République Algérienne Démocratique et Populaire

Ministère de L'enseignement Supérieur et de la Recherche Scientifique

Université de Sidi Bel Abbès

Faculté des Sciences Exactes

THÈSE DE DOCTORAT 3éme CYCLE (L.M.D)

Spécialité

Mathématiques

Option

Systémes Dynamiques et Applications

Sujet

UNE CONTRIBUTION À L'ÉTUDE QUALITATIVE DE CERTAINES ÉQUATIONS DIFFÉRENTIELLES FRACTIONNAIRES

Présentée par

Soufyane Bouriah

Thèse soutenue le 10/02/2016
devant le jury composé de

Encadreur

Mouffak BENCHOHRA : *Professeur à l'université de Sidi bel abbès, Algérie*

Président

Abdelghani OUAHAB : *Professeur à l'université de Sidi bel abbès, Algérie*

Examinateurs

Said ABBAS : *M.C.A à l'université de Saïda, Algérie*

Farida BERHOUN : *M.C.A à l'université de Sidi bel abbès, Algérie*

Boualem Attou SLIMANI : *M.C.A à l'université de Tlemcen, Algérie*

Remerciements

Je tiens à remercier en tout premier le *Prof. Mouffak Benchohra*, pour m'avoir fait confiance en Decembre 2012 quand il m'a accepté en thèse. Il m'a transmis ses connaissances et ses idées avec générosité. Je le remercie vivement pour m'avoir encouragé et guidé pendant ces trois années.

Je remercie très sincérement le *Prof. Ouahab Abdelghani* d'être le président de jury et pour sa lecture attentive et minutieuse du manuscrit. Je suis également heureux que Messieurs le *Dr.Abbas Said*, le *Dr.Berhoun Farida* et le *Dr.Slimani Boualem Attou* m'ont fait l'honneur d'accepter à joindre le jury, je leur dit un grand merci !

Mes plus sincères remerciements vont aussi à monsieur *Lazreg Jamal* pour les discussions que j'ai eu avec lui.

Je porte une attention toute particulière au laboratoire de Mathématiques de l'Université Djilali Liabes avec tous ses membres spécialement *madame Benyamina* .

Cette thèse n'aurait pas vu le jour sans la confiance, l'encouragement et le soutien indéfectibles de mes parents. Je remercie mon père qui a fait tous pour que je puisse continuer mes études. Maman je te remercie de tout mon coeur pour ton affection, pour le temps que tu m'as consacré pendant ces années. j'adresse un merci particulier à mon épouse pour son encouragement, sa patience et son soutien surtout pendant les moments de stress et de découragement sans oublier mon petit *" Youcef "*. Un grand merci à mes frères, mes soeurs et ma belle soeur, ce travail ne pourrait être accompli sans les prières de mes tantes que je remercie gravement.

Enfin, je saisis l'occasion pour remercier mon professeur *Kaynane Abdelatif* et mes amis *Aouad Djilali*, *Daoudi Khlifa* et *Hamdaoui Diaa-Eddine* pour leurs encouragements.

Bouriah Soufyane.

Publications

1. M. Benchohra, **S. Bouriah** and J.Henderson, Existence and Stability Results for Nonlinear Implicit Neutral Fractional Differential Equations with Finite Delay and Impulses, *Comm. Appl. Nonlinear Anal.* **22** (1), (2015), 46-67.

2. M. Benchohra and **S. Bouriah**, Existence and Stability Results for Nonlinear Boundary Value Problem for Implicit Differential Equations of Fractional Order, *Moroccan Journal Pure. Appl. Anal.* **1** (1), (2015),22-36.

3. W. Albarakati, M. Benchohra and **S. Bouriah**, Existence and stability results for nonlinear implicit fractional differential equations with delay and impulses, Differential Equations and Applications Vol. 8 No. 2 (2016), 273-293.

4. M. Benchohra, **S. Bouriah** and M.Darwish, Nonlinear Boundary Value Problem for Implicit Differential Equations of Fractional Order in Banach Spaces, *Fixed Point Theory* (to appear).

5. M. Benchohra and **S. Bouriah**, Existence and Stability Results for Nonlinear Implicit Fractional Differential Equations with Impulses, *Mem. Differential Equations Math. Phys* (to appear).

6. M. Benchohra and **S. Bouriah**, Existence and Stability Results for Neutral Functional Differential Equation of Fractional Order with Finite Delay, *Dynamics of Continuous, Discrete and Impulsive Systems* (to appear).

7. M. Benchohra and **S. Bouriah** , Boundary Value Problem for Nonlinear Implicit Fractional Differential Equations with Impulses in Banach Space (submitted).

Table des matières

Introduction **6**

1 Basic Ingredients **13**
1.1 A brief visit to the history of the Fractional Calculus 13
1.2 Applications of Fractional calculus 16
1.3 Some notations and definitions of Fractional Calculus Theory 18
1.4 Some definitions and proprieties of noncompactness measure 20
1.5 Some fixed point theorems 22

2 BVP for NIDE with Fractional Order **24**
2.1 Introduction and Motivations 24
2.2 Existence of solutions 26
2.3 Ulam-Hyers Rassias stability 30
2.4 Examples 36

3 BVP for NIFDE in Banach Space **39**
3.1 Introduction and Motivations 39
3.2 Existence Results for the BVP in Banach Space 40
3.2.1 Introduction 40
3.2.2 Existence of Solutions 41
3.2.3 An Example. 47
3.3 Existence Results for the non-local BVP in Banach Space 48
3.3.1 Introduction 48
3.3.2 Existence of Solutions 48
3.3.3 An Example. 49

4 BVP for Nonlinear IFDE with Impulses **51**
4.1 Introduction and Motivations 51
4.2 Existence Results for the BVP with Impulses 52
4.2.1 Introduction 53
4.2.2 Existence of Solutions 53
4.2.3 Ulam-Hyers Rassias stability 63
4.2.4 Examples 68

4.3 Existence Results for the BVP with Impulses in Banach Space 70
4.3.1 Introduction 70
4.3.2 Existence of Solutions 70
4.3.3 Examples 78

5 Nonlinear IFDE with finite delay and Impulses 81
5.1 Introduction and Motivations 81
5.2 Existence Results for the NIFDE with Finite Delay and Impulses . . . 83
5.2.1 Introduction 83
5.2.2 Existence of Solutions 84
5.2.3 Ulam-Hyers Rassias stability 93
5.2.4 Examples 98
5.3 Existence Results for the NIFDE with Finite Delay and Impulses in Banach Space 100
5.3.1 Introduction 100
5.3.2 Existence of Solutions 101
5.3.3 Examples 108

6 Neutral NIFDE with finite delay and Impulses 111
6.1 Introduction and Motivations 111
6.2 Existence and Stability Results for Neutral NIFDE with Finite Delay . 112
6.2.1 Introduction 112
6.2.2 Existence of Solutions 112
6.2.3 Ulam-Hyers Stability Results 115
6.2.4 Examples 119
6.3 Existence Results for the Neutral IFDE with Finite Delay and Impulses 121
6.3.1 Introduction 121
6.3.2 Existence of Solutions 122
6.3.3 Ulam-Hyers Stability Results 132
6.3.4 An Example. 137

Bibliographie 139

Introduction

The concept of fractional differential calculus has a long history. One may wonder what meaning may be ascribed to the derivative of a fractional order, that is $\frac{d^n y}{dx^n}$, where n is a fraction. In fact L'Hopital himself considered this very possibility in a correspondence with Leibniz, In 1695, L'Hopital wrote to Leibniz to ask, " *What if* n *be* $\frac{1}{2}$ *?*" From this question, the study of fractional calculus was born. Leibniz responded to the question, "$d^{\frac{1}{2}}x$ *will be equal to* $x\sqrt{dx : x}$. *This is an apparent paradox from which, one day, useful consequences will be drawn.*"

Many known mathematicians contributed to this theory over the years. Thus, 30 September 1695 is the exact date of birth of the "***fractional calculus***" ! Therefore, the fractional calculus it its origin in the works by *Leibnitz*, *L'Hopital* (1695), *Bernoulli* (1697), *Euler* (1730), and *Lagrange* (1772). Some years later, *Laplace* (1812), *Fourier* (1822), *Abel* (1823), *Liouville* (1832), *Riemann* (1847), *Grünwald* (1867), *Letnikov* (1868), *Nekrasov* (1888), *Hadamard* (1892), *Heaviside* (1892), *Hardy* (1915), *Weyl* (1917), *Riesz* (1922), *P. Levy*(1923), *Davis* (1924), *Kober* (1940), *Zygmund* (1945), *Kuttner* (1953), *J. L. Lions* (1959), and *Liverman* (1964)... have developed the basic concept of fractional calculus.

In June 1974, Ross has organized the "*First Conference on Fractional Calculus and its Applications*" at the University of New Haven, and edited its proceedings [128]; Thereafter, Spanier published the first monograph devoted to "*Fractional Calculus*" in 1974 [119]. The integrals and derivatives of non-integer order, and the fractional integrodifferential equations have found many applications in recent studies in theoretical physics, mechanics and applied mathematics. There exists the remarkably comprehensive encyclopedic-type monograph by *Samko*, *Kilbas* and *Marichev* which was published in Russian in 1987 and in English in 1993 [132]. (for more details see [111]) The works devoted substantially to fractional differential equations are : the book of *Miller* and *Ross* (1993) [114], of *Podlubny* (1999) [123], by *Kilbas* et al. (2006) [100], by *Diethelm* (2010) [72], by *Ortigueira* (2011) [120], by *Abbas et al.* (2012) [3], and by *Baleanu et al.* (2012) [32].

In recent years, there has been a significant development in the theory of fractional differential equations. It is caused by its applications in the modeling of many phenomena in various fields of science and engineering such as acoustic, control theory, chaos and fractals, signal processing, porous media, electrochemistry, viscoelasticity, rheology, polymer physics, optics, economics, astrophysics, chaotic dynamics, statistical physics, thermodynamics, proteins, biosciences, bioengineering, etc. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. See for example [33, 34, 86, 88, 111, 122, 131, 134].

Fractional calculus is a generalization of differentiation and integration to arbitrary order (non-integer) fundamental operator D^{α}_{a+} where $\alpha, a, \in \mathbb{R}$. Several approaches to fractional derivatives exist : Riemann-Liouville (RL), Hadamard, Grunwald-Letnikov (GL), Weyl and Caputo etc. The Caputo fractional derivative is well suitable to the physical interpretation of initial conditions and boundary conditions. We refer readers, for example, to the books such as [3, 22, 32, 72, 100, 104, 114, 119, 120, 123, 132] and the articles [4, 6, 7, 19, 26, 27, 38, 43, 45, 48, 50, 52, 56, 57, 58, 59, 99, 130], and references therein. ***In this thesis, we use always the Caputo's derivative***.

Fractional differential equations with nonlocal conditions have been discussed in [8, 10, 73, 82, 110, 117, 118] and references therein. Nonlocal conditions were initiated by Byszewski [65] when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. As remarked by Byszewski [63, 64], the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena.

There are two measures which are the most important ones. The Kuratowski measure of noncompactness $\alpha(B)$ of a bounded set B in a metric space is defined as infimum of numbers $r > 0$ such that B can be covered with a finite number of sets of diameter smaller than r. The Hausdorf measure of noncompactness $\chi(B)$ defined as infimum of numbers $r > 0$ such that B can be covered with a finite number of balls of radii smaller than r. Several authors have studied the measures of noncompactness in Banach spaces. See, for example, the books such as [18, 35, 135] and the articles [20, 36, 37, 49, 58, 60, 89, 115], and references therein.
Recently, considerable attention has been given to the existence of solutions of boundary value problem and boundary conditions for implicit fractional differential equations and integral equations with Caputo fractional derivative. See for example [7, 11, 16, 17, 28, 50, 51, 52, 58, 91, 97, 105, 106, 107, 108, 130, 148], and references therein.

During the last ten years, impulsive differential equations and impulsive differential inclusions with different conditions have intensely were studied by many mathematicians. The concept of differential equations with impulses are introduced by V. Milman and A. Myshkis in 1960 (see [113]). This subject was, thereafter, extensively investigated. Impulsive differential equations have become more important in recent years in some mathematical models of real phenomena, especially in biological or medical domains, in control theory, see for example the mongraphs of Graef et al. ([79]), Lakshmikantham et al ([103]), Perestyuk et al. ([121]), Samoilenko and Perestyuk [133], and several monographs have been published by many authors like the papers of Agarwal et al. ([5]), Ahmad and Sivasundaram ([16]), Benchohra et al. ([39, 40, 55]), Bainov and Simeonov ([30]), and ([42, 53, 60, 61, 84, 138, 139, 141, 144]).

On the other hand, anti-periodic problems constitute an important class of boundary value problems and have recently received considerable attention. See for example the papers of Ahmad and Nieto ([12, 13, 14]), Ahmad et al. ([15]), Wang and Liu ([142]). Anti-periodic boundary conditions appear in physics in a variety of situations (see for example, in ([1, 67]) and the references therein). For some recent work on anti-periodic boundary value problems of fractional differential equations with impulse, see ([41, 47]) and the references therein.

In the theory of ordinary differential equations, of partial differential equations, and in the theory of ordinary differential equations in a Banach space there are several types of data dependence . On the other hand, in the theory of functional equations there are some special kind of data dependence : Ulam-Hyers, Ulam-Hyers-Rassias, Ulam-Hyers- Bourgin, Aoki-Rassias [129].
The stability problem of functional equations originated from a question of Ulam [136, 137] concerning the stability of group homomorphisms : "*Under what conditions does there exist an additive mapping near an approximately additive mapping ?*" Hyers [90] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers Theorem was generalized by Aoki [24] for additive mappings and by Th.M. Rassias [124] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Gavruta [76]. After, many interesting results of the generalized Hyers-Ulam stability to a number of functional equations have been investigated by a number of mathematicians ; see [2, 21, 44, 46, 92, 93, 94, 95, 96, 98, 139, 143, 144] and the books [69, 125, 126] and references therein.

We have organized this thesis as follows :

Chapter 1.
This chapter consists of three Sections. In Section one, we present "*A brief visit to the history of the Fractional Calculus*", and in Section two, we present some "*Applications of Fractional calculus*".
Finally, in the last Section, we recall some preliminary : some basic concepts, and useful famous theorems and results (notations, definitions, lemmas and **fixed point theorems**) which are used throughout this thesis.

In **Chapter 2**, we discuss and establish the existence, the uniqueness and the Ulam-Hyers stability of solution for a class of boundary value problem for NIFDE and for

non-local boundary value problem with Caputo fractional derivative.
In Section 2.2, we will give existence and uniqueness results for the followings problems of implicit fractional differential equations :

$$^cD^\alpha y(t) = f(t, y(t), ^c D^\alpha y(t)), \quad \text{for every } t \in J := [0,T], T > 0, \quad 0 < \alpha \leq 1$$

$$ay(0) + by(T) = c$$

where $^cD^\alpha$ is the fractional derivative of Caputo, $f : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, and a, b, c are real constants with $a + b \neq 0$,
and

$$^cD^\alpha y(t) = f(t, y(t), ^c D^\alpha y(t)), \quad \text{for every } t \in J := [0,T], T > 0, \quad 0 < \alpha \leq 1$$

$$y(0) + g(y) = y_0$$

where $g : C([0,T], \mathbb{R}) \longrightarrow \mathbb{R}$ is a continuous function and y_0 a real constant.
In Section 2.3, we establish the stability results for the two previous problems.
Finally, in Section 2.4, two examples will be included to illustrate our main results.

Chapter 3 , here, two results for a class of boundary value problems for nonlinear implicit fractional differential equations and for non local boundary value problem in Banach space with Caputo fractional derivative are discussed.The arguments are based on Darbo's fixed point theorem combined with the technique of measures of noncompactness and on Mönch's fixed point theorems.
In Section 3.2, we establish existence and uniqueness results of the following boundary value problem for implicit fractional differential equation :

$$^cD^\nu y(t) = f(t, y(t), ^c D^\nu y(t)), \text{ for each, } t \in J := [0,T], \ T > 0, \ 0 < \nu \leq 1,$$

$$ay(0) + by(T) = c,$$

where $^cD^\nu$ is the Caputo fractional derivative, $(E, \|\cdot\|)$ is a real Banach space, $f : J \times E \times E \to E$ is a given function and a, b are real with $a + b \neq 0$ and $c \in E$.
Then, we will illustrate our result by an example.
In Section 3.3, we establish existence and uniqueness result to the following non local boundary value problem :

$$^cD^\nu y(t) = f(t, y(t), ^c D^\nu y(t)), \quad \text{for every } t \in J := [0,T], T > 0, \quad 0 < \nu \leq 1$$

$$y(0) + g(y) = y_0$$

where $g : C([0,T], E) \longrightarrow E$ is a continuous function and $y_0 \in E$.
Finally, an example is given to illustrate the applicability of our main results.

In **Chapter 4**, we establish the existence, the uniqueness and the Ulam-Hyers stability results to a class of boundary value problems for nonlinear implicit fractional

differential equations with impulses.Here two results are discussed, the first is based on the Banach contraction principle, and Schaefer's fixed point theorem, the second is based on the method associated with the technique of measures of noncompactness and the fixed point theorems of Darbo and Mönch.
In Section 4.2, we establish existence, uniqueness and stability results to the following boundary value problem with impulses :

$$^{c}D^{\alpha}_{t_k}y(t) = f(t, y, ^{c}D^{\alpha}_{t_k}y(t)), \text{ for each }, \ t \in (t_k, t_{k+1}], \ k = 0, \ldots, m, \ 0 < \alpha \leq 1,$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \ k = 1, \ldots, m,$$

$$ay(0) + by(T) = c,$$

where $^{c}D^{\alpha}_{t_k}$ is the Caputo fractional derivative, $f : J\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a given function, $I_k : \mathbb{R} \to \mathbb{R}$, and a, b, c are real constants with $a+b \neq 0$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k$.
At last, we present two examples to illustrate our results.
In Section 4.3, We discuss, existence and uniqueness of solutions to the following boundary value problem for nonlinear implicit fractional differential equations with impulses in Banach Space :

$$^{c}D^{\nu}_{t_k}y(t) = f(t, y, ^{c}D^{\nu}_{t_k}y(t)), \text{ for each }, \ t \in (t_k, t_{k+1}], \ k = 0, \ldots, m, \ 0 < \nu \leq 1,$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \ k = 1, \ldots, m,$$

$$ay(0) + by(T) = c,$$

where $^{c}D^{\nu}_{t_k}$ is the Caputo fractional derivative, $(E, \|\cdot\|)$ is a real Banach Space, $f : J \times E \times E \to E$ is a given function, $I_k : E \to E$, a, b are real constants with $a + b \neq 0$ and $c \in E$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k$.
Finally, we give two examples to illustrate our main results.

In **Chapter 5**, We establish the existence, the uniqueness and the Ulam-Hyers stability results to the implicit fractional-order differential equation with finite delay and impulse. Here two results are discussed, the first is based on the Banach contraction principle, and Schaefer's fixed point theorem, the second is based on the method associated with the technique of measures of noncompactness and the fixed point theorems of Darbo and Mönch.
In Section 5.2, we establish, existence, uniqueness and stability results to the following problem of implicit fractional differential equation with finite delay and impulses :

$$^{c}D^{\alpha}_{t_k}y(t) = f(t, y_t, ^{c}D^{\alpha}_{t_k}y(t)), \text{ for each }, \ t \in (t_k, t_{k+1}], \ k = 0, \ldots, m, \ 0 < \alpha \leq 1,$$

$$\Delta y|_{t=t_k} = I_k(y_{t_k^-}),\ k = 1, \dots, m,$$

$$y(t) = \varphi(t),\ t \in [-r, 0],\ r > 0,$$

where ${}^cD^{\alpha}_{t_k}$ is the Caputo fractional derivative, $f : J \times PC([-r, 0], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ is a given function, $I_k : PC([-r, 0], \mathbb{R}) \to \mathbb{R}$, and $\varphi \in PC([-r, 0], \mathbb{R}), 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$.
For each function y_t defined on $[-r, T]$ and for any $t \in J$, we denote by y_t the element of $PC([-r, 0], \mathbb{R})$ defined by :

$$y_t(\theta) = y(t + \theta),\quad \theta \in [-r, 0],$$

$y_t(\cdot)$ represent the history of the state from time $t - r$ up to time t.
Here $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, where $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^-} y(t_k + h)$ represent the right and left limits of y_t at $t = t_k$, respectively.
Finally, we give two examples to illustrate our results.
In Section 5.3,we establish, existence and uniqueness results to the following problem of implicit fractional differential equation with finite delay and impulses :

$${}^cD^{\nu}_{t_k} y(t) = f(t, y_t, {}^cD^{\nu}_{t_k} y(t)),\ \text{for each },\ t \in (t_k, t_{k+1}],\ k = 0, \dots, m,\ 0 < \nu \le 1,$$

$$\Delta y|_{t=t_k} = I_k(y_{t_k^-}),\ k = 1, \dots, m,$$

$$y(t) = \varphi(t),\ t \in [-r, 0],\ r > 0,$$

where ${}^cD^{\nu}_{t_k}$ is the Caputo fractional derivative, $(E, \|\cdot\|)$ is a real Banach space, $f : J \times PC([-r, 0], E) \times E \to E$ is a given function, $I_k : PC([-r, 0], E) \to E$, and $\varphi \in PC([-r, 0], E), 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$.
For each function y_t defined on $[-r, T]$ and for any $t \in J$, we denote by y_t the element of $PC([-r, 0], E)$ defined by :

$$y_t(\theta) = y(t + \theta),\quad \theta \in [-r, 0],$$

At last and as application, two examples are included.

