded sets.

Step 1. N maps bounded sets into equicontinuous sets of $PC\times PC$ into bounded sets. It suffices to prove that $N(\mathcal{B}_q\times\mathcal{B}_q)$ is relatively compact in $PC\times PC$, where $\mathcal{B}_q=\{z\in PC:\ \|z\|_{PC}\le q\}$. First, $N(\mathcal{B}_q\times\mathcal{B}_q)$ is an equicontinuous set of $PC\times PC$. To see this, let $0<\tau_1<\tau_2\le b$, $(x,y)\in\mathcal{B}_q\times\mathcal{B}_q$, and $(h_1,h_2)\in N(x,y)$. Then there exist $v_i\in S_{F_i,x,y}, i=1,2$ and $\mathcal{I}_k(x(t_k))\in I_k(x(t_k)),\overline{\mathcal{I}}_k(y(t_k))\in I_k(y(t_k)),\ k=1,\dots,m,$ such that

$$h_1(t)=T_1(t)x_0+\int_0^t T_1(t-s)v_1(s)ds+\sum_{0<t_k<t}T_1(t-t_k)\mathcal{I}_k(x(t_k)),\ t\in J,$$

and

$$h_2(t)=T_2(t)y_0+\int_0^t T_2(t-s)v_2(s)ds+\sum_{0<t_k<t}T_2(t-t_k)\overline{\mathcal{I}}_k(y(t_k)),\ t\in J.$$

Letting $d_k=\sup_{|r|\le q}\phi_k(r)$ and $\bar{d}_k=\sup_{|r|\le q}\overline{\phi}_k(r)$, we obtain the estimates

$$\begin{aligned}|h_1(\tau_2)-h_1(\tau_1)| &\le |T(\tau_2)x_0-T(\tau_1)x_0|\\ &+\int_0^{\tau_1}\|T_1(\tau_2-s)-T_1(\tau_1-s)\|_{B(E)}p_1(s)\psi_1(q)ds\\ &+\int_{\tau_1}^{\tau_2}\|T_1(\tau_2-s)\|_{B(E)}p_1(s)\psi_1(q)ds+\sum_{k=1}^m T_1(\tau_2-\tau_1)I_k(y_k)\\ &+\sum_{0<t_k<\tau_1}d_k\|T_1(\tau_1-t_k)-T_1(\tau_2-t_k)\|_{B(E)}.\end{aligned}$$

Hence

$$\begin{aligned}|h_1(\tau_2)-h_1(\tau_1)| &\le \|T_1(\tau_2-\tau_1)-Id\|_{B(E)}|x_0|\\ &+M_1\psi_1(q)\|T(\tau_2-\tau_1)-Id\|_{B(E)}\int_0^{\tau_1}p_1(s)ds\\ &+M_1\psi_1(q)\int_{\tau_1}^{\tau_2}p_1(s)ds+M_1\sum_{\tau_1<t_k<\tau_2}d_k\\ &+\|T_1(\tau_2-\tau_1)-Id\|_{B(E)}\sum_{0<t_k<\tau_1}d_k.\end{aligned}$$

The terms in the right-hand side tend to zero as $\tau_1 - \tau_2 \to 0$. Now we show that $H_1(t) = \{N_1(x(t), y(t)) : t \in J,\ (x, y) \in \mathcal{B}_q \times \mathcal{B}_q\}$ is precompact set in E. Let $0 < t \le b$ and $0 < \epsilon < t$ then for $(x, y) \in \mathcal{B}_q \times \mathcal{B}_q$ we have

$$\begin{aligned}(N_\epsilon(x(t), y(t))(t) &= \{T_1(t)x_0 + \textstyle\int_0^{t-\epsilon} T_1(t-s)v_1(s)ds + \displaystyle\sum_{0<t_k<t} T_1(t-t_k)I_k(x(t_k))\}\\ &= T_1(\epsilon)\{T(t-\epsilon)x_0 + \textstyle\int_0^{t-\epsilon} T_1(t-\epsilon-s)v_1(s)ds\\ &\quad + \displaystyle\sum_{0<t_k<t} T_1(t-\epsilon+t_k)I_k(x(t_k))\}.\end{aligned}$$

