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Introduction

0.1 Yamabe invariant

0.1.1 Yamabe problem

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. The Yamabe problem is the following : Does there exist a metric \bar{g} , conformal to g , such that the scalar curvature $S_{\bar{g}}$ of the metric \bar{g} is constant?

In 1960, H. Yamabe [43] formulated this problem and thought that he had solved it. However, eight years later N.S. Trudinger [42] pointed out a serious difficulty in the Yamabe's article. The problem is now completely solved, but it took many years to find the appropriate approach. Nowadays the problem of finding a metric \bar{g} with constant scalar curvature in the conformal class $[g]$ is known as the Yamabe problem. The first step of the solution was given by N.S. Trudinger [42] who had understood the gap of Yamabe's proof when the scalar curvature $S_g \geq 0$. In 1976, Aubin [6] solved the problem for any non locally conformally flat manifolds of dimension $n \geq 6$. The problem was completely solved in 1984 by R. Schoen [39] where the proof is based on the positive-mass theorem which had previously been proved by R. Schoen and S.T. Yau [[40], [41]]. The reader can be refereed to [33], [6] or [27] for more detail on the subject. The method to solve the Yamabe problem was the following. Let $u \in C^\infty(M), u > 0$ be a smooth function and $\bar{g} = u^{N-2}g$ a conformal metric to g , the real $N = \frac{2n}{n-2}$ is known as the critical Sobolev exponent. Then, we can check out that the scalar curvature S_g and $S_{\bar{g}}$ are related by the following equation (see [27]):

$$c_n \Delta_g u + S_g u = S_{\bar{g}} |u|^{N-2} u,$$

where $\Delta_g = -div_g(\nabla_g)$ is the Laplacian-Beltrami operator and $c_n = \frac{4(n-1)}{n-2}$. Let

$$L_g = c_n \Delta_g + S_g,$$

this operator is called the Yamabe operator. Now, solving the Yamabe problem is equivalent to finding a : $u \in C^\infty(M)$ and $u > 0$ solution of

$$L_g u = C_0 |u|^{N-2} u, \tag{0.1.1}$$

where C_0 is a constant. In other words, we prescribe the scalar curvature (we put $S_{\bar{g}} = C_0$) and we look for the solution u that is to say we look for the metric $\bar{g} = u^{N-2} g$. In order to obtain solutions of this equation, Yamabe defined the quantity

$$\mu(M, g) = \inf_{u \in C^\infty(M), u \neq 0} Y(u),$$

where

$$Y(u) = \frac{\int_M u L_g u dv_g}{\left(\int_M u^N dv_g\right)^{\frac{2}{N}}},$$

with

$$\int_M u L_g u dv_g = \int_M c_n |\nabla u|^2 + S_g u^2 dv_g.$$

The constant $\mu(M, g)$ is conformal invariant and it is known as the Yamabe invariant while Y is the Yamabe functional. If we write Euler-Lagrange equation associated to this functional, we will see that there is a bijection between critical points of Y and solutions of equation (0.1.1). It follows that, if u is a positive smooth function such that $Y(u) = \mu(M, g)$, then u is a solution of (0.1.1) and $\bar{g} = u^{N-2} g$ is the desired metric of constant scalar curvature C_0 . The key point of the resolution of the Yamabe problem is the following theorems, the first is due to Aubin [6]:

Theorem 0.1.1.

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ and \mathbb{S}^n stands for the standard unit n -sphere of \mathbb{R}^{n+1} . If $\mu(M, g) < \mu(\mathbb{S}^n)$, then there exists a positive smooth function u such that :

$$Y(u) = \mu(M, g).$$

The constant $\mu(\mathbb{S}^n) = n(n-1)\omega_n^{\frac{2}{n}}$ is the Yamabe invariant of the sphere and ω_n stands for the volume of this sphere.

The second theorem is due to Aubin [6] and Schoen [39]:

Theorem 0.1.2.

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Then $\mu(M, g) \leq \mu(\mathbb{S}^n)$. Moreover, we have equality in this inequality if and only if (M, g) is conformally diffeomorphic to the sphere.

For conformally diffeomorphic see (1.1.6).

0.1 Yamabe invariant

0.1.2 The second Yamabe invariant

Inspired by this approach, in their paper [1], B. Ammann and E. Humbert introduced and studied an invariant that they called the second Yamabe invariant. It is well known that the operator L_g is elliptic, self-adjoint with respect to the inner product in $L^2(M)$ and has a discrete spectrum :

$$\text{Spec}(L_g) = \{\lambda_1(g), \lambda_2(g), \dots\},$$

where the eigenvalues

$$\lambda_1(g) \leq \lambda_2(g) \leq \lambda_3(g) \leq \dots \leq \lambda_k(g) \rightarrow +\infty,$$

appear with their multiplicities. The variational characterization of $\lambda_1(g)$ is given by

$$\lambda_1(g) = \inf_{u \in C^\infty(M), u \neq 0} \frac{\int_M c_n |\nabla u|^2 + S_g u^2 dv_g}{\int_M u^2 dv_g}.$$

When the Yamabe invariant $\mu(M, g) \geq 0$, the authors showed that $\mu(M, g)$ can be given by :

$$\mu(M, g) = \inf_{\bar{g} \in [g]} \lambda_1(\bar{g}) \text{Vol}(M, \bar{g})^{\frac{2}{n}},$$

where

$$[g] = \{\bar{g} = u^{N-2}g, u \in C^\infty(M) \text{ and } u > 0\}, \quad (0.1.2)$$

is the conformal class of g and

$$\text{Vol}(M, \bar{g}) = \int_M dv_{\bar{g}} = \int_M u^N dv_g,$$

denotes the Riemannian volume of M with respect to the metric \bar{g} . The authors enlarged the definition of $\mu(M, g)$ by putting :

Definition 0.1.1.

Let $k \in \mathbb{N}^*$. Then, the k^{th} Yamabe invariant is defined by:

$$\mu_k(M, g) = \inf_{\bar{g} \in [g]} \lambda_k(\bar{g}) \text{Vol}(M, \bar{g})^{\frac{2}{n}},$$

where

$$\lambda_k(\bar{g}) = \inf_{v \in \text{Gr}_k^u(H_1^2(M))} \sup_{v \in V \setminus \{0\}} \frac{\int_M c_n |\nabla v|^2 + S_g v^2 dv_g}{\int_M u^{N-2} v^2 dv_g},$$

and the Grassmannian $\text{Gr}_k^u(H_1^2(M))$ is given in definition 1.2.2. With these notations, $\mu_1(M, g) = \mu(M, g)$ in the case $\mu(M, g) \geq 0$. The authors were interested in studying

the second Yamabe invariant $\mu_2(M, g)$ for manifolds such that the Yamabe invariant $\mu(M, g) \geq 0$, the most interesting case is when $\mu(M, g) > 0$. In particular, they proved that $\mu_2(M, g)$ cannot be attained by a conformal metric, but it is attained by a generalized metric, in other words, there exists a metric $\bar{g} = u^{N-2}g$ where u is no longer necessarily positive and smooth, but $u \in L^N(M)$, $u \geq 0$, $u \neq 0$ and such that

$$\mu_2(M, g) = \lambda_2(\bar{g}) \text{Vol}(M, \bar{g})^{\frac{2}{n}}.$$

They obtained the following results:

Theorem 0.1.3.

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ whose Yamabe invariant is non-negative. Then, $\mu_2(M, g)$ is attained by a generalized metric in the following cases:

- $\mu_1(M, g) > 0$ and $\mu_2(M, g) < [\mu_1(M, g)^{\frac{n}{2}} + (\mu_1(\mathbb{S}^n))^{\frac{n}{2}}]^{\frac{2}{n}}$.
- $\mu_1(M, g) = 0$ and $\mu_2(M, g) < \mu_1(\mathbb{S}^n)$.

Theorem 0.1.4.

The assumptions of Theorem 0.1.3 are satisfied in the following cases:

- $\mu_1(M, g) > 0$, (M, g) is not locally conformally flat and $n \geq 11$.
- $\mu_1(M, g) = 0$, (M, g) is not locally conformally flat and $n \geq 9$.

For conformally flat, the reader is referred to 1.1.3.

Theorem 0.1.5.

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Assume that $\mu_2(M, g)$ is attained by a generalized metric $\bar{g} = u^{N-2}g$. Then, there exists nodal (sign-changing) solution $w \in C^{3,\alpha}(M)$ ($\alpha \leq N - 2$) of equation 0.1.1 such that $|w| = u$ where the space $C^{3,\alpha}(M)$ is defined in 1.2.4.

Note that in the case of $\mu(M, g) < 0$, the authors showed that $\mu_1(M, g) = -\infty$. However, Safaa El Sayed in her article [21] completed the results of B. Ammann and E. Humbert [1] by studying how the sign of the second eigenvalue $\lambda_2(g)$ of the Yamabe operator can be related to the existence of nodal solutions (sign-changing solution) of the Yamabe equation (0.1.1).

0.2 Nodal solutions for a Paneitz-Branson type equation

0.2.1 Paneitz-Branson invariant, definitions and properties

In 1983, Paneitz [36] discovered a conformally invariant fourth-order operator on 4-dimensional Riemannian manifolds. In 1987, Branson [13] extended the notion to Riemannian manifolds of dimension $n \geq 5$. This operator has geometrical roots, it is associated to the notion of the Q-curvature which can be seen as the analogue of the scalar curvature for the conformal Laplacian in Yamabe problem. Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$, the Paneitz-Branson operator P_g^n is given by:

$$P_g^n v = \Delta_g^2 v - \operatorname{div}_g(\bar{A}(\nabla_g v)^\#) + \bar{a}v, \quad (0.2.1)$$

where $\Delta_g = -\operatorname{div}_g(\nabla_g)$ is the Laplacian-Beltrami operator, $\bar{A} = a_n S_g g + b_n \operatorname{Ric}_g$ a smooth symmetric (2,0)-tensor on M , and $\operatorname{Ric}_g, S_g$ denote respectively the Ricci curvature and the scalar curvature of g , the symbol $\#$ stands for the musical isomorphism (index are raised with the metric) and $\bar{A}(\nabla_g v)^\#$ is the (1,0)-tensor whose coordinates in local chart are

$$(\bar{A}(\nabla_g v)^\#)_i = \bar{A}_{ij}((\nabla_g v)^\#)^j = \bar{A}_{ij}g^{jk}(\nabla_g v)_k,$$

with

$$\bar{A}_{ij} = a_n S_g g_{ij} + b_n \operatorname{Ric}_{ij}, \quad a_n = \frac{(n-2)^2 + 4}{2(n-1)(n-2)}, \quad b_n = -\frac{4}{n-2},$$

and the function $\bar{a} = \frac{n-4}{2}Q_g^n$ where,

$$Q_g^n = \frac{1}{2(n-2)}\Delta_g S_g + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2}S_g^2 - \frac{2}{(n-2)^2}|\operatorname{Ric}_g|^2.$$

This operator is also conformally invariant. Indeed, let $u \in C^\infty(M), u > 0$, we consider the metric $\bar{g} = u^{\frac{N-2}{2}}g$ which is conformal to g and $N = \frac{2n}{n-4}$ is the critical Sobolev exponent. Then we have

$$\forall \varphi \in C^4(M), \quad P_g^n(u\varphi) = u^{N-1}P_{\bar{g}}^n(\varphi). \quad (0.2.2)$$

In particular, taking $\varphi \equiv 1$, one gets the equation:

$$P_g^n u = \frac{n-4}{2}Q_{\bar{g}}^n u^{N-1}. \quad (0.2.3)$$

The Paneitz-Branson operator can be seen as an extension of the conformal Laplacian and Q_g^n is the Q-curvature which has the same role like the scalar curvature S_g in the Yamabe problem. Now, if the metric is Einstein (see 1.1.4), the geometric Paneitz-Branson operator is reduced to:

$$P_g^n v = \Delta_g^2 v + \alpha_n S_g \Delta_g v + \beta_n S_g^2 v, \quad (0.2.4)$$

where

$$\alpha_n = \frac{n^2 - 2n - 4}{2n(n-1)} \quad \text{and} \quad \beta_n = \frac{(n-4)(n^2-4)}{16n(n-1)^2}.$$

Furthermore, the scalar curvature S_g is constant in this case. With a standard abuse of notation, we have

$$\int_M v P_g^n u dv_g = \int_M u P_g^n v dv_g = \int_M (\Delta_g u \Delta_g v + \bar{A}((\nabla_g u)^\#, (\nabla_g v)^\#) + \bar{a}uv) dv_g, \quad (0.2.5)$$

for all $u, v \in H_2^2(M)$ where $H_2^2(M)$ is the standard Sobolev space, which is the completed space $C^\infty(M)$ with respect to the norm :

$$\|v\| = \left(\int_M (|\Delta_g v|^2 + |\nabla_g v|^2 + v^2) dv_g \right)^{\frac{1}{2}}. \quad (0.2.6)$$

The fourth-order operator P_g^n is elliptic and also self-adjoint with respect to the inner product in $L^2(M)$ and has a discrete spectrum, $Spec(P_g^n) = \{\lambda_1(g), \lambda_2(g), \dots\}$ where the eigenvalues $\lambda_1(g) \leq \lambda_2(g) \leq \lambda_3(g) \dots \leq \lambda_k(g)$ appear with their multiplicities. In particular, by referring to [1], in the geometric case, one sees that for any generalized metric $\bar{g} = u^{\frac{N-2}{2}}$ of a Riemannian metric g where $u \in L^N(M)$ and $u \geq 0$, the k^{th} eigenvalue $\lambda_k(\bar{g})$ of the geometric Paneitz-Branson operator P_g^n is characterized by

$$\lambda_k(\bar{g}) = \inf_{V \in Gr_k^u(H_2^2(M))} \sup_{v \in V \setminus \{0\}} \frac{\int_M v P_g^n v dv_g}{\int_M u^{N-2} v^2 dv_g}, \quad (0.2.7)$$

where $k \in \mathbb{N}^*$ and $Gr_k^u(H_2^2(M))$ is given above. In [9], by analogy of the Yamabe invariant, M. Benalili and H. Boughazi defined the standard Paneitz-Branson $\mu(M, g)$ by:

$$\mu(M, g) = \inf_{\substack{v \in H_2^2(M) \\ v \neq 0}} \frac{\int_M v P_g^n v dv_g}{\left(\int_M v^N dv_g \right)^{\frac{2}{N}}}, \quad (0.2.8)$$

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and the Paneitz-Branson invariant of high order $\mu_k(M, g)$ by:

$$\mu_k(M, g) = \inf_{\bar{g} \in [g]} \lambda_k(\bar{g}) [Vol(M, \bar{g})]^{\frac{4}{n}}, \quad (0.2.9)$$

where $k \in \mathbb{N}^*$, $Vol(M, \bar{g}) = \int_M dv_{\bar{g}} = \int_M u^N dv_g$ denotes the Riemannian volume of M with respect to the metric \bar{g} and the set

$$[g] = \{\bar{g} = u^{\frac{N-2}{2}} g, \quad u \in C^\infty(M) \text{ and } u > 0\}, \quad (0.2.10)$$

is the conformal class of the metric g and $\lambda_k(\bar{g})$ are given by (0.2.7). The constants $\mu(M, g)$ and $\mu_k(M, g)$ are conformal invariants (see [1]).

0.2.2 Motivation

In 2010, the authors in [9] studied $\mu(M, g)$, $\mu_1(M, g)$ and $\mu_2(M, g)$ in the case $S_g > 0$. In particular, they proved the following theorem:

Theorem 0.2.1 (9).

Let (M, g) be a smooth compact Einstein manifold of dimension $n \geq 5$. Assume that the scalar curvature $S_g > 0$ and $n \geq 12$. Then $\mu_2(M, g) > 0$ and is attained by a generalized metric, in other words there exist $u \in L_+^N(M)$ and $w \in H_2^2(M)$ such that

$$P_g^n w = \mu_2(M, g) u^{N-2} w \quad \text{and} \quad \int_M u^{N-2} w^2 dv_g = 1.$$

In particular, w is nodal solution and $u = |w|$.

As we have mentioned before, in 2012 Safaa El Sayed in [21] studied the Yamabe equation when the Yamabe invariant $\mu(M, g) < 0$. More precisely, let (M, g) be a smooth compact manifold of dimension $n \geq 3$, the author was concerned by the following equation

$$L_g v = \epsilon |v|^{N-2} v, \quad (0.2.11)$$

where L_g is the Yamabe operator and $\epsilon \in \{-1, 0, 1\}$ and $N = \frac{2n}{n-2}$ is the critical Sobolev exponent of the embedding $H_1^2(M) \subset L^q(M)$. In particular, in Theorem 0.2.2, if the second eigenvalue of the Yamabe operator $\lambda_2(g) < 0$, the author proved that the

equation (2.2.1) has a nodal solution in the case $\epsilon = -1$.

Theorem 0.2.2. [21]

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ whose Yamabe invariant $\mu(M, g)$ is strictly negative, we denote by $\lambda_2(g)$ the second eigenvalue of L_g .

Then, if $\lambda_2(g) \leq 0$ or if $\lambda_2(g) > 0$, (M, g) not locally conformally flat and $n \geq 6$:

There exists a function v sign-changing solution to the equation

$$L_g v = \epsilon |v|^{N-2} v,$$

where $\epsilon = +1$ if $\lambda_2(g) > 0$, $\epsilon = -1$ if $\lambda_2(g) < 0$ and $\epsilon = 0$ if $\lambda_2(g) = 0$. Moreover, $v \in C^{3,\alpha}(M)$, for all $\alpha < N - 2$.

Inspired by the previous results, M. Benalili and H. Boughazi in [11] enlightened the role of S_g in the existence of nodal solutions and assumed that S_g is not necessarily positive. It should also be noted that when the scalar curvature $S_g < 0$, this implies that $\int_M v P_g^n v dv_g$ can be negative or positive and therefore the eigenvalues follow the same thing contrary to the case $S_g > 0$ which implies only the positivity of those eigenvalues. The authors obtained the following result:

Theorem 0.2.3. [11]

Let (M, g) be a smooth compact Einstein manifold of dimension $n \geq 5$. Assume that the scalar curvature $S_g < 0$ and $\mu(M, g) < 0$.

• If $\lambda_2(g) > 0$ and $\lambda_1(g) < 0$, then $\mu_2(M, g) > 0$ and is attained by a generalized metric, in other words there exist $u \in L_+^N(M)$ and $w \in H_2^2(M)$ such that

$$P_g^n w = \mu_2(M, g) u^{N-2} w \quad \text{and} \quad \int_M u^{N-2} w^2 dv_g = 1.$$

• If $\mu_2(M, g) < 0$ and is attained, then w is a nodal solution and $u = |w|$.

Here $\mu(M, g)$ is given by formula (2.1.10) and it is always attained by positive $C^{4,\beta}(M)$ function when it is negative and $0 < \beta < 1$.

Now in this work, we are interested by studying the sign of the latter solution w , then we have noted two problems in the latter theorem.

The first problem:

If we assume that $\lambda_2(g) > 0$, then $\mu_2(M, g) > 0$ and is attained by a generalized metric

0.2 Nodal solutions for a Paneitz-Branson type equation

but there is no information about the sign of w in this case. In fact, to prove that w changes the sign, we are going to use some ideas in [[9],[10]] in particular Theorem (4), but it does not work. Indeed, without loss of generality, we can assume that $\mu_2(M, g) = \lambda_2(\bar{g})$ and we know from [11] that for any generalized metric $\bar{g} = u^{\frac{N-2}{2}}g$, there exist two functions $v, w \in H_2^2(M)$ satisfying

$$P_g^n v = \lambda_1(\bar{g})u^{N-2}v, \quad (0.2.12)$$

and

$$P_g^n w = \lambda_2(\bar{g})u^{N-2}w. \quad (0.2.13)$$

However, the sign of the solution w depends on the positivity of the solution v and it is well known that the equation (0.2.12) has a positive solution if and only if $S_g > 0$, then it follows that it is impossible to find the sign of the solution w with this method.

The second problem:

If we assume that the second eigenvalue $\lambda_2(g) < 0$, which is the only the situation where we can get $\mu_2(M, g) < 0$. Unfortunately in this case, the second Paneitz-Branson invariant $\mu_2(M, g)$ is not well defined as shown below in the Proposition 0.2.1 and this situation is completely different to that made in [11], it seems that the second point in Theorem 0.2.3 is never realized and therefore we cannot talk about such solution.

To solve these problems, we will study a more general case, we will show a new result. The following theorem is our main result in this work :

Theorem 0.2.4.

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 5$. Assume that $\lambda_1(g) < 0$. Then, the equation

$$\Delta_g^2 v - \text{div}_g(A(\nabla_g v)^\#) + av = \epsilon|v|^{N-2}v, \quad (0.2.14)$$

has at least a nontrivial solution $v \in C^{4,\beta}(M)$ where A a smooth $(2,0)$ -tensor on M , the function a is smooth, $\epsilon < 0$ and $0 < \beta < 1$. In addition if the function $a \geq 0$, the solution v is nodal (sign-changing).

In fact, we will show that the method used in [21] can be extended to a more general setting such as fourth-order elliptic equations with more general coefficients and non Einstein manifolds and finally even on Einstein manifold whether $\lambda_2(g) > 0$ or not, in both cases the geometric equation $P_g^n v = \epsilon|v|^{N-2}v$ has at least one nodal solution where $\epsilon < 0$.

Note that the method used in [21] is different from the one used in [11]. More precisely,

we are going to show on a compact Riemannian manifold of dimension $n \geq 5$, that the first eigenvalue $\lambda_1(g)$ is finite and achieved and in the case of $\lambda_1(g) < 0$, the equation (0.2.14) has at least nontrivial solution. In order to obtain solutions of (0.2.14), the strategy we would like to apply is the following. For all $v \in C^\infty(M)$, we introduce the following operator :

$$L_g v = \Delta_g^2 v - \operatorname{div}_g(A(\nabla_g v)^\#) + av,$$

where A a smooth symmetric $(2, 0)$ -tensor field and $A(\nabla_g v)^\#$ is the $(1, 0)$ -tensor whose coordinates in local chart are

$$(A(\nabla_g v)^\#)_i = A_{ij}((\nabla_g v)^\#)^j = A_{ij}g^{jk}(\nabla_g v)_k,$$

and we define the quantity:

$$\alpha = \inf I_g(u), \tag{0.2.15}$$

where

$$I_g(u) = \frac{\left(\int_M |L_g u|^{\frac{2n}{n+4}} dv_g\right)^{\frac{n+4}{n}}}{\left|\int_M u L_g u dv_g\right|}.$$

Here the infimum is taken over functions u in $C^4(M)$ such that $\int_M u L_g u dv_g < 0$ and the constraint $\int_M u \varphi dv_g = 0$ for all function $\varphi \in \ker L_g$. We will show that the infimum α is attained by a function u , and when we put $v = |L_g u|^{\frac{-8}{n+4}} L_g u$, the function v will be solution of $L_g v = \epsilon |v|^{N-2} v$ where $\epsilon < 0$, we also note that many works are devoted to study this kind of equations which are very important and knows as non-linear fourth order elliptic equations with critical Sobolev growth.

0.2.3 Results

In this work, our results are like that :

1) On Einstein manifolds $\mu_2(M, g)$ is not well defined :

Proposition 0.2.1.

Let (M, g) be a compact Einstein manifold of dimension $n \geq 5$, assume that the scalar curvature S_g is negative. Suppose that the second eigenvalue $\lambda_2(g)$ of P_g^n is negative, then $\mu_2(M, g) = -\infty$.

2) We establish some results concerning the eigenvalues :

0.2 Nodal solutions for a Paneitz-Branson type equation

Proposition 0.2.2.

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 5$. Then, the first eigenvalue $\lambda_1(g)$ and the second eigenvalue $\lambda_2(g)$ of L_g are finite and achieved. In other words, there exist nontrivial functions $v, w \in H_2^2(M)$ solution of

$$L_g v = \lambda_1(g)v \quad \text{and} \quad L_g w = \lambda_2(g)w, \quad (0.2.16)$$

and such that

$$\int_M v^2 dv_g = \int_M w^2 dv_g = 1 \quad \text{and} \quad \int_M wv dv_g = 0.$$

Moreover $v, w \in C^{4,\beta}(M)$ with $0 < \beta < 1$. In particular, if $a \geq 0$ and $\lambda_1(g) < 0$, or if $a \leq 0$ and $\lambda_1(g) > 0$, v is nodal solution, the same thing goes with w .

3) The infimum $\alpha = \inf I_g(u)$ is attained :

Theorem 0.2.5.

Assume that $\lambda_1(g) < 0$. Then there exists a function $u \in H_2^2(M)$ which is not identically null and such that the infimum of the functional I_g is attained by u . In other words, we get

$$I_g(u) = \alpha.$$

The functional I_g is given by formula (0.2.15).