In **the last Chapter**, we establish sufficient conditions for the existence of solutions for a class of Problem for implicit neutral functional differential equations of fractional order for first, with finite delay, then, with finite delay and impulses using Caputo fractional derivative, also, the stability of this class of problem. The arguments are based upon the Banach's fixed point theorem and Schaefer's fixed point theorem.
Section 6.2 is devoted to the existence, uniqueness and stability results of solutions for the following problem for neutral NIFDE with finite delay :

$${}^cD^{\alpha}[y(t) - g(t, y_t)] = f(t, y_t, {}^cD^{\alpha} y(t)),\ t \in J = [0, T],\ T > 0\ , 0 < \alpha \le 1$$

$$y(t) = \varphi(t),\ t \in [-r, 0],\ r > 0,$$

where $f : J \times C([-r,0],\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ and $g : J \times C([-r,0],\mathbb{R}) \to \mathbb{R}$ are two given functions such that $g(0,\varphi) = 0$ and $\varphi \in C([-r,0],\mathbb{R})$.
For each function y_t defined on $[-r,T]$ and for any $t \in J$, we denote by y_t the element of $C([-r,0],\mathbb{R})$ defined by :

$$y_t(\theta) = y(t+\theta), \quad \theta \in [-r,0],$$

$y_t(\cdot)$ represent the evolution history of system state from time $t-r$ to time t.
At last, two examples are included to show the applicability of our results.
In Section 6.3, we establish existence, uniqueness and stability results for the following neutral NIFDE with finite delay and impulses :

$$^cD^{\alpha}_{t_k}[y(t)-\phi(t,y_t)] = f(t,y_t,{}^cD^{\alpha}_{t_k}y(t)),\ \text{for each } t \in (t_k,t_{k+1}],\ k=0,\dots,m,\ 0<\alpha\leq 1,$$

$$\Delta y|_{t=t_k} = I_k(y_{t_k^-}),\ k=1,\dots,m,$$
$$y(t) = \varphi(t),\ t \in [-r,0],\ r>0,$$

where $^cD^{\alpha}_{t_k}$ is the Caputo fractional derivative, $f : J \times PC([-r,0],\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$,
$\phi : J \times PC([-r,0],\mathbb{R}) \to \mathbb{R}$ are given functions with $\phi(0,\varphi) = 0, I_k : PC([-r,0],\mathbb{R})$
$\to \mathbb{R}$ and $\varphi \in PC([-r,0],\mathbb{R}), 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, and $PC([-r,0],\mathbb{R})$ is a space to be specified later.
For each function y defined on $[-r,T]$ and for any $t \in J$, we denote by y_t the element of $PC([-r,0],\mathbb{R})$ defined by :

$$y_t(\theta) = y(t+\theta), \quad \theta \in [-r,0],$$

that is, $y_t(\cdot)$ represents the history of the state from time $t-r$ up to time t.
And $\Delta y|_{t=t_k} = y(t_k^+)-y(t_k^-)$, where $y(t_k^+) = \lim_{h\to 0^+} y(t_k+h)$ and $y(t_k^-) = \lim_{h\to 0^-} y(t_k+h)$ represent the right and left limits of y_t at $t=t_k$, respectively.
And finally, we give an illustrative example.

Chapitre 1

Basic Ingredients

1.1 A brief visit to the history of the Fractional Calculus

In 1695, in a letter to the French mathematician *L'Hospital*, *Leibniz* raised the following question : "*Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders ?*" *L'Hospital* was somewhat curious about that question and replied by another question to *Leibniz* : "*What if the order will be* $\frac{1}{2}$?" *Leibnitz* in a letter dated September 30, replied : "*It will lead to a paradox, from which one day useful consequences will be drawn.*"
Many known mathematicians contributed to this theory over the years. Thus, September 30, 1695 is the exact date of birth of the "*fractional calculus*" ! Therefore, the fractional calculus it its origin in the works by *Leibnitz*, *L'Hopital* (1695), *Bernoulli* (1697), *Euler* (1730), and *Lagrange* (1772). Some years later, *Laplace* (1812), *Fourier* (1822), *Abel* (1823), *Liouville* (1832), *Riemann* (1847), Grünwald (1867), *Letnikov* (1868), *Nekrasov* (1888), *Hadamard* (1892), *Heaviside* (1892), *Hardy* (1915), *Weyl* (1917), *Riesz* (1922), *P. Levy* (1923), *Davis* (1924), *Kober* (1940), *Zygmund* (1945), *Kuttner* (1953), *J. L. Lions* (1959), and *Liverman* (1964)... have developed the basic concept of fractional calculus.

In 1783, *Leonhard Euler* made his first comments on fractional order derivative. He worked on progressions of numbers and introduced first time the generalization of factorials to *Gamma* function. A little more than fifty year after the death of *Leibniz*, *Lagrange*, in 1772, indirectly contributed to the development of exponents law for differential operators of integer order, which can be transferred to arbitrary order under certain conditions. In 1812, *Laplace* has provided the first detailed definition for fractional derivative. *Laplace* states that fractional derivative can be defined for functions with representation by an integral, in modern notation it can be written as $\int y(t)t^{-x}dt$. Few years after, *Lacroix* worked on generalizing the integer order derivative of function

$y(t) = t^m$ to fractional order, where m is some natural number. In modern notations, integer order n^{th} derivative derived by *Lacroix* can be given as

$$\frac{d^n y}{dt^n} = \frac{m!}{(m-n)!} t^{m-n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} t^{m-n}, \ m > n$$

where, Γ is the *Euler's Gamma function*.

Thus, replacing n with $\frac{1}{2}$ and letting $m = 1$, one obtains the derivative of order $\frac{1}{2}$ of the function t

$$\frac{d^{\frac{1}{2}} y}{dt^{\frac{1}{2}}} = \frac{\Gamma(2)}{\Gamma(\frac{3}{2})} t^{\frac{1}{2}} = \frac{2}{\sqrt{\pi}} \sqrt{t}$$

Euler's Gamma function (or ***Euler's integral*** of the second kind) has the same importance in the fractional-order calculus and it is basically given by integral

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

The exponential provides the convergence of this integral in ∞, the convergence at zero obviously occurs for all complex z from the right half of the complex plane ($Re(z) > 0$).

This function is generalization of a factorial in the following form :

$$\Gamma(n) = (n-1)!.$$

Other generalizations for values in the left half of the complex plane can be obtained in following way. If we substitute e^{-t} by the well-known limit

$$e^{-t} = \lim_{n \to \infty} \left(1 - \frac{t}{n}\right)^n$$

and then use n-times integration by parts, we obtain the following limit definition of the Gamma function

$$\Gamma(z) = \lim_{n \to \infty} \frac{n! n^z}{z(z+1)\dots(z+n)}.$$

Therefore, historically the first discussion of a derivative of fractional order appeared in a calculus written by Lacroix in 1819.

It was Liouville who engaged in the first major study of fractional calculus. Liouville's first definition of a derivative of arbitrary order ν involved an infinite series. Here, the series must be convergent for some ν. Liouville's second definition succeeded in giving a fractional derivative of x^{-a} whenever both x and are positive. Based on the definite integral related to Euler's gamma integral, the integral formula can be calculated for x^{-a}. Note that in the integral

$$\int_0^\infty u^{a-1} e^{-xu} du,$$

if we change the variables $t = xu$, then

$$\int_0^\infty u^{a-1}e^{-xu}du = \int \left(\frac{t}{x}\right)^{a-1} e^{-t}\frac{1}{x}dt = \frac{1}{x^a}\int_0^\infty t^{a-1}e^{-t}dt.$$

Thus,

$$\int_0^\infty u^{a-1}e^{-xu}du = \frac{1}{x^a}\int_0^\infty t^{a-1}e^{-t}dt.$$

Through the ***Gamma*** function, we obtain the integral formula

$$x^{-a} = \frac{1}{\Gamma(a)}\int_0^\infty u^{a-1}e^{-xu}du.$$

Consequently, by assuming that $\frac{d^\nu}{dx^\nu}e^{ax} = a^\nu e^{ax}$, for any $\nu > 0$, then

$$\frac{d^\nu}{dx^\nu}x^{-a} = \frac{\Gamma(a+\nu)}{\Gamma(a)}x^{-a-\nu} = (-1)^\nu\frac{\Gamma(a+\nu)}{\Gamma(a)}x^{-a-\nu}$$

In 1884 Laurent published what is now recognized as the definitive paper on the foundations of fractional calculus. Using Cauchy's integral formula for complex valued analytical functions and a simple change of notation to employ a positive ν rather than a negative ν will now yield Laurent's definition of integration of arbitrary order

$$_{x_0}D_x^\alpha h(x) = \frac{1}{\Gamma(\nu)}\int_{x_0}^x (x-t)^{\nu-1}h(t)dts.$$

The **Riemann-Liouville** differential operator of fractional calculus of order α defined as

$$(D_{a+}^\alpha f)(t) := \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\left(\dfrac{d}{dt}\right)^n \displaystyle\int_a^t (t-s)^{n-\alpha-1}f(s)ds & \text{if } n-1<\alpha<n, \\ \left(\dfrac{d}{dt}\right)^n f(t), & \text{if } \alpha = n, \end{cases}$$

where $\alpha, a, t \in \mathbb{R}$, $t > a$, $n = [\alpha]+1$; $[\alpha]$ denotes the integer part of the real number α, and Γ is the *Gamma* function.

The **Grünwald-Letnikov** differential operator of fractional calculus of order α defined as

$$(D_{a+}^\alpha f)(t) := \lim_{h\to 0} h^{-\alpha}\sum_{j=0}^{[\frac{t-a}{h}]}(-1)^j\binom{\alpha}{j}f(t-jh).$$

Binomial coefficients with alternating signs for positive value of n are defined as

$$\binom{n}{j} = \frac{n(n-1)(n-2)\cdots(n-j+1)}{j!} = \frac{n!}{j!(n-j)!}.$$

For binomial coefficients calculation we can use the relation between Euler's *Gamma* function and factorial, defined as

$$\binom{\alpha}{j} = \frac{\alpha!}{j!(\alpha-j)!} = \frac{\Gamma(\alpha)}{\Gamma(j+1)\Gamma(\alpha-j+1)}.$$

The Grünwald-Letnikov definition of differ-integral starts from classical definitions of derivatives and integrals based on infinitesimal division and limit. The disadvantages of this approach are its technical difficulty of the computations and the proofs and large restrictions on functions. (see [149])

The **Caputo** (**1967**) differential operator of fractional calculus of order α defined as

$$({}^cD^{\alpha}_{a+}f)(t) := \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_a^t (t-s)^{n-\alpha-1}f^{(n)}(s)ds & \text{if } n-1<\alpha<n, \\ \left(\dfrac{d}{dt}\right)^n f(t), & \text{if } \alpha = n, \end{cases}$$

where $\alpha, a, t \in \mathbb{R}$, $t > a$, $n = [\alpha]+1$. This operator is introduced in 1967 by the Italian Caputo.

This consideration is based on the fact that for a wide class of functions, the three best known definitions ((GL), (RL), and Caputo) are equivalent under some conditions. (see ([87])

Unfortunately, fractional calculus still lacks a geometric interpretation of integration or differentiation of arbitrary order. We refer readers, for example, to the books such as [3, 32, 88, 100, 104, 114, 119, 123, 132] and the articles [4, 6, 7, 26, 27, 38, 48, 50, 52, 56, 57, 58, 99, 130], and references therein.

1.2 Applications of Fractional calculus

The concept of fractional calculus has great potential to change the way we see, model and analyze the systems. It provides good opportunity to scientists and engineers for revisiting the origins. The theoretical and practical interests of using fractional order operators are increasing. The application domain of fractional calculus is ranging from accurate modeling of the microbiological processes to the analysis of astronomical images.
Next, we will present a brief survey of applications of fractional calculus in science and engineering.

The Tautochrone Problem (Historical Example) :
This example was studied, for the first time, by Abel in the early 19^{th} century. It was

one of the basic problems where the framework of the fractional calculus was used although it is not essentially necessary.

Signal and Image Processing :
In the last decade, the use of fractional calculus in *signal processing* has tremendously increased. In *signal processing*, the fractional operators are used in the design of differentiator and integrator of fractional order, fractional order differentiator **FIR** (finite impulse response), **IIR** type digital fractional order differentiator (infinite impulse response), a new **IIR** (infinite impulse response)-type digital fractional order differentiator (**DFOD**) and for modeling the speech signal. The fractional calculus allows the edge detection, enhances the quality of images, with interesting possibilities in various image enhancement applications such as image restoration image denoising and the texture enhancement. He is used, in particularly, in satellite image classification, and astronomical image processing.

Electromagnetic Theory :
The use of fractional calculus in electromagnetic theory has emerged in the last two decades. In 1998, Engheta [74] introduced the concept of fractional curl operators and this concept is extended by Naqvi and Abbas [116]. Engheta's work gave birth to the newfield of research in Electromagnetics, namely, "*Fractional Paradigms in Electromagnetic Theory*". Nowadays fractional calculus is widely used in Electromagnetics to explore new results ; for example, Faryad and Naqvi [75] have used fractional calculus for the analysis of a Rectangular Waveguide.

Control Engineering :
In industrial environments, robots have to execute their tasks quickly and precisely, minimizing production time, and the robustness of control systems is becoming imperative these days. This requires flexible robots working in large workspaces, which means that they are influenced by nonlinear and fractional order dynamic effects.

Biological Population Model
The problems of the diffusion of biological populations occur nonlinearly and the fractional order differential equations appear more and more frequently in different research areas.

Reaction-Diffusion Equations
Fractional equations can be used to describe some physical phenomenon more accurately than the classical integer order differential equation. The reaction-diffusion equations play an important role in dynamical systems of mathematics, physics, chemistry, bioinformatics, finance, and other research areas. There has been a wide variety of analytical and numerical methods proposed for fractional equations ([109, 147]), for example, finite difference method ([68]), finite element method, Adomian decompo-

sition method ([127]), and spectral technique ([112]). Interest in fractional reaction-diffusion equations has increased.

1.3 Some notations and definitions of Fractional Calculus Theory

In this chapter definitions and some auxiliary results are given regarding the main objects of the monograph : some notations and definitions of Fractional Calculus Theory, some definitions and proprieties of noncompactness measure, some fixed point theorems.

Definition 1.3.1 *([100, 123]). The fractional (arbitrary) order integral of the function $f \in L^1([0,T],\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,$$

where Γ is the gamma function.

Theorem 1.3.2 *[100]. For any $f \in C([a,b],\mathbb{R})$ the Riemann-Liouville fractional integral satisfies*

$$I^\alpha I^\beta f(t) = I^\beta I^\alpha f(t) = I^{\alpha+\beta} f(t),$$

for $\alpha, \beta > 0$.

Definition 1.3.3 *([99]). For a function $f \in AC^n(J)$, the Caputo fractional-order derivative of order α of h, is defined by*

$$({}^cD^\alpha f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds,$$

where $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of the real number α.

Lemma 1.3.4 *([114]) Let $\alpha \geq 0$ and $n = [\alpha]+1$. Then*

$$I^\alpha({}^cD^\alpha f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{f^k(0)}{k!} t^k.$$

Remark 1.3.5 *([114])The Caputo derivative of a constant is equal to zero.*

We need the following auxiliary lemmas.

Lemma 1.3.6 *([148]) Let $\alpha > 0$. Then the differential equation*

$$^{c}D^{\alpha} f(t) = 0$$

has solutions $f(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [\alpha] + 1$.

Lemma 1.3.7 *([148]) Let $\alpha > 0$. Then*

$$I^{\alpha c} D^{\alpha} f(t) = f(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [\alpha] + 1$.

We state the following generalization of ***Gronwall***'s lemma for singular kernels.

Lemma 1.3.8 *([145]) Let $v : [0, T] \to [0, +\infty)$ be a real function and $w(\cdot)$ is a nonnegative, locally integrable function on $[0, T]$. Assume that there are constants $a > 0$ and $0 < \alpha < 1$ such that*

$$v(t) \leq w(t) + a \int_0^t (t-s)^{-\alpha} v(s) ds,$$

Then, there exists a constant $K = K(\alpha)$ such that

$$v(t) \leq w(t) + Ka \int_0^t (t-s)^{-\alpha} w(s) ds, \text{ for every } t \in [0, T].$$

Bainov and Hristova [29] introduced the following integral inequality of Gronwall type for piecewise continuous functions which can be used in the sequel.

Lemma 1.3.9 *Let for $t \geq t_0 \geq 0$ the following inequality hold*

$$x(t) \leq a(t) + \int_{t_0}^t g(t,s) x(s) ds + \sum_{t_0 < t_k < t} \beta_k(t) x(t_k),$$

where $\beta_k(t) (k \in \mathbb{N})$ are nondecreasing functions for $t \geq t_0$, $a \in PC([t_0, \infty), \mathbb{R}_+)$, a is nondecreasing and $g(t,s)$ is a continuous nonnegative function for $t, s \geq t_0$ and nondecreasing with respect to t for any fixed $s \geq t_0$. Then, for $t \geq t_0$, the following inequality is valid :

$$x(t) \leq a(t) \prod_{t_0 < t_k < t} (1 + \beta_k(t)) exp \left(\int_{t_0}^t g(t,s) ds \right).$$

Theorem 1.3.10 *[83](theorem of Ascoli-Arzela). Let $A \subset C(J, \mathbb{R})$, A is relatively compact (i.e $\overline{A}$ is compact) if :*

1. *A is uniformly bounded i.e, there exists $M > 0$ such that*

$$|f(x)| < M \text{ for every } f \in A \text{ and } x \in J.$$

2. *A is equicontinuous i.e, for every* $\epsilon > 0$*, there exists* $\delta > 0$ *such that for each* $x, \overline{x} \in J$, $|x - \overline{x}| \leq \delta$ *implies* $|f(x) - f(\overline{x})| \leq \epsilon$*, for every* $f \in A$.

Theorem 1.3.11 *[80](theorem of Ascoli-Arzela). Let* $A \subset PC(J, E)$*, A is relatively compact (i.e* $\overline{A}$ *is compact) if :*

1. *A is uniformly bounded i.e, there exists* $M > 0$ *such that*

$$\|f(x)\| < M \text{ for every } f \in A \text{ and } x \in (t_k, t_{k+1}], k = 1, ..., m.$$

2. *A is equicontinuous on* $(t_k, t_{k+1}]$ *i.e, for every* $\epsilon > 0$*, there exists* $\delta > 0$ *such that for each* $x, \overline{x} \in (t_k, t_{k+1}]$, $|x - \overline{x}| \leq \delta$ *implies* $\|f(x) - f(\overline{x})\| \leq \epsilon$*, for every* $f \in A$.
3. *The set* $\{f(t) : f \in A; t \in (t_k, t_{k+1}], k = 1, ..., m\}$ *is relatively compact in E.*

1.4 Some definitions and proprieties of noncompactness measure

Next, we define in this Section the Kuratowski (1896-1980) and Hausdorf measures of noncompactness (**MNC** for short) and give their basic properties.

Definition 1.4.1 *([101]) Let* (X, d) *be a complete metric space and* $\mathcal{B}$ *the family of bounded subsets of* X*. For every* $B \in \mathcal{B}$ *we define the Kuratowski measure of noncompactness* $\alpha(B)$ *of the set B as the infimum of the numbers d such that B admits a finite covering by sets of diameter smaller than d.*

Remark 1.4.2 *The diameter of a set B is the number* $\sup\{d(x, y) : x \in B, y \in B\}$ *denoted by* $diam(B)$*, with* $diam(\emptyset) = 0$.
It is clear that $0 \leq \alpha(B) \leq diam(B) < +\infty$ *for each nonempty bounded subset B of X and that* $diam(B) = 0$ *if and only if B is an empty set or consists of exactly one point.*

Definition 1.4.3 *([35]) Let E be a Banach space and* Ω_E *the bounded subsets of E. The Kuratowski measure of noncompactness is the map* $\alpha : \Omega_E \to [0, \infty]$ *defined by*

$$\alpha(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^{n} B_i \text{ and } diam(B_i) \leq \epsilon\}; \text{ here } B \in \Omega_E,$$

where

$$diam(B_i) = sup\{||x - y|| : x, y \in B_i\}.$$

The Kuratowski measure of noncompactness satisfies the following properties :

Lemma 1.4.4 *([18, 35, 36, 101]) Let A and B bounded sets.*

(a) $\alpha(B) = 0 \Leftrightarrow \overline{B}$ *is compact (B is relatively compact), where* $\overline{B}$ *denotes the closure of B.*

(b) nonsingularity : α *is equal to zero on every one element-set.*

(c) If B is a finite set, then $\alpha(B) = 0$.
(d) $\alpha(B) = \alpha(\overline{B}) = \alpha(convB)$, where $convB$ is the convex hull of B.
(e) monotonicity : $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$.
(f) algebraic semi-additivity : $\alpha(A + B) \leq \alpha(A) + \alpha(B)$, where

$$A + B = \{x + y : x \in A, \quad y \in B\}.$$

(g) semi-homogencity : $\alpha(\lambda B) = |\lambda|\alpha(B)$; $\lambda \in \mathbb{R}$, where $\lambda(B) = \{\lambda x : x \in B\}$.
(h) semi-additivity : $\alpha(A \bigcup B) = max\{\alpha(A), \alpha(B)\}$.
(i) $\alpha(A \bigcap B) = min\{\alpha(A), \alpha(B)\}$.
(j) invariance under translations : $\alpha(B + x_0) = \alpha(B)$ for any $x_0 \in E$.

Remark 1.4.5 *The α-measure of noncompactness was introduced by Kuratowski in order to generalize the Cantor intersection theorem*

Theorem 1.4.6 *([101]) Let (X, d) be a complete metric space and $\{B_n\}$ be a decreasing sequence of nonempty, closed and bounded subsets of X and $\lim_{n\to\infty} \alpha(B_n) = 0$. Then the intersection B_∞ of all B_n is nonempty and compact.*

In [89], Horvath has proved the following generalization of the Kuratowski theorem :

Theorem 1.4.7 *([101]) Let (X, d) be a complete metric space and $\{B_i\}_{i\in I}$ be a family of nonempty of closed and bounded subsets of X having the finite intersection property. If $\inf_{i\in I} \alpha(B_i) = 0$ then the intersection B_∞ of all B_i is nonempty and compact.*

Lemma 1.4.8 *([81]) If $V \subset C(J, E)$ is a bounded and equicontinuous set, then*
(i) the function $t \to \alpha(V(t))$ is continuous on J, and

$$\alpha_c(V) = \sup_{0\leq t\leq T} \alpha(V(t)).$$

(ii) $\alpha\left(\int_0^T x(s)ds : x \in V\right) \leq \int_0^T \alpha(V(s))ds,$
where

$$V(s) = \{x(s) : x \in V\}, \ s \in J.$$

In the definition of the Kuratowski measure we can consider balls instead of arbitrary sets. In this way we get the definition of the Hausdorff measure :

Definition 1.4.9 *([101]) The Hausdorff measure of noncompactness $\chi(B)$ of the set B is the infimum of the numbers r such that B admits a finite covering by balls of radius smaller than r.*

Theorem 1.4.10 *([101]) Let $B(0, 1)$ be the unit ball in a Banach space X. Then*

$$\alpha(B(0, 1)) = \chi(B(0, 1)) = 0$$

if X is finite dimensional, and $\alpha(B(0, 1)) = 2$, $\chi(B(0, 1)) = 1$ otherwise.