Since $T(\epsilon)$ is compact then the set

$$\begin{aligned}\widetilde{H}_\epsilon(t) &= \{(N_\epsilon(x(t), y(t)) :\ (x, y) \in \mathcal{B}_q \times \mathcal{B}_q\}\\ &= T_1(\epsilon)\{T_1(t-\epsilon)x_0 + \textstyle\int_0^{t-\epsilon} T_1(t-\epsilon-s)v_1(s)ds\\ &\quad + \displaystyle\sum_{0<t_k<t} T_1(t-\epsilon+t_k)\mathcal{I}_k(x(t_k)),\\ &\quad (x, y) \in \mathcal{B}_q \times \mathcal{B}_q,\ v_1 \in S_{F,x,y},\ \mathcal{I}_k(x(t_k)) \in I_k(x(t_k))\},\end{aligned}$$

is precompact in E. Moreover for every $(h_1(t), h_\epsilon(t)) \in N_1(x(t), y(t)) \times N_\epsilon(x(t), y(t))$ such that

$$h_1(t) = T_1(t)x_0 + \textstyle\int_0^t T_1(t-s)v_1(s)ds + \displaystyle\sum_{0<t_k<t} T_1(t-t_k)\mathcal{I}_k(x(t_k))\}$$

and

$$h_\epsilon(t) = T_1(t)x_0 + \textstyle\int_0^{t-\epsilon} T_1(t-s)v_1(s)ds + \displaystyle\sum_{0<t_k<t} T_1(t-t_k)\mathcal{I}_k(x(t_k)),$$

we have

$$|h_1(t) - h_\epsilon(t)| \le M_1 \int_{t-\epsilon}^t p(s)\psi(r)ds$$

which tends to 0 as $\epsilon \to 0$. Therefore, there are precompact sets arbitrarily closed to the set $H(t)$. Then $H(t)$ is precompact in E. It is clear that $H(0) = \{u_0\}$ is precompact in X. Hence for each $t \in [0, b]$ the set $H(t)$ is precompact in E.

By the Arzelá-Ascoli theorem, we conclude that $N_i\colon PC \times PC \to \mathcal{P}_{cp,cv}(PC)$ are completely continuous operators.

Step 2. N has a closed graph.

Let $(x_n, y_n) \to (x, y), h_n \in N(x_n, y_n)$ and $h_n := (h_n^1, h_n^2) \to h := (h_1, h_2)$. We shall prove that $h_* \in N(x, y)$. Now $h_n \in N(x_n, y_n)$ means there exist $v_n^i \in S_{F_i,x,y_n}, i = 1, 2$ and $\mathcal{I}_k^n(x(t_k)) \in I_k(x_n(t_k)), \overline{\mathcal{I}}_k^n(y_n(t_k)) \in I_k(y_n(t_k)),\ k = 1, \ldots, m,$ such that

$$h_n^1(t) = T_1(t)x_0 + \int_0^t T_1(t-s)v_n^1(s)ds + \sum_{0<t_k<t} T_1(t-t_k)\mathcal{I}_k^n(x_n(t_k)),\ t \in J,$$

and

$$h_n^2(t) = T_2(t)y_0 + \int_0^t T_2(t-s)v_n^2(s)ds + \sum_{0<t_k<t} T_2(t-t_k)\overline{\mathcal{I}}_k^n(y_n(t_k)),\ t \in J.$$

Writing h_n^1 and h_n^2 in the form

$$h_n^1(t) = \begin{cases} L_0(v_n^1)(t), & \text{if } t \in [0, t_1], \\ L_1(v_n^1)(t), & \text{if } t \in (t_1, t_2], \\ \dots & \dots \\ L_{m-1}(v_n^1)(t), & \text{if } t \in (t_{m-1}, t_m], \end{cases} \tag{6.2.1}$$

where

$$\begin{aligned}
L_0(v_n^1)(t) &= T_1(t)x_0 + \int_0^t T_1(t-s)v_n^1(s)ds,\ t \in [0, t_1] \\
L_1(v_n^1)(t) &= T_1(t-t_1)[L_0(v_n^1)(t_1) + \mathcal{I}_1^n(L_0(v_n^1)(t_1))] + \int_{t_1}^t v_n^1(s)ds,\ t \in (t_1, t_2] \\
L_2(v_n^1)(t) &= T_1(t-t_2)[L_1(v_n^1)(t_2) + \mathcal{I}_2^n(L_1(v_n^1)(t_2))] + \int_{t_2}^t v_n^1(s)ds,\ t \in (t_2, t_3] \\
\dots & \dots \dots \\
L_{m-1}(v_n^1)(t) &= T_1(t-t_{m-1})[L_{m-2}(v_n^1)(t_{m-1}) + \mathcal{I}_{m-1}^n(L_{m-2}(v_n^1)(t_{m-1}))] \\
&\quad + \int_{t_{m-1}}^t v_n^1(s)ds,\ t \in (t_{m-1}, t_m].
\end{aligned}$$