4) Our main equation has at least a nontrivial solution:

Theorem 0.2.6.

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 5$. Assume that $\lambda_1(g) < 0$, then the following equation :

$$\Delta_g^2 v - \text{div}_g(A(\nabla_g v)^\#) + av = \epsilon|v|^{N-2}v, \quad (0.2.17)$$

has at least a nontrivial weak solution $v \in H_2^2(M)$ with $\epsilon < 0$.

5) Regularity and sign of the solution

Theorem 0.2.7.

The solution v of the equation (0.2.17) is in $C^{4,\beta}(M)$ and if we assume that the function $a \geq 0$, then v changes the sign.

6) In the geometrical case we have the following results:

Proposition 0.2.3.

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$. If L_g is the geometric Paneitz-Branson operator P_g^n , the infimum α of the functional I_g is a conformal invariant where α is given above.

Proposition 0.2.4.

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$ and P_g^n be the geometric Paneitz-Branson operator. Assume that $Q_g^n \geq 0$. If $\lambda_1(g) < 0$, the geometric equation $P_g^n v = \epsilon|v|^{N-2}v$ has a nontrivial nodal solution $v \in C^{4,\beta}(M)$ where $\epsilon < 0$ and $0 < \beta < 1$. In particular, on Einstein manifold we have always $Q_g^n \geq 0$.

A more interesting situation on non Einstein manifold with the standard Paneitz-Branson $\mu(M, g)$ when it is negative.

Theorem 0.2.8.

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$ and P_g^n be the geometric Paneitz-Branson operator. Assume that the standard Paneitz-Branson $\mu(M, g) < 0$ and $Q_g^n \geq 0$ where $\mu(M, g)$ is defined by formula (2.1.10). If $\lambda_1(g) < 0$, then $\mu(M, g)$ is attained by the generalized metric $\bar{g} = u^{\frac{N-2}{2}} g$.

0.3 The first GJMS invariant

0.3.1 Definitions and some properties of the GJMS operator

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$, and let k be an integer such that $k \geq 1$ and $2k \leq n$. In 1992, in [24] Graham-Jenne-Mason-Sparling have defined a family of conformally invariant differential operators defined for any Riemannian metric, they are called GJMS operators for short. More precisely, for any Riemannian metric g on M , there exists a local, formally self-adjoint, conformally covariant operator

$$P_g : C^\infty(M) \longrightarrow C^\infty(M),$$

such that for all $u \in C^\infty(M)$, the GJMS operator P_g is given by :

$$P_g u = \Delta_g^k u + lot, \tag{0.3.1}$$

where Δ_g is the Laplace-Beltrami operator, and

lot = lower order (differential) terms.

More details about P_g are found in the last chapter. This operator is conformally invariant in the following sens : let $\varphi \in C^\infty(M)$ be a positive function and $N = \frac{2n}{n-2k}$. If $n \neq 2k$, then for any conformal metric $\bar{g} = \varphi^{\frac{4}{n-2k}} g$ and for all $u \in C^\infty(M)$, we have :

$$P_g(u\varphi) = \varphi^{N-1} P_{\bar{g}}(u).$$

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By taking $u \equiv 1$, we get

$$P_g \varphi = \frac{n-2k}{2} Q_{\bar{g}} \varphi^{N-1}, \quad (0.3.2)$$

where

$$Q_{\bar{g}} = \frac{2}{n-2k} P_{\bar{g}}(1).$$

The scalar Q_g is called the Q -curvature and is a Riemannian invariant associated to this operator. The notion of the Q -curvature is due to Branson [14]. He also defined it in the critical case $n = 2k$. Now when $k = 1$, P_g is the conformal Laplacian operator and Q_g is the scalar curvature S_g (up to a constant). The problem of prescribing a constant scalar curvature is the Yamabe problem, the classical reference for this problem is the survey of Lee-Parker [33]. When $k = 2$, P_g is the Paneitz-Branson operator introduced by Paneitz in [36] and the Q -curvature was introduced by Branson-Ørsted [15]. Results for the prescription of the Q -curvature problem for the Paneitz operator are in Djadli-Hebey-Ledoux [19], Robert [37], Esposito-Robert [20], Hang-Yang [31], Gursky-Malchiodi [25] and Benalili-Boughazi [9]. Moreover, concerning fourth-order problems, there has been also an intensive literature on the question, we refer the reader to [9],[11],[18],[37]. Solving the problem of prescribing Q -curvature for the GJMS operator is very hard, we refer to Robert [38] and Mazumdar [35] for some particular situations. The simple case of these problems is prescribing constant Q -curvature which is equivalent to finding a positive smooth solution u of the following equation :

$$P_g u = C |u|^{N-2} u, \quad (0.3.3)$$

where C is a constant. In order to obtain solutions, we define the quantity

$$\mu = \inf_{u \in C^\infty(M), u > 0} I(u), \quad (0.3.4)$$

where

$$I(u) = \frac{\int_M u P_g u dv_g}{\left(\int_M |u|^N dv_g \right)^{\frac{2}{N}}}.$$

As in the first section, the constant μ will be called the (standard) GJMS invariant and in particular, if $u \in C^\infty(M)$, $u > 0$ and satisfy $I(u) = \mu$, clearly u is solution of (3.1.3) and $\bar{g} = u^{\frac{4}{n-2k}} g$ is the desired metric of constant Q -curvature.

It is also well known that the operator P_g is elliptic, self-adjoint with respect to the inner product in $L^2(M)$ [38] and has discrete spectrum with eigenvalues

$$\lambda_1(g) \leq \lambda_2(g) \leq \lambda_3(g) \dots \leq \lambda_k(g) \rightarrow +\infty,$$

appear with their multiplicities. The variational characterization of the first eigenvalue $\lambda_1(g)$ of P_g is given by:

$$\lambda_1(g) = \inf_{v \in H_k^2(M), v \neq 0} \frac{\int_M v P_g v dv_g}{\int_M v^2 dv_g}, \quad (0.3.5)$$

where the space $H_k^2(M)$ is the completion of $C^\infty(M)$ for the norm

$$\|u\|_{H_k^2} = \left(\int_M \sum_{l=0}^k |\nabla^l u|^2 dv_g \right)^{\frac{1}{2}}. \quad (0.3.6)$$

Now as above, we introduce an invariant μ_1 that we will call the first GJMS invariant and we will define it by:

$$\mu_1 = \inf_{\bar{g} \in [g]} \lambda_1(\bar{g}) Vol(M, \bar{g})^{\frac{2k}{n}}, \quad (0.3.7)$$

where $Vol(M, \bar{g}) = \int_M u^N dv_g$ denotes the Riemannian volume of M with respect to the metric \bar{g} and $[g]$ is the conformal class.

Along the same lines, we enlarge the conformal class $[g]$ to what we call the class of generalized metrics conformal to g . We say that $\bar{g} = u^{\frac{4}{n-2k}} g$ is a generalized metric of the Riemannian metric g if $u \in L^N(M)$, $u \geq 0$ and u is not identically null. By the standard min-max method, one sees that for any generalized metric $\bar{g} = u^{\frac{4}{n-2k}} g$, the first eigenvalue $\lambda_1(\bar{g})$ of the GJMS operator P_g is characterized by:

$$\lambda_1(\bar{g}) = \inf_{V \in Gr_1^u(H_k^2(M))} \sup_{v \in V \setminus \{0\}} \frac{\int_M v P_g v dv_g}{\int_M u^{N-2} v^2 dv_g}, \quad (0.3.8)$$

where the Grassmannian $Gr_1^u(H_k^2(M))$ is given in the definition 1.2.2.

0.3.2 Motivation

Motivating by the same first part's results, we want to study the first eigenvalue $\lambda_1(\bar{g})$ for any generalized metric \bar{g} and in particular we want to know when the first GJMS invariant μ_1 is attained by a generalized metric and what will be the relationship between μ_1 and the GJMS invariant μ . To solve this problem, we will use the ideas from [[1], [9]-[7], [21], [27]]. More precisely, the method we would like to apply is introduced in [[1], [9]].

Our main result in this part is the following generic theorem:

Theorem 0.3.1.

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Assume that $\lambda_1(g) > 0$ and $\mu < K_0^{-1}$ where K_0 is defined bellow. Then there exists a nontrivial function $v \in C^{2k}(M)$ which satisfies $P_g v = \mu_1 |v|^{N-2} v$. In other words, μ_1 is attained by the generalized metric $\bar{g} = |v|^{\frac{4}{n-2k}} g$ and in particular, if $Q_g \leq 0$, v is a nodal (sign-changing) solution. Moreover, if g is Einstein and $S_g > 0$, the solution $v > 0$ and

0.3 The first GJMS invariant

$v \in C^\infty(M)$ and this implies that $\mu = \mu_1$ and means that \bar{g} is a conformal metric. Consequently, in the latter case μ_1 is attained by the desired metric \bar{g} of constant Q -curvature: $Q_{\bar{g}} = \frac{2}{n-2k}\mu_1$.

K_0 is the best constant in the Sobolev's continuous embedding $D_k^2(\mathbb{R}^n) \subset L^N(\mathbb{R}^n)$.

$$\frac{1}{K_0} = K_0^{-1} = \inf_{u \in D_k^2(\mathbb{R}^n) - \{0\}} \frac{\int_{\mathbb{R}^n} |\Delta^{\frac{k}{2}} u|^2 dv_g}{\left(\int_{\mathbb{R}^n} |u|^N dv_g\right)^{\frac{2}{N}}}, \quad (0.3.9)$$

where $D_k^2(\mathbb{R}^n)$ be the space defined as the completion of $C_c^\infty(\mathbb{R}^n)$ for the norm $\|\Delta^{\frac{k}{2}} u\|_2$. K_0 is also noted by $K_0(n, k)$.

This theorem is a consequence of several results. Firstly, we establish some results concerning the eigenvalues, in particular, if $\lambda_1(g) > 0$, the first eigenvalue $\lambda_1(\bar{g})$ is achieved for all generalized metric $\bar{g} = u^{\frac{4}{n-2k}}g$ and the corresponding linear equation $P_g v = \lambda_1(\bar{g})u^{N-2}v$ has nodal (sign-changing) solution when $Q_g \leq 0$, however if $\lambda_1(g) < 0$, we show that there exists a generalized metric \bar{g} such that $\lambda_1(\bar{g}) = -\infty$ which implies that $\mu_1 = -\infty$. Secondly, we study our first GJMS invariant μ_1 in case $\lambda_1(g) > 0$, we will prove that μ_1 is attained by a generalized metric if $1 - \mu K_0 > 0$ and the corresponding equation $P_g v = \mu_1 |v|^{N-2}v$ has a nodal solution if $Q_g \leq 0$, we note that here this equation is nonlinear with critical growth. Finally, when the manifold is Einstein with positive scalar curvature, we will prove that the solution v of the latter equation is positive smooth, this implies the desired equality : $\mu_1 = \mu$ and therefore it is attained by a conformal metric \bar{g} which is of constant Q -curvature. The case $S_g < 0$ implies that the solution v is nodal.

0.3.3 Results

Our main results are:

1) Any minimizing sequence of $\lambda_1(\bar{g})$ is bounded :

Theorem 0.3.2.

For any generalized metric $\bar{g} = u^{\frac{4}{n-2k}}g$, assume that $u > 0$. Then any normalized minimizing sequence of $\lambda_1(\bar{g})$ is bounded in $H_k^2(M)$.

2) The case where $\lambda_1(\bar{g})$ can not be defined :

Proposition 0.3.1.

Assume that $\lambda_1(g) < 0$, then there exists $u \in L_+^N(M)$ such that $\lambda_1(\bar{g}) = -\infty$ where $\bar{g} = u^{\frac{4}{n-2k}}g$.

3) $\lambda_1(\bar{g})$ is attained by a generalized metric:

Theorem 0.3.3.

Let $\bar{g} = u^{\frac{4}{n-2k}}g$ be any generalized metric to g such that $u > 0$. Assume that $\lambda_1(g) > 0$. Then there exists a nontrivial function v in $H_k^2(M)$ such that, in the weak sense, v satisfy :

$$P_g v = \lambda_1(\bar{g})u^{N-2}v \quad \text{and} \quad \int_M u^{N-2}v^2 dv_g = 1. \quad (0.3.10)$$

Moreover, if $u \in C_+^\infty(M)$, then $v \in C^\infty(M)$ and if (M, g) is Einstein and $S_g > 0$, the solution $v > 0$.

4) Sign of the solution v

Proposition 0.3.2.

Assume that \bar{g} is a conformal metric that is to say $u \in C_+^\infty(M)$. If $Q_g \leq 0$, then the latter solution v is nodal (sign-changing).

5) The first relationship between μ_1 and μ

Lemma 0.3.1.

We have:

$$\mu_1 \leq \mu,$$

where μ is the GJMS invariant, see (3.1.4).

6) $\mu_1(M, g)$ is attained by a generalized metric :

Theorem 0.3.4.

Assume that $\lambda_1(g) > 0$ and $1 - \mu K_0 > 0$ where μ is the standard GJMS invariant. Then there exist two nontrivial functions $u \in L_+^N(M)$ and $v \in H_k^2(M)$ such that in the weak sense, we have

$$P_g v = \mu_1 u^{N-2}v \quad \text{and} \quad \int_M u^{N-2}v^2 dv_g = 1.$$

In other words, μ_1 is attained by a generalized metric.

7) Regularity and nodal solution :

Theorem 0.3.5.

Assume that μ_1 is attained by the generalized metric $\bar{g} = u^{\frac{4}{n-2k}}g$ where $u \in L_+^N(M)$. Then $u = |v|$ where u, v are as above and this means that v is a weak solution of the following non-linear equation with critical Sobolev growth:

$$P_g v = \mu_1 |v|^{N-2}v.$$

Moreover, the function $v \in C^{2k}(M)$ and if $Q_g \leq 0$, then v changes the sign.

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8) Einstein manifold and the second relationship between μ_1 and μ :

Theorem 0.3.6. *Assume that $S_g > 0$ and $1 - \mu K_0 > 0$ where μ is the GJMS invariant. Then μ_1 is attained by the conformal metric $u^{\frac{4}{n-2k}} g$. In other words, there exists $u \in C^\infty(M), u > 0$ solution to the following equation*

$$P_g u = \mu_1 u^{N-1} \quad \text{such that} \quad \int_M u^N dv_g = 1.$$

In this case we get that :

$$\mu_1 = \mu.$$

Therefore, the infimum μ_1 is achieved by the conformal metric $\bar{g} = u^{\frac{4}{n-2k}} g$ and this means that metric \bar{g} is such that the Q-curvature

$$Q_{\bar{g}} = \frac{2}{n-2k} \mu_1.$$

Corollary 0.3.1.

Assume that $S_g < 0$, $\lambda_1(g) > 0$ and $1 - \mu K_0 > 0$. If k is odd, the following equation

$$P_g v = \mu_1 |v|^{N-2} v,$$

has a nodal solution $v \in C^{2k}(M)$.

Chapter 1

Preliminaries and some definitions

In this chapter, we introduce all the definitions and theorems that are used in this thesis.

1.1 Curvatures on a Riemannian manifold

Definition 1.1.1. [27]

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 1$. There exists a unique torsion-free connection on M having the property that $\nabla g = 0$. This connection is called the Levi-Civita connection of g . Moreover, let i, j, k, l, α and β are integers then in local coordinates one has :

- The Christoffel symbols are given by the relations

$$\Gamma_{ij}^k = \frac{1}{2}g^{mk} \left(\frac{\partial g_{mj}}{\partial x_i} + \frac{\partial g_{mi}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_m} \right),$$

where g_{ij} are the components of the metric g and g^{mj} satisfies $g_{im}g^{mj} = \delta_i^j$.

- The curvature tensor R can be seen as the smooth $(3, 1)$ -tensor field on M , whose coordinates are given by the relation

$$R_{ijk}^l = \frac{\partial \Gamma_{ki}^l}{\partial x_j} - \frac{\partial \Gamma_{ji}^l}{\partial x_k} + \Gamma_{j\alpha}^l \Gamma_{ki}^\alpha - \Gamma_{k\alpha}^l \Gamma_{ji}^\alpha.$$

Definition 1.1.2. [27]

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 1$, and R be the curvature of the Levi-Civita connection. In local coordinates, one has :

- The Riemann curvature Rm_g of g is the smooth $(4, 0)$ -tensor field on M whose components are

$$R_{ijkl} = g_{i\alpha} R_{jkl}^\alpha.$$

- The Ricci curvature Ric_g of g is the smooth $(2, 0)$ -tensor field on M whose components are

$$R_{ij} = R_{\alpha i \beta j} g^{\alpha \beta}.$$

1.2 Sobolev spaces

- The scalar curvature $Scal_g$ of g is the smooth real-valued function on M whose expression is

$$Scal_g = S_g = R_{ij}g^{ij}.$$

Definition 1.1.3. [27]

A Riemannian manifold (M, g) is said to be conformally flat if for all x in M , there exists an open neighborhood Ω of x , and there exists a metric $\bar{g} \in [g]$ such as $Rm_{\bar{g}} = 0$ on Ω , then it follows that (M, g) is not locally conformally flat if there is a point $x \in M$ such that $Rm_{\bar{g}} \neq 0$ for all metric in $[g]$ and for all Ω containing this point.

Definition 1.1.4. [27]

A Riemannian manifold (M, g) is Einstein if and only if there exists a real number λ such that the Ricci tensor writes $Ric_g = \lambda g$. Here $\lambda = \frac{S_g}{n}$ where S_g is the scalar curvature and is constant in this case.

Definition 1.1.5. [27]

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 1$. The conformal class of g denoted by $[g]$ is the set of Riemannian metrics on M which are written in the form $\bar{g} = fg$, where $f \in C^\infty(M)$ and $f > 0$. We can write that

$$[g] = \{e^u g, u \in C^\infty(M)\}.$$

Definition 1.1.6. [27]

We say that (M_1, g_1) and (M_2, g_2) are conformally diffeomorphic, if there exists a conformal diffeomorphism $f : (M_1, g_1) \rightarrow (M_2, g_2)$ such that $f^*g_2 \in [g_1]$ where f^*g_2 is the pullback of g_2 and $[g_1]$ is the conformal class of g_1 .

1.2 Sobolev spaces

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 1$.

Definition 1.2.1. [27]

Let $p \geq 1$ a real number and let

$$\|\cdot\|_p = \left(\int_M |u|^p dv_g \right)^{\frac{1}{p}}.$$

The space $L^p(M)$ is the set of functions such that $\|\cdot\|_p < +\infty$, where dv_g is the Riemannian measure on M . In local coordinates

$$dv_g = \sqrt{|g|} dx = \sqrt{\det(g_{ij})} dx,$$

where dx is the Lebesgue measure.

Definition 1.2.2. [4]

Set $L_+^N(M) = \{u \in L^N(M), u \geq 0 \text{ and } u \neq 0\}$ and $C_+^\infty(M) = \{u \in C^\infty(M), u > 0\}$ where $N = \frac{2n}{n-2k}$, $n \neq 2k$ and $k \geq 1$. For all $u \in L_+^N(M)$, we define the Grassmannian $Gr_p^u(H_k^2(M))$ as the set of all p -dimensional subspaces ($p \geq 1$) of $H_k^2(M)$ such that the subspace $V = \text{span}(v_1, \dots, v_p) \in Gr_p^u(H_k^2(M))$ if and only if v_1, \dots, v_p are linearly independent on $M \setminus u^{-1}(0)$. Sometimes it will be convenient to use the equivalent statement that the functions $u^{\frac{N-2}{2}} v_1, u^{\frac{N-2}{2}} v_2, \dots, u^{\frac{N-2}{2}} v_p$ are linearly independent.

Definition 1.2.3. [27]

Let k be an integer and $p \geq 1$ a real number. Then, we defined the space of smooth functions

$$C_k^p(M) = \left\{ u \in C^\infty(M) \text{ such that } \int_M |\nabla_g^m u|^p dv_g < +\infty, \forall m = 0, 1, \dots, k. \right\},$$

where $\nabla_g^m u$ is the m^{th} covariant derivative of u .

The Sobolev space $H_k^p(M)$ is the completion of $C_k^p(M)$ with respect to the norm

$$\|u\|_{H_k^p} = \left(\sum_{m=0}^k \int_M |\nabla_g^m u|^p dv_g \right)^{\frac{1}{p}}.$$

Remarks 1.2.1. [26]

- Noting that a Cauchy sequence in $C_k^p(M)$ is also a Cauchy sequence $L^p(M)$, and that a Cauchy sequence in $C_k^p(M)$ which converges to 0 in $L^p(M)$ converges to 0 in $C_k^p(M)$, the Sobolev spaces $H_k^p(M)$ can be seen as a subspace of $L^p(M)$.
- If M is compact, one has that $C_k^p(M) = C^\infty(M)$.

Proposition 1.2.1. [26]

If $p = 2$, $H_k^2(M)$ is an Hilbert space equipped with the norm

$$\|u\|_{H_k^2} = \left(\sum_{m=0}^k \int_M |\nabla_g^m u|^2 dv_g \right)^{\frac{1}{2}}.$$

The scalar product (\cdot, \cdot) associated is defined by

$$(u, v) = \int_M uv dv_g + \sum_{m=1}^k \int_M \left(g^{i_1 j_1} \dots g^{i_m j_m} (\nabla_g^m u)_{i_1 \dots i_m} (\nabla_g^m v)_{j_1 \dots j_m} \right) dv_g,$$

where $i_1, \dots, i_m, j_1, \dots, j_m$ are integers and $(\nabla_g^m v)_{j_1 \dots j_m}$ is the component of the m^{th} covariant derivative $\nabla_g^m v$.

Theorem 1.2.1. [8]

Let E be a Banach space. Then E is reflexive if and only if the ball

$$B_E = \{x \in E, \|x\| \leq 1\},$$

is compact in the weak topology.

1.3 Sobolev embeddings

Theorem 1.2.2. [37]

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 1$, $k \in \mathbb{N}$ and $p > 1$. Then the unit ball of $H_k^p(M)$ is weakly compact. In other words, for any bounded sequence $(u_i)_{i \in \mathbb{N}} \in H_k^p(M)$, there exist a subsequence still labeled $(u_i)_{i \in \mathbb{N}} \in H_k^p(M)$ and $u \in H_k^p(M)$ such that u_i converges weakly to u in $H_k^p(M)$ and

$$\|u\|_{H_k^p} \leq \liminf \|u_i\|_{H_k^p}.$$

Properties 1.2.1. [26]

- When M is compact, $H_k^p(M)$ does not depend on the Riemannian metric.
- For all $p > 1$, $H_k^p(M)$ is reflexive.

Theorem 1.2.3. [27]

Let E be a Banach space and let $p \in]1, +\infty[$ et $(u_i)_{i \in \mathbb{N}}$ be a bounded sequence in $L^p(E)$ which converge almost everywhere to u . Then $u \in L^p(E)$ et (u_i) converges weakly to u in $L^p(E)$.

Definition 1.2.4. [27]

Let m a positive integer and $\alpha \in (0, 1)$.

- The Hölder space $C^{0,\alpha}(M)$ is the space of continuous functions with respect to the norm

$$\|u\|_{C^{0,\alpha}(M)} := \|u\|_\infty + \sup_{x \neq y \in M} \frac{|u(x) - u(y)|}{d_g(x, y)^\alpha},$$

where $d_g(\cdot, \cdot)$ is the geodesic distance.

- The Hölder space $C^{m,\alpha}(M)$ is the space of class C^m functions with $|\nabla_g^m u| \in C^{0,\alpha}(M)$ and respect to the norm

$$\|u\|_{C^{m,\alpha}(M)} := \|u\|_{C^m(M)} + \sup_{x \neq y \in M} \frac{|\nabla_g^m u(x) - \nabla_g^m u(y)|}{d_g(x, y)^\alpha}.$$

1.3 Sobolev embeddings

Theorem 1.3.1. [27]

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 1$.

$q \geq 1$ real and $m < k$ two integers such that $1/q > (k - m)/n$.

Then the embedding $H_k^q(M) \subset H_m^p(M)$ is continuous for all $p \geq 1$ such that $1/p \geq 1/q - (k - m)/n$.

Theorem 1.3.2. [27]

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 1$, $q \geq 1$ real, and $m < k$ two integers. If $1/q < (k - m)/n$, then $H_k^q(M) \subset C^m(M)$.

Theorem 1.3.3. [27] (**The Rellich-Kondrakov theorem**)

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 1$, $q \geq 1$ real

and $m < k$ two integers.

- If $1/q > (k - m)/n$:

Then the embedding $H_k^q(M) \subset H_m^p(M)$ is compact when $1 \geq 1/p > 1/q - (k - m)/n$.

- If $1/q < (k - m)/n$, then the embedding $H_k^q(M) \subset C^m(M)$ is compact.