Theorem 1.4.11 *([101]) Let $S(0,1)$ be the unit sphere in a Banach space X. Then $\alpha(S(0,1)) = \chi(S(0,1)) = 0$ if X is finite dimensional, and $\alpha(S(0,1)) = 2$, $\chi(S(0,1)) = 1$ otherwise.*

Theorem 1.4.12 *([101]) The Kuratowski and Hausdorff MNCs are related by the inequalities*

$$\chi(B) \leq \alpha(B) \leq 2\chi(B).$$

In the class of all infinite dimensional Banach spaces these inequalities are the best possible.

Example 1.4.13 *Let l^∞ be the space of all real bounded sequences with the supremum norm, and let A be a bounded set in l^∞. Then $\alpha(A) = 2\chi(A)$.*

For further facts concerning measures of noncompactness and their properties we refer to [18, 35, 36, 101, 135] and the references therein.

1.5 Some fixed point theorems

Theorem 1.5.1 (Banach's fixed point theorem (1922) [80]) *Let C be a non-empty closed subset of a Banach space X, then any contraction mapping T of C into itself has a unique fixed point.*

Theorem 1.5.2 (Schaefer's fixed point theorem [80]) *Let X be a Banach space, and $N : X \longrightarrow X$ completely continuous operator.*
If the set $\mathcal{E} = \{y \in X : \ y = \lambda N y, \ for some\ \lambda \in (0,1)\}$ is bounded, then N has fixed points.

Theorem 1.5.3 (Darbo's Fixed Point Theorem [77, 80]) *Let X be a Banach space and C be a bounded, closed, convex and nonempty subset of X. Suppose a continuous mapping $T : C \to C$ is such that for all closed subsets D of C,*

$$\alpha(T(D)) \leq k\alpha(D), \tag{1.1}$$

where $0 \leq k < 1$, and α is the Kuratowski measure of noncompactness. Then T has a fixed point in C.

Remark 1.5.4 *Mappings satisfying the Darbo-condition (1.1) have subsequently been called k-set contractions.*

Theorem 1.5.5 (Mönch's Fixed Point Theorem [9, 115]) *Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv} N(V) \quad or \quad V = N(V) \cup \{0\} \Rightarrow \alpha(V) = 0$$

holds for every subset V of D, then N has a fixed point.
Here α is the Kuratowski measure of noncompactness.

For more details see [9, 23, 78, 80, 101, 146]

Chapitre 2

Existence and Stability Results for Nonlinear BVP for Implicit Differential Equations of Fractional Order [1]

2.1 Introduction and Motivations

The purpose of this chapter, is to establish existence, uniqueness and stability results to the followings implicit fractional-order differential equations :

$$^cD^\alpha y(t) = f(t, y(t), ^cD^\alpha y(t)), \text{ for each } t \in J = [0, T], T > 0, \ 0 < \alpha \leq 1, \tag{2.1}$$

$$ay(0) + by(T) = c \tag{2.2}$$

where $^cD^\alpha$ is the fractional derivative of Caputo, $f : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, and a, b, c are real constants with $a + b \neq 0$.
and

$$^cD^\alpha y(t) = f(t, y(t), ^cD^\alpha y(t)), \text{ for each } t \in J = [0, T], T > 0, \ 0 < \alpha \leq 1, \tag{2.3}$$

$$y(0) + g(y) = y_0, \tag{2.4}$$

where $^cD^\alpha$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $g : C(J, \mathbb{R}) \to \mathbb{R}$ is a continuous function and $y_0 \in \mathbb{R}$.
In [52], Benchohra *et al.* studied the existence of solutions for boundary value problems, for following implicit fractional-order differential equation :

$$^cD^\alpha y(t) = f(t, y(t)), \text{ for each } t \in J = [0, T], T > 0, \ 0 < \alpha \leq 1,$$

1. M.Benchohra and S.Bouriah, Existence and Stability Results for Nonlinear Boundary Value Problem for Implicit Differential Equation of Fractional Order, *Moroccan Journal Pure. Appl.Anal.* **1** (1) 2015, 22-36.

$$ay(0) + by(T) = c,$$

where ${}^cD^\alpha$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \to \mathbb{R}$ is a given function and a, b, c are real constants with $a + b \neq 0$.

In [51], Benchohra and Hamani studied the existence of solutions for boundary value problems, for fractional order differential inclusions :

$${}^cD^\alpha y(t) \in F(t, y(t)) = 0, \text{ for each } t \in J = [0, T], \ 0 < \alpha \leq 1,$$

$$ay(0) + by(T) = c,$$

where ${}^cD^\alpha$ is the Caputo fractional derivative, $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a given multivalued function and a, b, c are real constants with $a + b \neq 0$.

In [54], by means of Krasnoselskii fixed-point theorem in cones, Benchohra and Hedia studied the existence of nonlinear fractional boundary value problem involving Caputo's derivative :

$${}^cD^\alpha y(t) + \varphi(t) f(t, y(t)) = 0, \text{ for each } t \in J = [0, 1], \ 0 < \alpha \leq 1,$$

$$ay(0) + by(1) = c,$$

where ${}^cD^\alpha$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \to [0, \infty)$ is a given function and a, b, c are real constants with $a + b \neq 0$, and $\varphi : [0, 1] \to \mathbb{R}$ is a given function.

In [97], Karthikeyan and Trujillo studied the existence of nonlinear fractional boundary value problem :

$${}^cD^\alpha y(t) = \varphi(t) f(t, y(t), (Sy)(t)), \text{ for each } t \in J = [0, T], \ 0 < \alpha < 1,$$

$$ay(0) + by(T) = c,$$

where ${}^cD^\alpha$ is the Caputo fractional derivative of order α, $f : J \times X \times X \to X$ is a given function, X is a Banach space and a, b, c are real constants with $a + b \neq 0$, and S is a nonlinear integral operator given by

$$(Sy)(t) = \int_0^t k(t, s) y(s) ds.$$

where $k \in C(J \times J, \mathbb{R}^+)$.
Fractional differential equations with nonlocal conditions have been discussed in [8, 10, 73, 82, 110, 117, 118] and references therein. Nonlocal conditions were initiated by Byszewski [65] when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. As remarked by Byszewski [63, 64], the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena. For example, in [71], the author used

$$g(y) = \sum_{i=1}^{p} c_i y(\tau_i) \tag{2.5}$$

where $c_i,\ i = 1, \ldots, p$, are given constants and $0 < \tau_1 < ... < \tau_p \leq T$, to describe the diffusion phenomenon of a small amount of gas in a transparent tube. In this case, (2.5) allows the additional measurements at $\tau_i,\ i = 1, \ldots, p$.

2.2 Existence of solutions

By $C(J, \mathbb{R})$ we denote the Banach space of continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\}.$$

Let us defining what we mean by a solution of problem (2.1) – (2.2) and (2.3) – (2.4).

Definition 2.2.1 *A function $u \in C^1(J, \mathbb{R})$ is said to be a solution of the problem* (2.1) – (2.2) *if u satisfied equation* (2.1) *and conditions* (2.2) *on J, and a function $y \in C^1(J, \mathbb{R})$ is called a solution of the problem* (2.3) – (2.4) *if y satisfied equation* (2.3) *and conditions* (2.4) *on J.*

For the existence of solutions for the problems (2.1) – (2.2) and (2.3) – (2.4), we need the following auxiliary lemmas :

Lemma 2.2.2 *Let $0 < \alpha \leq 1$ and $h : [0, T] \longrightarrow \mathbb{R}$ be a continuous function. Then the linear problem*

$$^cD^\alpha y(t) = h(t),\ t \in J \tag{2.6}$$

$$ay(0) + by(T) = c, \tag{2.7}$$

has a unique solution which is given by :

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds \\ &- \frac{1}{a+b} \left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s)ds - c \right]. \end{aligned} \tag{2.8}$$

Proof. By integration of formula (2.6) we obtain :

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds. \tag{2.9}$$

We use condition (2.7) to compute the constant y_0, so we have :

$$ay(0) = ay_0 \ \ and \ \ by(T) = by_0 + \frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s)ds,$$

then, $ay(0) + by(T) = c$, since

$$y_0 = \frac{-1}{(a+b)} \left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s)ds - c \right].$$

Substituting in equation (2.9)leads formula (2.8).

□

Lemma 2.2.3 *Let $f(t,u,v) : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function, then the problem* (2.1)-(2.2) *is equivalent to the problem :*

$$y(t) = \tilde{A} + I^{\alpha} g(t) \tag{2.10}$$

where $g \in C(J, \mathbb{R})$ satisfies the functional equation

$$g(t) = f(t, \tilde{A} + I^{\alpha} g(t), g(t))$$

and

$$\tilde{A} = \frac{1}{a+b}\left[c - \frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} g(s)ds\right].$$

Proof. Let y be solution of (2.10). We shall show that y is solution of (2.1)$-$(2.2). We have

$$y(t) = \tilde{A} + I^{\alpha} g(t).$$

So, $y(0) = \tilde{A}$ and $y(T) = \tilde{A} + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} g(s)ds$.

$$\begin{aligned} ay(0) + by(T) &= \frac{-ab}{(a+b)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} g(s)ds \\ &\quad + \frac{ac}{a+b} - \frac{b^2}{(a+b)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} g(s)ds \\ &\quad + \frac{bc}{a+b} + \frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} g(s)ds. \\ &= c. \end{aligned}$$

Then

$$ay(0) + by(T) = c.$$

On the other hand, we have

$$\begin{aligned} {}^cD^{\alpha} y(t) &= {}^cD^{\alpha}(\tilde{A} + I^{\alpha} g(t)) = g(t) \\ &= f(t, y(t), {}^cD^{\alpha} y(t)). \end{aligned}$$

Thus, y is solution of problem (2.1)-(2.2).

□

Lemma 2.2.4 *Let $0 < \alpha \leq 1$ and let $h : [0,T] \longrightarrow \mathbb{R}$ a continuous function. The linear problem*

$${}^cD^{\alpha} y(t) = h(t), \quad t \in J$$

$$y(0) + g(y) = y_0$$

has a unique solution which is given by :

$$y(t) = y_0 - g(y) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s)ds$$

Lemma 2.2.5 *Let $f : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function, then the problem (2.3)-(2.4) is equivalent to the following equation*

$$y(t) = y_0 - g(y) + I^{\alpha} K_y(t)$$

where $K_y \in C(J, \mathbb{R})$

$$K_y(t) = f(t, y(t), K_y(t)).$$

Theorem 2.2.6 *Assume assumption*

(H1) there exist two constants $K > 0$ et $0 < L < 1$ such that

$$|f(t,u,v) - f(t,\overline{u},\overline{v})| \leq K\,|u-\overline{u}| + L\,|v-\overline{v}| \quad \text{for each } t \in J \text{ and } u, \overline{u}, v, \overline{v} \in \mathbb{R}.$$

If

$$\frac{KT^{\alpha}}{(1-L)\,\Gamma(\alpha+1)}\left(1+\frac{|b|}{|a+b|}\right) < 1, \tag{2.11}$$

the problem (2.1)-(2.2) has a unique solution.

Proof. Let the operator

$$\begin{aligned} N \quad &: \quad C(J,\mathbb{R}) \longrightarrow C(J,\mathbb{R}) \\ Ny(t) \quad &= \quad \tilde{A}_y + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g_y(s)ds, \end{aligned}$$

where

$$g_y(t) = f(t, \tilde{A}_y + I^{\alpha} g_y(t), g_y(t)),$$

and

$$\tilde{A}_y = \frac{1}{a+b}\left[c - \frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} g_y(s)ds\right].$$

By Lemmas 2.2.2 and 2.2.3, it is clear that the fixed points of N are solutions of (2.1)-(2.2).

Let $y_1, y_2 \in C(J,\mathbb{R})$, and $t \in J$, then we have

$$\begin{aligned} |Ny_1(t) - Ny_2(t)| \;\leq\; & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|g_{y_1}(s) - g_{y_2}(s)\right| ds \\ & + \frac{|b|}{|a+b|\,\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\left|g_{y_1}(s) - g_{y_2}(s)\right| ds, \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} |g_{y_1}(t) - g_{y_2}(t)| \;&=\; |f(t, y_1(t), g_{y_1}(t)) - f(t, y_2(t), g_{y_2}(t))| \\ &\leq\; K\,|y_1(t) - y_2(t)| + L\left|g_{y_1}(t) - g_{y_2}(t)\right|. \end{aligned}$$

Thus

$$\left|g_{y_1}(t) - g_{y_2}(t)\right| \leq \frac{K}{1-L}\left|y_1(t) - y_2(t)\right|. \tag{2.13}$$

By replacing (2.13) in the inequality (2.12), we obtain

$$\begin{aligned}
|Ny_1(t) - Ny_2(t)| &\leq \frac{K}{(1-L)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|y_1(s) - y_2(s)\right| ds \\
&\quad + \frac{|b|K}{(1-L)|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\left|y_1(s) - y_2(s)\right| ds \\
&\leq \frac{KT^\alpha}{(1-L)\Gamma(\alpha+1)}\left\|y_1 - y_2\right\|_\infty \\
&\quad + \frac{|b|KT^\alpha}{(1-L)|a+b|\Gamma(\alpha+1)}\left\|y_1 - y_2\right\|_\infty.
\end{aligned}$$

Then

$$\left\|Ny_1 - Ny_2\right\|_\infty \leq \left[\frac{KT^\alpha}{(1-L)\Gamma(\alpha+1)}\left(1 + \frac{|b|}{|a+b|}\right)\right]\left\|y_1 - y_2\right\|_\infty.$$

From (2.11), it follows that N has a unique fixed point which is solution of problem (2.1)-(2.2).

□

Theorem 2.2.7 *Assume*

(P1) there exist $K > 0$, $0 < \overline{K} < 1$ and $0 < L < 1$ such that :

$$|f(t,u,v) - f(t,\overline{u},\overline{v})| \leq K|u - \overline{u}| + \overline{K}|v - \overline{v}| \text{ for any } u,\overline{u},v,\overline{v} \in \mathbb{R}$$

and

$$\|g(y) - g(\overline{y})\| \leq L\|y - \overline{y}\| \text{ for any } y,\overline{y} \in C(J,\mathbb{R}).$$

If

$$L + \frac{KT^\alpha}{(1-\overline{K})\Gamma(\alpha+1)} < 1 \tag{2.14}$$

then, the boundary value problem (2.3) -(2.4) has a unique solution on J.

Proof. Let the operator

$$\begin{aligned}
N &: C(J,\mathbb{R}) \to C(J,\mathbb{R}) \\
Ny(t) &= y_0 - g(y) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}K_y(s)ds
\end{aligned}$$

where

$$K_y(t) = f(t, y_0 - g(y) + I^\alpha K_y(t), K_y(t)).$$

By Lemmas 2.2.4 and 2.2.5, it is easy to see that the fixed points of N are the solutions of the problem (2.3)-(2.4). Let $y_1, y_2 \in C(J, \mathbb{R})$, we have for any $t \in J$

$$|Ny_1(t) - Ny_2(t)| \leq |g(y_1) - g(y_2)| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |K_{y_1}(s) - K_{y_2}(s)| \, ds$$

then

$$\begin{aligned} |Ny_1(t) - Ny_2(t)| &\leq L|y_1(t) - y_2(t)| \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |K_{y_1}(s) - K_{y_2}(s)| \, ds. \end{aligned} \tag{2.15}$$

On the other hand, we have for every $t \in J$

$$\begin{aligned} |K_{y_1}(t) - K_{y_2}(t)| &= |f(t, y_1(t), K_{y_1}(t)) - f(t, y_2(t), K_{y_2}(t))| \\ &\leq K|y_1(t) - y_2(t)| + \overline{K}|K_{y_1}(t) - K_{y_2}(t)|. \end{aligned}$$

Thus

$$|K_{y_1}(t) - K_{y_2}(t)| \leq \frac{K}{1-\overline{K}} |y_1(t) - y_2(t)|. \tag{2.16}$$

By replacing (2.16) in the inequality (2.15), we obtain

$$\begin{aligned} |Ny_1(t) - Ny_2(t)| &\leq L|y_1(t) - y_2(t)| \\ &\quad + \frac{K}{\left(1-\overline{K}\right)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |y_1(s) - y_2(s)| \\ &\leq \left[L + \frac{KT^\alpha}{\left(1-\overline{K}\right)\Gamma(\alpha+1)}\right] \|y_1 - y_2\|_\infty. \end{aligned}$$

Thus

$$\|Ny_1 - Ny_2\|_\infty \leq \left[L + \frac{KT^\alpha}{\left(1-\overline{K}\right)\Gamma(\alpha+1)}\right] \|y_1 - y_2\|_\infty$$

from which it follows that N is a contraction which implies that N admits a unique fixed point which is solution of the problem (2.3) -(2.4).

□

2.3 Ulam-Hyers Rassias stability

For the implicit fractional-order differential equation (2.1), we adopt the definition in Rus [129] for : Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability.

Definition 2.3.1 *The equation* (2.1) *is Ulam-Hyers stable if there exists a real number* $c_f > 0$ *such that for each* $\epsilon > 0$ *and for each solution* $z \in C^1(J, \mathbb{R})$ *of the inequality*

$$|^c D^\alpha z(t) - f(t, z(t), ^c D^\alpha z(t))| \le \epsilon, \ t \in J,$$

there exists a solution $y \in C^1(J, \mathbb{R})$ *of equation* (2.1) *with*

$$|z(t) - y(t)| \le c_f \epsilon, \ t \in J.$$

Definition 2.3.2 *The equation* (2.1) *is generalized Ulam-Hyers stable if there exists* $\psi_f \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\psi_f(0) = 0$, *such that for each solution* $z \in C^1(J, \mathbb{R})$ *of the inequality*

$$|^c D^\alpha z(t) - f(t, z(t), ^c D^\alpha z(t))| \le \epsilon, \ t \in J,$$

there exists a solution $y \in C^1(J, \mathbb{R})$ *of the equation* (2.1) *with*

$$|z(t) - y(t)| \le \psi_f(\epsilon), t \in J.$$

Definition 2.3.3 *The equation* (2.1) *is Ulam-Hyers-Rassias stable with respect to* $\varphi \in C(J, \mathbb{R}_+)$ *if there exists a real number* $c_f > 0$ *such that for each* $\epsilon > 0$ *and for each solution* $z \in C^1(J, \mathbb{R})$ *of the inequality*

$$|^c D^\alpha z(t) - f(t, z(t), ^c D^\alpha z(t))| \le \epsilon \varphi(t), \ t \in J,$$

there exists a solution $y \in C^1(J, \mathbb{R})$ *of equation* (2.1) *with*

$$|z(t) - y(t)| \le c_f \epsilon \varphi(t), \ t \in J.$$

Definition 2.3.4 *The equation* (2.1) *is generalized Ulam-Hyers-Rassias stable with respect to* $\varphi \in C(J, \mathbb{R}_+)$ *if there exists a real number* $c_{f,\varphi} > 0$ *such that for each solution* $z \in C^1(J, \mathbb{R})$ *of the inequality*

$$|^c D^\alpha z(t) - f(t, z(t), ^c D^\alpha z(t))| \le \varphi(t), \ t \in J,$$

there exists a solution $y \in C^1(J, \mathbb{R})$ *of equation* (2.1) *with*

$$|z(t) - y(t)| \le c_{f,\varphi} \varphi(t), \ t \in J.$$

Remark 2.3.5 *A function* $z \in C^1(J, \mathbb{R})$ *is a solution of the inequality*

$$|^c D^\alpha z(t) - f(t, z(t), ^c D^\alpha z(t))| \le \epsilon, \ t \in J,$$

if and only if there exists a function $g \in C(J, \mathbb{R})$ *(which depends on solution* y*) such that*

i) $|g(t)| \le \epsilon$, $\forall t \in J$.

ii) $^c D^\alpha z(t) = f(t, z(t), ^c D^\alpha z(t)) + g(t)$, $t \in J$.

Remark 2.3.6 *Clearly,*

i) *Definition* 2.3.1 $\Rightarrow$ *Definition* 2.3.2.

ii) *Definition* 2.3.3 $\Rightarrow$ *Definition* 2.3.4.

Remark 2.3.7 *A solution of the implicit differential inequality*

$$|^{c}D^{\alpha}z(t) - f(t, z(t),^{c} D^{\alpha}z(t))| \leq \epsilon,\ t \in J,$$

with fractional order is called an fractional $\epsilon-$*solution of the implicit fractional differential equation* (2.1).