and

$$h_n^2(t) = \begin{cases} \bar{L}_0(v_n^2)(t), & \text{if } t \in [0, t_1], \\ \bar{L}_1(v_n^2)(t), & \text{if } t \in (t_1, t_2], \\ \dots & \dots \\ \bar{L}_{m-1}(v_n^2)(t), & \text{if } t \in (t_{m-1}, t_m], \end{cases} \tag{6.2.2}$$

where

$$\begin{aligned}
\bar{L}_0(v_n^2)(t) &= T_2(t)y_0 + \int_0^t T_2(t-s)v_n^1(s)ds,\ t \in [0, t_1] \\
\bar{L}_1(v_n^2)(t) &= T_2(t-t_1)[L_0(v_n^2)(t_1) + \mathcal{I}_1^n(L_0(v_n^2)(t_1))] + \int_{t_1}^t v_n^2(s)ds,\ t \in (t_1, t_2] \\
\bar{L}_2(v_n^2)(t) &= T_2(t-t_2)[L_1(v_n^2)(t_2) + \mathcal{I}_2^n(L_1(v_n^2)(t_2))] + \int_{t_2}^t v_n^2(s)ds,\ t \in (t_2, t_3] \\
\dots & \dots \dots \\
\bar{L}_{m-1}(v_n^2)(t) &= T_2(t-t_{m-1})[L_{m-2}(v_n^2)(t_{m-1}) + \mathcal{I}_{m-1}^n(L_{m-2}(v_n^2)(t_{m-1}))] \\
&\quad + \int_{t_{m-1}}^t v_n^1(s)ds,\ t \in (t_{m-1}, t_m].
\end{aligned}$$

Consider the linear continuous operator

$$\Gamma_1 : L^1([0,t_1],E) \to C([0,t_1],E)$$

defined by

$$v \to \Gamma_1(v)(t) = \int_0^t T_1(t-s)v(s)ds,\ t \in [0,t_1],$$

and

$$\Gamma_1 : L^1([t_k,t_{k+1}],E) \to C_*([t_k,t_{k+1}],E)$$

defined by

$$v \to \Gamma_{k+1}(v)(t) = \int_{t_k}^t T_1(t-s)v(s)ds,\ t \in [t_k,t_{k+1}],\ k = 1,\dots,m$$

where

$$C_*([t_k,t_{k+1}],E) = \{y \in C((t_k,t_{k+1}],E) : y(t_k^+),\ \text{exists}\}$$

From Lemma 1.1.3, it follows that $\Gamma_k \circ S_{F_1^k}$ is closed graph, where

$$S_{F_1^k} = \{v \in L^1([t_k,t_{k+1}],E) \colon\ v(t) \in F_1(t,x(t),y(t)),\ \text{ a.e. }\ t \in [t_k,t_{k+1}]\},\ k = 0,1,\dots,m.$$

Moreover, we have that

$$\begin{array}{rcl} L_0(v_n^1)(t) - T_1(t)x_0 & \in & \Gamma_1 \circ S_{F_1^0,x(t),y(t)}, \\ L_1(v_n^1)(t) - T_1(t-t_1)[L_0(v_n^1)(t_1) + \mathcal{I}_1^n(L_0(v_n^1)(t_1))] & \in & \Gamma_1 \circ S_{F_1^1,x(t),y(t)}, \\ L_2(v_n^1)(t) - T_1(t-t_2)[L_1(v_n^1)(t_2) + \mathcal{I}_2^n(L_1(v_n^1)(t_2))] & \in & \Gamma_1 \circ S_{F_1^2,x(t),y(t)}, \\ \cdots & \cdots & \cdots \\ L_{m-1}(v_n^1)(t) - T_1(t-t_{m-1})[L_{m-2}(v_n^1)(t_{m-1}) + \mathcal{I}_{m-1}^n(L_{m-2}(v_n^1)(t_{m-1}))] & \in & \Gamma_m \circ S_{F_1^m,x(t),y(t)}. \end{array}$$