- For $q > n$, the embedding of $H_1^q(M) \subset C^\lambda(M)$ is compact for any $\lambda \in (0, 1)$ such that $(1 - \lambda)q > n$. In particular, the embedding $H_1^q(M) \subset C^0(M)$ is compact.

1.4 The best constant and Sobolev inequality

Definition 1.4.1. [26]

Let Δ_g be the Laplace-Beltrami operator associated to g acting on functions. In local coordinates,

$$\begin{aligned} \Delta_g u &= -\operatorname{div}_g(\nabla_g u) \\ &= -g^{ij}(\nabla_g^2 u)_{ij} \\ &= -g^{ij}(\partial_{ij} u - \Gamma_{ij}^k \partial_k u) \\ &= -\frac{1}{\sqrt{\det(g_{ij})}} \partial_m (\sqrt{\det(g_{ij})} g^{mk} \partial_k u), \end{aligned}$$

where i, j are integers, the Γ_{ij}^k are the Christoffel symbols of the Levi-Civita connection. The second covariant derivative :

$$\nabla_g^2 u = (\nabla_g^2 u)_{ij} dx^i \otimes dx^j = (\partial_{ij} u - \Gamma_{ij}^k \partial_k u) dx^i \otimes dx^j.$$

The first covariant derivative which is only the differential:

$$du = (\partial_i u) dx^i = \nabla_g u = (\nabla_g u)_i dx^i,$$

$\det(g_{ij})$ stands for the determinant of the matrix (g_{ij}) and \otimes is the tensorial product.

Theorem 1.4.1. [37]

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 1$ (it is obvious that M is without boundary). Let η be a smooth $(1, 0)$ -tensor. Then we have that

$$\int_M \operatorname{div}_g(\eta) dv_g = 0.$$

In particular, given $u, v \in C^\infty(M)$, we have that

$$\int_M u \Delta_g v dv_g = \int_M (\nabla u, \nabla v)_g dv_g = \int_M v \Delta_g u dv_g.$$

1.5 Lagrange multipliers theorem and regularity

Definition 1.4.2. [35]

Let $K_0(n, k)$ such that

$$K_0^{-1}(n, k) = \inf_{u \in D_k^2(\mathbb{R}^n) - \{0\}} \frac{\int_{\mathbb{R}^n} |\Delta_g^{\frac{k}{2}} u|^2 dv_g}{\left(\int_{\mathbb{R}^n} |u|^N dv_g \right)^{\frac{2}{N}}}, \quad (1.4.1)$$

$K_0(n, k)$ is the best constant in the Sobolev's continuous embedding $D_k^2(\mathbb{R}^n) \subset L^N(\mathbb{R}^n)$. Where $K_0(n, k)$ is the same constant introduced in 0.3.9 and $D_k^2(\mathbb{R}^n)$ be the space defined as the completion of $C_c^\infty(\mathbb{R}^n)$ for the norm $\|\Delta_g^{\frac{k}{2}} u\|_2$, $N = \frac{2n}{n-2k}$, $n \neq 2k$ and $k \in \mathbb{N}^*$. Here, we have adopted the following convention :

$$\Delta_g^{\frac{k}{2}} u = \begin{cases} \Delta_g^m u & \text{if } k = 2m \text{ is even} \\ \nabla \Delta_g^m u & \text{if } k = 2m + 1 \text{ is odd,} \end{cases}$$

and, when $k = 2m + 1$ is odd, $\Delta_g^{\frac{k}{2}} u \Delta_g^{\frac{k}{2}} v = (\nabla \Delta_g^m u, \nabla \Delta_g^m v)_g$.

Remarks 1.4.1. [2], [20]

- $K_0(n, 1) = \frac{4}{n(n-2)\omega_n^{\frac{2}{n}}}$ ($n \geq 3$).
- $K_0(n, 2) = \frac{16}{n(n-4)(n^2-4)\omega_n^{\frac{4}{n}}}$ ($n \geq 5$),

where ω_n denote the volume of the standard unit sphere S^n of \mathbb{R}^{n+1} .

Theorem 1.4.2. [35]

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 1$ and let k be a positive integer such that $n > 2k$. Then for any $\epsilon > 0$, there exists a real $B_\epsilon > 0$ such that for all $u \in H_k^2(M)$, one has

$$\left(\int_M |u|^N dv_g \right)^{\frac{2}{N}} \leq (K_0(n, k) + \epsilon) \int_M (\Delta_g^{\frac{k}{2}} u)^2 dv_g + B_\epsilon \|u\|_{H_{k-1}^2}^2.$$

1.5 Lagrange multipliers theorem and regularity

Theorem 1.5.1. [27]

Let $(E, \|\cdot\|)$ be a Banach space, Ω be an open subset of E , $f : \Omega \rightarrow \mathbb{R}$ be a differentiable function, and $\Phi : \Omega \rightarrow \mathbb{R}^n$ be of class C^1 where the Φ_i for $i = 1, \dots, n$ are the components of Φ . Let also $a \in \mathbb{R}^n$ be such that $H = \Phi^{-1}(a)$ is not empty. If $x_0 \in H$ is such that

$$f(x_0) = \min_{x \in H} f(x),$$

and $D\Phi(x_0)$ is surjective, then there exist $\lambda_i \in \mathbb{R}$, such that

$$Df(x_0) = \sum_{i=0}^n \lambda_i D\Phi_i(x_0).$$

The latter equation is called the Euler-Lagrange equation of the minimization problem $f(x_0) = \min_{x \in H} f(x)$. The λ_i are called the Lagrange multipliers of the equation.

Theorem 1.5.2. [27]

Let Ω be an open set of \mathbb{R}^n and

$$L(u) = \sum_{i,j=1}^n a_{ij}(x) D_{ij}u + \sum_{i=1}^n b_i(x) D_i u + c(x)u,$$

a linear elliptic operator of the second order with $C^\infty(\Omega)$ coefficients. Let $f \in L^1_{loc}(\Omega)$ and $u \in H^1_{1,loc}(\Omega)$ is a weak solution of the equation $L(u) = f$. Then one have,

- If $f \in C^{k,\alpha}(\Omega)$, then $u \in C^{k+2,\alpha}(\Omega)$ (with $k \in \mathbb{N}$, $0 < \alpha < 1$). In particular, if $f \in C^\infty(\Omega)$, then $u \in C^\infty(\Omega)$.
- If $f \in H^p_{k,loc}(\Omega)$, then $u \in H^p_{k+2,loc}(\Omega)$, where $k \in \mathbb{N}$, $p > 1$.

Chapter 2

Nodal solutions for a Paneitz-Branson type equation

2.1 Introduction and preliminaries

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$. Given A a smooth symmetric $(2, 0)$ -tensor field and $a \in C^\infty(M)$, we let the fourth-order operator L_g as follows:

$$L_g v = \Delta_g^2 v - \operatorname{div}_g(A(\nabla_g v)^\#) + av, \quad (2.1.1)$$

for all $v \in C^\infty(M, \mathbb{R})$ where $\Delta_g = -\operatorname{div}_g(\nabla_g)$ is the Laplacian-Beltrami operator, the symbol $\#$ stands for the musical isomorphism (index are raised with metric) and $A(\nabla_g v)^\#$ is the $(1, 0)$ -tensor whose coordinates in local chart are

$$(A(\nabla_g v)^\#)_i = A_{ij}((\nabla_g v)^\#)^j = A_{ij}g^{jk}(\nabla_g v)_k,$$

where A_{ij} are the component of A , g^{ij} the component of g^{-1} , $(\nabla_g v)_k$ the component of $\nabla_g v$ and i, j, k integers.

This part is concerned with the existence of nodal solutions (sign-changing solutions) to the nonlinear fourth-order elliptic equation

$$L_g v = \epsilon|v|^{N-2}v, \quad (2.1.2)$$

where $\epsilon < 0$ and $N = \frac{2n}{n-4}$. The number N is the critical Sobolev exponent, in fact the Sobolev embedding theorem asserts that the Sobolev space $H_2^2(M)$ is continuously embedded in the space $L^q(M)$ for $1 < q \leq N$, with the property that this embedding is compact when $q < N$. The fourth-order operator L_g is known as the Paneitz-Branson type operator with general coefficients. In 1983, Paneitz [36] discovered a conformally invariant fourth-order operator on 4-dimensional Riemannian manifolds. In 1987, Branson extended the notion to Riemannian manifolds of dimension $n \geq 5$. This operator has geometrical roots, it is associated to the notion of the Q-curvature

Nodal solutions for a Paneitz-Branson type equation

which can be seen as the analogue of the scalar curvature for the conformal Laplacian in Yamabe problem. The Paneitz-Branson operator P_g^n is given by:

$$P_g^n v = \Delta_g^2 v - \operatorname{div}_g(\bar{A}(\nabla_g v)^\#) + \bar{a}v, \quad (2.1.3)$$

where $\bar{A} = a_n S_g g + b_n \operatorname{Ric}_g$ a smooth $(2,0)$ -tensor on M , such that $\operatorname{Ric}_g, S_g$ are respectively the Ricci curvature and the scalar curvature of g , $\bar{A}_{ij} = a_n S_g g_{ij} + b_n \operatorname{Ric}_{ij}$, $a_n = \frac{(n-2)^2+4}{2(n-1)(n-2)}$, $b_n = -\frac{4}{n-2}$ and the function $\bar{a} = \frac{n-4}{2} Q_g^n$ where

$$Q_g^n = \frac{1}{2(n-2)} \Delta_g S_g + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} S_g^2 - \frac{2}{(n-2)^2} |\operatorname{Ric}_g|^2.$$

The Paneitz-branson operator is conformally invariant in the following sens : let $u \in C^\infty(M)$, $u > 0$, we consider the metric $\bar{g} = u^{\frac{N-2}{2}} g$ which is conformal to g and $N = \frac{2n}{n-4}$. Then we have

$$\forall \varphi \in C^4(M), \quad P_g^n(u\varphi) = u^{N-1} P_{\bar{g}}^n(\varphi). \quad (2.1.4)$$

In particular, taking $\varphi \equiv 1$, one gets the equation

$$P_g^n = \frac{n-4}{2} Q_{\bar{g}}^n u^{N-1}. \quad (2.1.5)$$

The Paneitz-Branson operator can be seen as an extension of the conformal Laplacian and Q_g^n is the Q-curvature. If the metric is Einstein, the geometric Paneitz-Branson operator is reduced to:

$$P_g^n v = \Delta_g^n v + \alpha_n S_g \Delta_g v + \beta_n S_g^2 v, \quad (2.1.6)$$

where

$$\alpha_n = \frac{n^2 - 2n - 4}{2n(n-1)} \quad \text{and} \quad \beta_n = \frac{(n-4)(n^2-4)}{16n(n-1)^2}.$$

Furthermore, the scalar curvature S_g is constant. With a standard abuse of notation, we have

$$\int_M v L_g u dv_g = \int_M u L_g v dv_g = \int_M (\Delta_g u \Delta_g v + A((\nabla_g u)^\#, (\nabla_g v)^\#) + auv) dv_g, \quad (2.1.7)$$

for all $u, v \in H_2^2(M)$.

It is well known that the fourth-order operator L_g is elliptic, self-adjoint with respect to the inner product in $L^2(M)$ and has a discrete spectrum,

$$\operatorname{Spec}(L_g) = \{\lambda_1(g), \lambda_2(g), \dots\},$$

2.1 Introduction and preliminaries

where the eigenvalues

$$\lambda_1(g) \leq \lambda_2(g) \leq \lambda_3(g) \dots \leq \lambda_k(g) \leq \dots$$

appear with their multiplicities. In particular, by referring to [1], in the geometric case, one sees that for any generalized metric $\bar{g} = u^{\frac{N-2}{2}}g$ of a Riemannian metric g where $u \in L^N(M)$ and $u \geq 0$, the k^{th} eigenvalue $\lambda_k(\bar{g})$ of the geometric Paneitz-Branson operator P_g^n is characterized by:

$$\lambda_k(\bar{g}) = \inf_{V \in Gr_k^u(H_2^2(M))} \sup_{v \in V \setminus \{0\}} \frac{\int_M v P_g^n v dv_g}{\int_M u^{N-2} v^2 dv_g}, \quad (2.1.8)$$

where $k \in N^*$ and $H_2^2(M)$ is the standard Sobolev space, which is the completed space $C^\infty(M)$ with respect to the equivalent norm:

$$\|v\| = \left(\int_M (|\Delta_g v|^2 + |\nabla_g v|^2 + v^2) dv_g \right)^{\frac{1}{2}}, \quad (2.1.9)$$

and the Grassmannian $Gr_k^u(H_2^2(M))$ is given in (1.2.2).

In [9], by analogy with the Yamabe invariant, the authors defined the standard Paneitz-Branson $\mu(M, g)$ by:

$$\mu(M, g) = \inf_{\substack{v \in H_2^2(M) \\ v \neq 0}} \frac{\int_M v P_g^n v dv_g}{\left(\int_M v^N dv_g \right)^{\frac{2}{N}}}, \quad (2.1.10)$$

and the Paneitz-Branson invariant of high order $\mu_k(M, g)$ by:

$$\mu_k(M, g) = \inf_{\bar{g} \in [g]} \lambda_k(\bar{g}) [Vol(M, \bar{g})]^{\frac{4}{n}}, \quad (2.1.11)$$

where $k \in N^*$, $Vol(M, g)$ denotes the Riemannian volume of M with respect to the metric g and the set

$$[g] = \left\{ \bar{g} = u^{\frac{N-2}{2}} g, u \in C_+^\infty(M) \right\}, \quad (2.1.12)$$

is the conformal class of the metric g and $\lambda_k(\bar{g})$ are given by (2.1.8).

The constants $\mu(M, g)$ and $\mu_k(M, g)$ are conformal invariant by definition.

For clarity purposes, we state here our main result:

Theorem 2.1.1.

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 5$. Assume that $\lambda_1(g) < 0$. Then, the equation

$$\Delta_g^2 v - \operatorname{div}_g(A(\nabla_g)^\#) + av = \epsilon|v|^{N-2}v,$$

has at least a nontrivial solution $v \in C^{4,\beta}(M)$ where $\epsilon < 0$ and $0 < \beta < 1$. In addition if the function $a \geq 0$, the solution v is nodal.

Notice that this theorem is regarded as combined results between Theorem 2.4.1, 2.5.1 and 2.6.1. This chapter is organized as follows. In section 2.2, we explain the motivation and we show that on Einstein manifolds $\mu_2(M, g)$ is not well defined where $\mu_2(M, g)$ is given by formula (2.1.11). In section 2.3, we establish some results concerning the eigenvalues, in particular $\lambda_1(g)$, $\lambda_2(g)$ are finite and achieved. In section 2.4 We show that the infimum $\alpha = \inf I_g(u)$ is attained by a function u where I_g is given by formula (2.2.6). In section 2.5, we prove that the main equation $L_g v = \epsilon|v|^{N-2}v$ has at least the solution $v = |L_g u|^{\frac{-8}{n+4}} L_g u$. In section 2.6, we establish the regularity of v and we show that v is a nodal solution and that the infimum α satisfy $I_g(v) = \alpha$. In section 2.7, we deal with the geometric operator P_g^n , we show that the infimum α is a conformal invariant and the geometric equation $P_g^n v = \epsilon|v|^{N-2}v$ has a nodal solution where $\epsilon < 0$ and at the end, we show that the conformal invariant $\mu(M, g)$ is attained by generalized metric when it is negative.

2.2 Motivation

In [9] the authors studied $\mu(M, g)$, $\mu_1(M, g)$ and $\mu_2(M, g)$ in the case $S_g > 0$. More precisely, they proved the following theorem:

Theorem 2.2.1.

Let (M, g) be a smooth compact Einstein manifold of dimension $n \geq 5$. Assume that the scalar curvature $S_g > 0$ and $n \geq 12$. Then $\mu_2(M, g) > 0$ and is attained by a generalized metric, in other words there exist $u \in L_+^N(M)$ and $w \in H_2^2(M)$ such that

$$P_g^n w = \mu_2(M, g)u^{N-2}w \quad \text{and} \quad \int_M u^{N-2}w^2 dv_g = 1.$$

In particular, w is nodal solution and $u = |w|$.

In [21] S. El Sayed studied the Yamabe equation when the Yamabe invariant $\mu(M, g) < 0$. More precisely, let (M, g) be a smooth compact manifold of dimension $n \geq 3$, the

2.2 Motivation

author was concerned by the following equation

$$C_n \Delta_g v + S_g v = \epsilon |v|^{N-2} u, \quad (2.2.1)$$

where $C_n = 4 \frac{(n-1)}{n-2}$, $\epsilon \in \{-1, 0, 1\}$ and $N = \frac{2n}{n-2}$ is the critical Sobolev exponent of the embedding $H_1^2(M) \subset L_q(M)$. In particular, in Theorem 0.2.2, if the second eigenvalue of the Yamabe operator $\lambda_2(g) < 0$, the author proved that the equation (2.2.1) has a nodal solution in the case $\epsilon = -1$. Inspired by the previous results, the authors in [11] enlightened the role of S_g in the existence of nodal solutions and assumed that S_g is not necessarily positive. It should also be noted that when the scalar curvature S_g is negative this implies that $\int_M v P_g^n v dv_g$ can be negative or positive and consequently the eigenvalues follow the same thing contrary to the case $S_g > 0$ which implies only the positivity of the eigenvalues. The authors obtained the following result:

Theorem 2.2.2.

Let (M, g) be a smooth compact Einstein manifold of dimension $n \geq 5$. Assume that the scalar curvature $S_g < 0$ and $\mu(M, g) < 0$. If $\lambda_2(g) > 0$ and $\lambda_1(g) < 0$, then $\mu_2(M, g) > 0$ and is attained by a generalized metric, in other words there exist $u \in L_+^N(M)$ and $w \in H_2^2(M)$ such that

$$P_g^n w = \mu_2(M, g) u^{N-2} w \quad \text{and} \quad \int_M u^{N-2} w^2 dv_g = 1.$$

Moreover, if $\mu_2(M, g) < 0$ and is attained, then w is a nodal solution and $u = |w|$. $\mu(M, g)$ is given by formula (2.1.10), it is always attained by positive $C^{4,\beta}(M)$ function when it is negative and $0 < \beta < 1$.

The first problem:

if we assume that $\lambda_2(g) > 0$ in the latter theorem, then $\mu_2(M, g) > 0$. Now to find nodal solutions, we would use some ideas in [[9],[10]] in particular Theorem (4), but it does not work. Indeed, without loss of generality, we can assume that $\mu_2(M, g) = \lambda_2(\bar{g})$ and we know from [11] that for any generalized metric $\bar{g} = u^{\frac{N-2}{2}} g$, there exist two functions $v, w \in H_2^2(M)$ satisfying

$$P_g^n v = \lambda_1(\bar{g}) u^{N-2} v, \quad (2.2.2)$$

and

$$P_g^n w = \lambda_2(\bar{g}) u^{N-2} w. \quad (2.2.3)$$

However, the sign of the solution w depend on the positivity of the solution v and it

is well known that the equation (2.2.2) has a positive solution if and only if $S_g > 0$, it follows that it is impossible to find the sign of the solution w with this method.

The second problem:

if we assume that $\lambda_2(g) < 0$, unfortunately in this case, the second Paneitz-Branson invariant $\mu_2(M, g)$ is not well defined as shown below in the Proposition 2.2.1 and the situation will be completely different to that made in [11], it seems that the second point in Theorem 2.2.2 is never realized and therefore, there is no nodal solution.

The strategy

To solve these problems, we will study a more general case, we will show a new result (Theorem 2.1.1). and we will prove that the method used in [21] can be extended to a more general setting such as fourth-order elliptic equations with more general coefficients and non Einstein manifolds and in particular we obtain the desired result to our problems on Einstein manifold, in other words : if $\lambda_2(g) > 0$ or $\lambda_2(g) < 0$, in both cases the geometric equation $P_g^n v = \epsilon|v|^{N-2}v$ has at least one nodal solution where $\epsilon < 0$. Note that the method used in [21] is different from the one used in [11]. More precisely, we are going to show on a smooth compact Riemannian manifold of dimension $n \geq 5$ that the first eigenvalue $\lambda_1(g)$ is finite and achieved and in the case of $\lambda_1(g) < 0$, the following equation

$$\Delta_g^2 v - \operatorname{div}_g(A(\nabla_g v)^\#) + av = \epsilon|v|^{N-2}v, \tag{2.2.4}$$

has at least a nontrivial solution $v \in C^{4,\beta}(M)$ where $\epsilon < 0$ and $0 < \beta < 1$.

In order to obtain solutions of (2.2.4), the strategy we would like to apply is the following: we define the quantity

$$\alpha = \inf I_g(u), \tag{2.2.5}$$

where the functional

$$I_g(u) = \frac{\left(\int_M |L_g u|^{\frac{2n}{n+4}} dv_g\right)^{\frac{n+4}{n}}}{\left|\int_M u L_g u dv_g\right|}, \tag{2.2.6}$$

and the infimum will be taken over all functions $u \in C^4(M)$ such that $\int_M u L_g u dv_g < 0$ and with the constraint $\int_M u \varphi dv_g = 0$ for all function $\varphi \in \ker L_g$. We will show that the infimum α is attained by a function u , when we put $v = |L_g u|^{\frac{-8}{n+4}} L_g u$, this function will be a weak solution of 2.2.4. Notice also that, the equation (2.2.4) is a critical nonlinear

2.2 Motivation

fourth-order elliptic equation with general coefficients. Many works are devoted to study this kind of equations, we refer the reader to [[9],[11],[12], [19],[20],[37],[23],[36]] and references therein.

Proposition 2.2.1.

Let (M, g) be a compact Einstein manifold of dimension $n \geq 5$, assume that the scalar curvature S_g is negative. Suppose that the second eigenvalue $\lambda_2(g) < 0$ of P_g^n , then $\mu_2(M, g) = -\infty$.

Proof

We know from [11] that for any generalized metric $\bar{g} = u^{\frac{N-2}{2}}g$ of a Riemannian metric g such that $u \in C^\infty(M)$ and $u > 0$, there exist two functions $v_1, v_2 \in C^\infty(M)$ fulfilling respectively (2.2.2), (2.2.3) and such that

$$\int_M u^{N-2} v_1 v_2 dv_g = 0, \quad \int_M u^{N-2} v_1^2 dv_g = 1 \quad \text{and} \quad \int_M u^{N-2} v_2^2 dv_g = 1.$$

Fix a point p in M . For $\epsilon > 0$, let ϕ_ϵ be a smooth cut-off function such that:

$$\begin{cases} 0 \leq \phi_\epsilon \leq 1, \\ \phi_\epsilon = 0 & \text{on } B_\epsilon(p), \\ \phi_\epsilon = 1 & \text{on } M \setminus B_{2\epsilon}(p), \\ |\nabla_g \phi_\epsilon| \leq \frac{c}{\epsilon} & \text{and } |\Delta_g \phi_\epsilon| \leq \frac{c}{\epsilon^2}, \end{cases}$$

where $B_\epsilon(p)$ is the open ball centered at p and of radius ϵ and $c > 0$ is a constant. Since $\lambda_1(\bar{g}) < \lambda_2(\bar{g}) < 0$, then we get:

$$\lim_{\epsilon \rightarrow 0} \int_M (\phi_\epsilon v_1) P_g^n (\phi_\epsilon v_1) dv_g = \int_M v_1 P_g^n v_1 dv_g = \lambda_1(\bar{g}) < 0.$$

Indeed:

Set the annulus $B(p) = B_{2\epsilon}(p) \setminus B_\epsilon(p)$, then one has

$$\int_M (\phi_\epsilon v_1) P_g^n (\phi_\epsilon v_1) dv_g = \underbrace{\int_{B_\epsilon(p)} (\phi_\epsilon v_1) P_g^n (\phi_\epsilon v_1) dv_g}_{I_1} + \underbrace{\int_{B(p)} (\phi_\epsilon v_1) P_g^n (\phi_\epsilon v_1) dv_g}_{I_2} + \underbrace{\int_{M \setminus B_{2\epsilon}(p)} (\phi_\epsilon v_1) P_g^n (\phi_\epsilon v_1) dv_g}_{I_3}.$$

Clearly the first integral $I_1 = 0$ (since $\phi_\epsilon = 0$ on the ball $B_\epsilon(p)$). For the second integral I_2 , we can find a constant $C > 0$ such that,

$$\begin{aligned}
 |I_2| &\leq \int_{B(p)} |\Delta_g(\phi_\epsilon v_1)|^2 + |a_n S_g| |\nabla_g(\phi_\epsilon v_1)|^2 + b_n S_g^2 (\phi_\epsilon v_1)^2 dv_g \\
 &\leq C \left(\int_{B(p)} |\Delta_g(\phi_\epsilon)|^2 + |\nabla_g(\phi_\epsilon)|^2 + (\phi_\epsilon)^2 dv_g \right).
 \end{aligned}$$

The proof of the latter inequality is given in appendix Lemma 2.8.1.