Theorem 2.3.8 *Assume that* $(H1)$ *and (2.11) are satisfied, then the problem (2.1)-(2.2) is Ulam-Hyers stable.*

Proof. Let $\epsilon > 0$ and let $z \in C^{1}(J, \mathbb{R})$ be a function which satisfies the inequality :

$$|^{c}D^{\alpha}z(t) - f(t, z(t),^{c} D^{\alpha}z(t))| \leq \epsilon \quad \text{for any } t \in J \tag{2.17}$$

and let $y \in C(J, \mathbb{R})$ be the unique solution of the following Cauchy problem

$$\begin{cases} {}^{c}D^{\alpha}y(t) = f(t, y(t),^{c} D^{\alpha}y(t))\,;\ \ t \in J;\ 0 < \alpha \leq 1 \\ y(0) = z(0), y(T) = z(T). \end{cases}$$

Using Lemmas 2.2.2 and 2.2.3, we obtain

$$y(t) = \tilde{A}_{y} + \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} g_{y}(s) ds.$$

On the other hand, if $y(T) = z(T)$ and $y(0) = z(0)$, then $\tilde{A}_{y} = \tilde{A}_{z}$. Indeed

$$\left|\tilde{A}_{y} - \tilde{A}_{z}\right| \leq \frac{|b|}{|a+b|\,\Gamma(\alpha)} \int_{0}^{T} (T-s)^{\alpha-1} |g_{y}(s) - g_{z}(s)|\, ds,$$

and by the inequality (2.13), we find

$$\begin{aligned} \left|\tilde{A}_{y} - \tilde{A}_{z}\right| &\leq \frac{|b|\,K}{(1-L)\,|a+b|\,\Gamma(\alpha)} \int_{0}^{T} (T-s)^{\alpha-1} |y(s) - z(s)|\, ds \\ &= \frac{|b|\,K}{(1-L)\,|a+b|} I^{\alpha} |y(T) - z(T)| = 0. \end{aligned}$$

Thus

$$\tilde{A}_{y} = \tilde{A}_{z}.$$

Thus, we have

$$y(t) = \tilde{A}_{z} + \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} g_{y}(s)\, ds.$$

By integration of the inequality (2.17), we obtain

$$\left|z(t)-\tilde{A}_z-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g_z(s)\,ds\right| \le \frac{\epsilon t^\alpha}{\Gamma(\alpha+1)} \le \frac{\epsilon T^\alpha}{\Gamma(\alpha+1)},$$

with

$$g_z(t)=f(t,\tilde{A}_z+I^\alpha g_z(t),g_z(t)).$$

We have for any $t\in J$

$$\begin{aligned}|z(t)-y(t)| &= \left|z(t)-\tilde{A}_z-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g_z(s)\,ds\right.\\ &\quad\left.+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(g_z(s)-g_y(s))\,ds\right|\\ &\le \left|z(t)-\tilde{A}_z-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g_z(s)\,ds\right|\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|g_z(s)-g_y(s)|\,ds.\end{aligned}$$

Using (2.13), we obtain

$$|z(t)-y(t)|\le\frac{\epsilon T^\alpha}{\Gamma(\alpha+1)}+\frac{K}{(1-L)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|z(s)-y(s)|\,ds,$$

and by the Gronwall's Lemma, we get

$$|z(t)-y(t)|\le\frac{\epsilon T^\alpha}{\Gamma(\alpha+1)}\left[1+\frac{\gamma K T^\alpha}{(1-L)\Gamma(\alpha+1)}\right]:=c\epsilon$$

where $\gamma=\gamma(\alpha)$ a constant, which completes the proof of the theorem. Moreover, if we set $\psi(\epsilon)=c\epsilon\,;\ \psi(0)=0$, then the problem (2.1)-(2.2) is generalized Ulam-Hyers stable.

□

Theorem 2.3.9 *Assume that (H1), (2.11) and*

(H2) there exists an increasing function $\varphi\in C(J,\mathbb{R}_+)$ and there exists $\lambda_\varphi>0$ such that for any $t\in J$

$$I^\alpha\varphi(t)\le\lambda_\varphi\varphi(t)$$

are satisfied, then, the problem (2.1)-(2.2) is Ulam-Hyers-Rassias stable.

Proof. Let $z\in C^1(J,\mathbb{R})$ be solution of the following inequality

$$|{}^cD^\alpha z(t)-f(t,z(t),{}^cD^\alpha z(t))|\le\epsilon\varphi(t)\ ,\ t\in J\ ,\ \epsilon>0 \tag{2.18}$$

and let $y \in C(J, \mathbb{R})$ be the unique solution of Cauchy problem :

$$\begin{cases} {}^cD^\alpha y(t) = f(t, y(t), {}^c D^\alpha y(t))\,; \;\; t \in J\,; \;\; 0 < \alpha \leq 1 \\ y(0) = z(0), \;\; y(T) = z(T). \end{cases}$$

By Lemmas 2.2.2 and 2.2.3, we have

$$y(t) = \tilde{A}_z + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_y(s)\, ds,$$

where $g_y \in C(J, \mathbb{R})$ satisfies the equation :

$$g_y(t) = f(t, \tilde{A}_z + I^\alpha g_y(t), g_y(t)),$$

and

$$\tilde{A}_z = \frac{1}{a+b}\left[c - \frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} g_z(s) ds\right].$$

By integration of (2.18), we obtain

$$\begin{aligned} \left| z(t) - \tilde{A}_z - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_z(s)\, ds \right| &\leq \frac{\epsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s)\, ds \\ &\leq \epsilon \lambda_\varphi \varphi(t). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} |z(t) - y(t)| &= \left| z(t) - \tilde{A}_z - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_z(s)\, ds \right. \\ &\quad \left. + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (g_z(s) - g_y(s))\, ds \right| \\ &\leq \left| z(t) - \tilde{A}_z - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_z(s)\, ds \right| \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |g_z(s) - g_y(s)|\, ds. \end{aligned}$$

Using (2.13), we have

$$|z(t) - y(t)| \leq \epsilon \lambda_\varphi \varphi(t) + \frac{K}{(1-L)\,\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z(s) - y(s)|\, ds.$$

By applying Gronwall's Lemma, we get that for any $t \in J$:

$$|z(t) - y(t)| \leq \epsilon \lambda_\varphi \varphi(t) + \frac{\gamma_1 \epsilon K \lambda_\varphi}{(1-L)\,\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s)\, ds$$

where $\gamma_1 = \gamma_1(\alpha)$ is constant, and by (H_2), we have :

$$|z(t)-y(t)| \leq \epsilon\lambda_\varphi\varphi(t) + \frac{\gamma_1\epsilon K\lambda_\varphi^2\varphi(t)}{(1-L)} = \left(1+\frac{\gamma_1 K\lambda_\varphi}{(1-L)}\right)\epsilon\lambda_\varphi\varphi(t).$$

Then for any $t \in J$:

$$|z(t)-y(t)| \leq \left[\left(1+\frac{\gamma_1 K\lambda_\varphi}{1-L}\right)\lambda_\varphi\right]\epsilon\varphi(t) = c\epsilon\varphi(t)$$

Which completes the proof of Theorem 2.3.9.

□

Theorem 2.3.10 *Assume that $(P1)$ and the inequality (2.14) are satisfied, then the problem (2.3)-(2.4) is Ulam-Hyers stable.*

Proof. Let $\epsilon > 0$ and let $z \in C^1(J,\mathbb{R})$ satisfying the inequality :

$$|{}^cD^\alpha z(t) - f(t, z(t), {}^cD^\alpha z(t))| \leq \epsilon \text{ for every } t \in J, \tag{2.19}$$

and let $y \in C(J,\mathbb{R})$ the unique solution of the Cauchy problem :

$$\begin{cases} {}^cD^\alpha y(t) = f(t, y(t), {}^cD^\alpha y(t)), \quad t \in J, \quad 0 < \alpha \leq 1 \\ z(0) + g(y) = y_0 \end{cases}$$

so

$$y(t) = y_0 - g(y) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} K_y(s)\,ds,$$

where

$$K_y(t) = f(t, y(t), K_y(t)).$$

By integration of the inequality (2.19), we find

$$\left|z(t) - y_0 + g(z) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} K_z(s)\,ds\right| \leq \frac{\epsilon T^\alpha}{\Gamma(\alpha+1)}$$

where $K_z(t) = f(t, z(t), K_z(t))$. For every $t \in J$, we have :

$$\begin{aligned} |z(t)-y(t)| &\leq \left|z(t) - y_0 + g(z) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} K_z(s)\,ds\right| \\ &\quad + \left|g(y) - g(z) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(K_z(s) - K_y(s))\,ds\right| \\ &\leq \frac{\epsilon T^\alpha}{\Gamma(\alpha+1)} + |g(z)-g(y)| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|K_z(s) - K_y(s)|\,ds. \end{aligned}$$

Using (2.16), we obtain

$$|z(t)-y(t)| \leq \frac{\epsilon T^{\alpha}}{\Gamma(\alpha+1)} + L|z(t)-y(t)| + \frac{K}{\left(1-\overline{K}\right)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|z(s)-y(s)|\,ds$$

thus

$$|z(t)-y(t)| \leq \frac{\epsilon T^{\alpha}}{(1-L)\Gamma(\alpha+1)} + \frac{K}{(1-L)\left(1-\overline{K}\right)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|z(s)-y(s)|\,ds.$$

Using Gronwall's Lemma, we obtain for every $t \in J$:

$$|z(t)-y(t)| \leq \frac{\epsilon T^{\alpha}}{(1-L)\Gamma(\alpha+1)}\left[1+\frac{\gamma K T^{\alpha}}{(1-L)\left(1-\overline{K}\right)\Gamma(\alpha+1)}\right] := c\epsilon$$

where $\gamma = \gamma(\alpha)$ a constant, so the problem (2.3)-(2.4) is Ulam-Hyers stable. If we set $\psi(\epsilon) = c\epsilon$; $\psi(0) = 0$, then the problem (2.3)$-$(2.4) is generalized Ulam-Hyers stable .

□

Theorem 2.3.11 *Assume that* $(P1)$*, the inequality* (2.14) *and*

(P2) there exists an increasing function $\varphi \in C(J, \mathbb{R}_+)$ *and* $\lambda_{\varphi} > 0$ *such that*

$$I^{\alpha}\varphi(t) \leq \lambda_{\varphi}\varphi(t) \text{ for each } t \in J$$

are satisfied, then the problem (2.3)-(2.4) *is Ulam-Hyers-Rassias stable.*

2.4 Examples

Example 1. Consider the following boundary value problem

$$^{c}D^{\frac{1}{2}}y(t) = \frac{1}{10e^{t+2}(1+|y(t)|+|^{c}D^{\frac{1}{2}}y(t)|)}, \text{ for each } t \in [0,1] \tag{2.20}$$

$$y(0)+y(1)=0. \tag{2.21}$$

Set

$$f(t,u,v) = \frac{1}{10e^{t+2}(1+|u|+|v|)}, \quad t \in [0,1],\ u,v \in \mathbb{R}.$$

Clearly, the function f is continuous.
For any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in [0,1]$

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \leq \frac{1}{10e^2}(|u-\bar{u}|+|v-\bar{v}|).$$

Hence condition $(H1)$ is satisfied with $K = L = \frac{1}{10e^2}$.
Thus condition

$$\frac{KT^{\alpha}}{(1-L)\Gamma(\alpha+1)}\left(1+\frac{|b|}{|a+b|}\right) = \frac{3}{2(10e^2-1)\Gamma(\frac{3}{2})} = \frac{3}{(10e^2-1)\sqrt{\pi}} < 1,$$

is satisfied with $a = b = T = 1$, $c = 0$, and $\alpha = \frac{1}{2}$. It follows from Theorem 2.2.6 that the problem (2.20)-(2.21) has a unique solution on J, Theorem 2.3.8 implies that the problem (2.20)-(2.21) is Ulam-Hyers stable.

Example 2. Consider the boundary value problem :

$$^{c}D^{\frac{1}{2}}y(t) = \frac{e^{-t}}{(9+e^t)}\left[\frac{|y(t)|}{1+|y(t)|} - \frac{\left|^{c}D^{\frac{1}{2}}y(t)\right|}{1+\left|^{c}D^{\frac{1}{2}}y(t)\right|}\right], \quad t \in J = [0,1] \tag{2.22}$$

$$y(0) + \sum_{i=1}^{n} c_i y(t_i) = 1, \tag{2.23}$$

where $0 < t_1 < t_2 < ... < t_n < 1$ and $c_i = 1, ..., n$ are positive constants with

$$\sum_{i=1}^{n} c_i < \frac{1}{3}.$$

Set

$$f(t,u,v) = \frac{e^{-t}}{(9+e^t)}\left[\frac{u}{1+u} - \frac{v}{1+v}\right], \ t \in [0,1], \ u,v \in [0,+\infty).$$

Clearly, the function f is continuous. For each $u, \bar{u}, v, \bar{v} \in \mathbb{R}$ and $t \in [0,1]$:

$$\begin{aligned}
|f(t,u,v) - f(t,\bar{u},\bar{v})| &\leq \frac{e^{-t}}{(9+e^t)}(|u-\bar{u}| + |v-\bar{v}|) \\
&\leq \frac{1}{10}|u-\bar{u}| + \frac{1}{10}|v-\bar{v}|.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
|g(u) - g(\bar{u})| &= \left|\sum_{i=1}^{n} c_i u - \sum_{i=1}^{n} c_i \bar{u}\right| \\
&\leq \sum_{i=1}^{n} c_i |u-\bar{u}| \\
&< \frac{1}{3}|u-\bar{u}|.
\end{aligned}$$

Hence condition $(P1)$ is satisfied with $K = \overline{K} = \dfrac{1}{10}$ and $L = \dfrac{1}{3}$. We have

$$L + \frac{KT^{\alpha}}{\left(1 - \overline{K}\right)\Gamma\left(\alpha + 1\right)} = \frac{1}{3} + \frac{1}{9\Gamma\left(\dfrac{3}{2}\right)} = \frac{9\sqrt{\pi} + 6}{27\sqrt{\pi}} < 1.$$

It follows from Theorem 2.2.7 that the problem (2.22)- (2.23) has a unique solution on J and by Theorem 2.3.10, the problem (2.22)-(2.23) is Ulam-Hyers stable.

Remark 2.4.1 *The main results of **Example 2** stay available when*

$$g(t) = \frac{1}{4}\left(\frac{|y(t)|}{1 + |y(t)|}\right)$$

and

$$L + \frac{KT^{\alpha}}{\left(1 - \overline{K}\right)\Gamma\left(\alpha + 1\right)} = \frac{1}{4} + \frac{1}{9\Gamma\left(\dfrac{3}{2}\right)} = \frac{9\sqrt{\pi} + 8}{36\sqrt{\pi}} < 1.$$

Chapitre 3

Existence Results for Nonlinear BVP for Implicit Fractional Differential Equations in Banach Space [1]

3.1 Introduction and Motivations

The purpose of this chapter, is to establish existence and uniqueness results to the followings problems of implicit fractional differential equations in Banach Space :

$$^cD^\nu y(t) = f(t, y(t), ^cD^\nu y(t)), \text{ for each, } t \in J := [0, T], \ T > 0, \ 0 < \nu \leq 1,$$

$$ay(0) + by(T) = c,$$

where $^cD^\nu$ is the Caputo fractional derivative, $(E, ||\cdot||)$ is a real Banach space, $f : J \times E \times E \to E$ is a given function and a, b are real with $a + b \neq 0$ and $c \in E$.
and

$$^cD^\nu y(t) = f(t, y(t), ^cD^\nu y(t)), \quad \text{for every } t \in J := [0, T], T > 0, \quad 0 < \nu \leq 1$$

$$y(0) + g(y) = y_0$$

where $^cD^\nu$ is the Caputo fractional derivative, $(E, ||\cdot||)$ is a real Banach space, $f : J \times E \times E \to E$ is a given function, $g : C(J, E) \to E$ is a continuous function and $y_0 \in E$.
Recently, fractional differential equations have been studied by Abbes *et al* [3, 6], Baleanu *et al* [32, 34], Diethelm [72], Kilbas and Marzan [99], Srivastava *et al* [100],

1. M. Benchohra, S.Bouriah and M.Darwish, Existence Results for Nonlinear BVP for Implicit Differential Equations of Fractional Order in Banach Space, *Fixed Point Theory.*

Lakshmikantham *et al* [104], Samko *et al* [132]. More recently, some mathematicians have considered boundary value problems and boundary conditions for implicit fractional differential equations.
In [91], Hu and Wang investigated the existence of solution of the nonlinear fractional differential equation with integral boundary condition :

$$D^{\alpha}u(t) = f(t, u(t), D^{\beta}u(t)),\ t \in (0,1),\ 1 < \alpha \leq 2,\ 0 < \beta < 1,$$

$$u(0) = u_0,\ u(1) = \int_0^1 g(s)u(s)ds,$$

where D^{α} is the Riemann-Liouville fractional derivative, $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, is a continuous function and g be an integrable function.
In [130], by means of Schauder fixed-point theorem, Su and Liu studied the existence of nonlinear fractional boundary value problem involving Caputo's derivative :

$${}^{c}D^{\alpha}u(t) = f(t, u(t), {}^{c}D^{\beta}u(t)),\ \text{for each } t \in (0,1),\ 1 < \alpha \leq 2,\ 0 < \beta \leq 1,$$

$$u(0) = u^{'}(1) = 0,\ \textit{or}\ u^{'}(1) = u(1) = 0, \textit{or}\ u(0) = u(1) = 0,$$

where $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Many techniques have been developed for studying the existence and uniqueness of solutions of initial and boundary value problem for fractional differential equations. Several authors tried to develop a technique that depends on the Darbo or the Mönch fixed point theorems with the Hausdorff or Kuratowski measure of noncompactness. The notion of the measure of noncompactness was defined in many ways. In 1930, Kuratowski [102] defined the measure of non-compactness, $\alpha(A)$, of a bounded subset A of a metric space (X, d), and in 1955, Darbo [70] introduced a new type of fixed point theorem for set contractions.

In this Chapter, the results are based on Darbo's fixed point theorem combined with the technique of measures of noncompactness and on Mönch's fixed point theorem.

3.2 Existence Results for the BVP in Banach Space

3.2.1 Introduction

The purpose of this Section, is to establish sufficient conditions for the existence of solutions for the following problem of implicit fractional differential equations with Caputo fractional derivative :

$${}^{c}D^{\nu}y(t) = f(t, y(t), {}^{c}D^{\nu}y(t)),\ \text{for each},\ t \in J := [0,T],\ T > 0,\ 0 < \nu \leq 1, \tag{3.1}$$

$$ay(0) + by(T) = c, \tag{3.2}$$

where ${}^cD^\nu$ is the Caputo fractional derivative, $(E, ||\cdot||)$ is a real Banach space, $f : J \times E \times E \to E$ is a given function and a, b are real with $a + b \neq 0$ and $c \in E$.
At last,As application, an example is included to show the applicability of our results.

3.2.2 Existence of Solutions

Let $(E; \|\cdot\|)$ be a valued-Banach space, and $t \in J$. We denote by $C(J, E)$ the space of E valued continuous functions on J with the usual supremum norm

$$\|y\|_\infty = \sup\{||y(t)|| : t \in J\}$$

for any $y \in C(J, E)$.
Moreover, for a given set V of functions $v : J \to E$ let us denote by

$$V(t) = \{v(t), v \in V\},\ t \in J$$

and

$$V(J) = \{v(t) : v \in V,\ t \in J\}.$$

Let us defining what we mean by a solution of problem (3.1)-(3.2).

Definition 3.2.1 *A function $y \in C^1(J, E)$ is said to be a solution of the problem (3.1)–(3.2) if y satisfied equation (3.1) on J and conditions (3.2).*

For the existence of solutions for the problem (3.1)–(3.2), we need the following auxiliary lemma :

Lemma 3.2.2 *([33]) Let $0 < \nu \leq 1$ and $h : [0, T] \longrightarrow E$ be a continuous function. The linear problem*

$$^cD^\nu y(t) = h(t),\ t \in J$$

$$ay(0) + by(T) = c,$$

has a unique solution which is given by :

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}h(s)ds \\ &- \frac{1}{a+b}\left[\frac{b}{\Gamma(\nu)}\int_0^T (T-s)^{\nu-1}h(s)ds - c\right]. \end{aligned}$$

Lemma 3.2.3 *Let $f : J \times E \times E \longrightarrow E$ be a continuous function, then the problem* (3.1)-(3.2) *is equivalent to the problem :*

$$y(t) = \tilde{A} + I^\nu g(t) \tag{3.3}$$

where $g \in C(J, E)$ satisfies the functional equation

$$g(t) = f(t, \tilde{A} + I^\nu g(t), g(t))$$

and

$$\tilde{A} = \frac{1}{a+b}\left[c - \frac{b}{\Gamma(\nu)}\int_0^T (T-s)^{\nu-1} g(s)ds\right].$$

Proof. Let y be solution of (3.3). We shall show that y is solution of (3.1)$-$(3.2). We have

$$y(t) = \tilde{A} + I^{\nu} g(t).$$

So, $y(0) = \tilde{A}$ and $y(T) = \tilde{A} + \dfrac{1}{\Gamma(\nu)}\displaystyle\int_0^T (T-s)^{\nu-1} g(s)ds.$

$$\begin{aligned}
ay(0) + by(T) &= \frac{-ab}{(a+b)\Gamma(\nu)}\int_0^T (T-s)^{\alpha-1} g(s)ds \\
&\quad + \frac{ac}{a+b} - \frac{b^2}{(a+b)\Gamma(\nu)}\int_0^T (T-s)^{\nu-1} g(s)ds \\
&\quad + \frac{bc}{a+b} + \frac{b}{\Gamma(\nu)}\int_0^T (T-s)^{\nu-1} g(s)ds. \\
&= c.
\end{aligned}$$

Then

$$ay(0) + by(T) = c.$$

On the other hand, we have

$$\begin{aligned}
{}^cD^{\nu} y(t) &= {}^cD^{\nu}(\tilde{A} + I^{\nu} g(t)) = g(t) \\
&= f(t, y(t), {}^c D^{\nu} y(t)).
\end{aligned}$$

Thus, y is solution of problem (3.1)-(3.2).

$\square$

First we list the following hypotheses :

(H1) The function $f : J \times E \times E \to E$ is continuous.

(H2) There exist constants $K > 0$ and $0 < L < 1$ such that

$$||f(t,u,v) - f(t,\bar{u},\bar{v})|| \le K||u-\bar{u}|| + L||v-\bar{v}||$$

for any $u, v, \bar{u}, \bar{v} \in E$ and $t \in J$.

Remark 3.2.4 *[25] Condition (H2) is equivalent to the inequality*

$$\alpha\Big(f(t, B_1, B_2)\Big) \le K\alpha(B_1) + L\alpha(B_2),$$

for any bounded sets $B_1, B_2 \subseteq E$ and for each $t \in J$.

Theorem 3.2.5 *Assume (H1),(H2) hold. If*

$$\frac{(|b|+|a+b|)T^{\nu}K}{|a+b|\Gamma(\nu+1)(1-L)} < 1 \tag{3.4}$$

then the IVP (3.1)-(3.2) has at least one solution on J.