Then there exists $v_0 \in S_{F_1^0,x,y}$ such that

$$h_1(t) = T_1(t)x_0 + \int_0^t T_1(t-s)v_0(s)ds, \quad t \in [0,t_1].$$

Using the fact that $\mathcal{I}_1$ has a closed graph, then there exist $I_1(x) \in \mathcal{I}_1(x)$ and $v_1 \in S_{F_1^1,x,y}$ such that

$$h_1(t) = T_1(t-t_1)[L_0(v_0)(t_1) + I_1(L_0(v_n^1)(t_1)) + \int_{t_1}^t T(t-s)v_1(s)ds, \quad t \in (t_1,t_2].$$

We continuous this process, we get

$$\begin{array}{rcl} h_1(t) & = & T_1(t-t_{m-1})[L_{m-2}(v_{m-1})(t_{m-1}) + \mathcal{I}_{m-1}(L_{m-2}(v_{m-1})(t_{m-1}))] \\ & & + \displaystyle\int_{t_m}^t T(t-s)v_m(s)ds, \quad t \in (t_m,b]. \end{array}$$

Set

$$v(t)=\begin{cases} v_0(t) & t\in[0,t_1]\\ v_1(t) & t\in(t_1,t_2]\\ \dots & \\ v_m(t) & t\in(t_m,b].\end{cases}$$

Hence

$$h_1(t)=T_1(t)x_0+\int_0^t v(s)ds+\sum_{0<t_k<t}T_1(t-t_k)I_k(x(t_k)),\quad t\in J.$$

Similarly we can prove that there exist $\bar{v}\in S_{F_2,x,y}$ and $\bar{I}_k(y)\in\mathcal{I}_k(x),\ k=1,\dots,m$ such that

$$h_2(t)=T_2(t)y_0+\int_0^t \bar{v}(s)ds+\sum_{0<t_k<t}T_2(t-t_k)\bar{I}_k(y(t_k)),\quad t\in J.$$

Then we have $(x,y,h)\in Graph(N)$.

Step 7. A priori bounds on solutions.
Let $(x,y)\in PC\times PC$ be such that $(x,y)\in N(x,y)$. Then there exist $(v_1,v_2)\in S_{F_1,x,y}\times S_{F_2,x,y}$ and $\mathcal{I}_k(x(t_k))\in I_k(x(t_k)),\overline{\mathcal{I}}_k(y(t_k))\in I_k(y(t_k)),\ k=1,\dots,m$, such that

$$x(t)=T_1(t)x_0+\int_0^t T_1(t-s)v_1(s)ds+\sum_{0<t_k<t}T_1(t-t_k)\mathcal{I}_k(x(t_k)),\ t\in J$$

and

$$y(t)=T_2(t)y_0+\int_0^t T_2(t-s)v_2(s)ds+\sum_{0<t_k<t}T_2(t-t_k)\overline{\mathcal{I}}_k(y(t_k)),\ t\in J.$$

- For $t\in[0,t_1]$, we have

$$E|x(t)|\ \leq\ M|x_0|+M\int_0^t |v_1(s)|ds.$$

Hence

$$|x(t)|\ \leq\ M|x_0|+M\int_0^t p_1(s)\psi_1(|x(s)|+|y(s)|)ds,$$

and

$$|y(t)|\ \leq\ M|y_0|+M\int_0^t p_2(s)\psi_2(|x(s)|+|y(s)|)ds.$$

Therefore

$$|x(t)|+|y(t)|\ \leq\ M|x_0|+M|y_0|+\int_0^t p(s)\phi(|x(s)|+|y(s)|)ds,$$

where
$$\gamma_0 = M(|x_0| + |y_0|),\ p(t) = p_1(t) + p_2(t),\ t \in [0, t_1].$$
By Lemma 6.2.1, we have

$$|x(t)| + |y(t)| \ \leq \ \Psi_0^{-1}\left(\int_0^{t_1} p(s)ds\right) := K_0,\ \text{for each } t \in [0, t_1],$$

where
$$\Psi_0(z) = \int_{\gamma_0}^{z} \frac{du}{\psi(u)}.$$
Consequently
$$\|x\|_\infty \leq K_0 \ \text{ and } \|y\|_\infty \leq K_0.$$