Now since $|\Delta_g(\phi_\epsilon)| \leq \frac{c}{\epsilon^2}$ and after using polar coordinates, there exists a constant $C_1 > 0$ such that,

$$\begin{aligned}
 \int_{B(p)} |\Delta_g(\phi_\epsilon)|^2 dv_g &\leq \left(\frac{c^2}{\epsilon^4}\right) \int_{B(p)} dv_g \\
 &= \left(\frac{c^2}{\epsilon^4}\right) \int_{B(0)} \sqrt{|g|} dx \\
 &= \left(\frac{c^2}{\epsilon^4}\right) \int_\epsilon^{2\epsilon} \sqrt{|g|} r^{n-1} dr \left(\int_{S^{n-1}} d\sigma \right) \\
 &\leq \left(\frac{c^2 \max(\sqrt{|g|})}{\epsilon^4}\right) \left(\int_{S^{n-1}} d\sigma \right) \int_\epsilon^{2\epsilon} r^{n-1} dr \\
 &= \underbrace{c^2 \max(\sqrt{|g|}) \left(\int_{S^{n-1}} d\sigma \right) \frac{1}{n} (2^n - 1) \frac{\epsilon^n}{\epsilon^4}}_{C_1} \\
 &= C_1 \epsilon^{n-4} \xrightarrow{\epsilon \rightarrow 0} 0 \quad \text{since } n \geq 5,
 \end{aligned}$$

where $|g|$ denotes the determinant of g , $d\sigma$ the area element on the standard unit sphere S^{n-1} and $B(0)$ is the euclidean ball. By the same way, there exist $C_2, C_3 > 0$ such that,

$$\int_{B(p)} |\nabla_g(\phi_\epsilon)|^2 dv_g \leq \left(\frac{c^2}{\epsilon^2}\right) \int_{B(p)} dv_g \leq C_2 \epsilon^{n-2} \xrightarrow{\epsilon \rightarrow 0} 0,$$

and

$$\int_{B(p)} \phi_\epsilon^2 dv_g \leq \int_{B(p)} dv_g \leq C_3 \epsilon^n \xrightarrow{\epsilon \rightarrow 0} 0,$$

which means that the second integral:

2.3 First and the second eigenvalues

$$I_2 \xrightarrow{\epsilon \rightarrow 0} 0.$$

And finally since $\phi_\epsilon = 1$ on $M \setminus B_{2\epsilon}(p)$, the third integral

$$I_3 = \int_{M \setminus B_{2\epsilon}(p)} v_1 P_g^n v_1 dv_g.$$

This implies that:

$$\lim_{\epsilon \rightarrow 0} \int_M (\phi_\epsilon v_1) P_g^n (\phi_\epsilon v_1) dv_g = \lim_{\epsilon \rightarrow 0} (I_1 + I_2 + I_3) = \lim_{\epsilon \rightarrow 0} I_3 = \int_M v_1 P_g^n v_1 dv_g = \lambda_1(\bar{g}) < 0.$$

If we put $w_1 = \phi_\epsilon v_1$, for ϵ small enough, we still have $\int_M w_1 P_g^n w_1 dv_g < 0$. The same thing goes with v_2 . Now, let $u_\epsilon \geq 0$ be a smooth function with support in $B_\epsilon(p)$ and $V = \text{span}(w_1, w_2)$, then for all $v \in V$ and for $s > 0$, we have $u_\epsilon + s > 0$ and

$$\lim_{s \rightarrow 0} \int_M (u_\epsilon + s)^{N-2} v^2 dv_g = 0,$$

and in particular for $\bar{g} = u_\epsilon^{\frac{N-2}{2}} g = \lim_{s \rightarrow 0} (u_\epsilon + s)^{\frac{N-2}{2}} g$, one has

$$\lambda_2(u_\epsilon^{\frac{N-2}{2}} g) \leq \lim_{s \rightarrow 0} \left(\sup_{v \in V} \frac{\int_M v P_g^n(v) dv_g}{\int_M (u_\epsilon + s)^{N-2} v^2 dv_g} \right) = -\infty,$$

and since $\mu_2(M, g)$ is the infimum over $[g]$, it follows that

$$\mu_2(M, g) = \inf_{\bar{g} \in [g]} \lambda_2(\bar{g}) [\text{vol}(M, \bar{g})]^{\frac{4}{n}} = -\infty. \quad \blacksquare$$

2.3 First and the second eigenvalues

In this section, we quote some facts which will be used in the sequel. The following Lemma is proved in [37].

Lemma 2.3.1.

Let (M, g) be a compact Riemannian manifold. Then for any $\epsilon > 0$, there exists $C_\epsilon > 0$

such that

$$\|\nabla_g u\|_2 \leq \epsilon \|\Delta_g u\|_2 + C_\epsilon \|u\|_2, \quad (2.3.1)$$

for all $u \in H_2^2(M)$.

In fact, since A is smooth, there exists a constant $C > 0$ such that for all $u \in H_2^2(M)$, we have

$$\int_M A((\nabla_g u)^\#, (\nabla_g u)^\#) dv_g \leq C \int_M |\nabla_g u|^2 dv_g.$$

Consequently, the previous lemma implies the existence of a constant $C > 0$ such that

$$\left| \int_M A((\nabla_g u)^\#, (\nabla_g u)^\#) dv_g \right| \leq \int_M \frac{1}{2} (\Delta_g u)^2 + C u^2 dv_g. \quad (2.3.2)$$

Now as already mentioned in the introduction, the operator L_g has a discrete spectrum. Before we get in the proof of the main result, we begin by showing that the first eigenvalue $\lambda_1(g)$ and the second eigenvalue $\lambda_2(g)$ of the operator L_g are finite and achieved and in particular $\lambda_1(g) \neq -\infty$. The following proposition will be essential for the existence of the infimum α where α is given by (2.2.5).

Proposition 2.3.1.

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 5$. Then, the first eigenvalue $\lambda_1(g)$ and the second eigenvalue $\lambda_2(g)$ of L_g are finite and achieved. In other words, there exist nontrivial functions $v, w \in H_2^2(M)$ solution of

$$L_g v = \lambda_1(g) v \quad \text{and} \quad L_g w = \lambda_2(g) w, \quad (2.3.3)$$

and such that

$$\int_M v^2 dv_g = \int_M w^2 dv_g = 1 \quad \text{and} \quad \int_M v w dv_g = 0,$$

where $\lambda_1(g)$, $\lambda_2(g)$ are given by 2.1.8. Moreover $v, w \in C^{4,\beta}(M)$ with $0 < \beta < 1$. In particular, if $a \geq 0$ and $\lambda_1(g) < 0$, or if $a \leq 0$ and $\lambda_1(g) > 0$, v is nodal solution, the same thing goes with w .

Proof

We first prove that $\lambda_1(g) > -\infty$. Let

$$J_g(v) = \frac{\int_M v L_g v dv_g}{\int_M v^2 dv_g}. \quad (2.3.4)$$

2.3 First and the second eigenvalues

Using the inequality (2.3.2), then for all $v \in H_2^2(M) \setminus \{0\}$, we get that

$$\begin{aligned}
 J_g(v) &= \frac{\int_M (\Delta_g v)^2 + A((\nabla_g v)^\#, (\nabla_g v)^\#) + av^2 dv_g}{\int_M v^2 dv_g} \\
 &\geq \frac{\int_M \frac{1}{2} (\Delta_g v)^2 dv_g - (C + \|a\|_\infty) \|v\|_2^2}{\int_M v^2 dv_g} \\
 &\geq \frac{\int_M \frac{1}{2} (\Delta_g v)^2 dv_g}{\int_M v^2 dv_g} - \frac{(C + \|a\|_\infty) \|v\|_2^2}{\|v\|_2^2} \\
 &\geq -(C + \|a\|_\infty) > -\infty.
 \end{aligned} \tag{2.3.5}$$

Let $(v_m) \in H_2^2(M)$ be a minimizing sequence for $\lambda_1(g)$, that is to say :

$$\lim_{m \rightarrow +\infty} \frac{\int_M v_m L_g v_m dv_g}{\int_M v_m^2 dv_g} = \lambda_1(g),$$

without loss of generality, we can assume that the sequence $(v_m)_m$ is such that

$$\int_M v_m^2 dv_g = 1. \tag{2.3.6}$$

For m large enough, one has $J_g(v_m) \leq \lambda_1(g) + 1$, then from (2.3.5), we get that

$$\int_M (\Delta_g v_m)^2 dv_g \leq 2((C + \|a\|_\infty) + \lambda_1(g) + 1). \tag{2.3.7}$$

By (2.3.6), (2.3.7) and (2.3.1), we then get that there exists $B > 0$ such that $\|v_m\| \leq B$ this means that the sequence $(v_m)_m$ is bounded in $H_2^2(M)$ and after restriction to a subsequence still labeled $(v_m)_m$ we may assume that there exist $v \in H_2^2(M)$, such that the sequence $v_m \rightharpoonup v$ weakly in $H_2^2(M)$ and $v_m \rightarrow v$ strongly in $H_l^2(M)$ for all $l \in \{0, 1\}$. It follows that

$$\lim_{m \rightarrow +\infty} \int_M v_m^2 dv_g = \int_M v^2 dv_g = 1, \tag{2.3.8}$$

which implies that $v \neq 0$.

Nodal solutions for a Paneitz-Branson type equation

Now let $\varphi_m = v_m - v$, then for all m , one gets :

$$\begin{aligned}
\int_M v_m L_g v_m dv_g &= \int_M (\varphi_m + v) L_g v_m dv_g \\
&= \int_M (\varphi_m L_g v_m + v L_g v_m) dv_g \\
&= \int_M (v_m L_g \varphi_m + v_m L_g v) dv_g \\
&= \int_M ((v + \varphi_m) L_g v + (v + \varphi_m) L_g \varphi_m) dv_g \\
&= \int_M v L_g v dv_g + \int_M \varphi_m L_g \varphi_m dv_g + \int_M 2v L_g \varphi_m dv_g \\
&= \int_M v L_g v dv_g + \int_M (\Delta_g \varphi_m)^2 + A((\nabla_g \varphi_m)^\#, (\nabla_g \varphi_m)^\#) + a\varphi_m^2 dv_g \\
&\quad + 2 \int_M (\Delta_g v \Delta_g \varphi_m + A((\nabla_g v)^\#, (\nabla_g \varphi_m)^\#) + av\varphi_m) dv_g.
\end{aligned}$$

Since $(\varphi_m)_m$ goes to 0, weakly in $H_2^2(M)$, we get

$$\int_M (\Delta_g v \Delta_g \varphi_m + A((\nabla_g v)^\#, (\nabla_g \varphi_m)^\#) + av\varphi_m) dv_g \longrightarrow 0, \quad (2.3.9)$$

and from the strong convergence of $(\varphi_m)_m$ to 0 in $H_1^2(M)$, we get

$$\int_M A((\nabla_g \varphi_m)^\#, (\nabla_g \varphi_m)^\#) + a\varphi_m^2 dv_g \longrightarrow 0. \quad (2.3.10)$$

(2.3.9) and (2.3.10) imply that,

$$\int_M v_m L_g v_m dv_g = \int_M v L_g v dv_g + \int_M (\Delta_g \varphi_m)^2 dv_g + o(1). \quad (2.3.11)$$

Again since m is large enough, one can write

$$\int_M v_m L_g v_m dv_g = \lambda_1(g) + o(1),$$

2.3 First and the second eigenvalues

it follows from (2.3.11) that

$$\lambda_1(g) = \int_M v L_g v dv_g + \int_M (\Delta_g \varphi_m)^2 dv_g + o(1). \quad (2.3.12)$$

Now with (2.3.8) and as $\lambda_1(g)$ is the infimum, we get that

$$\lambda_1(g) \leq \int_M v L_g v dv_g. \quad (2.3.13)$$

Plugging (2.3.12) and (2.3.13) together, we get that

$$\lambda_1(g) = \int_M v L_g v dv_g,$$

and

$$\lim_{m \rightarrow +\infty} \int_M (\Delta_g \varphi_m)^2 dv_g = 0.$$

Consequently v is a nontrivial weak minimizer of the functional $J_g(v)$ associated to $\lambda_1(g)$. In other words, the first eigenvalue $\lambda_1(g)$ is achieved at v . Note that the functional J_g is differentiable in $H_2^2(M)$, by writing the Euler-Lagrange equation, we find that v satisfies in the weak sense the following equation : for all $\varphi \in H_2^2(M)$, we have

$$\int_M (\Delta_g \varphi \Delta_g v + A((\nabla_g \varphi)^\#, (\nabla_g v)^\#) + a \varphi v) dv_g = \lambda_1(g) \int_M \varphi v dv_g,$$

by self-adjointness of L_g , we get that

$$\int_M \varphi (\Delta_g^2 v - \operatorname{div}_g(A(\nabla_g v)^\#) + av) dv_g = \lambda_1(g) \int_M \varphi v dv_g,$$

and at the end, we write

$$\Delta_g^2 v - \operatorname{div}_g(A(\nabla_g v)^\#) + av = \lambda_1(g)v. \quad (2.3.14)$$

With standard argument of regularity theorem for fourth-order elliptic equations, the function $v \in C^{4,\beta}(M)$ where $0 < \beta < 1$. In particular, if $a \geq 0$ and $\lambda_1(g) < 0$, the function v changes the sign. Indeed, if we integrate the equation (2.3.14) we find,

$$\underbrace{\int_M \Delta_g(\Delta_g v) dv_g}_{=0} - \underbrace{\int_M \operatorname{div}_g(A(\nabla_g v)^\#) dv_g}_{=0} + \int_M av dv_g = \lambda_1(g) \int_M v dv_g.$$

Now assume that $v \geq 0$, then we get that

$$\underbrace{\int_M avdv_g}_{\geq 0} = \lambda_1(g) \underbrace{\int_M vdv_g}_{< 0},$$

and since $v \neq 0$, this makes a contradiction. If we assume that $v \leq 0$, it follows that

$$\underbrace{\int_M avdv_g}_{\leq 0} = \lambda_1(g) \underbrace{\int_M vdv_g}_{> 0}.$$

And this is also a contradiction. Moreover, if $a \leq 0$ and $\lambda_1(g) > 0$, we get the same thing. Consequently, in both cases, the function v changes the sign.

Now we define

$$\lambda'_2(g) = \inf_M \frac{\int_M vL_g vdv_g}{\int_M v^2 dv_g},$$

where the infimum is taken over the set

$$E = \{w \in H^2_2(M) \text{ such that } w \neq 0, \int_M w^2 dv_g = 1 \text{ and } \int_M wvdv_g = 0\}.$$

Let (w_m) be a minimizing sequence for $\lambda'_2(g)$, with the same method as above, we find non trivial minimizer w to $\lambda'_2(g)$ such that $L_g(w) = \lambda'_2(g)w$ in the weak sense with $\int_M w^2 dv_g = 1$. Now writing

$$\begin{aligned} \int_M wvdv_g &= \int_M w_m v - w_m v + wvdv_g \\ &= \int_M v(w - w_m)dv_g + \int_M w_m vdv_g = 0. \end{aligned}$$

Indeed, since the sequence $w_m \in E$, $\int_M w_m vdv_g = 0$ and by the weak convergence of w_m to w in $L^2(M)$, we get that $\int_M v(w - w_m)dv_g \rightarrow 0$.

Now as in [9], we show that $\lambda'_2(g) = \lambda_2(g)$. ■

2.4 Existence of the infimum α

Theorem 2.4.1.

Assume that $\lambda_1(g) < 0$. Then there exists a function $u \in H_2^2(M)$ which is not identically null and such that the infimum of the functional I_g is attained by u where I_g is given by formula (2.2.6). In other words, we get

$$I_g(u) = \alpha.$$

Proof

Let $(u_m)_m$ be minimizing sequence for α , that is

$$\alpha = \lim_{m \rightarrow +\infty} I_g(u_m) = \lim_{m \rightarrow +\infty} \frac{M \left(\int |L_g u_m|^{\frac{2n}{n+4}} dv_g \right)^{\frac{n+4}{n}}}{\left| \int_M u_m L_g u_m dv_g \right|}, \quad (2.4.1)$$

with the constraint

$$\int_M u_m \varphi dv_g = 0, \quad (2.4.2)$$

for any function $\varphi \in \ker L_g$ and such that

$$\int_M u_m L_g u_m dv_g = -1. \quad (2.4.3)$$

Firstly, the condition $\lambda_1(g) < 0$ guarantees the non-emptiness of the constraint. Indeed, as shown in the Proposition (2.3.1), the first eigenvalue $\lambda_1(g)$ is finite and achieved. In other words, there exists v such that $L_g v = \lambda_1(g)v$ and for all $\varphi \in \ker L_g$, one has

$$\begin{aligned} v L_g \varphi = 0 &\implies \int_M v L_g \varphi = 0 \\ &\implies \int_M \varphi L_g v = 0 \\ &\implies \lambda_1(g) \int_M \varphi v dv_g = 0 \\ &\implies \int_M v \varphi dv_g = 0. \end{aligned}$$

Secondly, we have $I_g(\lambda u_m) = I_g(u_m)$, (with $\lambda \in \mathbb{R}^*$) this means that the sequence $h_m = \lambda u_m$ is also a minimizing sequence, if we choose λ such that $\lambda = \left(\left| \int_M u_m L_g u_m dv_g \right| \right)^{\frac{1}{2}}$,

it follows that

$$\int_M h_m L_g h_m dv_g = \lambda^2 \int_M u_m L_g u_m dv_g = \frac{1}{\left| \int_M u_m L_g u_m dv_g \right|} \int_M u_m L_g u_m dv_g = -1.$$

Hence without loss of generality, we can assume that $\int_M u_m L_g u_m dv_g = -1$.

Thirdly, we begin by showing that $(u_m)_m$ is a bounded sequence in $H_4^{\frac{2n}{n+4}}(M)$, then we are going to proceed by contradiction. Suppose that $(u_m)_m$ is not bounded in $H_4^{\frac{2n}{n+4}}(M)$ and we assume that, up to a subsequence that

$$\lim_{m \rightarrow +\infty} \|u_m\|_{H_4^{\frac{2n}{n+4}}} = +\infty.$$

Let

$$v_m = \frac{u_m}{\|u_m\|_{H_4^{\frac{2n}{n+4}}}}.$$

Clearly $\|v_m\|_{H_4^{\frac{2n}{n+4}}} = 1$, then the sequence $(v_m)_m$ is a bounded sequence in the space $H_4^{\frac{2n}{n+4}}(M)$ and therefore, there exists $v \in H_4^{\frac{2n}{n+4}}(M)$ such that after restriction to a subsequence we may assume that $v_m \rightharpoonup v$ weakly in $H_4^{\frac{2n}{n+4}}(M)$ and with the Sobolev embedding $H_4^{\frac{2n}{n+4}}(M) \subset H_2^2(M)$ we get that

$$v_m \rightharpoonup v \quad \text{weakly in } H_2^2(M). \quad (2.4.4)$$

It follows from (2.4.4) that, for all $\varphi \in H_2^2(M)$ one has

$$\int_M \varphi L_g v_m dv_g \longrightarrow \int_M \varphi L_g v dv_g. \quad (2.4.5)$$

In particular $\varphi \in L^N(M)$ which is the dual space of $L^{\frac{2n}{n+4}}(M)$, then (2.4.5) means that

$$L_g v_m \rightharpoonup L_g v \quad \text{weakly in } L^{\frac{2n}{n+4}}(M), \quad (2.4.6)$$

and here by standard arguments, one has

$$\left(\int_M |L_g v|^{\frac{2n}{n+4}} dv_g \right)^{\frac{n+4}{n}} \leq \liminf \left(\int_M |L_g v_m|^{\frac{2n}{n+4}} dv_g \right)^{\frac{n+4}{n}}, \quad (2.4.7)$$

thus

$$\left(\int_M |L_g v|^{\frac{2n}{n+4}} dv_g \right)^{\frac{n+4}{4}} \leq \frac{1}{\|u_m\|_{H_4^{\frac{2n}{n+4}}}^2} \liminf \left(\int_M |L_g u_m|^{\frac{2n}{n+4}} dv_g \right)^{\frac{n+4}{n}}.$$

2.4 Existence of the infimum α

Now for a large enough m , one gets

$$\left(\int_M \underbrace{|L_g v|}_{\geq 0}^{\frac{2n}{n+4}} dv_g \right)^{\frac{n+4}{4}} \leq \frac{1 + \alpha}{\|u_m\|_{H_4^{\frac{2n}{n+4}}}^2} \rightarrow 0,$$

which implies that $L_g v = 0$, in other words $v \in \ker L_g$.

From (2.4.2) one gets

$$\int_M v_m v dv_g = \frac{\int u_m v dv_g}{\|u_m\|_{H_4^{\frac{2n}{n+4}}}} = 0,$$

and since

$$\int_M v^2 dv_g = \lim_{m \rightarrow +\infty} \int_M v_m v dv_g,$$

one has

$$v = 0. \quad (2.4.8)$$

According to the regularity Theorem (1.7) in [37] applied to v_m , (for $p = \frac{2n}{n+4}$, $k = 0$) we have

$$1 = \|v_m\|_{H_4^{\frac{2n}{n+4}}} \leq c(\|L_g v_m\|_{L^{\frac{2n}{n+4}}} + \|v_m\|_{L^{\frac{2n}{n+4}}}), \quad (c > 0). \quad (2.4.9)$$

Passing to the limit, we obtain $\|v\|_{L^{\frac{2n}{n+4}}} \geq \frac{1}{c}$ which gives contradiction with (2.4.8).

Finally we deduce that $(u_m)_m$ is a bounded sequence in $H_4^{\frac{2n}{n+4}}(M)$ and there exists a function u in $H_4^{\frac{2n}{n+4}}(M)$ such that $u_m \rightarrow u$ weakly in $H_4^{\frac{2n}{n+4}}(M)$. As above, we deduce that

$$\left(\int_M |L_g u|^{\frac{2n}{n+4}} dv_g \right)^{\frac{n+4}{n}} \leq \liminf \left(\int_M |L_g u_m|^{\frac{2n}{n+4}} dv_g \right)^{\frac{n+4}{n}}. \quad (2.4.10)$$

Now with the embedding $H_4^{\frac{2n}{n+4}}(M) \subset H_2^2(M)$, we deduce that

$$u_m \rightarrow u \quad \text{weakly in } H_2^2(M), \quad (2.4.11)$$

and then it follows that $u_m \rightarrow u$ strongly in $H_1^2(M)$ and $L^2(M)$. From the strong convergence of $(u_m)_m$ in $H_1^2(M)$, as in formula (2.3.10), one has

$$\int_M A((\nabla_g u)^\#, (\nabla_g u)^\#) + au^2 dv_g = \lim_m \int_M A((\nabla_g u_m)^\#, (\nabla_g u_m)^\#) + au_m^2 dv_g.$$

Together with the weak convergence of $(u_m)_m$ and (2.4.3), one gets

$$\begin{aligned}
 \int_M u L_g u dv_g &= \int_M (\Delta_g u)^2 dv_g + \int_M A((\nabla_g u)^\#, (\nabla_g u)^\#) + au^2 dv_g \\
 &\leq \liminf \int_M (\Delta_g u_m)^2 dv_g + \lim \int_M A((\nabla_g u_m)^\#, (\nabla_g u_m)^\#) + au_m^2 dv_g \\
 &\leq \liminf \int_M (\Delta_g u_m)^2 + A((\nabla_g u_m)^\#, (\nabla_g u_m)^\#) + au_m^2 dv_g \\
 &= \liminf \int_M u_m L_g u_m dv_g \\
 &= -1.
 \end{aligned} \tag{2.4.12}$$

This implies that $\int_M u L_g u dv_g < 0$, so we deduce that the function $u \neq 0$. Using the definition of I_g (2.2.6), (2.4.12) and (2.4.10), one gets

$$I_g(u) \leq \left(\int_M |L_g u|^{\frac{2n}{n+4}} dv_g \right)^{\frac{n+4}{n}} \leq \liminf \left(\int_M |L_g u_m|^{\frac{2n}{n+4}} dv_g \right)^{\frac{n+4}{n}} = \alpha,$$

and since α is the infimum, we obtain the desired equality $I_g(u) = \alpha$. Again by the weak convergence of $(u_m)_m$ and (2.4.2), we also have

$$0 = \lim_{m \rightarrow +\infty} \int_M u_m \varphi dv_g = \int_M u \varphi dv_g,$$

for any function $\varphi \in L^2(M)$ and in particular for any $\varphi \in \ker L_g$. Finally with (2.4.11), since $u \neq 0$ and using formula (2.3.11) with u_m , u and $\varphi_m = u_m - u$ and passing to the limit, we get that

$$\int_M u L_g u dv_g = -1.$$

This means that α is achieved by the non-identical null function u . ■

2.5 Paneitz-Branson type equation

2.5 Paneitz-Branson type equation

Theorem 2.5.1.