This theorem will be proved in two ways : the first is based on Darbo's fixed point theorem combined with the technique of measures of noncompactness and the second on *Mönch's* fixed point theorem.

Proof 1.

Transform the problem (3.1)-(3.2) into a fixed point problem. Define the operator $N : C(J,E) \to C(J,E)$ by :

$$N(y)(t) = \tilde{A} + I^{\nu}g(t) \tag{3.5}$$

where $g \in C(J,E)$ satisfies the functional equation

$$g(t) = f(t, y(t), g(t))$$

and

$$\tilde{A} = \frac{1}{a+b}\left[c - \frac{b}{\Gamma(\nu)}\int_0^T (T-s)^{\nu-1}g(s)ds\right].$$

Clearly, the fixed points of operator N are solutions of problem (3.1)-(3.2). We shall show that N satisfies the assumption of *Darbo's* fixed point Theorem. The proof will be given in several claims.

Claim 1 : *N is continuous.*

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C(J,E)$. Then for each $t \in J$

$$\begin{aligned}||N(u_n)(t) - N(u)(t)|| &\leq \frac{|b|}{|a+b|\Gamma(\nu)}\int_0^T (T-s)^{\nu-1}\|g_n(s)-g(s)\|ds \\ &+ \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}||g_n(s)-g(s)||ds,\end{aligned} \tag{3.6}$$

where $g_n, g \in C(J,E)$ such that

$$g_n(t) = f(t, u_n(t), g_n(t)),$$

and

$$g(t) = f(t, u(t), g(t)).$$

By $(H2)$ we have, for each $t \in J$,

$$||g_n(t) - g(t)|| = ||f(t, u_n(t), g_n(t)) - f(t, u(t), g(t))||$$

$$\leq \quad K||u_n(t) - u(t)|| + L||g_n(t) - g(t)||.$$

Then

$$||g_n(t) - g(t)|| \leq \frac{K}{1-L}||u_n(t) - u(t)||.$$

Since $u_n \to u$, then we get $g_n(t) \to g(t)$ as $n \to \infty$ for each $t \in J$.
Let $\eta > 0$ be such that, for each $t \in J$, we have $||g_n(t)|| \leq \eta$ and $||g(t)|| \leq \eta$.
Then we have,

$$\begin{aligned}(t-s)^{\nu-1}||g_n(s) - g(s)|| &\leq (t-s)^{\nu-1}[||g_n(s)|| + ||g(s)||]\\ &\leq 2\eta(t-s)^{\nu-1}.\end{aligned}$$

For each $t \in J$, the function $s \to 2\eta(t-s)^{\nu-1}$ is integrable on $[0,t]$, then by means of the Lebesgue Dominated Convergence Theorem and (3.6) has that

$$||N(u_n)(t) - N(u)(t)|| \to 0 \ as \ n \to \infty.$$

Then

$$||N(u_n) - N(u)||_\infty \to 0 \ as \ n \to \infty.$$

Consequently, N is continuous.
Let the constant R such that

$$R \geq \frac{\|c\|\Gamma(\nu+1)(1-L) + (|b| + |a+b|)T^\nu f^*}{|a+b|\Gamma(\nu+1)(1-L) - (|b| + |a+b|)T^\nu K}, \tag{3.7}$$

where $f^* = \sup_{t\in J} \|f(t,0,0)\|$.
Define

$$D_R = \{u \in C(J,E) : \|u\|_\infty \leq R\}.$$

It is clear that D_R is a bounded, closed and convex subset of $C(J,E)$.

Claim 2 : $N(D_R) \subset D_R$.

Let $u \in D_R$ we show that $Nu \in D_R$. We have, for each $t \in J$

$$\begin{aligned}||Nu(t)|| &\leq \frac{\|c\|}{|a+b|} + \frac{|b|}{|a+b|\Gamma(\nu)}\int_0^T (T-s)^{\nu-1}||g(s)||ds\\ &+ \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}||g(s)||ds.\end{aligned} \tag{3.8}$$

By (H2) we have for each $t \in J$,

$$||g(t)|| \quad = \quad ||f(t,u(t),g(t)) - f(t,0,0) + f(t,0,0)||$$

$$\begin{aligned}&\leq \|f(t,u(t),g(t))-f(t,0,0)\|+\|f(t,0,0)\|\\&\leq K\|u(t)\|+L\|g(t)\|+f^*\\&\leq KR+L||g(t)||+f^*.\end{aligned}$$

Then

$$||g(t)|| \leq \frac{f^*+KR}{1-L} := M$$

Thus, (3.7) and (3.8) implies that

$$\begin{aligned}||Nu(t)|| &\leq \frac{||c||}{|a+b|}+\left[\frac{|b|}{|a+b|}+1\right]\frac{T^\nu}{\Gamma(\nu+1)}\left(\frac{f^*+KR}{1-L}\right)\\&\leq \frac{||c||}{|a+b|}+\frac{(|b|+|a+b|)T^\nu f^*}{|a+b|\Gamma(\nu+1)(1-L)}\\&+\frac{(|b|+|a+b|)T^\nu KR}{|a+b|\Gamma(\nu+1)(1-L)}\\&\leq R.\end{aligned}$$

Consequently,

$$N(D_R)\subset D_R.$$

Claim 3 : *$N(D_R)$ is bounded and equicontinuous.*

By Claim 2 we have $N(D_R)=\{N(u) : u\in D_R\}\subset D_R$. Thus, for each $u\in D_R$ we have $\|N(u)\|_\infty \leq R$ which means that $N(D_R)$ is bounded. Let $t_1,t_2\in J,\ t_1<t_2$, and let $u\in D_R$. Then

$$\begin{aligned}\|N(u)(t_2)-N(u)(t_1)\| &= \Big\|\frac{1}{\Gamma(\nu)}\int_0^{t_1}[(t_2-s)^{\nu-1}-(t_1-s)^{\nu-1}]g(s)ds\\&\quad+\frac{1}{\Gamma(\nu)}\int_{t1}^{t_2}(t_2-s)^{\nu-1}g(s)ds\Big\|\\&\leq \frac{M}{\Gamma(\nu+1)}(t_2^\nu-t_1^\nu+2(t_2-t_1)^\nu).\end{aligned}$$

As $t_1\to t_2$, the right-hand side of the above inequality tends to zero.

Claim 4 : *The operator $N : D_R\to D_R$ is a strict set contraction.*

Let $V\subset D_R$ and $t\in J$, then we have,

$$\begin{aligned}\alpha(N(V)(t)) &= \alpha((Ny)(t),y\in V)\\&\leq \frac{1}{\Gamma(\nu)}\Big\{\int_0^t(t-s)^{\nu-1}\alpha(g(s))ds, y\in V\Big\}.\end{aligned}$$

Then Remark 3.2.4 implies that, for each $s \in J$,

$$\begin{aligned}\alpha(\{g(s), y \in V\}) &= \alpha(\{f(s, y(s), g(s)), y \in V\}) \\ &\leq K\alpha(\{y(s), y \in V\}) + L\alpha(\{g(s), y \in V\}).\end{aligned}$$

Thus

$$\alpha\left(\{g(s), y \in V\}\right) \leq \frac{K}{1-L}\alpha\{y(s), y \in V\}.$$

Then

$$\begin{aligned}\alpha(N(V)(t)) &\leq \frac{K}{(1-L)\Gamma(\nu)}\left\{\int_0^t (t-s)^{\nu-1}\{\alpha(y(s))\}ds, y \in V\right\} \\ &\leq \frac{K\alpha_c(V)}{(1-L)\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}ds \\ &\leq \frac{KT^\nu}{(1-L)\Gamma(\nu+1)}\alpha_c(V).\end{aligned}$$

Therefore

$$\alpha_c(NV) \leq \frac{KT^\nu}{(1-L)\Gamma(\nu+1)}\alpha_c(V).$$

So, by (3.4), the operator N is a set contraction. As a consequence of Theorem 1.5.3, we deduce that N has a fixed point which is solution to the problem (3.1)-(3.2). This completes the proof.

□

Proof 2. Consider the operator N defined in (3.5). We shall show that N satisfies the assumption of *Mönch's* fixed point theorem. We know that $N : D_R \to D_R$ is bounded and continuous, we need to prove that the implication

$$V = \overline{conv}N(V) \quad \text{or} \quad V = N(V) \cup \{0\} \Rightarrow \alpha(V) = 0$$

holds for every subset V of D_R. Now let V be a subset of D_R such that $V \subset \overline{conv}(N(V) \cup \{0\})$. V is bounded and equicontinuous and therefore the function $t \to v(t) = \alpha(V(t))$ is continuous on J. By Remark 3.2.4, Lemma 1.4.8 and the properties of the measure α we have for each $t \in J$

$$\begin{aligned}v(t) &\leq \alpha(N(V)(t) \cup \{0\}) \\ &\leq \alpha(N(V)(t)) \\ &\leq \alpha\{(Ny)(t), y \in V\} \\ &\leq \frac{K}{(1-L)\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}\{\alpha(y(s))ds, y \in V\}\end{aligned}$$

$$\leq \frac{K}{(1-L)\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}v(s)ds.$$

Lemma 1.3.8 implies that $v(t)=0$ for each $t\in J$, and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzelà theorem, V is relatively compact in D_R. Applying now Theorem 1.5.5 we conclude that N has a fixed point $y\in D_R$. Hence N has a fixed point which is solution to the problem (3.1)-(3.2). This completes the proof. □

3.2.3 An Example.

Consider the following infinite system

$$^cD^{\frac{1}{2}}y_n(t)=\frac{(3+||y_n(t)||+||^cD^{\frac{1}{2}}y_n(t)||)}{3e^{t+2}(1+||y_n(t)||+||^cD^{\frac{1}{2}}y_n(t)||)},\ \text{for each},\ t\in[0,1], \tag{3.9}$$

$$y_n(0)+y_n(1)=0. \tag{3.10}$$

Set

$$E=l^1=\{y=(y_1,y_2,...,y_n,...),\sum_{n=1}^{\infty}|y_n|<\infty\},$$

and

$$f(t,u,v)=\frac{(3+||u||+||v||)}{3e^{t+2}(1+||u||+||v||)},\quad t\in[0,1],\ u,v\in E.$$

E is a Banach space with the norm $||y||=\sum_{n=1}^{\infty}|y_n|$.

Clearly, the function f is jointly continuous.
For any $u,v,\bar{u},\bar{v}\in E$ and $t\in[0,1]$:

$$||f(t,u,v)-f(t,\bar{u},\bar{v})||\leq\frac{1}{3e^2}(||u-\bar{u}||+||v-\bar{v}||).$$

Hence condition (H2) is satisfied with $K=L=\frac{1}{3e^2}$.
And the conditions

$$\frac{(|b|+|a+b|)T^{\nu}K}{|a+b|\Gamma(\nu+1)(1-L)}=\frac{1}{\sqrt{\pi}(e^2-\frac{1}{3})}<1$$

$$\frac{KT^{\nu}}{(1-L)\Gamma(\nu+1)}=\frac{2}{(3e^2-1)\sqrt{\pi}}<1$$

are satisfied with $a=b=T=1, c=0$ and $\nu=\frac{1}{2}$.
It follows from Theorem 3.2.5 that the problem (3.9)-(3.10) has at least one solution on J.

3.3 Existence Results for the non-local BVP in Banach Space

3.3.1 Introduction

The purpose of this Section, is to establish sufficient conditions for the existence of solutions for the following boundary value problem for implicit fractional differential equations with Caputo fractional derivative :

$$^{c}D^{\nu}y(t) = f(t, y(t), ^{c}D^{\nu}y(t)), \quad \text{for every } t \in J := [0,T], T > 0, \quad 0 < \nu \leq 1 \tag{3.11}$$

$$y(0) + g(y) = y_0 \tag{3.12}$$

where $^{c}D^{\nu}$ is the Caputo fractional derivative, $(E, ||\cdot||)$ is a real Banach space, $f : J \times E \times E \to E$ is a given function, $g : C(J,E) \to E$ is a continuous function and $y_0 \in E$. Finally, an example is given to demonstrate the application of our main results.

3.3.2 Existence of Solutions

Let $(E; \|\cdot\|)$ be a valued-Banach space, and $t \in J$. We denote by $C(J,E)$ the space of E valued continuous functions on J with the usual supremum norm

$$\|y\|_{\infty} = \sup\{||y(t)|| : t \in J\}$$

for any $y \in C(J,E)$.

Definition 3.3.1 *A function $y \in C^1(J,E)$ is called solution of problem (3.11)-(3.12) if it satisfies the equation (3.11) on J and the condition (3.12).*

Lemma 3.3.2 *Let $0 < \nu \leq 1$ and let $h : [0,T] \longrightarrow E$ be a continuous function. The linear problem*

$$^{c}D^{\nu}y(t) = h(t), \quad t \in J$$

$$y(0) + g(y) = y_0$$

has a unique solution which is given by :

$$y(t) = y_0 - g(y) + \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} h(s)ds$$

Lemma 3.3.3 *Let $f : J \times E \times E \longrightarrow E$ be a continuous function, then the problem* (3.11)-(3.12) *is equivalent to the following problem*

$$y(t) = y_0 - g(y) + I^{\nu}H(t)$$

where $H \in C(J, \mathbb{R})$

$$H(t) = f(t, y(t), H(t)).$$

Set the following hypothesis :

(H3) there exist $0 < \overline{K}$ such that

$$\|g(u) - g(\overline{u})\| \leq \overline{K}\|u - \overline{u}\| \text{ for any } u, \overline{u} \in C(J, E).$$

Remark 3.3.4 *[25] Condition $(H3)$ is equivalent to the inequality*

$$\alpha\Big(g(B)\Big) \leq \overline{K}\alpha(B),$$

for any bounded sets $B \subseteq E$.

Theorem 3.3.5 *Assume (H1)-(H3) hold.*
If

$$\overline{K} + \frac{KT^{\nu}}{(1-L)\Gamma(\nu+1)} < 1, \tag{3.13}$$

then the IVP (3.11)-(3.12) has at least one solution on J.

3.3.3 An Example.

Consider the boundary value problem :

$${}^cD^{\frac{1}{2}}y_n(t) = \frac{e^{-t}}{(9+e^t)}\left[1 + \frac{\|y_n(t)\|}{1+\|y_n(t)\|} - \frac{\|{}^cD^{\frac{1}{2}}y_n(t)\|}{1+\|{}^cD^{\frac{1}{2}}y_n(t)\|}\right], \quad t \in J = [0,1] \tag{3.14}$$

$$y_n(0) + \sum_{i=1}^{m} c_i y_n(t_i) = 1, \tag{3.15}$$

where $0 < t_1 < t_2 < ... < t_m < 1$ and $c_i = 1, ..., m$ are positif constants with

$$\sum_{i=1}^{m} c_i < \frac{1}{3}.$$

Set

$$E = l^1 = \{y = (y_1, y_2, ..., y_n, ...), \sum_{n=1}^{\infty} |y_n| < \infty\},$$

and

$$f(t,u,v) = \frac{e^{-t}}{(9+e^t)}\left[1 + \frac{\|u\|}{1+\|u\|} - \frac{\|v\|}{1+\|v\|}\right], \; t \in [0,1], \; u, v \in E.$$

E is a Banach space with the norm $||y|| = \sum_{n=1}^{\infty} |y_n|$.

Clearly, the function f is continuous.
For each $u, \bar{u}, v, \bar{v} \in E$ and $t \in [0,1]$:

$$\|f(t,u,v) - f(t,\bar{u},\bar{v})\| \leq \frac{e^{-t}}{9+e^t}(\|u-\bar{u}\| + \|v-\bar{v}\|)$$

$$\leq \frac{1}{10}\|u-\bar{u}\| + \frac{1}{10}\|v-\bar{v}\|.$$

Hence condition (H2) is satisfied with $K = L = \dfrac{1}{10}$.
On the other hand, we have for any $u, \bar{u} \in E$

$$\|g(u) - g(\bar{u})\| \leq \frac{1}{3}\|u-\bar{u}\|.$$

Hence conditions $(H3)$ is satisfied with $\overline{K} = \dfrac{1}{3}$. And the condition

$$\overline{K} + \frac{KT^{\nu}}{(1-L)\Gamma(\nu+1)} = \frac{9\sqrt{\pi}+6}{27\sqrt{\pi}} < 1,$$

is satisfied with $T = 1$ and $\nu = \dfrac{1}{2}$.
It follows from Theorem 3.3.5 that the problem (3.14)-(3.15) has at least one solution on J.

Chapitre 4

Boundary Value Problem for Nonlinear Implicit Fractional Differential Equations with Impulses

4.1 Introduction and Motivations

In this chapter, we establish, existence, uniqueness and stability results to the following boundary value problems for nonlinear implicit fractional differential equations with impulses

$$^cD^{\alpha}_{t_k}y(t) = f(t, y, ^cD^{\alpha}_{t_k}y(t)), \text{ for each }, \ t \in (t_k, t_{k+1}], \ k = 0, \dots, m, \ 0 < \alpha \leq 1,$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \ k = 1, \dots, m,$$

$$ay(0) + by(T) = c,$$

where $^cD^{\alpha}_{t_k}$ is the Caputo fractional derivative, $f : J\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a given function, $I_k : \mathbb{R} \to \mathbb{R}$, and a, b, c are real constants with $a+b \neq 0$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k$.
An extension of this problem is given in Section 4.3. More precisely, we shall present a result of existence and uniqueness for the following boundary value problems for nonlinear implicit fractional differential equations with impulses in Banach space.

$$^cD^{\nu}_{t_k}y(t) = f(t, y, ^cD^{\nu}_{t_k}y(t)), \text{ for each }, \ t \in (t_k, t_{k+1}], \ k = 0, \dots, m, \ 0 < \nu \leq 1,$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \ k = 1, \dots, m,$$

$$ay(0) + by(T) = c,$$

where ${}^cD^{\nu}_{t_k}$ is the Caputo fractional derivative, $(E, ||\cdot||)$ is a real Banach space, $f : J \times E \times E \to E$ is a given function, $I_k : E \to E$, a, b are real constants with $a + b \neq 0$ and $c \in E$.
The theory of impulsive differential equations of integer order has found its extensive applications in realistic mathematical modelling of a wide variety of practical situations and has emerged as an important area of investigation in recent years. See [30, 31, 55, 79, 103, 121, 133], and [66, 84, 85, 138], the references therein.
Very recently, anti-periodic boundary value problems of fractional differential equations have received considerable attention because they occur in the mathematical modeling of a variety of physical processes ; See for example [1, 13, 41, 47, 67, 140, 150].

In [142], F. Wang and Z. Liu, by using Schauder's fixed point theorem and the contraction mapping principle, considered the existence of solutions for the following nonlinear fractional differential equations with fractional anti-periodic boundary conditions :

$$\begin{gathered} {}^cD^{\alpha}y(t) = f(t, y(t), {}^cD^{\beta}y(t)), \quad t \in [0, T], \\ y(0) = -y(T), \quad {}^cD^{\gamma}y(0) = -{}^cD^{\gamma}(T),\ 0 < \beta < 1. \end{gathered}$$

Where denotes the Caputo fractional derivative of order $1 < \alpha \leq 2$, $0 < \gamma$, $\beta < 1$, $\alpha - \beta \geq 1$ and f is a given continuous function.
In [12], B. Ahmad and J.J. Nieto, studied the existence and uniqueness of solutions for impulsive differential equations of fractional order $1 < \alpha \leq 2$, with anti-periodic boundary conditions in a Banach space :

$$\begin{gathered} {}^cD^{\alpha}y(t) = f(t, y(t)), \quad t \in J = [0, T],\ t \neq t_k,\ k = 1, \ldots, m,\ 1 < \alpha \leq 2, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)),\ t_k \in (0, T), \quad k = 1, \ldots, m, \\ \Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)),\ t_k \in (0, T), \quad k = 1, \ldots, m, \\ y(0) = -y(T), \quad y'(0) = -y'(T), \end{gathered}$$

where $k = 1, \ldots, m$, ${}^cD^{\alpha}$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \to \mathbb{R}$ is a given function, $I_k, \overline{I}_k : \mathbb{R} \to \mathbb{R}$ and $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $\Delta y'|_{t=t_k} = y'(t_k^+) - y'(t_k^-)$. Their study is based on the contraction mapping principle and Krasnoselskii's fixed point theorem.

4.2 Existence Results for the BVP with Impulses

[1]

1. M. Benchohra and S. Bouriah, Existence and Stability Results for Nonlinear Implicit Fractional Differential Equations with Impulses, *Mem. Differential Equations Math. Phys.*

4.2.1 Introduction

In this Section, we establish, existence, uniqueness and stability results of solutions for the following boundary value problem for nonlinear implicit fractional differential equations with impulse and Caputo fractional derivative :

$$^{c}D^{\alpha}_{t_k}y(t) = f(t, y, ^{c}D^{\alpha}_{t_k}y(t)), \text{ for each }, \ t \in (t_k, t_{k+1}], \ k = 0, \dots, m, \ 0 < \alpha \leq 1, \quad (4.1)$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \ k = 1, \dots, m, \quad (4.2)$$

$$ay(0) + by(T) = c, \quad (4.3)$$

where $^{c}D^{\alpha}_{t_k}$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, $I_k : \mathbb{R} \to \mathbb{R}$, and a, b, c are real constants with $a+b \neq 0$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k$.
The arguments are based upon the Banach contraction principle, and Schaefer's fixed point theorem. At last, we present two examples to show the applicability of our results.

4.2.2 Existence of Solutions

Denote by $C(J, \mathbb{R})$ the Banach space of continuous functions from J into $\mathbb{R}$, with the usual supremum norm

$$\|y\|_\infty = sup\{|y(t)|, \quad t \in J\}.$$

Consider the set of functions

$$\begin{aligned} PC(J, \mathbb{R}) \ &= \{y : J \to \mathbb{R} : y \in C((t_k, t_{k+1}], \mathbb{R}), \ k = 0, \dots, m \text{ and there exist } y(t_k^-) \text{ and} \\ &\quad y(t_k^+), \ k = 1, \dots, m \text{ with } y(t_k^-) = y(t_k)\}. \end{aligned}$$

$PC(J, \mathbb{R})$ is a Banach space with the norm

$$\|y\|_{PC} = \sup_{t\in J} |y(t)|.$$

Let $J_0 = [t_0, t_1]$ and $J_k = (t_k, t_{k+1}]$ where $k = 1, \dots, m$.