- For $t \in (t_1, t_2]$, we have

$$|x(t)| \ \leq \ M\|I_1(x(t_1))\|_{\mathcal{P}} + M\int_{t_1}^{t} |v_1(s)|ds.$$

Hence

$$|x(t)| \ \leq \ M\phi_1(K_0) + M\int_{t_1}^{t} p_1(s)\psi_1(|x(s)| + |y(s)|)ds,$$

and

$$|y(t)| \ \leq \ M\bar{\phi}(K_0) + M\int_{t_1}^{t} p_2(s)\psi_2(|x(s)| + |y(s)|)ds.$$

Therefore

$$|x(t)| + |y(t)| \ \leq \ \gamma_1 + \int_{t_1}^{t} p(s)\psi(|x(s)| + |y(s)|)ds,$$

where
$$\gamma_1 = M(\phi_1(K_0) + \bar{\phi}_1(K_0)),\ p(t) = p_1(t) + p_2(t),\ t \in [t_1, t_2].$$
By Lemma 6.2.1, we have

$$|x(t)| + |y(t)| \ \leq \ \Psi_1^{-1}\left(\int_{t_1}^{t_2} p(s)ds\right) := K_1,\ \text{for each } t \in [t_1, t_2],$$

where
$$\Psi_1(z) = \int_{\gamma_1}^{z} \frac{du}{\psi(u)}.$$
This implies that
$$\|x\|_\infty \leq K_1 \ \text{ and } \|y\|_\infty \leq K_1.$$

- We continuous this process, we get

$$|x(t)| + |y(t)| \leq \Psi_m^{-1}\left(\int_{t_m}^{b} p(s)ds\right) := K_1, \text{ for each } t \in [t_m, b],$$

where

$$\Psi_m(z) = \int_{\gamma_m}^{z} \frac{du}{\psi(u)}, \ \gamma_m = M(\phi_m(K_{m-1}) + \bar{\phi}_m(K_{m-1})).$$

Then

$$\|x\|_\infty \leq K_m \text{ and } \|y\|_\infty \leq K_m.$$

This shows that

$$\mathcal{E} = \{(x, y) \in PC \times PC : (x, y) \in \lambda N(x, y), \ \lambda \in (0, 1)\}$$

is bounded. As a consequence of Theorem 2.3.4 we deduce that N has a fixed point $(x, y) \in PC \times PC$ which is a solution to the problem (6.0.1)-(6.0.3). □

6.3 An example

In this section we present an example to illustrate the usefulness and applicability of our results.

Example 6.3.1. *Consider the following couple of partial differential inclusions with impulsive effects*

$$\begin{cases} u'(t,\xi) & \in & \frac{\partial^2}{\partial \xi^2}u(t,\xi) + F(t,u(t,\xi),v(t,\xi)),, \quad t \geq 0, \quad t \neq t_k, \quad 0 \leq \xi \leq \pi, \\ v'(t,\xi) & \in & \frac{\partial^2}{\partial \xi^2}v(t,\xi) + G(t,u(t,\xi),v(t,\xi)), \quad t \geq 0, \quad t \neq t_k, \quad 0 \leq \xi \leq \pi, \\ u(t_k^+,\xi) & - & u(t_k^-,\xi) = \alpha_k u(t_k^-,\xi), \quad k = 1, \cdots, m, \\ v(t_k^+,\xi) & - & v(t_k^-,\xi) = \bar{\alpha}_k v(t_k^-,\xi), \quad k = 1, \cdots, m, \\ u(t,0) & = & u(t,\pi) = 0, t \geq 0, \\ v(t,0) & = & v(t,\pi) = 0, t \geq 0, \\ u(0,\xi) & = & u_0(\xi), \ 0 \leq \xi \leq \pi, \\ v(0,\xi) & = & v_0(\xi), \ 0 \leq \xi \leq \pi, \end{cases} \tag{6.3.1}$$

where $\alpha_k > 0$, *and* $G, F : [0, \pi] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R})$ *are multivalued maps.*