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 5$. Then, the equation

$$\Delta_g^2 v - \operatorname{div}_g(A(\nabla_g v)^\#) + av = \epsilon|v|^{N-2}v,$$

has at least nontrivial weak solution $v \in H_2^2(M)$.

Proof

Let u as in the Theorem 2.4.1 and let u_1, \dots, u_k be a base of $\ker L_g$. Now, by the Lagrange multipliers theorem, there exist real numbers $\lambda_1, \dots, \lambda_k$ such that for all function $\varphi \in C^\infty(M)$, we get

$$DI_g(u) \cdot \varphi = \sum_{i=1}^k \lambda_i DJ_g(u) \cdot \varphi,$$

where $J_g(u) = \int_M (u + t\varphi)u_i dv_g$, that is to say :

$$(I(u + t\varphi))'_{t=0} = \sum_{i=1}^k \lambda_i (J(u + t\varphi)u_i)'_{t=0}.$$

With a simple calculation, one can obtain the following equation

$$2\delta^{\frac{4}{n+4}} \int_M L_g u |L_g u|^{\frac{-8}{n+4}} L_g \varphi dv_g + 2\delta \int_M \varphi L_g u dv_g = \sum_{i=1}^k \lambda_i \int_M \varphi u_i dv_g,$$

where we have put

$$\delta = \left(\int_M |L_g u|^{\frac{2n}{n+4}} dv_g \right)^{\frac{n+4}{n}}. \quad (2.5.1)$$

Indeed, we have $I_g(v) = \frac{I_1(v)}{I_2(v)}$ where $I_1(v) = \left(\int_M |L_g v|^{\frac{2n}{n+4}} dv_g \right)^{\frac{n+4}{n}}$ and $I_2(v) = \left| \int_M v L_g v dv_g \right|$,

then

$$\begin{aligned}
 (I_1(u + t\varphi))'_{t=0} &= \left(\frac{n+4}{n}\right) \left(\int_M |L_g(u)|^{\frac{2n}{n+4}} dv_g\right)^{\frac{4}{n}} \left(\int_M |L_g(u + t\varphi)|^{\frac{2n}{n+4}} dv_g\right)'_{t=0} \\
 &= \left(\frac{n+4}{n}\right) \delta^{\frac{4}{n+4}} \left(\frac{2n}{n+4}\right) \int_M |L_g u|^{\frac{n-4}{n+4}} (|L_g(u + t\varphi)|)'_{t=0} dv_g \\
 &= 2\delta^{\frac{4}{n+4}} \int_M |L_g u|^{\frac{n-4}{n+4}} (\sqrt{(L_g(u + t\varphi))^2})'_{t=0} dv_g \\
 &= 2\delta^{\frac{4}{n+4}} \int_M |L_g u|^{\frac{n-4}{n+4}} \frac{L_g u L_g \varphi}{|L_g u|} dv_g \\
 &= 2\delta^{\frac{4}{n+4}} \int_M |L_g u|^{\frac{-8}{n+4}} L_g \varphi L_g u dv_g.
 \end{aligned}$$

On the other hand and since $\int_M u L_g u dv_g = -1$, we have

$$\begin{aligned}
 (I_2(u + t\varphi))'_{t=0} &= \left(\sqrt{\int_M (u + t\varphi) L_g(u + t\varphi) dv_g}\right)'_{t=0} \\
 &= \frac{(\int_M u L_g u dv_g) (\int_M (u + t\varphi) L_g(u + t\varphi) dv_g)'_{t=0}}{|\int_M u L_g u dv_g|} \\
 &= \left(\int_M u L_g u dv_g\right) \left(\int_M (u L_g \varphi + \varphi L_g u) dv_g\right) \\
 &= 2 \left(\int_M u L_g u dv_g\right) \left(\int_M \varphi L_g u dv_g\right) \quad (L_g \text{ is self-adjoint}) \\
 &= -2 \int_M \varphi L_g u dv_g.
 \end{aligned}$$

2.5 Paneitz-Branson type equation

By applying the derivative of the quotient, we have

$$\begin{aligned}
(I(u + t\varphi))'_{t=0} &= \frac{(I_1(u + t\varphi))'_{t=0}I_2(u) - (I_2(u + t\varphi))'_{t=0}I_1(u)}{I_2(u)^2} \\
&= \frac{(2\delta^{\frac{4}{n+4}} \int_M |L_g u|^{\frac{-8}{n+4}} L_g \varphi L_g u dv_g)(\int_M u L_g u dv_g) + 2\delta \int_M \varphi L_g u dv_g}{|\int_M u L_g u dv_g|^2} \\
&= 2\delta^{\frac{4}{n+4}} \int_M L_g u |L_g u|^{\frac{-8}{n+4}} L_g \varphi dv_g + 2\delta \int_M \varphi L_g u dv_g \\
&= \sum_{i=1}^k \lambda_i \left(\int_M (u + t\varphi) u_i dv_g \right)'_{t=0} \\
&= \sum_{i=1}^k \lambda_i \int_M \varphi u_i dv_g,
\end{aligned}$$

and as the operator L_g is self-adjoint, we get that

$$2\delta^{\frac{4}{n+4}} \int_M (L_g u |L_g u|^{\frac{-8}{n+4}}) L_g \varphi dv_g + 2\delta \int_M u L_g \varphi dv_g = \sum_{i=1}^k \lambda_i \int_M \varphi u_i dv_g.$$

If $\varphi \in \ker L_g$, then the left-hand side member in the latter equality is zero, this means that for all $\varphi \in \ker L_g$, one gets

$$\sum_{i=1}^k \lambda_i \int_M \varphi u_i dv_g = 0.$$

In particular, it follows that for $\varphi = \sum_{i=1}^k \lambda_i u_i \in \ker L_g$, one has

$$0 = \sum_{i=1}^k \lambda_i \int_M \varphi u_i dv_g = \int_M \left(\sum_{i=1}^k \lambda_i u_i \right) \varphi dv_g = \int_M \varphi^2 dv_g,$$

this implies that $\varphi^2 = 0$ which gives $\varphi = 0$. Hence, we deduce that

$$\sum_{i=1}^k \lambda_i u_i = 0,$$

this gives, for any function $\varphi \in \ker L_g$ or not, we have always

$$\sum_{i=1}^k \lambda_i \int_M \varphi u_i dv_g = \int_M \left(\sum_{i=1}^k \lambda_i u_i \right) \varphi dv_g = 0.$$

The following equation

$$2\delta^{\frac{4}{n+4}} \int_M (L_g u |L_g u|^{\frac{-8}{n+4}}) L_g \varphi dv_g + 2\delta \int_M u L_g \varphi dv_g = 0,$$

we deduce that the function u satisfies in the weak sense the following equation

$$L_g [L_g u |L_g u|^{\frac{-8}{n+4}}] = \epsilon L_g u, \quad (2.5.2)$$

where $\epsilon = -\delta^{\frac{n}{n+4}} < 0$. Now, we set

$$v = L_g u |L_g u|^{\frac{-8}{n+4}}, \quad (2.5.3)$$

then v is nontrivial and the absolute value of v satisfies

$$|v| = |L_g u|^{\frac{n-4}{n+4}}, \quad (2.5.4)$$

which implies that $|v|^{\frac{n+4}{n-4}} = |L_g u|$, therefore, $|v|^{N-2}|v| = |L_g u|$. That is

$$L_g u = |v|^{N-2}v, \quad (2.5.5)$$

and finally, by plugging (2.5.3) in (2.5.2) and (2.5.5) in (2.5.2), we deduce that the function v satisfies the equation:

$$L_g v = \epsilon |v|^{N-2}v, \quad (2.5.6)$$

where $N - 2 = \frac{8}{n-4}$. ■

2.6 Regularity and sign of the solution

Theorem 2.6.1.

Assume that the function $a \geq 0$ then the solution v of the latter equation (2.5.6) is in $C^{4,\beta}(M)$ and changes the sign.

Proof

2.6 Regularity and sign of the solution

We have

$$\begin{aligned}
 \int_M |v|^N dv_g &= \int_M |L_g u| |L_g u|^{\frac{-8}{n+4}} |v|^N dv_g \\
 &= \int_M |L_g u|^{\frac{n-4}{n+4}N} dv_g \\
 &= \int_M |L_g u|^{\frac{2n}{n+4}} dv_g \\
 &= \alpha^{\frac{n}{n+4}} < +\infty,
 \end{aligned}$$

this means that $v \in L^N(M)$. Now, we show that $L_g v \in L^{\frac{N}{N-1}}(M) = L^{\frac{2n}{n+4}}(M)$. In fact, from 2.5.6 we have

$$\begin{aligned}
 |L_g v| &= |\epsilon| |v|^{N-1} \implies |L_g v|^{\frac{N}{N-1}} = |\epsilon|^{\frac{N}{N-1}} |v|^N \\
 &\implies \int_M |L_g v|^{\frac{N}{N-1}} dv_g = |\epsilon|^{\frac{N}{N-1}} \int_M |v|^N dv_g < +\infty \\
 &\implies L_g v \in L^{\frac{N}{N-1}}(M) = L^{\frac{2n}{n+4}}(M),
 \end{aligned}$$

which implies that

$$v \in H_4^{\frac{2n}{n+4}}(M) \subset H_2^2(M),$$

and with a standard argument of regularity theorem for fourth-order elliptic equations (see [20], Proposition (3)), the function $v \in C^{4,\beta}(M)$ with $0 < \beta < 1$.

If we integrate the equation (2.5.6) we find:

$$\underbrace{\int_M \Delta_g(\Delta_g v) dv_g}_{=0} - \underbrace{\int_M \operatorname{div}_g(A(\nabla_g u)^\#) dv_g}_{=0} + \int_M a v dv_g = \int_M \epsilon |v|^{N-2} v dv_g.$$

Assume that $v \geq 0$, then

$$\underbrace{\int_M a v dv_g}_{\geq 0} = \underbrace{\int_M \epsilon |v|^{N-2} v dv_g}_{< 0},$$

and since $v \neq 0$, this makes a contradiction.

Moreover, if we assume that $v \leq 0$, then

$$\underbrace{\int_M a v dv_g}_{\leq 0} = \underbrace{\int_M \epsilon |v|^{N-2} v dv_g}_{> 0}.$$

And this is also a contradiction. Then in both cases, the function v changes the sign.

■

Proposition 2.6.1.

The infimum α is also achieved by v , that is to say that the functional I_g satisfy the following equality:

$$I_g(v) = \alpha.$$

Proof

Calculate the functional I_g at v . From the main equation 2.5.6 one get

$$|L_g v| = |\epsilon| |v|^{N-1} \quad \text{and} \quad v L_g v = \epsilon |v|^N.$$

Then

$$\begin{aligned} I_g(v) &= \frac{\left(\int_M |L_g v|^{\frac{2n}{n+4}} dv_g \right)^{\frac{n+4}{n}}}{\left| \int_M v L_g v dv_g \right|} \\ &= \frac{\epsilon^2 \left(\int_M |v|^{\frac{(N-1)2n}{n+4}} dv_g \right)^{\frac{n+4}{n}}}{|\epsilon| \int_M |v|^N dv_g} \\ &= \frac{|\epsilon| \left(\int_M |v|^{\frac{(N-1)2n}{n+4}} dv_g \right)^{\frac{n+4}{n}}}{\int_M |v|^N dv_g}. \end{aligned}$$

And as

$$\frac{(N-1)2n}{n+4} = \frac{\left(\frac{2n}{n-4} - 1\right)2n}{n+4} = \frac{\left(\frac{n+4}{n-4}\right)2n}{n+4},$$

it follows that

$$I_g(v) = \delta^{\frac{n}{n+4}} \frac{\left(\int_M |v|^{\frac{2n}{n-4}} dv_g \right)^{\frac{n+4}{n}}}{\int_M |v|^{\frac{2n}{n-4}} dv_g} = \delta^{\frac{n}{n+4}} \left(\int_M |v|^{\frac{2n}{n-4}} dv_g \right)^{\frac{4}{n}}.$$

Now with (2.5.4), we get that

$$I_g(v) = \delta^{\frac{n}{n+4}} \left(\int_M |L_g u|^{\frac{2n}{n+4}} dv_g \right)^{\frac{4}{n}},$$

2.7 Geometric case

and finally by (2.5.1), we obtain

$$I_g(v) = \delta^{\frac{n}{n+4}} \delta^{\frac{4}{n+4}} = \delta = \alpha.$$

Moreover, by (2.5.5) for any $\varphi \in \ker L_g$, one has

$$\int_M |v|^{N-2} v \varphi dv_g = \int_M \varphi L_g u dv_g = \int_M u L_g \varphi dv_g = 0.$$

■

2.7 Geometric case

In this section, we deal with the geometric Paneitz-Branson operator P_g^n . We have the nice following results.

Proposition 2.7.1.

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$. If L_g is the geometric Paneitz-Branson operator P_g^n , the infimum α of the functional I_g is a conformal invariant. α is given by (2.2.5).

Proof

Let $\bar{g} = \varphi^{\frac{N-2}{2}} g$ be a conformal metric, φ a smooth positive function. Then

$$dv_{\bar{g}} = \varphi^N dv_g, \tag{2.7.1}$$

and by the conformal invariance of P_g^n formula (2.1.4), we get that

$$P_{\bar{g}}^n u = \varphi^{1-N} P_g^n(u\varphi), \quad \forall u \in C^4(M). \tag{2.7.2}$$

Plugging the expression of $P_{\bar{g}}^n$ in I_g , one can write

$$I_{\bar{g}}(u) = \frac{(\int_M |P_{\bar{g}}^n u|^{\frac{2n}{n+4}} dv_{\bar{g}})^{\frac{n+4}{n}}}{|\int_M u P_{\bar{g}}^n u dv_{\bar{g}}|},$$

by using (2.7.1) and (2.7.2), a direct calculation gives

$$I_{\bar{g}}(u) = \frac{(\int_M |\varphi^{1-N} P_g^n(u\varphi)|^{\frac{2n}{n+4}} \varphi^N dv_g)^{\frac{n+4}{n}}}{|\int_M u \varphi^{1-N} P_g^n(u\varphi) \varphi^N dv_g|}.$$

Nodal solutions for a Paneitz-Branson type equation

Since $(1 - N)\frac{2n}{n+4} = (1 - \frac{2n}{n-4})\frac{2n}{n+4} = -\frac{2n}{n-4} = -N$, we deduce that

$$I_{\bar{g}}(u) = \frac{\left(\int_M |P_g^n(u\varphi)|^{\frac{2n}{n+4}} dv_g\right)^{\frac{n+4}{n}}}{\left|\int_M u\varphi P_g^n(u\varphi) dv_g\right|},$$

thus

$$I_{\bar{g}}(u) = I_g(\varphi u).$$

From the proposition 2.6.1, there exists $v \in C^4(M)$ such that $\int_M |v|^{N-2} v v' dv_g = 0$ for all $v' \in \ker P_g^n$ and after using $P_{\bar{g}}^n(v'\varphi^{-1}) = \varphi^{1-N} P_g^n(v') = 0$, the function $v'\varphi^{-1} \in \ker P_{\bar{g}}^n$. Since $\varphi \in C^\infty(M)$, it follows that for any $v' \in \ker P_g^n$, the condition

$$\int_M |\varphi v|^{N-2} (v\varphi) v' dv_g = 0,$$

becomes

$$\int_M |v\varphi|^{N-2} (v\varphi) v' \varphi^{-N} dv_{\bar{g}} = 0, \quad (\text{because } dv_{\bar{g}} = \varphi^N dv_g),$$

which implies that

$$\int_M |v|^{N-2} (v) (v'\varphi^{-1}) dv_{\bar{g}} = 0, \quad (\varphi > 0),$$

for any function $v'\varphi^{-1} \in \ker P_{\bar{g}}^n$.

Note that the infimum α is achieved by a function $v \in C^4(M)$ such that

$$\int_M v L_g v dv_g = \epsilon \int_M |v|^N dv_g < 0,$$

and

$$\int_M |v|^{N-2} v \varphi dv_g = \int_M \varphi L_g v dv_g = \int_M v L_g \varphi dv_g = 0,$$

for any $\varphi \in \ker L_g$ as shown in the previous proposition and this explains why the infimum α can be taken over functions such as v and therefore it will be conformal invariant. ■

Proposition 2.7.2.

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$ and P_g^n be the geometric Paneitz-Branson operator. Assume that $Q_g^n \geq 0$. If $\lambda_1(g) < 0$, the geometric equation $P_g^n v = \epsilon |v|^{N-2} v$ has a nontrivial nodal solution $v \in C^{4,\beta}(M)$ where $\epsilon < 0$ and $0 < \beta < 1$. In particular, on Einstein manifold we have always $Q_g^n \geq 0$.

2.7 Geometric case

Proof

As explained in the section motivation, if we assume $\lambda_2(g) > 0$, the second Paneitz-Branson invariant $\mu_2(M, g) \geq 0$ and it is impossible to find the sign of the solution and if we assume that $\lambda_2(g) < 0$, the second Paneitz-Branson invariant $\mu_2(M, g)$ is not well defined as shown in Proposition 2.2.1. Now by the main theorem Theorem 2.1.1 applied to P_g^n and since $\bar{a} = \frac{n-4}{2}Q_g^n \geq 0$, in all of cases, $P_g^n v = \epsilon|v|^{N-2}v$ has a nontrivial nodal solution $v \in C^{4,\beta}(M)$ where $\epsilon < 0$ and $0 < \beta < 1$.

In fact on Einstein manifold, we can easily get

$$\begin{aligned} Q_g^n &= \left(\frac{2}{n-4}\right)\bar{a} \\ &= \left(\frac{2}{n-4}\right)b_n S_g^2 \\ &= \left(\frac{2}{n-4}\right)\frac{(n-4)(n^2-4)}{16n(n-1)^2}S_g^2 \\ &= \frac{n^2-4}{8n(n-1)^2}S_g^2 \geq 0. \end{aligned}$$

■

A more interesting situation on non Einstein manifold with the standard Paneitz-Branson $\mu(M, g)$ when it is negative.

Theorem 2.7.1.

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$ and P_g^n be the geometric Paneitz-Branson operator. Assume that the standard Paneitz-Branson $\mu(M, g) < 0$ and $Q_g^n \geq 0$ where $\mu(M, g)$ is defined by formula (2.1.10). If $\lambda_1(g) < 0$, then $\mu(M, g)$ is attained by the generalized metric $\bar{g} = u^{\frac{N-2}{2}}g$.

Proof

Since $\mu(M, g) < 0$ and $Q_g^n \geq 0$, it follows from the latter proposition (Proposition 2.7.2) that the geometric equation

$$P_g^n v = \mu(M, g)|v|^{N-2}v, \tag{2.7.3}$$

has a nontrivial nodal solution $v \in C^{4,\beta}(M)$ where in this case $\epsilon = \mu(M, g) < 0$ and $0 < \beta < 1$. Now we multiply the equation by v and we integrate, we then get

$$\int_M v P_g^n v dv_g = \mu(M, g) \int_M |v|^{N-2}v^2 dv_g = \mu(M, g) \int_M |v|^N dv_g.$$

On the other hand, if we put $v = \lambda h$, one has

$$\begin{aligned} \int_M \lambda h P_g^n(\lambda h) dv_g &= \mu(M, g) \int_M |\lambda h|^N dv_g \implies \lambda^2 \int_M h P_g^n(h) dv_g = \mu(M, g) |\lambda|^N \int_M |h|^N dv_g \\ &\implies \int_M h P_g^n(h) dv_g = \mu(M, g) |\lambda|^{N-2} \int_M |h|^N dv_g, \end{aligned}$$

then one can choose $\lambda = \pm \left(\int_M |h|^N dv_g \right)^{\frac{1}{2-N}}$, which implies that $\int_M h P_g^n h dv_g = \mu(M, g)$, hence we can assume that $\int_M |h|^N dv_g = 1$. Note that by formula (2.1.10), $\mu(M, g)$ is a conformal invariant which means that $\mu(M, g)$ is attained. Now if we set $u = |v|$, we can also write

$$\mu(M, g) = \frac{\int_M v P_g^n v dv_g}{\int_M u^{N-2} v^2 dv_g},$$

this implies that $\mu(M, g)$ is attained by the generalized metric $\bar{g} = u^{\frac{N-2}{2}} g$. In addition, if we put $v = (\mu(M, g)^{\frac{4-n}{8}}) f$ in (2.7.3), one get

$$\mu(M, g)^{\frac{4-n}{8}} P_g^n f = \mu(M, g) |(\mu(M, g)^{\frac{4-n}{8}} f|^{N-2} \mu(M, g)^{\frac{4-n}{8}} f,$$

then it follows that

$$\begin{aligned} P_g^n f &= \mu(M, g) |\mu(M, g)|^{-1} |f|^{N-2} f, \quad (\text{where } N-2 = \frac{8}{n-4}) \\ &= -|f|^{N-2} f \quad (\text{because } \mu < 0). \end{aligned}$$

So f satisfies the equation

$$P_g^n f = -|f|^{N-2} f. \quad \blacksquare$$

2.8 Appendix

Let I_2 be the integral given in the proof of Proposition 2.2.1. More precisely,

$$I_2 = \int_{B(p)} (\phi_\epsilon v_1) P_g^n(\phi_\epsilon v_1) dv_g = \int_{B(p)} |\Delta_g(\phi_\epsilon v_1)|^2 + a_n S_g |\nabla_g(\phi_\epsilon v_1)|^2 + b_n S_g^2 (\phi_\epsilon v_1)^2 dv_g,$$

where $v \in C^\infty(M)$, ϕ_ϵ a cut-off function, $B_\epsilon(p)$ is the open ball centered at p and $B(p) = B_{2\epsilon}(p) \setminus B_\epsilon(p)$.

2.8 Appendix

Lemma 2.8.1.

There exists a constant $C > 0$ such that

$$\begin{aligned} |I_2| &\leq \int_{B(p)} |\Delta_g(\phi_\epsilon v_1)|^2 + |a_n S_g| |\nabla_g(\phi_\epsilon v_1)|^2 + b_n S_g^2 (\phi_\epsilon v_1)^2 dv_g \\ &\leq C \left(\int_{B(p)} |\Delta_g(\phi_\epsilon)|^2 + |\nabla_g(\phi_\epsilon)|^2 + (\phi_\epsilon)^2 dv_g \right). \end{aligned}$$

Proof

By definition we have :

$$\begin{aligned} \Delta_g(\phi_\epsilon v_1) &= -\operatorname{div}_g(\nabla_g(\phi_\epsilon v_1)) \\ &= -\operatorname{div}_g(\phi_\epsilon \nabla_g(v_1) + v_1 \nabla_g(\phi_\epsilon)) \\ &= \phi_\epsilon \Delta_g v_1 + v_1 \Delta_g \phi_\epsilon - 2(\nabla_g v_1, \nabla_g \phi_\epsilon), \end{aligned}$$

and we have: $|\Delta_g(\phi_\epsilon v_1)| \leq |\phi_\epsilon \Delta_g v_1| + |v_1 \Delta_g \phi_\epsilon| + 2|(\nabla_g v_1, \nabla_g \phi_\epsilon)|$.