Definition 4.2.1 A function $y \in PC(J, \mathbb{R})$ whose α-derivative exists on J_k is said to be a solution of (4.1)–(4.3) if y satisfies the equation $^{c}D^{\alpha}_{t_k}y(t) = f(t, y(t), ^{c}D^{\alpha}_{t_k}y(t))$ on J_k, and satisfy the conditions

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \ k = 1, \dots, m,$$
$$ay(0) + by(T) = c.$$

To prove the existence of solutions to (4.1)–(4.3), we need the following auxiliary Lemma.

Lemma 4.2.2 *Let $0 < \alpha \leq 1$ and let $\sigma : J \to \mathbb{R}$ be continuous. A function y is a solution of the fractional integral equation*

$$y(t) = \begin{cases} \frac{-1}{a+b}\left[b\sum_{i=1}^{m} I_i(y(t_i^-)) + \frac{b}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\sigma(s)ds\right. \\ \left. + \frac{b}{\Gamma(\alpha)}\int_{t_m}^{T}(T-s)^{\alpha-1}\sigma(s)ds - c\right] + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sigma(s)ds \qquad \text{if } t \in [0,t_1] \\ \\ \frac{-1}{a+b}\left[b\sum_{i=1}^{m} I_i(y(t_i^-)) + \frac{b}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\sigma(s)ds\right. \\ \left. + \frac{b}{\Gamma(\alpha)}\int_{t_m}^{T}(T-s)^{\alpha-1}\sigma(s)ds - c\right] + \sum_{i=1}^{k} I_i(y(t_i^-)) + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\sigma(s)ds \\ + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\sigma(s)ds, \qquad \text{if } t \in (t_k, t_{k+1}], \end{cases} \tag{4.4}$$

where $k = 1, \ldots, m$, if and only if y is a solution of the fractional BVP

$$^cD^\alpha y(t) = \sigma(t), \quad t \in J_k, \tag{4.5}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{4.6}$$

$$ay(0) + by(T) = c. \tag{4.7}$$

Proof. Assume y satisfies (4.5)-(4.7). If $t \in [0, t_1]$ then

$$^cD^\alpha y(t) = \sigma(t).$$

Lemma 1.3.7 implies

$$y(t) = c_0 + I^\alpha\sigma(t) = c_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sigma(s)ds$$

for $c_0 \in \mathbb{R}$. If $t \in (t_1, t_2]$ then Lemma 1.3.7 implies

$$\begin{aligned} y(t) &= y(t_1^+) + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}\sigma(s)ds \\ &= \Delta y|_{t=t_1} + y(t_1^-) + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}\sigma(s)ds \\ &= I_1(y(t_1^-)) + \left[c_0 + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\sigma(s)ds\right] \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}\sigma(s)ds. \end{aligned}$$

$$
\begin{aligned}
&= c_0 + I_1(y(t_1^-)) + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1}\sigma(s)ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t - s)^{\alpha-1}\sigma(s)ds.
\end{aligned}
$$

If $t \in (t_2, t_3]$, then from Lemma 1.3.7, we get

$$
\begin{aligned}
y(t) &= y(t_2^+) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t - s)^{\alpha-1}\sigma(s)ds \\
&= \Delta y|_{t=t_2} + y(t_2^-) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t - s)^{\alpha-1}\sigma(s)ds \\
&= I_2(y(t_2^-)) + \Big[c_0 + I_1(y(t_1^-)) + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1}\sigma(s)ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1}\sigma(s)ds\Big] + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t - s)^{\alpha-1}\sigma(s)ds. \\
&= c_0 + \big[I_1(y(t_1^-)) + I_2(y(t_2^-))\big] + \Big[\frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1}\sigma(s)ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1}\sigma(s)ds\Big] + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t - s)^{\alpha-1}\sigma(s)ds.
\end{aligned}
$$

Repeating the process in this way, the solution $y(t)$ for $t \in (t_k, t_{k+1}]$ where $k = 1, \dots, m$, can be written as

$$
\begin{aligned}
y(t) &= c_0 + \sum_{i=1}^{k} I_i(y(t_i^-)) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1}\sigma(s)ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1}\sigma(s)ds.
\end{aligned}
$$

Applying the boundary conditions $ay(0) + by(T) = c$, we get

$$
\begin{aligned}
c &= c_0(a + b) + b\sum_{i=1}^{m} I_i(y(t_i^-)) + \frac{b}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1}\sigma(s)ds \\
&+ \frac{b}{\Gamma(\alpha)} \int_{t_m}^{T} (T - s)^{\alpha-1}\sigma(s)ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
c_0 &= \frac{-1}{a+b}\Big[b\sum_{i=1}^{m} I_i(y(t_i^-)) + \frac{b}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1}\sigma(s)ds \\
&+ \frac{b}{\Gamma(\alpha)} \int_{t_m}^{T} (T - s)^{\alpha-1}\sigma(s)ds - c\Big].
\end{aligned}
$$

Thus, if $t \in (t_k, t_{k+1}]$ where $k = 1, \dots, m$, then

$$\begin{aligned} y(t) &= \frac{-1}{a+b}\left[b\sum_{i=1}^{m} I_i(y(t_i^-)) + \frac{b}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\sigma(s)ds\right. \\ &+ \left.\frac{b}{\Gamma(\alpha)}\int_{t_m}^{T}(T-s)^{\alpha-1}\sigma(s)ds - c\right] + \sum_{i=1}^{k} I_i(y(t_i^-)) \\ &+ \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\sigma(s)ds + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\sigma(s)ds. \end{aligned}$$

Conversely, assume that y satisfies the impulsive fractional integral equation (4.4). If $t \in [0, t_1]$ then $ay(0) + by(T) = c$ and using the fact that ${}^cD^\alpha$ is the left inverse of I^α we get

$$ {}^cD^\alpha y(t) = \sigma(t), \quad \text{for each } t \in [0, t_1]. $$

If $t \in (t_k, t_{k+1}]$, $k = 1, \dots, m$ and using the fact that ${}^cD^\alpha C = 0$, where C is a constant, we get

$$ {}^cD^\alpha y(t) = \sigma(t), \text{for each } t \in (t_k, t_{k+1}]. $$

Also, we can easily show that

$$ \Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m. $$

□

We are now in a position to state and prove our existence result for the problem (4.1)−(4.3) based on Banach's fixed point.

Theorem 4.2.3 *Assume*

(H1) The function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous.

(H2) There exist constants $K > 0$ and $0 < L < 1$ such that

$$ |f(t, u, v) - f(t, \bar{u}, \bar{v})| \le K|u - \bar{u}| + L|v - \bar{v}| $$

for any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in J$.

(H3) There exists a constant $\widetilde{l} > 0$ such that

$$ |I_k(u) - I_k(\overline{u})| \le \widetilde{l}|u - \overline{u}|, $$

for each $u, \overline{u} \in \mathbb{R}$ and $k = 1, \dots, m$.

If

$$ \left(\frac{|b|}{|a+b|} + 1\right)\left[m\widetilde{l} + \frac{(m+1)KT^\alpha}{(1-L)\Gamma(\alpha+1)}\right] < 1, \tag{4.8} $$

then there exists a unique solution for BVP (4.1)-(4.3) on J.

Proof. Transform the problem (4.1)-(4.3) into a fixed point problem. Consider the operator $N : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ defined by

$$\begin{aligned} N(y)(t) &= \frac{-1}{a+b}\left[b\sum_{i=1}^{m} I_i(y(t_i^-)) + \frac{b}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}g(s)ds\right.\\ &+ \left.\frac{b}{\Gamma(\alpha)}\int_{t_m}^{T}(T-s)^{\alpha-1}g(s)ds - c\right] + \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}g(s)ds\\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}g(s)ds + \sum_{0<t_k<t} I_k(y(t_k^-)), \end{aligned} \tag{4.9}$$

where $g \in C(J,\mathbb{R})$ be such that

$$g(t) = f(t, y(t), g(t)).$$

Clearly, the fixed points of operator N are solutions of problem (4.1)–(4.3).
Let $u, w \in PC(J,\mathbb{R})$. Then for $t \in J$, we have

$$\begin{aligned} |N(u)(t) - N(w)(t)| &\leq \frac{|b|}{|a+b|}\left[\sum_{i=1}^{m}|I_i(u(t_i^-)) - I_i(w(t_i^-))|\right.\\ &+ \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}|g(s)-h(s)|ds\\ &+ \left.\frac{1}{\Gamma(\alpha)}\int_{t_m}^{T}(T-s)^{\alpha-1}|g(s)-h(s)|ds\right]\\ &+ \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|g(s)-h(s)|ds\\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|g(s)-h(s)|ds\\ &+ \sum_{0<t_k<t}|I_k(u(t_k^-)) - I_k(w(t_k^-))|, \end{aligned}$$

where $g, h \in C(J,\mathbb{R})$ be such that

$$g(t) = f(t, u(t), g(t)),$$

and

$$h(t) = f(t, w(t), h(t)).$$

By $(H2)$ we have

$$|g(t) - h(t)| = |f(t, u(t), g(t)) - f(t, w(t), h(t))|$$

$$\leq \quad K|u(t)-w(t)|+L|g(t)-h(t)|.$$

Then

$$|g(t)-h(t)| \leq \frac{K}{1-L}|u(t)-w(t)|.$$

Therefore, for each $t \in J$

$$\begin{aligned}
|N(u)(t)-N(w)(t)| \quad &\leq \quad \frac{|b|}{|a+b|}\left[\sum_{k=1}^{m}\widetilde{l}|u(t_k^-)-w(t_k^-)|\right. \\
&+ \quad \frac{K}{(1-L)\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|u(s)-w(s)|ds \\
&+ \quad \left.\frac{K}{(1-L)\Gamma(\alpha)}\int_{t_m}^{T}(T-s)^{\alpha-1}|u(s)-w(s)|ds\right] \\
&+ \quad \frac{K}{(1-L)\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|u(s)-w(s)|ds \\
&+ \quad \frac{K}{(1-L)\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|u(s)-w(s)|ds \\
&+ \quad \sum_{k=1}^{m}\widetilde{l}|u(t_k^-)-w(t_k^-)|. \\
&\leq \quad \left(\frac{|b|}{|a+b|}+1\right)\left[m\widetilde{l}+\frac{mKT^\alpha}{(1-L)\Gamma(\alpha+1)}\right. \\
&\quad\quad \left.+\frac{KT^\alpha}{(1-L)\Gamma(\alpha+1)}\right]\|u-w\|_{PC}.
\end{aligned}$$

Thus

$$\|N(u)-N(w)\|_{PC} \leq \left(\frac{|b|}{|a+b|}+1\right)\left[m\widetilde{l}+\frac{(m+1)KT^\alpha}{(1-L)\Gamma(\alpha+1)}\right]\|u-w\|_{PC}.$$

By (4.8), the operator N is a contraction. Hence, by Banach's contraction principle, N has a unique fixed point which is a unique solution of the problem (4.1)–(4.3).

□

Our second result is based on Schaefer's fixed point theorem.

Theorem 4.2.4 *Assume (H1), (H2) and*

(H4) There exist $p, q, r \in C(J, \mathbb{R}_+)$ *with* $r^* = \sup_{t\in J} r(t) < 1$ *such that*

$$|f(t,u,w)| \leq p(t)+q(t)|u|+r(t)|w| \text{ for } t \in J \text{ and } u, w \in \mathbb{R}.$$

(H5) The functions $I_k : \mathbb{R} \to \mathbb{R}$ are continuous and there exist constants M^*, $N^* > 0$ such that

$$|I_k(u)| \leq M^*|u| + N^* \text{ for each } u \in \mathbb{R},\ k = 1, \dots, m.$$

If

$$\left(\frac{|b|}{|a+b|} + 1\right)\left(mM^* + \frac{(m+1)q^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\right) < 1, \tag{4.10}$$

then the BVP (4.1)-(4.3) *has at least one solution on J.*

Proof. Let the operator N defined in (4.9). We shall use Schaefer's fixed point theorem to prove that N has a fixed point. The proof will be given in several steps.

Step 1 : *N is continuous.* Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $PC(J, \mathbb{R})$. Then for each $t \in J$,

$$\begin{aligned}
|N(u_n)(t) - N(u)(t)| \leq\ & \frac{|b|}{|a+b|}\left[\sum_{i=1}^{m} |I_k(u_n(t_k^-)) - I_k(u(t_k^-))|\right. \\
+\ & \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i - s)^{\alpha-1}|g_n(s) - g(s)|ds \\
+\ & \left.\frac{1}{\Gamma(\alpha)}\int_{t_m}^{T}(T-s)^{\alpha-1}|g_n(s) - g(s)|ds\right] \\
+\ & \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k - s)^{\alpha-1}|g_n(s) - g(s)|ds \\
+\ & \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|g_n(s) - g(s)|ds \\
+\ & \sum_{0<t_k<t}|I_k(u_n(t_k^-)) - I_k(u(t_k^-))|,
\end{aligned} \tag{4.11}$$

where $g_n, g \in C(J, \mathbb{R})$ such that

$$g_n(t) = f(t, u_n(t), g_n(t)),$$

and

$$g(t) = f(t, u(t), g(t)).$$

By $(H2)$, we have

$$\begin{aligned}
|g_n(t) - g(t)| &= |f(t, u_n(t), g_n(t)) - f(t, u(t), g(t))| \\
&\leq K|u_n(t) - u(t)| + L|g_n(t) - g(t)|.
\end{aligned}$$

Then

$$|g_n(t) - g(t)| \leq \frac{K}{1-L}|u_n(t) - u(t)|.$$

Since $u_n \to u$, then we get $g_n(t) \to g(t)$ as $n \to \infty$ for each $t \in J$. And let $\eta > 0$ be such that, for each $t \in J$, we have $|g_n(t)| \leq \eta$ and $|g(t)| \leq \eta$. Then, we have

$$\begin{aligned}(t-s)^{\alpha-1}|g_n(s)-g(s)| &\leq (t-s)^{\alpha-1}[|g_n(s)|+|g(s)|]\\ &\leq 2\eta(t-s)^{\alpha-1},\end{aligned}$$

and

$$\begin{aligned}(t_k-s)^{\alpha-1}|g_n(s)-g(s)| &\leq (t_k-s)^{\alpha-1}[|g_n(s)|+|g(s)|]\\ &\leq 2\eta(t_k-s)^{\alpha-1}.\end{aligned}$$

For each $t \in J$, the functions $s \to 2\eta(t-s)^{\alpha-1}$ and $s \to 2\eta(t_k-s)^{\alpha-1}$ are integrable on $[0,t]$, then the Lebesgue Dominated Convergence Theorem and (4.11) imply that

$$|N(u_n)(t)-N(u)(t)| \to 0 \; as \; n \to \infty,$$

and hence

$$\|N(u_n)-N(u)\|_{PC} \to 0 \text{ as } n \to \infty.$$

Consequently, N is continuous.

Step 2 : *N maps bounded sets into bounded sets in $PC(J,\mathbb{R})$.* Indeed, it is enough to show that for any $\eta^* > 0$, there exists a positive constant ℓ such that for each $u \in B_{\eta^*} = \{u \in PC(J,\mathbb{R}) : ||u||_{PC} \leq \eta^*\}$, we have $\|N(u)\|_{PC} \leq \ell$. We have for each $t \in J$,

$$\begin{aligned}N(u)(t) &= \frac{-1}{a+b}\left[b\sum_{i=1}^{m} I_i(u(t_i^-)) + \frac{b}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}g(s)ds\right.\\ &+ \left.\frac{b}{\Gamma(\alpha)}\int_{t_m}^{T}(T-s)^{\alpha-1}g(s)ds - c\right] + \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}g(s)ds\\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}g(s)ds + \sum_{0<t_k<t} I_k(u(t_k^-)),\end{aligned} \tag{4.12}$$

where $g \in C(J,\mathbb{R})$ be such that

$$g(t) = f(t,u(t),g(t)).$$

By (H4), we have for each $t \in J$,

$$\begin{aligned}|g(t)| &= |f(t,u(t),g(t))|\\ &\leq p(t)+q(t)|u(t)|+r(t)|g(t)|\\ &\leq p(t)+q(t)\eta^*+r(t)|g(t)|\\ &\leq p^*+q^*\eta^*+r^*|g(t)|,\end{aligned}$$

where $p^* = \sup_{t\in J} p(t)$, and $q^* = \sup_{t\in J} q(t)$.
Then

$$|g(t)| \leq \frac{p^* + q^*\eta^*}{1 - r^*} := M.$$

Thus (4.12) implies

$$\begin{aligned}
|N(u)(t)| &\leq \frac{|b|}{|a+b|}\left[m(M^*|u| + N^*) + \frac{mMT^\alpha}{\Gamma(\alpha+1)} + \frac{MT^\alpha}{\Gamma(\alpha+1)}\right] \\
&+ \frac{|c|}{|a+b|} + \frac{mMT^\alpha}{\Gamma(\alpha+1)} + \frac{MT^\alpha}{\Gamma(\alpha+1)} + m(M^*|u| + N^*) \\
&\leq \left(\frac{|b|}{|a+b|} + 1\right)\left[m(M^*|u| + N^*) + \frac{(m+1)MT^\alpha}{\Gamma(\alpha+1)}\right] + \frac{|c|}{|a+b|}.
\end{aligned}$$

Then

$$\|N(u)\|_{PC} \leq \left(\frac{|b|}{|a+b|} + 1\right)\left[m(M^*\eta^* + N^*) + \frac{(m+1)MT^\alpha}{\Gamma(\alpha+1)}\right] + \frac{|c|}{|a+b|} := \ell.$$

Step 3 : *N maps bounded sets into equicontinuous sets of $PC(J, \mathbb{R})$.*

Let $\tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$, B_{η^*} be a bounded set of $PC(J, \mathbb{R})$ as in Step 2, and let $u \in B_{\eta^*}$. Then

$$\begin{aligned}
&|N(u)(\tau_2) - N(u)(\tau_1)| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{\tau_1} |(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}||g(s)|ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2} |(\tau_2 - s)^{\alpha-1}||g(s)|ds + \sum_{0<t_k<\tau_2-\tau_1} |I_k(u(t_k^-))| \\
&\leq \frac{M}{\Gamma(\alpha+1)}[2(\tau_2 - \tau_1)^\alpha + (\tau_2^\alpha - \tau_1^\alpha)] + (\tau_2 - \tau_1)(M^*|u| + N^*) \\
&\leq \frac{M}{\Gamma(\alpha+1)}[2(\tau_2 - \tau_1)^\alpha + (\tau_2^\alpha - \tau_1^\alpha)] + (\tau_2 - \tau_1)(M^*\eta^* + N^*).
\end{aligned}$$

As $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Ascoli-Arzela theorem, we can conclude that $N : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ is completely continuous.