Let

$$x(t)(\xi) = u(t, \xi), y(t)(\xi) = v(t, \xi) \quad t \in J, \quad \xi \in [0, \pi],$$

$$I_k(x(t_k))(\xi) = \alpha_k u(t_k^-, \xi), \bar{I}_k(y(t_k))(\xi) = \alpha_k v(t_k^-, \xi) \quad \xi \in [0, \pi], \quad k = 1, \cdots, m,$$

$$f(t, x(t), y(t))(\xi) = F(t, u(t, \xi), v(t, \xi)), \quad , \quad \xi \in [0, \pi],$$

$$g(t, x(t), y(t))(\xi) = G(t, u(t, \xi), v(t, \xi)), \quad , \quad \xi \in [0, \pi].$$

$$u_0(\xi) = u(0,\xi),\ v_0(\xi) = v(0,\xi)\quad,\quad \xi \in [0,\pi],$$

Take $\mathcal{K} = \mathcal{H} = L^2([0,\pi])$. *We define the operator* A *by* $Au = u''$, *with domain* $D(A) = \{u \in \mathcal{H}, u', u'' \in \mathcal{H} \quad \text{and} \quad u(0) = u(\pi) = 0\}$.

Then, it is well known that

$$Az = -\sum_{n=1}^{\infty} e^{-n^2 t}\langle z, e_n\rangle e_n, \quad z \in \mathcal{H},$$

and A *is the infinitesimal generator of an analytic semigroup* $\{S(t)\}_{t\geq 0}$ *on* $\mathcal{H}$, *which is given by*

$S(t)u = \sum_{n=1}^{\infty} e^{-n^2 t}\langle u, e_n\rangle e_n$, $u \in \mathcal{H}$, *and* $e_n(u) = (2/\pi)^{1/2}\sin(nu), n = 1, 2, \cdots,$ *is the orthogonal set of eigenvectors of* A. *The analytic semigroup* $\{S(t)\}_{t>0}$, $t \in J$, *is compact, and there exists a constant* $M \geq 1$ *such that* $\|S(t)\|^2 \leq M$.

Assume now that

(i) *There exist some positive number* $d_k, \bar{d}_k \quad k \in \{1, \cdots, m\}$ *such that*

$$|I_k(\xi)| \leq d_k, \quad |\bar{I}_k(\xi)| \leq \bar{d}_k$$

for any $\xi \in \mathbb{R}$.

(ii) *The functions* $F, G : [0,T] \times \mathcal{H} \longrightarrow \mathcal{P}_{cp,cv}(\mathcal{H})$ *defined by* $F_*(t,u)(.) = F(t,u(.)), G_*(t,u)(.) = G(t,u(.)), u(.) = (x(.), y(.))$ *are u.s.c. and we impose suitable conditions on* F *and* G *to verify assumption* $(\mathcal{G}_1)$.

(iii) *Assume that there exists an integrable function* $\eta : [0,T] \longrightarrow \mathbb{R}^+$ *such that*

$$|F(t,x,y)|^2 \leq \eta(t)\psi(|x|^2 + |y|^2), \quad |G(t,x,y)|^2 \leq \eta(t)\psi(|x|^2 + |y|^2)$$

for any $t \in [0,T]$ *and* $x, y \in \mathbb{R}$, *where* $\psi : [0,\infty) \longrightarrow (0,\infty)$ *is continuous, nondecreasing and concave with*

$$\int_1^{\infty} \frac{ds}{\psi(s)} = +\infty.$$

Thus, problem (6.3.1) can be written in the abstract form

$$\begin{cases} x'(t) \in Ax(t) + F(t,x,y), \quad t \in J := [0,T], \\ y'(t) \in Ay(t) + Gt,x,y), \quad t \in J := [0,T], \\ x(t_k^+) - x(t_k) = I_k(x(t_k)),\ k = 1, \ldots, m; \\ y(t_k^+) - y(t_k) = I_k(y(t_k)),\ k = 1, \ldots, m; \\ x(0) = x_0, \\ y(0) = y_0. \end{cases} \tag{6.3.2}$$

Thanks to these assumptions, it is straightforward to check that $(\mathcal{G}_1) - (\mathcal{G}_3)$ hold true and, then, assumptions in Theorem 6.2.1 are fulfilled, and we can conclude that system (6.3.1) possesses a mild solution on $[0,T]$.

Conclusion and Perspective

Our main goal in this thesis is to present several results of existence and stability for some classes of impulsive differential equations and inclusion in Banach space.
These result were obtained by using the krasnoselskii fixed point theorem.
we plan in the future the study of other classes, in particular, discrete differential equations will be examined.

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