Then

$$\begin{aligned} |\Delta_g(\phi_\epsilon v_1)|^2 &\leq (|\phi_\epsilon \Delta_g v_1| + |v_1 \Delta_g \phi_\epsilon|)^2 + 4(\nabla_g v_1, \nabla_g \phi_\epsilon)^2 \\ &\quad + 4(|\phi_\epsilon \Delta_g v_1| + |v_1 \Delta_g \phi_\epsilon|)|(\nabla_g v_1, \nabla_g \phi_\epsilon)| \\ &\leq v_1^2 |\Delta_g(\phi_\epsilon)|^2 + \phi_\epsilon^2 |\Delta_g(v_1)|^2 + 4(\nabla_g v_1, \nabla_g \phi_\epsilon)^2 \\ &\quad + 4|v_1 \Delta_g(\phi_\epsilon)(\nabla_g v_1, \nabla_g \phi_\epsilon)| + 4|\phi_\epsilon \Delta_g(v_1)(\nabla_g v_1, \nabla_g \phi_\epsilon)| \\ &\quad + 2|v_1 \phi_\epsilon \Delta_g(\phi_\epsilon) \Delta_g(v_1)|. \end{aligned}$$

By calculating each term of $|\Delta_g(\phi_\epsilon v_1)|^2$, one has,

$$\begin{aligned} \int_{B(p)} |\Delta_g(\phi_\epsilon v_1)|^2 dv_g &\leq \int_{B(p)} v_1^2 |\Delta_g(\phi_\epsilon)|^2 dv_g + \int_{B(p)} \phi_\epsilon^2 |\Delta_g(v_1)|^2 dv_g + 4 \int_{B(p)} (\nabla_g v_1, \nabla_g \phi_\epsilon)^2 dv_g \\ &\quad + 4 \int_{B(p)} |v_1 \Delta_g(\phi_\epsilon)(\nabla_g v_1, \nabla_g \phi_\epsilon)| dv_g + 4 \int_{B(p)} |\phi_\epsilon \Delta_g(v_1)(\nabla_g v_1, \nabla_g \phi_\epsilon)| dv_g \\ &\quad + 2 \int_{B(p)} |v_1 \phi_\epsilon \Delta_g(\phi_\epsilon) \Delta_g(v_1)| dv_g. \end{aligned}$$

Nodal solutions for a Paneitz-Branson type equation

Now since $v_1 \in C^\infty(M)$, we set: $C = \max(\|v_1\|_\infty, \|\Delta_g v_1\|_\infty, \|\nabla_g v_1\|_\infty)$ and let us deal with the various terms of $\int_{B(p)} |\Delta_g(\phi_\epsilon v_1)|^2 dv_g$. As is easily checked, one has:

$$\int_{B(p)} v_1^2 |\Delta_g(\phi_\epsilon)|^2 dv_g \leq C^2 \int_{B(p)} |\Delta_g(\phi_\epsilon)|^2 dv_g,$$

$$\int_{B(p)} \phi_\epsilon^2 |\Delta_g(v_1)|^2 dv_g \leq C^2 \int_{B(p)} \phi_\epsilon^2 dv_g.$$

By using the Cauchy-Schwarz inequality, one gets

$$\int_{B(p)} (\nabla_g v_1, \nabla_g \phi_\epsilon)^2 dv_g \leq \int_{B(p)} |\nabla_g v_1|^2 |\nabla_g \phi_\epsilon|^2 dv_g \leq C^2 \int_{B(p)} |\nabla_g \phi_\epsilon|^2 dv_g,$$

$$\begin{aligned} \int_{B(p)} |v_1 \Delta_g(\phi_\epsilon) (\nabla_g v_1, \nabla_g \phi_\epsilon)| dv_g &\leq C \int_{B(p)} |\Delta_g(\phi_\epsilon)| |(\nabla_g v_1, \nabla_g \phi_\epsilon)| dv_g \\ &\leq C \int_{B(p)} |\Delta_g(\phi_\epsilon)| |\nabla_g v_1| |\nabla_g \phi_\epsilon| dv_g \\ &\leq C^2 \int_{B(p)} |\Delta_g(\phi_\epsilon)| |\nabla_g \phi_\epsilon| dv_g \\ &\leq \frac{C^2}{2} \int_{B(p)} (|\Delta_g(\phi_\epsilon)|^2 + |\nabla_g \phi_\epsilon|^2) dv_g, \end{aligned}$$

$$\begin{aligned} \int_{B(p)} |\phi_\epsilon \Delta_g(v_1) (\nabla_g v_1, \nabla_g \phi_\epsilon)| dv_g &\leq C \int_{B(p)} |\phi_\epsilon| |\nabla_g v_1| |\nabla_g \phi_\epsilon| dv_g \\ &\leq C^2 \int_{B(p)} |\phi_\epsilon| |\nabla_g \phi_\epsilon| dv_g \\ &\leq \frac{C^2}{2} \int_{B(p)} (\phi_\epsilon^2 + |\nabla_g \phi_\epsilon|^2) dv_g, \end{aligned}$$

where we have used $|a||b| \leq \frac{1}{2}(a^2 + b^2)$.

Similarly,

$$\int_{B(p)} |v_1 \phi_\epsilon \Delta_g(\phi_\epsilon) \Delta_g(v_1)| \leq \frac{C^2}{2} \int_{B(p)} (\phi_\epsilon^2 + |\Delta_g \phi_\epsilon|^2) dv_g.$$

2.8 Appendix

Hence we can find a constant $C_0 > 0$ such that

$$\int_{B(p)} |\Delta_g(\phi_\epsilon v_1)|^2 dv_g \leq C_0 \left(\int_{B(p)} |\Delta_g(\phi_\epsilon)|^2 + |\nabla_g(\phi_\epsilon)|^2 + (\phi_\epsilon)^2 dv_g \right).$$

Let us deal with the various terms of $\int_{B(p)} |\nabla_g(\phi_\epsilon v_1)|^2 dv_g$:

$$\int_{B(p)} |\nabla_g(\phi_\epsilon v_1)|^2 dv_g \leq \int_{B(p)} \phi_\epsilon^2 |\nabla_g v_1|^2 dv_g + \int_{B(p)} v_1^2 |\nabla_g(\phi_\epsilon)|^2 dv_g + 2 \int_{B(p)} |\phi_\epsilon v_1 (\nabla_g v_1, \nabla_g \phi_\epsilon)| dv_g.$$

Following the same thing, one has:

$$\begin{aligned} \int_{B(p)} \phi_\epsilon^2 |\nabla_g v_1|^2 dv_g &\leq C^2 \int_{B(p)} \phi_\epsilon^2 dv_g, \\ \int_{B(p)} v_1^2 |\nabla_g(\phi_\epsilon)|^2 dv_g &\leq C^2 \int_{B(p)} |\nabla_g(\phi_\epsilon)|^2 dv_g, \end{aligned}$$

and

$$\begin{aligned} 2 \int_{B(p)} |\phi_\epsilon v_1 (\nabla_g v_1, \nabla_g \phi_\epsilon)| dv_g &\leq 2C \int_{B(p)} |\phi_\epsilon| |(\nabla_g v_1, \nabla_g \phi_\epsilon)| dv_g \\ &\leq 2C \int_{B(p)} |\phi_\epsilon| |\nabla_g v_1| |\nabla_g \phi_\epsilon| dv_g \\ &\leq 2C^2 \int_{B(p)} |\phi_\epsilon| |\nabla_g \phi_\epsilon| dv_g \\ &\leq C^2 \int_{B(p)} (\phi_\epsilon^2 + |\nabla_g \phi_\epsilon|^2) dv_g. \end{aligned}$$

Then we can also find a constant $C_1 > 0$ such that

$$\int_{B(p)} |\nabla_g(\phi_\epsilon v_1)|^2 dv_g \leq C_1 \left(\int_{B(p)} |\nabla_g(\phi_\epsilon)|^2 + (\phi_\epsilon)^2 dv_g \right).$$

By combining the two estimates, we obtain the desired result. In other words, we can find a constant $C > 0$ such that :

$$|I_2| \leq C \left(\int_{B(p)} |\Delta_g(\phi_\epsilon)|^2 + |\nabla_g(\phi_\epsilon)|^2 + (\phi_\epsilon)^2 dv_g \right).$$

■

Chapter 3

The first GJMS invariant

3.1 Introduction

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$, and let k be an integer such that $k \geq 1$ and $2k \leq n$. In 1992, in [24] Graham-Jenne-Mason-Sparling have defined a family of conformally invariant differential operators defined for any Riemannian metric (GJMS operators for short). The construction of these operators is based on the ambient metric of Fefferman-Graham [22]. More precisely, for any Riemannian metric g on M , there exists a local, formally self-adjoint, conformally covariant operator

$$P_g : C^\infty(M) \longrightarrow C^\infty(M),$$

such that for all $u \in C^\infty(M)$, the GJMS operator P_g is given by :

$$P_g u = \Delta_g^k u + \text{lot}, \tag{3.1.1}$$

where Δ_g is the Laplace-Beltrami operator, and *lot* denotes differential terms of lower order. For more detail about P_g , we refer the reader to Robert [38]. This operator enjoys nice conformal invariance properties. Indeed, let $\varphi \in C^\infty(M)$ be a positive function and $N = \frac{2n}{n-2k}$. If $n \neq 2k$, then any metric \bar{g} written in the form $\varphi^{\frac{4}{n-2k}} g$ is a conformal metric to g and therefore, for any metric \bar{g} conformal to g , the operator P_g is conformally invariant in the following sense: for all $u \in C^\infty(M)$, we have $P_g(u\varphi) = \varphi^{N-1} P_{\bar{g}}(u)$. By taking $u \equiv 1$, we get

$$P_g \varphi = \frac{n-2k}{2} Q_{\bar{g}} \varphi^{N-1}, \tag{3.1.2}$$

where

$$Q_{\bar{g}} = \frac{2}{n-2k} P_{\bar{g}}(1).$$

The scalar Q_g is called the Q -curvature and is a Riemannian invariant associated to this operator. Historically, the notion of the Q -curvature is due to Branson [14]. He

3.1 Introduction

also defined it in the critical case $n = 2k$. Now when $k = 1$, P_g is the conformal Laplacian operator and Q_g is the scalar curvature S_g (up to a constant). The problem of prescribing a constant scalar curvature is known as the Yamabe problem, the classical reference for this problem is a survey by Lee-Parker [33]. When $k = 2$, P_g is the Paneitz-Branson operator introduced by Paneitz in [36] and the Q -curvature was introduced by Branson-Ørsted [15]. Results for the prescription of the Q -curvature problem for the Paneitz operator are in Djadli-Hebey-Ledoux [19], Robert [37], Esposito-Robert [20], Hang-Yang [31], Gursky-Malchiodi [25] and Benalili-Boughazi [9]. Moreover, concerning fourth-order problems, there has been also an intensive literature on the question, we refer the reader to [9],[11],[18],[37]. Solving the problem of prescribing Q -curvature for the GJMS operator is a very difficult problem and its underlying analysis is intricate, we refer to Robert [38] and Mazumdar [35] for some particular situations. The simple case of these problems is prescribing constant Q -curvature which is equivalent to finding a positive smooth solution u of the following equation

$$P_g u = C|u|^{N-2}u, \quad (3.1.3)$$

where C is a constant. In order to obtain solutions, we define the quantity

$$\mu = \inf_{u \in C^\infty(M), u > 0} I(u), \quad (3.1.4)$$

where

$$I(u) = \frac{\int_M u P_g u dv_g}{\left(\int_M |u|^N dv_g\right)^{\frac{2}{N}}}.$$

As in the Yamabe problem, the constant μ will be called the GJMS invariant. In particular, if $u \in C^\infty(M)$, $u > 0$ and satisfy $I(u) = \mu$, clearly u is solution of (3.1.3) and $\bar{g} = u^{\frac{4}{n-2k}}g$ is the desired metric of constant Q -curvature. It is well known that the operator P_g is elliptic, self-adjoint with respect to the inner product in $L^2(M)$ [38] and has discrete spectrum with eigenvalues

$$\lambda_1(g) \leq \lambda_2(g) \leq \lambda_3(g) \dots \leq \lambda_k(g) \rightarrow +\infty,$$

appear with their multiplicities. The variational characterization of the first eigenvalue $\lambda_1(g)$ of P_g is given by:

$$\lambda_1(g) = \inf_{v \in H_k^2(M), v \neq 0} \frac{\int_M v P_g v dv_g}{\int_M v^2 dv_g},$$

where the space $H_k^2(M)$ is the completion of $C^\infty(M)$ for the norm

$$\|u\|_{H_k^2} = \left(\int_M \sum_{l=0}^k |\nabla^l u|^2 dv_g \right)^{\frac{1}{2}}.$$

Now by referring to Ammann-Humbert [1], we introduce an invariant μ_1 that we will call the first GJMS invariant and we will define it by:

$$\mu_1 = \inf_{\bar{g} \in [g]} \lambda_1(\bar{g}) Vol(M, \bar{g})^{\frac{2k}{n}}, \quad (3.1.5)$$

where the set $[g] = \{\bar{g} = u^{\frac{4}{n-2k}}g, u \in C^\infty(M) \text{ and } u > 0\}$ is the conformal class of the metric g and $Vol(M, \bar{g}) = \int_M u^N dv_g$ denotes the Riemannian volume of M with respect to the metric \bar{g} .

In order to find minimizers, we enlarge the conformal class $[g]$ to what we call the class of generalized metrics conformal to g . We say that $\bar{g} = u^{\frac{4}{n-2k}}g$ is a generalized metric of the Riemannian metric g if $u \in L^N(M), u \geq 0$ and u is not identically null. By the standard min-max method via Rayleigh quotients for defining eigenvalues combined with conformal covariance of P_g , one sees that for any generalized metric $\bar{g} = u^{\frac{4}{n-2k}}g$, the first eigenvalue $\lambda_1(\bar{g})$ of the GJMS operator $P_{\bar{g}}$ is characterized by:

$$\lambda_1(\bar{g}) = \inf_{V \in Gr_1^u(H_k^2(M))} \sup_{v \in V \setminus \{0\}} \frac{\int_M v P_{\bar{g}} v dv_{\bar{g}}}{\int_M u^{N-2} v^2 dv_g}, \quad (3.1.6)$$

where the Grassmannian $Gr_1^u(H_k^2(M))$ is given by 1.2.2.

Motivating by the same first part's results, we want to study the first eigenvalue $\lambda_1(\bar{g})$ for any generalized metric \bar{g} and after that our main problem will be when the first GJMS invariant μ_1 is attained by a generalized metric (or conformal metric) and what is the relationship between μ_1 and the standard GJMS invariant μ . To solve this problem, we will use the ideas from [1,27,9-7,21]. More precisely, the method we would like to apply is introduced in [1] for studying the second Yamabe invariant μ_2 (see Definition 3.2.1 for $p = 2$) and generalized for the Paneitz-Branson operator on Einstein manifolds by Benalili and Boughazi in [9]. For clarity purposes, we state our main generic theorem and after this we give more details about this method :

Theorem 3.1.1.

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Assume that $\lambda_1(g) > 0$ and $\mu < K_0^{-1}$ where K_0 is defined bellow. Then there exists a nontrivial function $v \in C^{2k}(M)$ which satisfies $P_g v = \mu_1 |v|^{N-2} v$. In other words, μ_1 is attained by the generalized metric $\bar{g} = |v|^{\frac{4}{n-2k}} g$ and in particular, if $Q_g \leq 0$, v is a nodal (sign-changing) solution. Moreover, if g is Einstein and $S_g > 0$, the solution $v > 0$ and $v \in C^\infty(M)$ and this implies that $\mu = \mu_1$ and means that \bar{g} is a conformal metric. Consequently, in the latter case μ_1 is attained by the desired metric \bar{g} of constant Q -curvature: $Q_{\bar{g}} = \frac{2}{n-2k} \mu_1$,

where K_0 is defined in 0.3.9.

3.2 Motivation

This theorem is a consequence of several results. Firstly, we establish some results concerning the eigenvalues, in particular, if $\lambda_1(g) > 0$, the first eigenvalue $\lambda_1(\bar{g})$ is achieved for all generalized metric $\bar{g} = u^{\frac{4}{n-2k}}g$. We also show that the linear equation $P_g v = \lambda_1(\bar{g})u^{N-2}v$ has nodal (sign-changing) solution if $Q_g \leq 0$ and if $\lambda_1(g) < 0$, we show that there exists a generalized metric \bar{g} such that $\lambda_1(\bar{g}) = -\infty$ which implies that $\mu_1 = -\infty$. Secondly, we study our first GJMS invariant μ_1 in case $\lambda_1(g) > 0$, we will prove that μ_1 is attained by a generalized metric if $1 - \mu K_0 > 0$ and the corresponding the nonlinear GJMS equation $P_g v = \mu_1|v|^{N-2}v$ has a nodal solution if $Q_g \leq 0$. Finally, when the manifold is Einstein with $S_g > 0$, we will prove that the solution of the latter equation is $v > 0$ smooth, this implies: $\mu_1 = \mu$ and is attained by a conformal metric \bar{g} which is of constant Q -curvature. In particular, when $S_g < 0$, the solution v is nodal.

3.2 Motivation

We start by giving a short motivation by recalling some results. Indeed, in [1] Ammann and Humbert defined the Yamabe invariant of high order μ_p by

Definition 3.2.1.

$$\mu_p = \inf_{\bar{g} \in [g]} \lambda_p(\bar{g}) [Vol(M, \bar{g})]^{\frac{2}{n}},$$

where

$$\lambda_p(\bar{g}) = \inf_{V \in Gr_p^u(H_1^2(M))} \sup_{v \in V \setminus \{0\}} \frac{\int_M v P_g v dv_g}{\int_M u^{N-2} v^2 dv_g},$$

is the p^{th} eigenvalue of the conformal Laplacian P_g , $\bar{g} = u^{\frac{4}{n-2}}g$ is a generalized metric, $p \in \mathbb{N}^*$ and the Grassmannian $Gr_p^u(H_1^2(M))$ is given by 1.2.2.

The authors studied the second Yamabe invariant μ_2 in the case $\mu \geq 0$ where μ is the Yamabe invariant. In particular, they obtained the following theorem:

Theorem 3.2.1.

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Assume that μ_2 is attained by a generalized metric. Then the following equation $P_g w = \mu_2 u^{N-2} w$ has a nodal solution $w \in C^2(M)$ such that $u = |w|$.

Inspired by the previous results. In [9], Benalili and Boughazi generalize this method to the Paneitz-Branson operator on Einsteinian manifolds. Under some assumptions, they studied μ , μ_1 and μ_2 in the case $S_g > 0$ and after ten years the authors in [11] extend these results to the case $S_g < 0$. For more detail and similar work, we refer the readers to Benalili-Boughazi [10], Boughazi [7] and S. Elsayed [21]. We also specify a

very interesting result proven in [21] which states that the sign of eigenvalue $\lambda_p(\bar{g})$ is conformal invariant. We remind the readers that we are looking for similar results with the GJMS operator, we want to study μ_1 and when we can get $\mu_1 = \mu$ and finally, we point out that the study of μ_2 seems to be much more difficult than expected.

3.3 Some specific properties of the GJMS

In this section, we give some properties of the GJMS operator. For the proofs, the reader is referred to Robert [38] and the references therein. The operator P_g can be written (partially) in divergence form, we precise this divergence form that will be useful in the sequel:

Proposition 3.3.1.

Let P_g be the conformal GJMS operator. Then for any $l \in \{1, \dots, k-1\}$, there exists $A_{(l)}(g)$ a smooth T_{2l}^0 -tensor field on M such that

$$P_g v = \Delta_g^k v + \sum_{l=1}^{k-1} A_{l,g}(v) + \frac{n-2k}{2} Q_g v, \quad (3.3.1)$$

where

$$A_{l,g}(v) = (-1)^l \nabla^{j_1 \dots j_l} (A_{(l)}(g)_{i_1 \dots i_l j_1 \dots j_l} \nabla^{i_1 \dots i_l} v).$$

Indices are raised via the musical isomorphism. In addition for any $l \in \{1, \dots, k-1\}$, $A_{(l)}(g)$ is symmetric in the following sense: $A_{(l)}(g)(X, Y) = A_{(l)}(g)(Y, X)$ for all T_0^l -tensors X, Y on M . In particular, for all $u, v \in C^\infty(M)$ we have

$$\int_M v P_g u dv_g = \int_M u P_g v dv_g = \int_M \Delta_g^{\frac{k}{2}} u \Delta_g^{\frac{k}{2}} v + \sum_{l=0}^{k-1} A_{(l)}(g)(\nabla^l u, \nabla^l v) dv_g, \quad (3.3.2)$$

where for $l=0$, $A_{(0)}(g)(\nabla^0 u, \nabla^0 v) = \frac{n-2k}{2} Q_g uv$. Here, we have adopted the convention

$$\Delta_g^{\frac{k}{2}}(u) = \begin{cases} \Delta_g^m(u) & \text{if } k = 2m \text{ is even} \\ \nabla \Delta_g^m(u) & \text{if } k = 2m + 1 \text{ is odd,} \end{cases}$$

and, when $k = 2m + 1$ is odd, $\Delta_g^{\frac{k}{2}} u \Delta_g^{\frac{k}{2}} v = (\nabla \Delta_g^m u, \nabla \Delta_g^m v)$.

Since $A_{(l)}(g)$ are smooth, then for any $l \in \{0, \dots, k-1\}$, there exist $C_l > 0$ such that for all $u \in H_k^2(M)$, one has

$$\left| \int_M \sum_{l=0}^{k-1} A_{(l)}(g)(\nabla^l u, \nabla^l u) dv_g \right| \leq \sum_{l=0}^{k-1} C_l \int_M |\nabla^l u|^2 dv_g \leq \max(C_l) \|u\|_{H_{k-1}^2}^2. \quad (3.3.3)$$

3.3 Some specific properties of the GJMS

As a consequence of (3.3.2), we get that the bilinear form $(u, v) \mapsto \int_M u P_g v dv_g$ extends to a continuous symmetrical bilinear form on the space $H_k^2(M) \times H_k^2(M)$.

We say that P_g is coercive if there exists $C > 0$ such that

$$\int_M v P_g v dv_g \geq C \|v\|_2^2, \quad \forall v \in H_k^2(M).$$

Proposition 3.3.2.

For all $u \in H_k^2(M)$, we define the norm $\|u\|_{P_g}$ by:

$$\|u\|_{P_g} = \left(\int_M u P_g u dv_g \right)^{\frac{1}{2}}.$$

Assume that P_g is coercive. Then $\|\cdot\|_{P_g}$ is a norm on $H_k^2(M)$ equivalent to the standard norm $\|\cdot\|_{H_k^2}$. In addition, if $(v_m)_m$ is a sequence in $H_k^2(M)$ such that $v_m \rightarrow 0$ weakly in $H_k^2(M)$, and $v_m \rightarrow 0$ strongly in $H_{k-1}^2(M)$, then from Bochner-Lichnerowicz-Weitzenbock type formula, one gets that

$$\int_M |\Delta_g^{\frac{k}{2}}(v_m)|^2 dv_g = \int_M |\nabla^k v_m|^2 dv_g + o(1). \quad (3.3.4)$$

The reader is referred to [22] for the two following propositions:

Proposition 3.3.3.

Assume that (M, g) is Einstein, then P_g expresses as an explicit product of second-order operators with constant coefficients that depend only on the scalar curvature. In other words, the GJMS operator P_g is given by:

$$P_g = \prod_{l=1}^k (\Delta_g + c_l S_g) \quad \text{where} \quad c_l = \frac{(n+2l-2)(n-2l)}{4n(n-1)}.$$

Moreover, by calculating one can write

$$P_g = \Delta_g^k + \sum_{l=0}^{k-2} b_{k-l-1} (S_g)^{l+1} \Delta_g^{k-l-1} + b_0 (S_g)^k, \quad (3.3.5)$$

where b_{k-1}, \dots, b_1, b_0 are positive real numbers obtained from c_l .

In addition, formula (3.3.2) implies that

$$\int_M u P_g u dv_g = \int_M (|\Delta_g^{\frac{k}{2}}(u)|^2 + \sum_{l=0}^{k-1} b_{k-l-1} (S_g)^{l+1} |\nabla^{k-l-1} u|^2) dv_g.$$

Proposition 3.3.4.

Assume that the metric g is Einstein with $S_g > 0$ and $n > 2k$, then P_g is coercive and for all $u \in C^{2k}(M)$ such that $P_g u \geq 0$, either $u > 0$ or $u \equiv 0$. This statement is a direct consequence of k applications of the second-order comparison principle (see [20] Proposition 4).

We also introduce the following results. For the proofs, the reader is referred to Mazumdar [35].

Theorem 3.3.1.

Let (M, g) be a compact Riemannian manifold of dimension n and let k be a positive integer such that $2k < n$. For any $\epsilon > 0$, there exists $B_\epsilon > 0$ such that for all $u \in H_k^2(M)$ one has

$$\|u\|_N^2 \leq (K_0 + \epsilon) \int_M |\Delta_g^{\frac{k}{2}}(u)|^2 dv_g + B_\epsilon \|u\|_{H_{k-1}^2}^2, \quad (3.3.6)$$

where K_0 is given by formula (0.3.9).

Moreover, for all $u, v \in C^\infty(M)$, there exists $C > 0$ (depend on $\|v\|_\infty$) such that

$$\int_M |\Delta_g^{\frac{k}{2}}(vu)|^2 dv_g \leq C \left(\int_M |\Delta_g^{\frac{k}{2}}(u)|^2 dv_g + \|u\|_{H_{k-1}^2}^2 \right). \quad (3.3.7)$$

Proposition 3.3.5.

Let (M, g) be a closed manifold of dimension n and let k be a positive integer such that $2k < n$. Let $f \in C^{0,\alpha}(M)$ a Hölder continuous function. Suppose that $u \in H_k^2(M)$ be a weak solution of $P_g u = f|u|^{N-2}u$. Then $u \in C^{2k}(M)$, and is a classical solution of the above equation. Further if $u > 0$ and $f \in C^\infty(M)$, then $u \in C^\infty(M)$.

3.4 Generalized metrics and the first eigenvalue

Theorem 3.4.1.

For any generalized metric $\bar{g} = u^{\frac{4}{n-2k}}g$, assume that $u > 0$. Then any normalized minimizing sequence of $\lambda_1(\bar{g})$ is bounded in $H_k^2(M)$.