Step 4 : *A priori bounds.* Now it remains to show that the set

$$E = \{u \in PC(J, \mathbb{R}) : u = \lambda N(u) \text{ for some } 0 < \lambda < 1\}$$

is bounded. Let $u \in E$, then $u = \lambda N(u)$ for some $0 < \lambda < 1$. Thus, for each $t \in J$ we have

$$\begin{aligned} u(t) &= \frac{-1}{a+b}\left[b\lambda\sum_{i=1}^{m} I_i(u(t_i^-)) + \frac{b\lambda}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}g(s)ds\right.\\ &+ \left.\frac{b\lambda}{\Gamma(\alpha)}\int_{t_m}^{T}(T-s)^{\alpha-1}g(s)ds - c\lambda\right] + \frac{\lambda}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}g(s)ds\\ &+ \frac{\lambda}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}g(s)ds + \lambda\sum_{0<t_k<t} I_k(u(t_k^-)). \end{aligned} \tag{4.13}$$

And, by (H4), we have for each $t \in J$,

$$\begin{aligned} |g(t)| &= |f(t,u(t),g(t))|\\ &\leq p(t) + q(t)|u(t)| + r(t)|g(t)|\\ &\leq p^* + q^*|u(t)| + r^*|g(t)|. \end{aligned}$$

Thus

$$\begin{aligned} |g(t)| &\leq \frac{1}{1-r^*}(p^* + q^*|u(t)|)\\ &\leq \frac{1}{1-r^*}(p^* + q^*\|u\|_{PC}). \end{aligned}$$

This implies, by (4.13) and (H5), that for each $t \in J$ we have

$$\begin{aligned} |u(t)| &\leq \frac{|b|}{|a+b|}\left[m(M^*\|u\|_{PC} + N^*) + \frac{mT^\alpha(p^*+q^*\|u\|_{PC})}{(1-r^*)\Gamma(\alpha+1)} + \frac{T^\alpha(p^*+q^*\|u\|_{PC})}{(1-r^*)\Gamma(\alpha+1)}\right]\\ &+ \frac{|c|}{|a+b|} + \frac{mT^\alpha(p^*+q^*\|u\|_{PC})}{(1-r^*)\Gamma(\alpha+1)} + \frac{T^\alpha(p^*+q^*\|u\|_{PC})}{(1-r^*)\Gamma(\alpha+1)} + m(M^*\|u(t)\|_{PC} + N^*). \end{aligned}$$

Then

$$\begin{aligned} \|u\|_{PC} &\leq \left(\frac{|b|}{|a+b|}+1\right)\left[m(M^*\|u(t)\|_{PC} + N^*) + \frac{(m+1)\,(p^*+q^*\|u\|_{PC})\,T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\right]\\ &+ \frac{|c|}{|a+b|}\\ &\leq \left(\frac{|b|}{|a+b|}+1\right)\left(mN^* + \frac{(m+1)p^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\right) + \frac{|c|}{|a+b|}\\ &+ \left(\frac{|b|}{|a+b|}+1\right)\left(mM^* + \frac{(m+1)q^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\right)\|u\|_{PC}. \end{aligned}$$

Thus

$$\left[1 - \left(\frac{|b|}{|a+b|}+1\right)\left(mM^* + \frac{(m+1)q^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\right)\right]\|u\|_{PC} \;\leq\; \left(\frac{|b|}{|a+b|}+1\right)\left[\frac{|c|}{|a+b|}\right.$$

$$+mN^* + \frac{(m+1)p^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\Bigg].$$

Finally , by (4.10) we have

$$\|u\|_{PC} \leq \frac{\left(\frac{|b|}{|a+b|}+1\right)\left[mN^* + \frac{(m+1)p^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)} + \frac{|c|}{|a+b|}\right]}{\left[1-\left(\frac{|b|}{|a+b|}+1\right)\left(mM^* + \frac{(m+1)q^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\right)\right]} := R.$$

This shows that the set E is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that N has a fixed point which is a solution of the problem $(4.1)-(4.3)$. □

4.2.3 Ulam-Hyers Rassias stability

Here, we adopt the concepts in Wang et it [139] and introduce Ulam's type stability concepts for the problem (4.1)-(4.2).
Let $z \in PC^1(J,\mathbb{R}), \epsilon > 0, \psi > 0$ and $\varphi \in PC(J,\mathbb{R}_+)$ is nondecreasing. We consider the set of inequalities

$$\begin{cases} |{}^cD^\alpha z(t) - f(t,z(t),{}^cD^\alpha z(t))| \leq \epsilon, \ t \in (t_k,t_{k+1}], k=1,...,m \\ |\Delta z(t_k) - I_k(z(t_k^-))| \leq \epsilon, \ k=1,...,m; \end{cases} \tag{4.14}$$

the set of inequalities

$$\begin{cases} |{}^cD^\alpha z(t) - f(t,z(t),{}^cD^\alpha z(t))| \leq \varphi(t), \ t \in (t_k,t_{k+1}], k=1,...,m \\ |\Delta z(t_k) - I_k(z(t_k^-))| \leq \psi, \ k=1,...,m; \end{cases} \tag{4.15}$$

and the set of inequalities

$$\begin{cases} |{}^cD^\alpha z(t) - f(t,z(t),{}^cD^\alpha z(t))| \leq \epsilon\varphi(t), \ t \in (t_k,t_{k+1}], k=1,...,m \\ |\Delta z(t_k) - I_k(z(t_k^-))| \leq \epsilon\psi, \ k=1,...,m. \end{cases} \tag{4.16}$$

Definition 4.2.5 *The problem* (4.1)-(4.2) *is Ulam-Hyers stable if there exists a real number $c_{f,m} > 0$ such that for each $\epsilon > 0$ and for each solution $z \in PC^1(J,\mathbb{R})$ of the inequality (4.14) there exists a solution $y \in PC^1(J,\mathbb{R})$ of the problem* (4.1)-(4.2) *with*

$$|z(t) - y(t)| \leq c_{f,m}\epsilon, \ t \in J.$$

Definition 4.2.6 *The problem* (4.1)-(4.2) *is generalized Ulam-Hyers stable if there exists $\theta_{f,m} \in C(\mathbb{R}_+,\mathbb{R}_+), \theta_{f,m}(0) = 0$ such that for each solution $z \in PC^1(J,\mathbb{R})$ of the inequality (4.14) there exists a solution $y \in PC^1(J,\mathbb{R})$ of the problem* (4.1)-(4.2) *with*

$$|z(t) - y(t)| \leq \theta_{f,m}(\epsilon), \ t \in J.$$

Definition 4.2.7 *The problem* (4.1)-(4.2) *is Ulam-Hyers-Rassias stable with respect to* (φ,ψ) *if there exists* $c_{f,m,\varphi} > 0$ *such that for each* $\epsilon > 0$ *and for each solution* $z \in PC^1(J,\mathbb{R})$ *of the inequality (4.16) there exists a solution* $y \in PC^1(J,\mathbb{R})$ *of the problem* (4.1)-(4.2) *with*

$$|z(t) - y(t)| \leq c_{f,m,\varphi}\epsilon(\varphi(t) + \psi),\ t \in J.$$

Definition 4.2.8 *The problem* (4.1)-(4.2) *is generalized Ulam-Hyers-Rassias stable with respect to* (φ,ψ) *if there exists* $c_{f,m,\varphi} > 0$ *such that for each solution* $z \in PC^1(J,\mathbb{R})$ *of the inequality (4.15) there exists a solution* $y \in PC^1(J,\mathbb{R})$ *of the problem* (4.1)-(4.2) *with*

$$|z(t) - y(t)| \leq c_{f,m,\varphi}(\varphi(t) + \psi),\ t \in J.$$

Remark 4.2.9 *It is clear that : (i) Definition 4.2.5 implies Definition 4.2.6 ; (ii) Definition 4.2.7 implies Definition 4.2.8 ; (iii) Definition 4.2.7 for* $\varphi(t) = \psi = 1$ *implies Definition 4.2.5.*

Remark 4.2.10 *A function* $z \in PC^1(J,\mathbb{R})$ *is a solution of the inequality (4.16) if and only if there is* $\sigma \in PC(J,\mathbb{R})$ *and a sequence* $\sigma_k, k = 1,...,m$ *(which depend on z) such that*

i) $|\sigma(t)| \leq \epsilon\varphi(t),\ t \in (t_k, t_{k+1}], k = 1,...,m$ *and* $|\sigma_k| \leq \epsilon\psi, k = 1,...,m;$

ii) ${}^cD^\alpha z(t) = f(t, z(t), {}^c D^\alpha z(t)) + \sigma(t),\ t \in (t_k, t_{k+1}], k = 1,...,m\,;$

iii) $\Delta z(t_k) = I_k(z(t_k^-)) + \sigma_k,\ k = 1,...,m.$

One can have similar remarks for inequalities 4.15 and 4.14. Now, we state the following Ulam-Hyers-Rassias stable result.

Theorem 4.2.11 *Assume (H1)-(H3), (4.8) and*

(H6) there exists a nondecreasing function $\varphi \in PC\,(J,\mathbb{R}_+)$ *and there exists* $\lambda_\varphi > 0$ *such that for any* $t \in J$:

$$I^\alpha\varphi\,(t) \leq \lambda_\varphi\varphi\,(t)$$

are satisfied, then, the problem (4.1)-(4.2) *is Ulam-Hyers-Rassias stable with respect to* (φ,ψ).

proof. Let $z \in PC^1(J,\mathbb{R})$ be a solution of the inequality (4.16). Denote by y the unique solution of the BVP :

$$\begin{cases} {}^cD^\alpha_{t_k} y(t) = f(t, y(t), {}^c D^\alpha_{t_k} y(t)),\ t \in (t_k, t_{k+1}], k = 1,...,m\ ; \\ \Delta y(t_k) = I_k(y(t_k^-)),\ k = 1,...,m\ ; \\ ay(0) + by(T) = c\ ; \\ y(0) = z(0). \end{cases}$$

Using Lemma 4.2.2, we obtain for each $t \in (t_k, t_{k+1}]$

$$\begin{aligned} y(t) &= y(0) + \sum_{i=1}^{k} I_i(y(t_i^-)) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} g(s)ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} g(s)ds, \ t \in (t_k, t_{k+1}], \end{aligned}$$

where $g \in C(J, \mathbb{R})$ be such that

$$g(t) = f(t, y(t), g(t)).$$

Since z solution of the inequality (4.16) and by Remark 4.2.10, we have

$$\begin{cases} {}^c D_{t_k}^{\alpha} z(t) = f(t, z(t), {}^c D_{t_k}^{\alpha} z(t)) + \sigma(t), \ t \in (t_k, t_{k+1}], k = 1, ..., m \ ; \\ \Delta z(t_k) = I_k(z(t_k^-)) + \sigma_k, \ k = 1, ..., m \ . \end{cases} \tag{4.17}$$

Clearly, the solution of (4.17) is given by

$$\begin{aligned} z(t) &= z(0) + \sum_{i=1}^{k} I_i(z(t_i^-)) + \sum_{i=1}^{k} \sigma_i + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} h(s)ds \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} \sigma(s)ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} h(s)ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \sigma(s)ds, \ t \in (t_k, t_{k+1}], \end{aligned}$$

where $h \in C(J, \mathbb{R})$ be such that

$$h(t) = f(t, z(t), h(t)).$$

Hence for each $t \in (t_k, t_{k+1}]$, it follows that

$$\begin{aligned} |z(t) - y(t)| &\leq \sum_{i=1}^{k} |\sigma_i| + \sum_{i=1}^{k} |I_i(z(t_i^-)) - I_i(y(t_i^-))| \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |h(s) - g(s)|ds \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |\sigma(s)|ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |h(s) - g(s)|ds \end{aligned}$$

$$+ \quad \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|\sigma(s)|.$$

Thus

$$\begin{aligned}
|z(t)-y(t)| &\leq m\epsilon\psi+(m+1)\epsilon\lambda_\varphi\varphi(t)+\sum_{i=1}^{k}\widetilde{l}|z(t_i^-)-y(t_i^-)| \\
&+ \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}|h(s)-g(s)|ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|h(s)-g(s)|ds.
\end{aligned}$$

By (H2), we have

$$\begin{aligned}
|h(t)-g(t)| &= |f(t,z(t),h(t))-f(t,y(t),g(t))| \\
&\leq K|z(t)-y(t)|+L|g(t)-h(t)|.
\end{aligned}$$

Then

$$|h(t)-g(t)|\leq \frac{K}{1-L}|z(t)-y(t)|.$$

Therefore, for each $t\in J$

$$\begin{aligned}
|z(t)-y(t)| &\leq m\epsilon\psi+(m+1)\epsilon\lambda_\varphi\varphi(t)+\sum_{i=1}^{k}\widetilde{l}|z(t_i^-)-y(t_i^-)| \\
&+ \frac{K}{(1-L)\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}|z(s)-y(s)|ds \\
&+ \frac{K}{(1-L)\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|z(s)-y(s)|ds.
\end{aligned}$$

Thus

$$\begin{aligned}
|z(t)-y(t)| &\leq \sum_{i=1}^{k}\widetilde{l}|z(t_i^-)-y(t_i^-)|+\epsilon(\psi+\varphi(t))(m+(m+1)\lambda_\varphi) \\
&+ \frac{K(m+1)}{(1-L)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|z(s)-y(s)|ds.
\end{aligned}$$

Applying Lemma 1.3.9, we get

$$\begin{aligned}
|z(t)-y(t)| &\leq \epsilon(\psi+\varphi(t))(m+(m+1)\lambda_\varphi) \\
&\times \left[\prod_{0<t_k<t}(1+\widetilde{l})\exp\left(\int_0^t\frac{K(m+1)}{(1-L)\Gamma(\alpha)}(t-s)^{\alpha-1}ds\right)\right]
\end{aligned}$$

$$\leq c_\varphi \epsilon(\psi + \varphi(t)),$$

where

$$\begin{aligned} c_\varphi &= (m + (m+1)\lambda_\varphi) \left[\prod_{k=1}^{m} (1 + \widetilde{l}) \exp\left(\frac{K(m+1)T^\alpha}{(1-L)\Gamma(\alpha+1)} \right) \right] \\ &= (m + (m+1)\lambda_\varphi) \left[(1 + \widetilde{l}) \exp\left(\frac{K(m+1)T^\alpha}{(1-L)\Gamma(\alpha+1)} \right) \right]^m . \end{aligned}$$

Thus, the problem (4.1)-(4.2) is Ulam-Hyers-Rassias stable with respect to (φ, ψ). The proof is complete.

□

Next, we present the following Ulam-Hyers stable result.

Theorem 4.2.12 *Assume that (H1)-(H3) and* (4.8) *are satisfied, then, the problem* (4.1)-(4.2) *is Ulam-Hyers stable*

proof. Let $z \in PC^1(J, \mathbb{R})$ be a solution of the inequality (4.14). Denote by y the unique solution of the BVP :

$$\begin{cases} {}^cD^\alpha_{t_k} y(t) = f(t, y(t), {}^cD^\alpha_{t_k} y(t)), \ t \in (t_k, t_{k+1}], k = 1, ..., m \ ; \\ \Delta y(t_k) = I_k(y(t_k^-)), \ k = 1, ..., m \ ; \\ ay(0) + by(T) = c \ ; \\ y(0) = z(0). \end{cases}$$

From the proof of Theorem 4.2.11, we get the inequality

$$\begin{aligned} |z(t) - y(t)| &\leq \sum_{i=1}^{k} \widetilde{l} |(z(t_i^-)) - (y(t_i^-))| + m\epsilon + \frac{T^\alpha \epsilon(m+1)}{\Gamma(\alpha+1)} \\ &+ \frac{K(m+1)}{(1-L)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z(s) - y(s)| ds. \end{aligned}$$

Applying Lemma 1.3.9, we get

$$\begin{aligned} |z(t) - y(t)| &\leq \epsilon \left(\frac{m\Gamma(\alpha+1) + T^\alpha(m+1)}{\Gamma(\alpha+1)} \right) \\ &\times \left[\prod_{0 < t_k < t} (1 + \widetilde{l}) \exp\left(\int_0^t \frac{K(m+1)}{(1-L)\Gamma(\alpha)} (t-s)^{\alpha-1} ds \right) \right] \\ &\leq c_\varphi \epsilon, \end{aligned}$$

where

$$c_\varphi = \left(\frac{m\Gamma(\alpha+1) + T^\alpha(m+1)}{\Gamma(\alpha+1)} \right) \left[\prod_{k=1}^{m} (1 + \widetilde{l}) \exp\left(\frac{K(m+1)T^\alpha}{(1-L)\Gamma(\alpha+1)} \right) \right]$$

$$= \left(\frac{m\Gamma(\alpha+1)+T^{\alpha}(m+1)}{\Gamma(\alpha+1)}\right)\left[(1+\widetilde{l})\exp\left(\frac{K(m+1)T^{\alpha}}{(1-L)\Gamma(\alpha+1)}\right)\right]^{m}.$$

Which completes the proof of the Theorem.

□

Moreover, if we set $\theta(\epsilon)=c\epsilon;\theta(0)=0$, then, the problem (4.1)-(4.2) is generalized Ulam-Hyers stable.

4.2.4 Examples

Example 1. Consider the following impulsive boundary value problem

$$^{c}D^{\frac{1}{2}}_{t_k}y(t)=\frac{1}{5e^{t+2}(1+|y(t)|+|^{c}D^{\frac{1}{2}}_{t_k}y(t)|)},\ \text{for each, } t\in J_0\cup J_1. \tag{4.18}$$

$$\Delta y|_{t=\frac{1}{2}}=\frac{|y(\frac{1}{2}^{-})|}{10+|y(\frac{1}{2}^{-})|}. \tag{4.19}$$

$$2y(0)-y(1)=3, \tag{4.20}$$

where $J_0=\left[0,\frac{1}{2}\right]$, $J_1=\left(\frac{1}{2},1\right]$, $t_0=0$, and $t_1=\frac{1}{2}$.
Set

$$f(t,u,v)=\frac{1}{5e^{t+2}(1+|u|+|v|)},\quad t\in[0,1],\ u,v\in\mathbb{R}$$

Clearly, the function f is jointly continuous.
For each $u,v,\bar{u},\bar{v}\in\mathbb{R}$ and $t\in[0,1]$:

$$|f(t,u,v)-f(t,\bar{u},\bar{v})|\leq\frac{1}{5e^{2}}(|u-\bar{u}|+|v-\bar{v}|).$$

Hence condition (H2) is satisfied with $K=L=\frac{1}{5e^2}$.
And let

$$I_1(u)=\frac{u}{10+u},\quad u\in[0,\infty).$$

Let $u,v\in[0,\infty)$. Then we have

$$|I_1(u)-I_1(v)|=\left|\frac{u}{10+u}-\frac{v}{10+v}\right|=\frac{10|u-v|}{(10+u)(10+v)}\leq\frac{1}{10}|u-v|.$$

Thus condition

$$\begin{aligned}\left(\frac{|b|}{|a+b|}+1\right)\left[m\widetilde{l}+\frac{(m+1)KT^{\alpha}}{(1-L)\Gamma(\alpha+1)}\right] &= 2\left[\frac{1}{10}+\frac{\frac{2}{5e^2}}{(1-\frac{1}{5e^2})\Gamma(\frac{3}{2})}\right]\\ &= 2\left[\frac{4}{(5e^2-1)\sqrt{\pi}}+\frac{1}{10}\right]<1,\end{aligned}$$

is satisfied with $T=1$, $a=2$, $b=-1$, $c=3$, $m=1$ and $\widetilde{l}=\frac{1}{10}$. It follows from Theorem 4.2.3 that the problem (4.18)–(4.20) has a unique solution on J.
Set for any $t\in[0,1]$, $\varphi(t)=t, \psi=1$.
Since

$$I^{\frac{1}{2}}\varphi(t)=\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^t (t-s)^{\frac{1}{2}-1}sds\leq\frac{2t}{\sqrt{\pi}},$$

then, condition (H6) is satisfied with $\lambda_\varphi=\frac{2}{\sqrt{\pi}}$. From which it follows that the problem (4.18)-(4.19) is Ulam-Hyers-Rassias stable with respect to (φ,ψ).

Example 2. Consider the following impulsive anti-periodic problem

$$^{c}D^{\frac{1}{2}}_{t_k}y(t)=\frac{2+|y(t)|+|^{c}D^{\frac{1}{2}}_{t_k}y(t)|}{108e^{t+3}(1+|y(t)|+|^{c}D^{\frac{1}{2}}_{t_k}y(t)|)},\ \text{for each},\ t\in J_0\cup J_1. \tag{4.21}$$

$$\Delta y|_{t=\frac{1}{3}}=\frac{|y(\frac{1}{3}^{-})|}{6+|y(\frac{1}{3}^{-})|}, \tag{4.22}$$

$$y(0)=-y(1), \tag{4.23}$$

where $J_0=\left[0,\frac{1}{3}\right]$, $J_1=\left(\frac{1}{3},1\right]$, $t_0=0$, and $t_1=\frac{1}{3}$.
Set

$$f(t,u,v)=\frac{2+|u|+|v|}{108e^{t+3}(1+|u|+|v|)},\quad t\in[0,1],\ u,v\in\mathbb{R}.$$

Clearly, the function f is jointly continuous.
For any $u,v,\bar{u},\bar{v}\in\mathbb{R}$ and $t\in[0,1]$:

$$|f(t,u,v)-f(t,\bar{u},\bar{v})|\leq\frac{1}{108e^3}(|u-\bar{u}|+|v-\bar{v}|).$$

Hence condition (H2) is satisfied with $K=L=\frac{1}{108e^3}$.
We have, for each $t\in[0,1]$,

$$|f(t,u,v)|\leq\frac{1}{108e^{t+3}}(2+|u|+|v|).$$

Thus condition (H4) is satisfied with $p(t)=\frac{1}{54e^{t+3}}$ and $q(t)=r(t)=\frac{1}{108e^{t+3}}$.
And let

$$I_1(u)=\frac{u}{6+u},\quad u\in[0,\infty).$$

We have, for each $u\in[0,\infty)$,

$$|I_1(u)|\leq\frac{1}{6}u+1$$

Thus condition (H5) is satisfied with $M^*=\frac{1}{6}$ and $N^*=1$. Thus condition

$$\left(\frac{|b|}{|a+b|}+1\right)\left(mM^*+\frac{(m+1)q^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\right)=\frac{3}{2}\left(\frac{1}{6}+\frac{4}{(108e^3-1)\sqrt{\pi}}\right)<1,$$

is satisfied with $T = 1$, $a = 1$, $b = 1$, $c = 0$, $m = 1$ and $q^*(t) = r^*(t) = \frac{1}{108e^3}$. It follows from Theorem 4.2.4 that the problem (4.21)–(4.23) has at least one solution on J.

4.3 Existence Results for the BVP with Impulses in Banach Space

[2]

4.3.1 Introduction

The purpose of this Section, is to establish existence and uniqueness results to the following boundary value problems for nonlinear implicit fractional differential equations with impulses in Banach space :

$$^cD^{\nu}_{t_k}y(t) = f(t, y, ^cD^{\nu}_{t_k}y(t)), \text{ for each , } t \in (t_k, t_{k+1}],\ k = 0, \dots, m,\ 0 < \nu \leq 1, \quad (4.24)$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)),\ k = 1, \dots, m, \quad (4.25)$$

$$ay(0) + by(T) = c, \quad (4.26)$$

where $^cD^{\nu}_{t_k}$ is the Caputo fractional derivative, $(E, ||\cdot||)$ is a real Banach space, $f : J \times E \times E \to E$ is a given function, $I_k : E \to E$, a, b are real constants with $a + b \neq 0$ and $c \in E$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k$.
In this Section, two results are discussed ; the first is based on Darbo's fixed point theorem combined with the technique of measures of noncompactness, the second on Mönch's fixed point theorem. At last, two examples are given to demonstrate the application of our main results.

4.3.2 Existence of Solutions

Let us defining what we mean by a solution of problem (4.24)-(4.26).

Definition 4.3.1 *A function $y \in PC(J, E)$ whose ν-derivative exists on J_k is said to be a solution of (4.24)–(4.26) if y satisfies the equation $^cD^{\nu}_{t_k}y(t) = f(t, y(t), ^cD^{\nu}_{t_k}y(t))$ on J_k, and satisfy the conditions*

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)),\ k = 1, \dots, m,$$
$$ay(0) + by(T) = c.$$

2. M. Benchohra and S. Bouriah , Boundary Value Problem for Nonlinear Implicit Fractional Differential Equations with Impulses in Banach Space.

To prove the existence of solutions to (4.24)–(4.26), we need the following auxiliary Lemma.