Proof

Let $(v_m)_m$ be a minimizing sequence of $\lambda_1(\bar{g})$, in other words

$$\lambda_1(\bar{g}) = \lim_{m \rightarrow +\infty} \lambda_{1,m} \quad \text{where} \quad \lambda_{1,m} = \frac{\int_M (|\Delta_{\bar{g}}^{\frac{k}{2}}(v_m)|^2 + \sum_{l=0}^{k-1} A_{(l)}(g)(\nabla^l v_m, \nabla^l v_m)) dv_g}{\int_M u^{N-2} v_m^2 dv_g}.$$

It is easy to see that $(\lambda v_m)_m$ (with $\lambda \neq 0$) is also a minimizing sequence, then if we choose $\lambda = (\int_M u^{N-2} v_m^2 dv_g)^{-\frac{1}{2}}$, it follows that $\int_M u^{N-2} (\lambda v_m)^2 dv_g = \lambda^2 \int_M u^{N-2} v_m^2 dv_g = 1$, hence the sequence $(\lambda v_m)_m$ is renormalized. Without loss of generality, we assume that the sequence $(v_m)_m$ is such that

$$\int_M u^{N-2} v_m^2 dv_g = 1. \quad (3.4.1)$$

1) If $\lambda_1(\bar{g}) > 0$, then for all v in $H_k^2(M) \setminus \{0\}$, one has

$$\begin{aligned} \int_M v P_g v dv_g &\geq \lambda_1(\bar{g}) \int_M u^{N-2} v^2 dv_g \\ &\geq \underbrace{\lambda_1(\bar{g}) \min_{x \in M} u(x)^{N-2}}_C \int_M v^2 dv_g, \quad (\text{since } u > 0) \\ &\geq C \|v\|_2^2. \end{aligned}$$

This means that P_g is coercive and we know from Proposition 3.3.2 that $\|\cdot\|_{P_g}$ is a norm on $H_k^2(M)$ equivalent to the standard norm $\|\cdot\|_{H_k^2}$. Then for m large enough, one has

$$\lambda_{1,m} = \int_M v_m P_g v_m dv_g = \|v_m\|_{P_g}^2 \leq \lambda_1(\bar{g}) + 1,$$

hence the sequence $(v_m)_m$ is bounded in $H_k^2(M)$ and $\lambda_{1,m} \geq 0$.

2) If $\lambda_1(\bar{g}) < 0$, the GJMS operator is not necessarily coercive, then we will assume that $(v_m)_m$ is not bounded in $H_k^2(M)$, in other words $\|v_m\|_{H_k^2} \rightarrow +\infty$ and we let

$$v'_m = \frac{v_m}{\|v_m\|_{H_k^2}}.$$

Clearly $\|v'_m\|_{H_k^2} = 1$, this means that the sequence $(v'_m)_m$ is bounded in $H_k^2(M)$ and after restriction to a subsequence still labeled $(v'_m)_m$, we may assume that there exists $v' \in H_k^2(M)$ such that $v'_m \rightharpoonup v'$ weakly in $H_k^2(M)$ and $v'_m \rightarrow v'$ strongly in $H_{k-1}^2(M)$. On the other hand, the sequence $(v'_m)_m$ satisfies the following equation:

$$\int_M (|\Delta_g^{\frac{k}{2}}(v'_m)|^2) dv_g + \sum_{l=0}^{k-1} \int_M A_{(l)}(g)(\nabla^l v'_m, \nabla^l v'_m) dv_g = \lambda_{1,m} \int_M u^{N-2} v'^2_m dv_g. \quad (3.4.2)$$

Now from the weak convergence, we have

$$\lim \int_M u^{N-2} v' v'_m dv_g = \int_M u^{N-2} (v')^2 dv_g,$$

and since

$$\|v_m\| \rightarrow +\infty \quad \text{as } m \rightarrow +\infty$$

, it follows that $v'_m \rightarrow 0$ almost everywhere in M .

Moreover, as

$$0 \leq \int_M u^{N-2} (v' - v'_m)^2 dv_g = \int_M u^{N-2} (v')^2 dv_g - 2 \int_M u^{N-2} v' v'_m dv_g + \int_M u^{N-2} (v'_m)^2 dv_g,$$

one has,

$$\int_M u^{N-2} (v')^2 dv_g \leq \int_M u^{N-2} (v'_m)^2 dv_g = \frac{\int_M u^{N-2} v'^2_m dv_g}{\|v_m\|_{H_k^2}^2} = \frac{1}{\|v_m\|_{H_k^2}^2} \rightarrow 0. \quad (3.4.3)$$

Consequently,

$$\int_M u^{N-2} (v')^2 dv_g = 0,$$

and since $u > 0$, it is easy to see that

$$v' \equiv 0.$$

It follows that $v'_m \rightharpoonup 0$ weakly in $H_k^2(M)$ and $v'_m \rightarrow 0$ strongly in $H_{k-1}^2(M)$ therefore,

$$\int_M \sum_{l=0}^{k-1} |\nabla^l v'_m|^2 dv_g \rightarrow 0. \quad (3.4.4)$$

Then by (3.4.4) and using (3.3.3), one has also

$$\sum_{l=0}^{k-1} \int_M A_{(l)}(g)(\nabla^l v'_m, \nabla^l v'_m) dv_g \rightarrow 0. \quad (3.4.5)$$

3.4 Generalized metrics and the first eigenvalue

Again by (3.4.4), the following equality

$$1 = \|v'_m\|_{H_k^2}^2 = \int_M |\nabla^k v'_m|^2 dv_g + \int_M \sum_{l=0}^{k-1} |\nabla^l v'_m|^2 dv_g,$$

leads necessarily to

$$\int_M |\nabla^k v'_m|^2 dv_g \longrightarrow 1.$$

Independently, from formula (3.3.4) i.e

$$\int_M (|\Delta_g^{\frac{k}{2}}(v'_m)|^2) dv_g = \int_M |\nabla^k v'_m|^2 dv_g + o(1),$$

this implies that

$$\int_M (|\Delta_g^{\frac{k}{2}}(v'_m)|^2) dv_g \longrightarrow 1. \quad (3.4.6)$$

Since $\lambda_1(\bar{g}) < 0$, then for m large enough $\lambda_{1,m} < 0$, it follows from (3.4.2), (3.4.3), (3.4.5) and (3.4.6) that the sequence $(v'_m)_m$ is such that

$$\underbrace{\int_M (|\Delta_g^{\frac{k}{2}}(v'_m)|^2) dv_g}_{\rightarrow 1} + \underbrace{\sum_{l=0}^{k-1} \int_M A_{(l)}(g)(\nabla^l v'_m, \nabla^l v'_m) dv_g}_{\rightarrow 0} = \lambda_{1,m} \underbrace{\int_M u^{N-2}(v'_m)^2 dv_g}_{\rightarrow a},$$

where $a \leq 0$ or does not exist, in all cases this gives a contradiction. This proves that $(v_m)_m$ is bounded in $H_k^2(M)$.

Moreover, we have

$$- \int_M |\nabla^k v_m|^2 dv_g \leq \int_M (|\Delta_g^{\frac{k}{2}}(v_m)|^2) dv_g,$$

which lead to

$$- \int_M |\nabla^k v_m|^2 dv_g - \max(C_l) \|v_m\|_{H_{k-1}^2}^2 \leq \int_M (|\Delta_g^{\frac{k}{2}}(v_m)|^2) dv_g + \sum_{l=0}^{k-1} \int_M A_{(l)}(g)(\nabla^l v_m, \nabla^l v_m) dv_g,$$

where $\max(C_l)$ is given by (3.3.3), this means that

$$\min(-1, -\max(C_l)) \|v_m\|_{H_k^2}^2 \leq \int_M (|\Delta_g^{\frac{k}{2}}(v_m)|^2) + \sum_{l=0}^{k-1} \int_M A_{(l)}(g)(\nabla^l v_m, \nabla^l v_m) dv_g.$$

In other words,

$$\lambda_{1,m} \geq \min(-1, -\max(C_l)) \|v_m\|_{H_k^2}^2,$$

and since $(v_m)_m$ is bounded, then there exists $M > 0$ such that

$$\lambda_{1,m} \geq \min(-1, -\max(C_l))M > -\infty.$$

■

Proposition 3.4.1.

Assume that $\lambda_1(g) < 0$, then there exists $u \in L_+^N(M)$ such that $\lambda_1(\bar{g}) = -\infty$ where $\bar{g} = u^{\frac{4}{n-2k}}g$.

Proof

If $\lambda_1(g) = -\infty$, there is nothing to prove. If not, since $\lambda_1(g) < 0$, there exist a function $v \in C^\infty(M)$ such that $\int_M v P_g v dv_g < 0$. Fix a point p in M . For $\epsilon > 0$, let ϕ_ϵ be a cut-off function adapted to our context, in other words a smooth function such that :

$$\begin{cases} 0 \leq \phi_\epsilon \leq 1 \\ \phi_\epsilon = 0 \text{ on } B_\epsilon(p) \text{ and } \phi_\epsilon = 1 \text{ on } M \setminus B_{2\epsilon}(p) \\ |\nabla^l \phi_\epsilon| \leq \frac{c_l}{\epsilon^l} \text{ for all } l \text{ in } \{1, 2, \dots, k-1\} \\ |\Delta_g^{\frac{k}{2}} \phi_\epsilon| \leq \frac{c_k}{\epsilon^k}, \end{cases} \quad \begin{matrix} (28_1) \\ (28_2) \end{matrix}$$

where $B_\epsilon(p)$ is the open ball centered at p and of radius ϵ and $c_l > 0$ are constants that do not depend on ϵ .

We claim that:

$$\lim_{\epsilon \rightarrow 0} \int_M (\phi_\epsilon v) P_g(\phi_\epsilon v) dv_g = \int_M v P_g v dv_g < 0.$$

Indeed:

Set $A_\epsilon(p) = B_{2\epsilon}(p) \setminus B_\epsilon(p)$, then one has

$$\int_M (\phi_\epsilon v) P_g(\phi_\epsilon v) dv_g = \underbrace{\int_{B_\epsilon(p)} (\phi_\epsilon v) P_g(\phi_\epsilon v) dv_g}_{I_1} + \underbrace{\int_{A_\epsilon(p)} (\phi_\epsilon v) P_g(\phi_\epsilon v) dv_g}_{I_2} + \underbrace{\int_{M \setminus B_{2\epsilon}(p)} (\phi_\epsilon v) P_g(\phi_\epsilon v) dv_g}_{I_3}.$$

Clearly the first integral $I_1 = 0$ (since $\phi_\epsilon = 0$ on the ball $B_\epsilon(p)$). For the second integral I_2 since $v \in C^\infty(M)$, we can find a constant $C > 0$ such that

$$\begin{aligned} |I_2| &\leq \int_{A_\epsilon(p)} (|\Delta_g^{\frac{k}{2}}(\phi_\epsilon v)|^2 + \sum_{l=0}^{k-1} |A_{(l)}(g)(\nabla^l \phi_\epsilon v, \nabla^l \phi_\epsilon v)|) dv_g \\ &\leq C \left(\int_{A_\epsilon(p)} (|\Delta_g^{\frac{k}{2}}(\phi_\epsilon)|^2 + \sum_{l=0}^{k-1} |\nabla^l(\phi_\epsilon)|^2) dv_g \right). \end{aligned} \quad (3.4.7)$$

3.4 Generalized metrics and the first eigenvalue

The latter inequality (3.4.7) is a direct consequence of formula (3.3.7).

Using (28₁), (28₂), (3.4.7) and passing to the polar coordinates, we can easily find constants $C_k, C_{k-1}, \dots, C_0 > 0$ such that,

$$|I_2| \leq \frac{C_k}{\epsilon^{2k}} \int_{\epsilon}^{2\epsilon} r^{n-1} dr + \left(\sum_{l=0}^{k-1} \frac{C_l}{\epsilon^{2l}} \right) \int_{\epsilon}^{2\epsilon} r^{n-1} dr \xrightarrow{\epsilon \rightarrow 0} 0 \quad \text{since } n > 2k,$$

which means that the second integral:

$$I_2 \xrightarrow{\epsilon \rightarrow 0} 0.$$

And finally since $\phi_{\epsilon} = 1$ on $M \setminus B_{2\epsilon}(p)$, the third integral $I_3 = \int_{M \setminus B_{2\epsilon}(p)} v P_g v dv_g$, this implies that:

$$\lim_{\epsilon \rightarrow 0} \int_M (\phi_{\epsilon} v) P_g (\phi_{\epsilon} v) dv_g = \lim_{\epsilon \rightarrow 0} (I_1 + I_2 + I_3) = \lim_{\epsilon \rightarrow 0} I_3 = \int_M v P_g v dv_g < 0.$$

If we put $w = \phi_{\epsilon} v$, for ϵ small enough, we still have $\int_M w P_g w dv_g < 0$.

Now, let $u_{\epsilon} \geq 0$ be a smooth function with support in $B_{\epsilon}(p)$ and let $\bar{g} = u_{\epsilon}^{\frac{N-2}{k}} g$ since

$$\lambda_1(\bar{g}) = \inf_{v \in H_k^2(M), v \neq 0} \frac{\int_M v P_g v dv_g}{\int_M u_{\epsilon}^{N-2} v^2 dv_g},$$

it follows that for any real $\alpha > 0$, one has

$$\lambda_1(\bar{g}) \leq \lim_{\alpha \rightarrow 0} \frac{\int_M (w + \alpha) P_g (w + \alpha) dv_g}{\int_M u_{\epsilon}^{N-2} (w + \alpha)^2 dv_g} = -\infty.$$

Indeed

$$\lim_{\alpha \rightarrow 0} \int_M u_{\epsilon}^{N-2} (w + \alpha)^2 dv_g = 0,$$

and

$$\lim_{\alpha \rightarrow 0} \left(\int_M (w + \alpha) P_g (w + \alpha) dv_g \right) = \int_M w P_g w dv_g < 0. \quad \blacksquare$$

Theorem 3.4.2.

Let $\bar{g} = u^{\frac{4}{n-2k}} g$ be any generalized metric to g such that $u > 0$. Assume that $\lambda_1(g) > 0$. Then there exists a nontrivial function v in $H_k^2(M)$ such that, in the weak sense, satisfy :

$$P_g v = \lambda_1(\bar{g}) u^{N-2} v \quad \text{and} \quad \int_M u^{N-2} v^2 dv_g = 1. \quad (3.4.8)$$

Moreover, if $u \in C^{\infty}(M)$, then $v \in C^{\infty}(M)$ and if (M, g) is Einstein and $S_g > 0$, the solution $v > 0$.

Proof

Let $(v_m)_m$ be a minimizing sequence for $\lambda_1(\bar{g})$. In other words, the sequence $(v_m)_m \in H_k^2(M)$, $u^{\frac{N-2}{2}} v_m \neq 0$ and such that

$$\lim_m \frac{\int_M v_m P_g v_m dv_g}{\int_M u^{N-2} v_m^2 dv_g} = \lambda_1(\bar{g}). \quad (3.4.9)$$

Without loss of generality, we can always normalize v_m by $\int_M u^{N-2} v_m^2 dv_g = 1$. Since $\lambda_1(g) > 0$, P_g is coercive. Then Theorem 3.4.1 implies that the sequence (v_m) is bounded in $H_k^2(M)$, and after restriction to a subsequence we may assume that there exists v in $H_k^2(M)$ such that $v_m \rightarrow v$ weakly in $H_k^2(M)$, strongly in $H_{k-1}^2(M)$ and almost everywhere in M . Again since P_g is coercive, Proposition 3.3.2 implies that $\|\cdot\|_{P_g}$ is a norm on $H_k^2(M)$ equivalent to the standard norm $\|\cdot\|_{H_k^2}$, then by standard argument, one has

$$\int_M v P_g v dv_g \leq \liminf \int_M v_m P_g v_m dv_g = \lambda_1(\bar{g}),$$

as in [7] from (Lemma (4)), we get

$$\int_M u^{N-2} |v^2 - v_m^2| dv_g \rightarrow 0 \quad \text{i.e.} \quad \int_M u^{N-2} v^2 dv_g = 1,$$

and since $\lambda_1(\bar{g})$ is the infimum, one gets

$$\int_M v P_g v dv_g = \lambda_1(\bar{g}).$$

Consequently v is a non-trivial weak minimizer of the functional associated to $\lambda_1(\bar{g})$. Writing the Euler-Lagrange equation, we find that v satisfies in the weak sense the equation

$$P_g v = \lambda_1(\bar{g}) u^{N-2} v.$$

Moreover, since v is nontrivial, we have

$$\lambda_1(\bar{g}) = \int_M v P_g v dv_g = \|v\|_{P_g}^2 > 0. \quad (3.4.10)$$

If $u \in C_+^\infty(M)$, we get $\lambda_1(\bar{g}) u^{N-2} v \in H_k^2(M)$, then $P_g v \in H_k^2(M)$ and by regularity theorems $v \in H_{3k}^2(M)$, it follows by successive iterations that $v \in H_l^2(M)$ where l is large enough and finally if $\frac{1}{2} < \frac{l-m}{n}$, one gets

$$H_l^2(M) \subset C^m(M).$$

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So we can take $m = 2k$ i.e

$$v \in C^{2k}(M), \text{ therefore } v \in C^\infty(M).$$

In particular, if (M, g) is Einstein and $S_g > 0$, from [7] (Proposition (7)), one has

$$v > 0.$$

■

Remark 3.4.1.

Let v be the solution of the equation (3.4.8). Then there exists a nontrivial function w in $H_k^2(M)$ such that, in the weak sense one has :

$$P_g w = \lambda'_2(\bar{g}) u^{N-2} w,$$

with the constraints $\int_M u^{N-2} w^2 dv_g = 1$ and $\int_M u^{N-2} v w dv_g = 0$, where

$$\lambda'_2(\bar{g}) = \inf \frac{\int_M v P_g v dv_g}{\int_M u^{N-2} v^2 dv_g},$$

and the infimum is taken over the set

$$E = \{w \in H_k^2(M) \text{ such that } u^{\frac{N-2}{2}} w \neq 0, \int_M u^{N-2} w^2 dv_g = 1 \text{ and } \int_M u^{N-2} v w dv_g = 0\}.$$

Proof

Let $(w_m)_m$ be a minimizing sequence for $\lambda'_2(\bar{g})$, with the same method as above, we find non trivial minimizer w to $\lambda'_2(\bar{g})$ such that $P_g w = \lambda'_2(\bar{g}) u^{N-2} w$ in the weak sens with $\int_M u^{N-2} w^2 dv_g = 1$. Now writing

$$\begin{aligned} \int_M u^{N-2} v w dv_g &= \int_M (u^{N-2} w_m v - u^{N-2} w_m v + u^{N-2} w v) dv_g \\ &= \int_M u^{N-2} v (w - w_m) dv_g + \int_M u^{N-2} w_m v dv_g = 0. \end{aligned}$$

As the sequence $w_m \in E$, $\int_M u^{N-2} w_m v dv_g = 0$, and by using the weak convergence of w_m to w in $L^N(M)$ and since $u^{N-2} v \in L^{\frac{N}{N-1}}(M)$ where $L^{\frac{N}{N-1}}(M)$ is the dual space of $L^N(M)$, we get $\int_M u^{N-2} v (w - w_m) dv_g \rightarrow 0$ thus,

$$\int_M u^{N-2} v w dv_g = 0.$$

If $u \in C_+^\infty(M)$, one also gets $w \in C^{2k}(M)$ and finally, as in [7] it follows that $\lambda_2(\bar{g}) = \lambda_2(\bar{g})$. ■

Proposition 3.4.2.

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Assume that \bar{g} is a conformal metric and $\lambda_1(\bar{g}) > 0$. If $Q_g \leq 0$, then the solution v of (3.4.8) is nodal.

Proof

By Theorem 3.4.2, v satisfies the equation $P_g v = \lambda_1(\bar{g})u^{N-2}v$, then from (3.3.1), one can write

$$\Delta_g^k v + \sum_{l=1}^{k-1} A_{l,g}(v) + \frac{n-2k}{2} Q_g v = \lambda_1(\bar{g})u^{N-2}v.$$

Integrating over M , we get that

$$\int_M \Delta_g^k v dv_g + \sum_{l=1}^{k-1} \int_M A_{l,g}(v) dv_g + \int_M \frac{n-2k}{2} Q_g v dv_g = \lambda_1(\bar{g}) \int_M u^{N-2} v dv_g.$$

Since \bar{g} is conformal, again from Theorem 3.4.2, $v \in C^\infty(M)$ and this implies that

$$\underbrace{\int_M \Delta_g^k v dv_g}_{=0} + \underbrace{\sum_{l=1}^{k-1} \int_M A_{l,g}(v) dv_g}_{=0} + \frac{n-2k}{2} \int_M Q_g v dv_g = \lambda_1(\bar{g}) \int_M u^{N-2} v dv_g,$$

since $\lambda_1(\bar{g}) > 0$ and $Q_g \leq 0$, hence if $v \geq 0$, one has

$$\underbrace{\frac{n-2k}{2} \int_M Q_g v dv_g}_{\leq 0} = \underbrace{\int_M \lambda_1(\bar{g})u^{N-2}v dv_g}_{> 0}.$$

This makes a contradiction, if $v \leq 0$, one has

$$\underbrace{\frac{n-2k}{2} \int_M Q_g v dv_g}_{\geq 0} = \underbrace{\int_M \lambda_1(\bar{g})u^{N-2}v dv_g}_{< 0},$$

and this is also a contradiction. Consequently, v changes the sign.

If $\lambda_1(\bar{g}) < 0$ and $Q_g \geq 0$. With the same method, we get the same thing. ■

3.5 Existence of a minimum of μ_1

In this section, we study the first GJMS invariant μ_1 in case $\lambda_1(g) > 0$. We will prove that μ_1 is attained by a generalized metric. However, if $\lambda_1(g) < 0$, Proposition 3.4.1 implies that μ_1 is not well defined. In other words, from the variational characterization of μ_1 , one has

$$\mu_1 = -\infty.$$

In order to prove Theorem 3.5.1, we prove some useful lemmas.

Definition 3.5.1.

In this definition, we precise the formula (3.1.5). Indeed, by using the definition of $\lambda_1(\bar{g})$ formula (3.1.6), the first GJMS invariant μ_1 is given by

$$\begin{aligned} \mu_1 &= \inf_{\bar{g} \in [g]} \lambda_1(\bar{g}) \text{Vol}(M, \bar{g})^{\frac{2k}{n}} \\ &= \inf_{\substack{u \in C_+^\infty(M) \\ v \in Gr_1^u(H_k^2(M))}} \sup_{v \in V^*} \frac{\int_M v P_g v dv_g}{\int_M u^{N-2} v^2 dv_g} \left(\int_M u^N dv_g \right)^{\frac{2k}{n}}. \end{aligned}$$

Lemma 3.5.1.

We have:

$$\mu_1 \leq \mu, \tag{3.5.1}$$

where μ is the GJMS invariant, see (3.1.4).

Proof

$$\begin{aligned} \mu_1 &= \inf_{\bar{g} \in [g]} \lambda_1(\bar{g}) \text{Vol}(M, \bar{g})^{\frac{2k}{n}} \\ &= \inf_{\bar{g} \in [g]} \lambda_1(u^{\frac{N-2}{k}} g) \text{Vol}(M, \bar{g})^{\frac{2k}{n}} \\ &= \inf_{u \in C_+^\infty(M)} \left(\inf_{V \in Gr_1^u(H_k^2(M))} \sup_{v \in V^*} \frac{\int_M v P_g v dv_g}{\int_M u^{N-2} v^2 dv_g} \right) \left(\int_M u^N dv_g \right)^{\frac{2k}{n}}, \end{aligned}$$

where $V^* = V \setminus \{0\}$.

From the embedding $C_+^\infty(M) \subset H_k^2(M)$, one can write

$$\mu_1 \leq \inf_{\substack{u \in C_+^\infty(M) \\ v \in Gr_1^u(C_+^\infty(M))}} \sup_{v \in V^*} \frac{\int_M v P_g v dv_g}{\int_M u^{N-2} v^2 dv_g} \left(\int_M u^N dv_g \right)^{\frac{2k}{n}}.$$

In particular for $u = v$, one has

$$\begin{aligned}
 \mu_1 &\leq \inf_{\substack{u \in C_+^\infty(M) \\ V \in Gr_1^u(C_+^\infty(M))}} \sup_{v \in V^*} \frac{\int_M v P_g v dv_g}{\int_M v^{N-2} v^2 dv_g} \left(\int_M v^N dv_g \right)^{\frac{2k}{n}} \\
 &\leq \inf_{\substack{u \in C_+^\infty(M) \\ V \in Gr_1^u(C_+^\infty(M))}} \sup_{v \in V^*} \frac{\int_M v P_g v dv_g}{\left(\int_M v^N dv_g \right)^{1-\frac{2k}{n}}} \\
 &\leq \inf_{\substack{u \in C_+^\infty(M) \\ V \in Gr_1^u(C_+^\infty(M))}} \sup_{v \in V^*} \frac{\int_M v P_g v dv_g}{\left(\int_M v^N dv_g \right)^{\frac{2}{N}}}.
 \end{aligned}$$

Since $V \in Gr_1^u(C_+^\infty(M))$, $V = \{\lambda v, \lambda \in \mathbb{R}^*\}$ where $v \in C_+^\infty(M)$, then we deduce that:

$$\sup_{v \in V^*} \frac{\int_M v P_g v dv_g}{\left(\int_M v^N dv_g \right)^{\frac{2}{N}}} = \sup_{\lambda \in \mathbb{R}^*} \frac{\int_M (\lambda v) P_g (\lambda v) dv_g}{\left(\int_M (\lambda v)^N dv_g \right)^{\frac{2}{N}}} = \frac{\int_M v P_g v dv_g}{\left(\int_M v^N dv_g \right)^{\frac{2}{N}}},$$

this implies that

$$\mu_1 \leq \inf_{v \in C_+^\infty(M)} \frac{\int_M v P_g v dv_g}{\left(\int_M v^N dv_g \right)^{\frac{2}{N}}} = \mu.$$

■

Lemma 3.5.2.

Let (v_m) and (u_m) be two sequences such that $v_m \rightarrow v$ weakly in $H_k^2(M)$, $u_m \rightarrow u$ weakly in $L^N(M)$ and checking $\int_M u_m^{N-2} v_m^2 dv_g = 1$. Then

$$\int_M u_m^{N-2} (v_m - v)^2 dv_g = 1 - \int_M u^{N-2} v^2 dv_g + o(1).$$

Proof

Writing

$$\begin{aligned}
 \int_M u_m^{N-2} (v_m - v)^2 dv_g &= \int_M u_m^{N-2} v_m^2 dv_g + \int_M u_m^{N-2} v^2 dv_g - \int_M 2u_m^{N-2} v_m v dv_g \\
 &= 1 + \int_M u_m^{N-2} v^2 dv_g - \int_M 2u_m^{N-2} v_m v dv_g.
 \end{aligned}$$

The sequence u_m^{N-2} is bounded in $L^{\frac{N}{N-2}}(M)$ and converges almost everywhere to u^{N-2} on M then by using (1.2.3), one has $u_m^{N-2} \rightarrow u^{N-2}$ weakly in $L^{\frac{N}{N-2}}(M)$.

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This means that for all ϕ in $L^{\frac{N}{2}}(M)$, one gets $\int_M u_m^{N-2} \phi dv_g \rightarrow \int_M u^{N-2} \phi dv_g$. In particular for $\phi = v^2$, we obtain

$$\int_M u_m^{N-2} v^2 dv_g \rightarrow \int_M u^{N-2} v^2 dv_g.$$

On the other hand since

$$\int_M u_m^{N-2} v_m^{\frac{N}{N-1}} dv_g \leq \left(\int_M u_m^N dv_g \right)^{\frac{N-2}{N-1}} \left(\int_M v_m^N dv_g \right)^{\frac{1}{N-1}},$$

this means that the sequence $u_m^{N-2} v_m$ is also bounded in $L^{\frac{N}{N-1}}(M)$ and since $u_m^{N-2} v_m$ goes to $u^{N-2} v$ almost everywhere, one has $u_m^{N-2} v_m \rightarrow u^{N-2} v$ weakly in $L^{\frac{N}{N-1}}(M)$, then for all $\phi \in L^N(M)$, one has $\int_M u_m^{N-2} v_m \phi dv_g \rightarrow \int_M u^{N-2} v \phi dv_g$. In particular for $\phi = v \in L^N(M)$, we obtain

$$\int_M u_m^{N-2} v_m v dv_g \rightarrow \int_M u^{N-2} v^2 dv_g.$$

Consequently,

$$\int_M u_m^{N-2} (v_m - v)^2 dv_g = 1 - \int_M u^{N-2} v^2 dv_g + o(1). \quad (3.5.2)$$

■

Theorem 3.5.1.

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Assume that $\lambda_1(g) > 0$ and

$$1 - \mu K_0 > 0, \quad (3.5.3)$$

where μ is the GJMS invariant and K_0 is given by (1.4.1). Then there exist two nontrivial functions $u \in L_+^N(M)$ and $v \in H_k^2(M)$ such that in the weak sense, we have

$$P_g v = \mu_1 u^{N-2} v \quad \text{and} \quad \int_M u^{N-2} v^2 dv_g = 1. \quad (3.5.4)$$

In other words, μ_1 is attained by a generalized metric.

Proof

let $g_m = u_m^{\frac{4}{n-2k}} g$ be a minimizing sequence of conformal metrics of μ_1 , a sequence of metrics such that $u_m \in C^\infty(M)$, $u_m > 0$ and

$$\mu_1 = \lim_m \lambda_1(g_m) \text{vol}(M, g_m)^{\frac{2k}{n}}.$$

For more clarity we set : $\lambda_1(g_m) = \lambda_{1,m}$.

Without loss of generality, we may assume that

$$\text{vol}(M, g_m) = \int_M u_m^N dv_g = 1. \quad (3.5.5)$$

Indeed, since

$$\frac{2kN}{n} = \frac{2k2n}{n(n-2k)} = \frac{2n}{n-2k} - 2 = N - 2,$$

it follows that for any $\lambda > 0$, one gets

$$I(\lambda u, v) = \frac{\int_M v P_g v dv_g}{\int_M (\lambda u)^{N-2} v^2 dv_g} \left(\int_M (\lambda u)^N dv_g \right)^{\frac{2k}{n}} = I(u, v).$$

This means that if (u_m) is a minimizing sequence, $(\lambda u_m)_m$ is also is a minimizing sequence, just choose $\lambda = \left(\int_M u_m^N dv_g \right)^{-\frac{1}{N}}$. i.e

$$\mu_1 = \lim_m \lambda_{1,m}.$$

Step 1: (3.5.5) implies that the sequence $(u_m)_m$ is bounded in $L^N(M)$, hence there exists $u \in L^N(M)$, $u \geq 0$ such that $u_m \rightarrow u$ weakly in $L^N(M)$ and by standard argument, we get

$$\int_M u^N dv_g \leq \liminf \int_M u_m^N dv_g = 1. \quad (3.5.6)$$

Now, we are going to prove that the generalized metric $u^{\frac{4}{n-2k}} g$ with $u \in L^N(M)$, $u \geq 0$ and $u \neq 0$ minimizes μ_1 .

Since $\lambda_1(g) > 0$, P_g is coercive. Then for all $u_m \in C^\infty(M)$, Theorem 3.4.2 implies the existence of $v_m \in C^\infty(M)$ such that

$$P_g v_m = \lambda_{1,m} u_m^{N-2} v_m \quad \text{and} \quad \int_M u_m^{N-2} v_m^2 dv_g = 1.$$

Now for m large enough, we may assume that

$$\lambda_{1,m} \leq \mu_1 + 1,$$

which implies that

$$\|v_m\|_{P_g}^2 = \int_M v_m P_g v_m dv_g = \lambda_{1,m} \leq \mu_1 + 1.$$

Hence the sequence $(v_m)_m$ is bounded in $H_k^2(M)$, then there exists $v \in H_k^2(M)$ such that $v_m \rightarrow v$ weakly in $H_k^2(M)$ and $v_m \rightarrow v$ strongly in $H_{k-1}^2(M)$. This, together

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with the weak convergence of $(u_m)_m$, imply that the function v is a weak solution of the following equation

$$P_g v = \mu_1 u^{N-2} v. \quad (3.5.7)$$

Step 2: we show that u, v are not identically null.

Letting

$$\varphi_m = v_m - v.$$

Clearly $\varphi_m \rightarrow 0$ weakly in $H_k^2(M)$, and the strongly in $H_{k-1}^2(M)$.

Now for all m , one gets :

$$\begin{aligned} \int_M v_m P_g v_m dv_g &= \int_M (\varphi_m + v) P_g v_m dv_g \\ &= \int_M (\varphi_m P_g v_m + v P_g v_m) dv_g \\ &= \int_M (v_m P_g \varphi_m + v_m P_g v) dv_g \\ &= \int_M ((v + \varphi_m) P_g v + (v + \varphi_m) P_g \varphi_m) dv_g \\ &= \int_M v P_g v dv_g + \int_M \varphi_m P_g \varphi_m dv_g + \int_M 2v P_g \varphi_m dv_g \\ &= \int_M v P_g v dv_g + \int_M |\Delta_g^{\frac{k}{2}} \varphi_m|^2 dv_g + \sum_{l=0}^{k-1} \int_M A_{(l)}(g) (\nabla^l \varphi_m, \nabla^l \varphi_m) dv_g \\ &\quad + \int_M 2v P_g \varphi_m dv_g. \end{aligned}$$

Since $(\varphi_m)_m$ goes to 0 weakly in $H_k^2(M)$, we get

$$\int_M v P_g \varphi_m dv_g \longrightarrow 0, \quad (3.5.8)$$

and from the strong convergence of $(\varphi_m)_m$ to 0 in $H_{k-1}^2(M)$, we get

$$\int_M \sum_{l=0}^{k-1} |\nabla^l \varphi_m|^2 dv_g = o(1) \quad \text{and} \quad \sum_{l=0}^{k-1} \int_M A_{(l)}(g) (\nabla^l \varphi_m, \nabla^l \varphi_m) dv_g = o(1). \quad (3.5.9)$$

(3.5.8) and (3.5.9) imply that,

$$\int_M v_m P_g v_m dv_g - \int_M v P_g v dv_g = \int_M |\Delta_g^{\frac{k}{2}} \varphi_m|^2 dv_g + o(1). \quad (3.5.10)$$

Now let,

$$A = \int_M u_m^{N-2} \varphi_m^2 dv_g. \quad (3.5.11)$$

Then by applying Hölder inequality, Theorem 3.3.1 and from (3.5.9), one has

$$\begin{aligned} A &\leq \left(\int_M (u_m)^{N-2 \frac{N}{N-2}} dv_g \right)^{\frac{N-2}{N}} \left(\int_M (\varphi_m)^{\frac{N}{2} \cdot 2} dv_g \right)^{\frac{2}{N}} \\ &\leq \|\varphi_m\|_N^2 \\ &\leq (K_0 + \varepsilon) \int_M |\Delta_g^{\frac{k}{2}} \varphi_m|^2 dv_g + B_\varepsilon \int_M \sum_{l=0}^{k-1} |\nabla^l \varphi_m|^2 dv_g \\ &\leq (K_0 + \varepsilon) \int_M |\Delta_g^{\frac{k}{2}} \varphi_m|^2 + o(1) \\ &\leq (K_0 + \varepsilon) \left(\int_M v_m P_g v_m - v P_g v dv_g \right) + o(1) \\ &\leq (K_0 + \varepsilon) \left(\lambda_{1,m} - \int_M v P_g v dv_g \right) + o(1) \\ &\leq (K_0 + \varepsilon) \left(\lambda_{1,m} - \mu_1 \int_M u^{N-2} v^2 dv_g \right) + o(1). \end{aligned}$$

Independently, with Lemma 3.5.2 formula (3.5.2), we have

$$A = 1 - \int_M u^{N-2} v^2 dv_g + o(1),$$

then it follows that

$$1 - \int_M u^{N-2} v^2 dv_g \leq (K_0 + \varepsilon) \left(\lambda_{1,m} - \mu_1 \int_M u^{N-2} v^2 dv_g \right) + o(1).$$

Now when $m \rightarrow +\infty$, one gets

$$1 - \int_M u^{N-2} v^2 dv_g \leq (K_0 + \varepsilon) (\mu_1 - \mu_1 \int_M u^{N-2} v^2 dv_g),$$

therefore,

$$1 - (K_0 + \varepsilon) \mu_1 \leq \int_M u^{N-2} v^2 dv_g - (K_0 + \varepsilon) \mu_1 \int_M u^{N-2} v^2 dv_g,$$

and this leads to

$$1 - K_0 \mu_1 \leq (1 - K_0 \mu_1) \int_M u^{N-2} v^2 dv_g + \varepsilon \mu_1 (1 - \int_M u^{N-2} v^2 dv_g). \quad (3.5.12)$$

3.6 Nonlinear GJMS equation and nodal solution

Now, by using Lemma 3.5.1 formula (3.5.1) and the assumption (3.5.3), one easily has $1 - K_0\mu_1 > 0$ and as we can choose ε sufficiently small enough, (3.5.12) necessarily one has

$$\int_M u^{N-2}v^2 dv_g \geq 1.$$

On the other hand, by the weak convergence one gets that,

$$\int_M u^{N-2}v^2 dv_g \leq \liminf \int_M u_m^{N-2}v_m^2 dv_g = 1,$$

which yields to

$$\int_M u^{N-2}v^2 dv_g = 1.$$

This implies that v and u are not identically null which means that μ_1 is attained by the generalized metric $u^{\frac{4}{n-2k}}g$. Moreover, we obtain

$$\mu_1 = \|v\|_{P_g}^2 = \int_M vP_gv dv_g > 0. \quad (3.5.13)$$

■

3.6 Nonlinear GJMS equation and nodal solution

In this section, we show that the equation $P_gv = \mu_1|v|^{N-2}v$ has a nodal solution if $Q_g \leq 0$.

Theorem 3.6.1.

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Assume that μ_1 is attained by the generalized metric $u^{\frac{4}{n-2k}}g$ where $u \in L_+^N(M)$. Then $u = |v|$ where $v \in H_k^2(M)$, v is a solution weak of $P_gv = \mu_1u^{N-2}v$ and such that $\int_M u^{N-2}v^2 dv_g = 1$. Moreover, the function $v \in C^{2k}(M)$ and if $Q_g \leq 0$, then v changes the sign.

Proof

Let the function $h = a|v| \in L_+^N(M)$ with $a > 0$ chosen such that $\int_M h^N dv_g = 1$, by definition

$$\begin{aligned} \mu_1 &\leq \frac{\int_M vP_gv dv_g}{\int_M h^{N-2}v^2 dv_g} = \frac{\mu_1 \int_M u^{N-2}v^2 dv_g}{\int_M h^{N-2}v^2 dv_g} = \frac{a^2 \mu_1 \int_M u^{N-2}v^2 dv_g}{a^2 \int_M h^{N-2}v^2 dv_g} \\ &= \frac{\mu_1 \int_M u^{N-2}(av)^2 dv_g}{\int_M (a|v|)^{N-2}(av)^2 dv_g} = \frac{\mu_1 \int_M u^{N-2}(av)^2 dv_g}{\int_M h^N dv_g} \\ &= \mu_1 \int_M u^{N-2}(a|v|)^2 dv_g. \end{aligned}$$

By using (3.5.6) and Hölder's inequality, it follows that

$$\begin{aligned}
 \mu_1 &\leq \mu_1 \left(\int_M u^{(N-2)\frac{N}{N-2}} dv_g \right)^{\frac{N-2}{N}} \left(\int_M (a|v|)^{2\frac{N}{2}} dv_g \right)^{\frac{2}{N}} \\
 &\leq \mu_1 \left(\int_M u^N dv_g \right)^{\frac{N-2}{N}} \left(\int_M h^N dv_g \right)^{\frac{2}{N}} \\
 &\leq \mu_1 \left(\int_M u^N dv_g \right)^{\frac{N-2}{N}} \leq \mu_1,
 \end{aligned} \tag{3.6.1}$$

this implies that we have both equality in the Hölder inequality. The equality in the Hölder inequality implies that there exists a constant $b > 0$ such that :

$$u = b|v|.$$

From the equality $1 = \int_M u^{N-2} v^2 dv_g = b^{N-2} \int_M |v|^N dv_g$, we obtain

$$\frac{1}{b^{N-2}} = \int_M |v|^N dv_g.$$

(3.6.1) implies that $\int_M u^N dv_g = 1$, then it follows that

$$b^N \int_M |v|^N dv_g = 1,$$

which leads to

$$\frac{1}{b^{N-2}} = \int_M |v|^N dv_g = \frac{1}{b^N},$$

therefore

$$b = 1 \quad \text{and} \quad u = |v|.$$

Hence, v is a weak solution of

$$P_g v = \mu_1 |v|^{N-2} v, \tag{3.6.2}$$

and from standard regularity see Proposition 3.3.5, we get that $v \in C^{2k}(M)$. In addition, since $\mu_1 > 0$ and $Q_g \leq 0$, by following the same proof of Proposition 3.4.2, we deduce that the function v changes the sign. ■

3.7 Case of Einsteinian manifold and positive solution

In this section, on Einstein manifold when $S_g > 0$, we will prove that the solution v of equation (3.6.2) is positive, $\mu_1 = \mu$ and is attained by a conformal metric which leads to the existence of a metric \bar{g} conformal to g such that the Q -curvature is constant. In the case $S_g < 0$, and k is odd, the solution is nodal.

3.7 Case of Einsteinian manifold and positive solution

Theorem 3.7.1.

Let (M, g) be a compact Einstein manifold of dimension $n \geq 3$. Assume that $S_g > 0$ and $1 - \mu K_0 > 0$ where μ is the GJMS invariant and K_0 is given by formula (1.4.1). Then μ_1 is attained by the conformal metric $u^{\frac{4}{n-2k}}g$. In other words, there exists $u \in C^\infty(M)$, $u > 0$ solution to the following equation

$$P_g u = \mu_1 u^{N-1} \quad \text{such that} \quad \int_M u^N dv_g = 1.$$

Proof

We follow the same proof of Theorem 3.5.1.

Let $g_m = u_m^{\frac{4}{n-2k}}g$ be a minimizing sequence of conformal metrics of μ_1 , a sequence of metrics such that $u_m \in C^\infty(M)$, $u_m > 0$ and

$$\mu_1 = \lim_m \lambda_{1,m} \quad \text{and} \quad \int_M u_m^N dv_g = 1. \quad (3.7.1)$$

Firstly: (3.7.1) implies that the sequence $(u_m)_m$ is bounded in $L^N(M)$, hence there exists $u \in L^N(M)$, $u \geq 0$ such that $u_m \rightarrow u$ weakly in $L^N(M)$.

Since (M, g) is Einstein and $S_g > 0$, P_g is coercive. Then for all $u_m \in C^\infty(M)$, Theorem 3.4.2 implies the existence of $v_m \in C^\infty(M)$ such that $v_m > 0$ and

$$P_g v_m = \lambda_{1,m} u_m^{N-2} v_m \quad \text{and} \quad \int_M u_m^{N-2} v_m^2 dv_g = 1.$$

Now for m large enough, we may assume that

$$\lambda_{1,m} \leq \mu_1 + 1,$$

which implies that

$$\|v_m\|_{P_g}^2 = \int_M v_m P_g(v_m) dv_g = \lambda_{1,m} \leq \mu_1 + 1.$$

Hence the sequence $(v_m)_m$ is bounded in $H_k^2(M)$, then there exists $v \in H_k^2(M)$ such that $v \geq 0$, $v_m \rightarrow v$ weakly in $H_k^2(M)$ and $v_m \rightarrow v$ strongly in $H_{k-1}^2(M)$. This, together with the weak convergence of $(u_m)_m$, imply that the function v is a weak solution of the following equation

$$P_g v = \mu_1 u^{N-2} v. \quad (3.7.2)$$

And in particular

$$v \geq 0.$$

Secondly: Since $1 - \mu K_0 > 0$, by step (2) of the poof of Theorem 3.5.1, the functions u, v satisfy $\int_M u^{N-2} v^2 dv_g = 1$ and are not identically null. Since $v \geq 0$, we let the

function $h = av \in L_+^N(M)$ where $a > 0$ chosen such that $\int_M h^N dv_g = 1$ and by following the same proof of Theorem 3.6.1, one has

$$u = v.$$

Therefore, v is a weak solution of

$$P_g v = \mu_1 v^{N-1},$$

and from standard regularity see Proposition 3.3.5, we get that $v \in C^{2k}(M)$. In particular since $v \geq 0$ and $\mu_1 > 0$, one has $P_g v \geq 0$ and since $v \neq 0$, it follows from Proposition 3.3.4 that $v > 0$ and again by regularity $v \in C^\infty(M)$.

Now since $\int_M v P_g v dv_g = \mu_1$, $\int_M |v|^N dv_g = 1$ and from the definition of μ , one has

$$\mu \leq \frac{\int_M v P_g v dv_g}{\left(\int_M |v|^N dv_g\right)^{\frac{2}{N}}} = \mu_1. \quad (3.7.3)$$

It follows that

$$\mu \leq \mu_1,$$

and by Lemma 3.5.1 formula (3.5.1), we get that

$$\mu_1 = \mu.$$

Therefore, the infimum μ_1 is achieved by the conformal metric $\bar{g} = u^{\frac{4}{n-2k}} g$ and this means that metric \bar{g} is such that the Q -curvature

$$Q_{\bar{g}} = \frac{2}{n-2k} \mu_1. \quad \blacksquare$$

A more interesting situation on Einstein manifold is when $S_g < 0$, this implies that $\int_M v P_g v dv_g$ can be negative or positive and consequently the eigenvalues follow same thing contrary to the case $S_g > 0$ which implies only the positivity of eigenvalues.

Corollary 3.7.1.

Let (M, g) be a smooth compact Einstein manifold of dimension $n \geq 3$, assume that $S_g < 0$, $\lambda_1(g) > 0$ and $1 - \mu K_0 > 0$. If k is odd, the following equation:

$$P_g v = \mu_1 |v|^{N-2} v, \quad (3.7.4)$$

has a nodal solution $v \in C^{2k}(M)$.

3.7 Case of Einsteinian manifold and positive solution

Proof

Since $\lambda_1(g) > 0$ and $1 - \mu K_0 > 0$, Theorem 3.5.1 implies that μ_1 is attained by a generalized metric and since (M, g) is Einstein, by using (3.3.5), (3.7.4) can be written as

$$\Delta_g^k u + \sum_{l=0}^{k-2} b_{k-l-1} (S_g)^{l+1} \Delta_g^{k-l-1} u + b_0 (S_g)^k u = \mu_1 u^{N-2} v,$$

where b_{k-1}, \dots, b_1, b_0 are positive real numbers. Therefore, if k is odd, $b_0 (S_g)^k < 0$ and by applying Theorem 3.6.1 with $\frac{n-2k}{2} Q_g = b_0 (S_g)^k$, we get the result. ■

Perspective

In the same spirit, we would like to study the second GJMS invariant μ_2 and of course we will look for the general conditions where it can be attained by a generalized metric. In other words, we will study the corresponding GJMS equation and maybe we can discuss the existence and the sign of its solution.

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Abstract

The purpose of our thesis is to study the existence of certain conformal invariants GJMS type on a compact Riemannian manifold of dimension $n \geq 3$. The GJMS operator can be seen as a generalization of the Yamabe operator and the Paneitz-Branson operator. In particular, we study when those invariants are attained, these problems are equivalent to solve some nonlinear elliptic equations with critical Sobolev growth. Our work is based on the variational methods.

Key words and phrases:

- 1) Fourth-order elliptic equation of Paneitz-Branson type operator.
- 2) Eigenvalues.
- 3) Nodal solution.
- 4) GJMS type equation.
- 5) Generalized metric.

Résumé

L'objectif de notre thèse est d'étudier l'existence de certains invariants conformes de type GJMS sur une variété riemannienne compacte de dimension $n \geq 3$. Le GJMS est un opérateur qui est une généralisation du fameux opérateur de Yamabe et aussi celui de Paneitz-Branson. En particulier, nous étudions quand est ce que ces invariants sont atteints. Ces problèmes sont équivalents à résoudre quelques équations elliptiques non linéaire avec exposant critique de Sobolev. Notre travail est basé sur les méthodes variationnelles.

Mots clés et phrases :

- 1) Équation elliptique du quatrième ordre de type Paneitz-Branson.
- 2) Les valeurs propres.
- 3) Solution nodale.
- 4) Équation de type GJMS.
- 5) Métrique généralisée.

ملخص

الهدف من أطروحتنا هو دراسة وجود بعض ثوابت توافق من نوع GJMS على منوعة ريمان المترابطة ذات البعد $n \geq 3$. مؤثر GJMS هو تعميم لمؤثر يماي الشهير وكذلك لبانتش-برانسون. بالخصوص ندرس الحالات التي تتحقق فيها هذه المؤثرات. هذه المسائل تكافئ حل بعض المعادلات الإهليجية الغير خطية ذات الأس الحرج لسو بولاف. يعتمد عملنا على طرق المتغيرات.

كلمات وجمل مفتاحية:

- 1) معادلة إهليجية من الدرجة الرابعة لمؤثر باننش-برانسون.
- 2) القيم الذاتية
- 3) حل يغير إشارته.
- 4) معادلة من نوع GJMS
- 5) المقياس المعمم