Lemma 4.3.2 *Let $0 < \nu \leq 1$ and let $\sigma : J \to E$ be a continuous function. A function y is a solution of the fractional integral equation*

$$y(t) = \begin{cases} \dfrac{-1}{a+b}\left[b\displaystyle\sum_{i=1}^{m} I_i(y(t_i^-)) + \dfrac{b}{\Gamma(\nu)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\nu-1}\sigma(s)ds \right. \\ \left. + \dfrac{b}{\Gamma(\nu)}\displaystyle\int_{t_m}^{T}(T-s)^{\nu-1}\sigma(s)ds - c\right] + \dfrac{1}{\Gamma(\nu)}\displaystyle\int_0^t (t-s)^{\nu-1}\sigma(s)ds \qquad \text{if } t \in [0,t_1] \\ \\ \dfrac{-1}{a+b}\left[b\displaystyle\sum_{i=1}^{m} I_i(y(t_i^-)) + \dfrac{b}{\Gamma(\nu)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\nu-1}\sigma(s)ds \right. \\ \left. + \dfrac{b}{\Gamma(\nu)}\displaystyle\int_{t_m}^{T}(T-s)^{\nu-1}\sigma(s)ds - c\right] + \displaystyle\sum_{i=1}^{k} I_i(y(t_i^-)) + \dfrac{1}{\Gamma(\nu)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\nu-1}\sigma(s)ds \\ +\dfrac{1}{\Gamma(\nu)}\displaystyle\int_{t_k}^{t}(t-s)^{\nu-1}\sigma(s)ds, \qquad \text{if } t \in (t_k, t_{k+1}], \end{cases} \tag{4.27}$$

where $k = 1, \ldots, m$, if and only if y is a solution of the fractional BVP

$$\begin{aligned} {}^cD^{\nu}y(t) &= \sigma(t), \quad t \in J_k, \\ \Delta y|_{t=t_k} &= I_k(y(t_k^-)), \quad k = 1, \ldots, m, \\ ay(0) + by(T) &= c. \end{aligned}$$

This lemma was already proved in the previous section.

First we list the following hypotheses :

(P1) The function $f : J \times E \times E \to E$ is continuous.

(P2) There exist constants $K > 0$ and $0 < L < 1$ such that

$$||f(t,u,v) - f(t,\bar{u},\bar{v})|| \leq K||u-\bar{u}|| + L||v-\bar{v}||$$

for any $u, \bar{u}, v, \bar{v} \in E$ and $t \in J$.

(P3) There exists a constant $\widetilde{l} > 0$ such that

$$\|I_k(u) - I_k(\overline{u})\| \leq \widetilde{l}\|u - \overline{u}\|,$$

for each $u, \overline{u} \in E$ and $k = 1, \ldots, m$.

We are now in a position to state and prove our existence result for the problem (4.24)-(4.26) based on concept of measures of noncompactness and *Darbo's* fixed point theorem.

Remark 4.3.3 *[25] Conditions $(P2)$ and $(P3)$ are respectively equivalent to the inequalities*

$$\alpha\Big(f(t,B_1,B_2)\Big) \leq K\alpha(B_1) + L\alpha(B_2)$$

$$\alpha\Big(I_k(B_1)\Big) \leq \widetilde{l}\alpha(B_1),$$

for any bounded sets $B_1, B_2 \subseteq E$, for each $t \in J$ and $k = 1, \ldots, m$.

Theorem 4.3.4 *Assume (P1)–(P3) hold.*
If

$$\left(\frac{|b|}{|a+b|}+1\right)\left(m\widetilde{l}+\frac{(m+1)KT^{\nu}}{(1-L)\Gamma(\nu+1)}\right) < 1, \tag{4.28}$$

then the BVP (4.24)-(4.26) has at least one solution on J.

Proof.

Transform the problem (4.24)-(4.26) into a fixed point problem. Consider the operator $N : PC(J,E) \to PC(J,E)$ defined by

$$\begin{aligned} N(y)(t) &= \frac{-1}{a+b}\left[b\sum_{i=1}^{m} I_i(y(t_i^-)) + \frac{b}{\Gamma(\nu)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\nu-1}g(s)ds\right. \\ &+ \left.\frac{b}{\Gamma(\nu)}\int_{t_m}^{T}(T-s)^{\nu-1}g(s)ds - c\right] + \frac{1}{\Gamma(\nu)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\nu-1}g(s)ds \\ &+ \frac{1}{\Gamma(\nu)}\int_{t_k}^{t}(t-s)^{\nu-1}g(s)ds + \sum_{0<t_k<t} I_k(y(t_k^-)), \end{aligned} \tag{4.29}$$

where $g \in C(J,E)$ be such that

$$g(t) = f(t, y(t), g(t)).$$

Clearly, the fixed points of operator N are solutions of problem (4.24)–(4.26).
We shall show that N satisfies the assumption of *Darbo's* fixed point Theorem. The proof will be given in several claims.

Claim 1 : *N is continuous.*
Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $PC(J,E)$. Then for each $t \in J$,

$$\begin{aligned}
\|N(u_n)(t) - N(u)(t)\| &\leq \frac{|b|}{|a+b|}\left[\sum_{i=1}^{m} \|I_k(u_n(t_k^-)) - I_k(u(t_k^-))\|\right. \\
&+ \frac{1}{\Gamma(\nu)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\nu-1}\|g_n(s)-g(s)\|ds \\
&+ \left.\frac{1}{\Gamma(\nu)}\int_{t_m}^{T}(T-s)^{\nu-1}\|g_n(s)-g(s)\|ds\right] \\
&+ \frac{1}{\Gamma(\nu)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\nu-1}\|g_n(s)-g(s)\|ds \\
&+ \frac{1}{\Gamma(\nu)}\int_{t_k}^{t}(t-s)^{\nu-1}\|g_n(s)-g(s)\|ds \\
&+ \sum_{0<t_k<t}\|I_k(u_n(t_k^-)) - I_k(u(t_k^-))\|,
\end{aligned} \tag{4.30}$$

where $g_n, g \in C(J,E)$ such that

$$g_n(t) = f(t, u_n(t), g_n(t)),$$

and

$$g(t) = f(t, u(t), g(t)).$$

By $(P2)$, we have

$$\begin{aligned}
\|g_n(t) - g(t)\| &= \|f(t,u_n(t),g_n(t)) - f(t,u(t),g(t))\| \\
&\leq K\|u_n(t)-u(t)\| + L\|g_n(t)-g(t)\|.
\end{aligned}$$

Then

$$\|g_n(t) - g(t)\| \leq \frac{K}{1-L}\|u_n(t)-u(t)\|.$$

Since $u_n \to u$, then we get $g_n(t) \to g(t)$ as $n \to \infty$ for each $t \in J$. And let $\eta > 0$ be such that, for each $t \in J$, we have $\|g_n(t)\| \leq \eta$ and $\|g(t)\| \leq \eta$. Then, we have

$$\begin{aligned}
(t-s)^{\nu-1}\|g_n(s)-g(s)\| &\leq (t-s)^{\nu-1}[\|g_n(s)\| + \|g(s)\|] \\
&\leq 2\eta(t-s)^{\nu-1},
\end{aligned}$$

and

$$\begin{aligned}
(t_k-s)^{\nu-1}\|g_n(s)-g(s)\| &\leq (t_k-s)^{\nu-1}[\|g_n(s)\| + \|g(s)\|] \\
&\leq 2\eta(t_k-s)^{\nu-1}.
\end{aligned}$$

For each $t \in J$, the functions $s \to 2\eta(t-s)^{\nu-1}$ and $s \to 2\eta(t_k-s)^{\nu-1}$ are integrable on $[0,t]$, then the Lebesgue Dominated Convergence Theorem and (4.30) imply that

$$\|N(u_n)(t) - N(u)(t)\| \to 0 \; as \; n \to \infty,$$

and hence

$$\|N(u_n) - N(u)\|_{PC} \to 0 \text{ as } n \to \infty.$$

Consequently, N is continuous.
Let the constant R such that :

$$R \geq \frac{\|c\|\Gamma(\nu+1)(1-L) + (|b| + |a+b|)\left[mc_1\Gamma(\nu+1)(1-L) + (m+1)T^\nu f^*\right]}{|a+b|\Gamma(\nu+1)(1-L) - (|b| + |a+b|)\left[m\widetilde{l}\Gamma(\nu+1)(1-L) + (m+1)T^\nu K\right]}, \tag{4.31}$$

where $c_1 = \sup\limits_{v\in E}\|I(v)\|$ and $f^* = \sup\limits_{t\in J}\|f(t,0,0)\|$.
Define

$$D_R = \{u \in PC(J,E) : \|u\|_{PC} \leq R\}.$$

It is clear that D_R is a bounded, closed and convex subset of $PC(J,E)$.

Claim 2 : $N(D_R) \subset D_R$.

Let $u \in D_R$ we show that $Nu \in D_R$.We have, for each $t \in J$

$$\begin{aligned}
\|N(y)(t)\| &\leq \frac{\|c\|}{|a+b|} + \frac{|b|}{|a+b|}\left[\sum_{i=1}^{m}\|I_i(y(t_i^-))\| + \frac{1}{\Gamma(\nu)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\nu-1}\|g(s)\|ds\right.\\
&+ \left.\frac{1}{\Gamma(\nu)}\int_{t_m}^{T}(T-s)^{\nu-1}\|g(s)\|ds\right] + \frac{1}{\Gamma(\nu)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\nu-1}\|g(s)\|ds\\
&+ \frac{1}{\Gamma(\nu)}\int_{t_k}^{t}(t-s)^{\nu-1}\|g(s)\|ds + \sum_{0<t_k<t}\|I_k(y(t_k^-))\|.
\end{aligned} \tag{4.32}$$

By (P2) we have for each $t \in J$,

$$\begin{aligned}
||g(t)|| &\leq \|f(t,u(t),g(t)) - f(t,0,0)\| + \|f(t,0,0)\|\\
&\leq K\|u(t)\| + L\|g(t)\| + f^*\\
&\leq K\|u(t)\|_{PC} + L\|g(t)\| + f^*\\
&\leq KR + L||g(t)|| + f^*
\end{aligned}$$

Then

$$||g(t)|| \leq \frac{f^* + KR}{1-L} := M.$$

Thus, $(4.31), (4.32)$ and (P3) implies that

$$\begin{aligned}
||Nu(t)|| &\leq \frac{\|c\|}{|a+b|} + \left(\frac{|b|}{|a+b|} + 1\right)\left(\sum_{i=1}^{m}\|I_i(y(t_i^-)) - I_i(0)\| + \sum_{i=1}^{m}\|I_i(0)\|\right)\\
&+ \left(\frac{|b|}{|a+b|} + 1\right)\frac{(m+1)T^\nu M}{\Gamma(\nu+1)}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\|c\|}{|a+b|} + \left(\frac{|b|}{|a+b|} + 1\right)\left[m(\widetilde{l}R + c_1) + \frac{(m+1)T^\nu M}{\Gamma(\nu+1)}\right] \\
&\leq R,
\end{aligned}$$

from which it follows that for each $t \in J$, we have $\|Nu(t)\| \leq R$.
Which implies that $\|Nu\|_{PC} \leq R$.
Consequently,

$$N(D_R) \subset D_R.$$

Claim 3 : *$N(D_R)$ is bounded and equicontinuous.*

By Claim 2 we have $N(D_R) = \{N(u) : u \in D_R\} \subset D_R$. Thus, for each $u \in D_R$ we have $\|N(u)\|_{PC} \leq R$ which means that $N(D_R)$ is bounded. Let $t_1, t_2 \in (0, T],\ t_1 < t_2$, and let $u \in D_R$. Then

$$\begin{aligned}
&\|N(u)(t_2) - N(u)(t_1)\| \\
&\leq \frac{1}{\Gamma(\nu)} \int_0^{t_1} |(t_2-s)^{\nu-1} - (t_1-s)^{\nu-1}| \|g(s)\| ds + \frac{1}{\Gamma(\nu)} \int_{t_1}^{t_2} |(t_2-s)^{\nu-1}| \|g(s)\| ds \\
&\quad + \sum_{0<t_k<t_2-t_1} \|I_k(u(t_k^-)) - I_k(0)\| + \sum_{0<t_k<t_2-t_1} \|I_k(0)\| \\
&\leq \frac{M}{\Gamma(\nu+1)}[2(t_2-t_1)^\nu + (t_2^\nu - t_1^\nu)] + (t_2-t_1)(\widetilde{l}\|u(t_k^-)\| + c_1) \\
&\leq \frac{M}{\Gamma(\nu+1)}[2(t_2-t_1)^\nu + (t_2^\nu - t_1^\nu)] + (t_2-t_1)(\widetilde{l}\|u\|_{PC} + c_1) \\
&\leq \frac{M}{\Gamma(\nu+1)}[2(t_2-t_1)^\nu + (t_2^\nu - t_1^\nu)] + (t_2-t_1)(\widetilde{l}R + c_1).
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero.

Claim 4 : *The operator $N : D_R \to D_R$ is a strict set contraction.*

Let $V \subset D_R$ and $t \in J$, then we have,

$$\begin{aligned}
\alpha(N(V)(t)) &= \alpha((Ny)(t), y \in V) \\
&\leq \frac{|b|}{|a+b|}\Bigg[\sum_{i=1}^{m}\Big\{\alpha(I_i(y(t_i^-))), y \in V\Big\} \\
&\qquad + \frac{1}{\Gamma(\nu)}\sum_{i=1}^{m}\Big\{\int_{t_{i-1}}^{t_i}(t_i-s)^{\nu-1}\alpha(g(s))ds, y \in V\Big\} \\
&\qquad + \frac{1}{\Gamma(\nu)}\Big\{\int_{t_m}^{T}(T-s)^{\nu-1}\alpha(g(s))ds, y \in V\Big\}\Bigg] \\
&\quad + \frac{1}{\Gamma(\nu)}\sum_{0<t_k<t}\Big\{\int_{t_{k-1}}^{t_k}(t_k-s)^{\nu-1}\alpha(g(s))ds, y \in V\Big\}
\end{aligned}$$

$$\begin{aligned}
&+ \frac{1}{\Gamma(\nu)}\left\{\int_{t_k}^{t}(t-s)^{\nu-1}\alpha(g(s))ds, y\in V\right\}\\
&+ \sum_{0<t_k<t}\left\{\alpha(I_k(y(t_k^-))), y\in V\right\}.
\end{aligned}$$

Then Remark 4.3.3 and Lemma 1.4.4 imply that, for each $s \in J$,

$$\begin{aligned}
\alpha(\{g(s), y\in V\}) &= \alpha(\{f(s, y(s), g(s)), y\in V\})\\
&\leq K\alpha(\{y(s), y\in V\}) + L\alpha(\{g(s), y\in V\}).
\end{aligned}$$

Thus

$$\alpha\left(\{g(s), y\in V\}\right) \leq \frac{K}{1-L}\alpha\{y(s), y\in V\}.$$

On the other hand, for each $t \in J$ and $k = 1, \ldots, m$, we have

$$\sum_{0<t_k<t}\alpha\left(\left\{I_k(y(t_k^-)), y\in V\right\}\right) \leq m\widetilde{l}\alpha(\{y(t), y\in V\}).$$

Then

$$\begin{aligned}
\alpha(N(V)(t)) \leq\ & \frac{|b|}{|a+b|}\Big[m\widetilde{l}\alpha(\{y(t), y\in V\})\\
&+\frac{mK}{\Gamma(\nu)(1-L)}\left\{\int_0^t(t-s)^{\nu-1}\{\alpha(y(s))\}ds, y\in V\right\}\\
&+\frac{K}{\Gamma(\nu)(1-L)}\left\{\int_0^T(T-s)^{\nu-1}\{\alpha(y(s))\}ds, y\in V\right\}\Big]\\
&+ \frac{mK}{\Gamma(\nu)(1-L)}\left\{\int_0^t(t-s)^{\nu-1}\{\alpha(y(s))\}ds, y\in V\right\}\\
&+ \frac{K}{\Gamma(\nu)(1-L)}\left\{\int_0^t(t-s)^{\nu-1}\{\alpha(y(s))\}ds, y\in V\right\}\\
&+ m\widetilde{l}\alpha(\{y(t), y\in V\})\\
\leq\ & \left(\frac{|b|}{|a+b|}+1\right)\left(m\widetilde{l}+\frac{(m+1)KT^{\nu}}{(1-L)\Gamma(\nu+1)}\right)\alpha_c(V).
\end{aligned}$$

Therefore

$$\alpha_c(NV) \leq \left(\frac{|b|}{|a+b|}+1\right)\left(m\widetilde{l}+\frac{(m+1)KT^{\nu}}{(1-L)\Gamma(\nu+1)}\right)\alpha_c(V).$$

So, by (4.28), the operator N is a set contraction. As a consequence of Theorem 1.5.3, we deduce that N has a fixed point which is solution to the problem (4.24)-(4.26).

□

Our next existence result for the problem (4.24)-(4.26) is based on concept of measures of noncompactness and *Mönch's* fixed point theorem.

Theorem 4.3.5 *Assume (P1)-(P3) and (4.28) hold.*
If

$$m\widetilde{l} < 1$$

Then the BVP (4.24)-(4.26) has at least one solution.

Proof. Consider the operator N defined in (4.29). We shall show that N satisfies the assumption of *Mönch's* fixed point theorem. We know that $N : D_R \to D_R$ is bounded and continuous, we need to prove that the implication

$$V = \overline{conv}N(V) \quad \text{or} \quad V = N(V) \cup \{0\} \Rightarrow \alpha(V) = 0$$

holds for every subset V of D_R. Now let V be a subset of D_R such that $V \subset \overline{conv}(N(V) \cup \{0\})$. V is bounded and equicontinuous and therefore the function $t \to v(t) = \alpha(V(t))$ is continuous on $[0, T]$.
Using Lemma 4.3.2, we can write for each $t \in J$ and $k = 0, \ldots, m$,

$$\begin{aligned} N(y(t)) &= y(0) + \sum_{i=1}^{k} I_i(y(t_i^-)) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} g(s) ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} g(s) ds, \end{aligned}$$

where $g \in C(J, \mathbb{R})$ be such that

$$g(t) = f(t, y(t), g(t)).$$

And by Remark 4.3.3, Lemma 1.4.8 and the properties of the measure α we have for each $t \in J$

$$\begin{aligned} v(t) &\leq \alpha(N(V)(t) \cup \{0\}) \\ &\leq \alpha(N(V)(t)) \\ &\leq \alpha\{(Ny)(t), y \in V\} \\ &\leq \alpha(y(0)) + \frac{mK}{\Gamma(\nu)(1-L)} \Big\{ \int_0^t (t-s)^{\nu-1} \{\alpha(y(s))\} ds, y \in V \Big\} \\ &+ \frac{K}{\Gamma(\nu)(1-L)} \Big\{ \int_0^t (t-s)^{\nu-1} \{\alpha(y(s))\} ds, y \in V \Big\} \\ &+ m\widetilde{l}\alpha(\{y(t), y \in V\}) \\ &\leq m\widetilde{l}\alpha(\{y(t), y \in V\}) + \frac{(m+1)K}{(1-L)\Gamma(\nu)} \left\{ \int_0^t (t-s)^{\nu-1} \{\alpha(y(s))\} ds, y \in V \right\} \end{aligned}$$

$$= m\widetilde{l}v(t) + \frac{(m+1)K}{(1-L)\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} v(s)ds.$$

Then

$$v(t) \leq \frac{(m+1)K}{(1-m\widetilde{l})(1-L)\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} v(s)ds.$$

Lemma 1.3.8 implies that $v(t) = 0$ for each $t \in J$.
Then $V(t)$is relatively compact in E. In view of the Ascoli-Arzelà theorem, V is relatively compact in D_R. Applying now Theorem 1.5.5 we conclude that N has a fixed point $y \in D_R$. Hence N has a fixed point which is solution to the problem (4.24)-(4.26). $\square$

Remark 4.3.6 *Our results for the boundary value problem* (4.24)-(4.26) *are appropriate for the following problems :*

- *Initial value problem :* $a = 1, b = 0, c = 0$.
- *Terminal value Problem :* $a = 0, b = 1, c$ *arbitrary.*
- *Anti-periodic problem :* $a = 1, b = 1, c = 0$.

However, our results are not applicable for the periodic problem, i.e. for $a = 1$, $b = -1$, $c = 0$.

4.3.3 Examples

Example 1. Consider the following infinite system

$$^cD_{t_k}^{\frac{1}{2}} y_n(t) = \frac{e^{-t}}{(11+e^t)} \left[\frac{y_n(t)}{1+y_n(t)} - \frac{^cD_{t_k}^{\frac{1}{2}} y_n(t)}{1+^c D_{t_k}^{\frac{1}{2}} y_n(t)} \right], \text{ for each, } t \in J_0 \cup J_1. \tag{4.33}$$

$$\Delta y_n|_{t=\frac{1}{2}} = \frac{y_n(\frac{1}{2}^-)}{10 + y_n(\frac{1}{2}^-)}. \tag{4.34}$$

$$2y_n(0) - y_n(1) = 3, \tag{4.35}$$

where $J_0 = \left[0, \frac{1}{2}\right]$, $J_1 = \left(\frac{1}{2}, 1\right]$, $t_0 = 0$, and $t_1 = \frac{1}{2}$.
Set

$$E = l^1 = \{y = (y_1, y_2, ..., y_n, ...), \sum_{n=1}^{\infty} |y_n| < \infty\}$$

$$f = (f_1, f_2, ..., f_n, ...),$$

such that

$$f(t,u,v) = \frac{e^{-t}}{(11+e^t)} \left[\frac{u}{1+u} - \frac{v}{1+v} \right], \ t \in [0,1], \ u, v \in E.$$

Clearly, the function f is jointly continuous.
E is a Banach space with the norm $||y|| = \sum_{n=1}^{\infty} |y_n|$.
For any $u, \bar{u}, v, \bar{v} \in E$ and $t \in [0,1]$:

$$||f(t,u,v) - f(t,\bar{u},\bar{v})|| \leq \frac{1}{12}(||u-\bar{u}|| + ||v-\bar{v}||).$$

Hence condition (P2) is satisfied with $K = L = \dfrac{1}{12}$.
And let

$$I_1(u) = \frac{u}{10+u}, \quad u \in E.$$

Let $u, v \in E$. Then we have

$$\|I_1(u) - I_1(v)\| = \Big\|\frac{u}{10+u} - \frac{v}{10+v}\Big\| \leq \frac{1}{10}\|u-v\|.$$

Hence the condition (P3) is satisfied with $\widetilde{l} = \dfrac{1}{10}$.
And the conditions

$$\begin{aligned}\left(\frac{|b|}{|a+b|}+1\right)\left(m\widetilde{l} + \frac{(m+1)KT^{\nu}}{(1-L)\Gamma(\nu+1)}\right) &= \frac{1}{10} + \frac{\frac{2}{12}}{(1-\frac{1}{12})\Gamma(\frac{3}{2})} \\ &= \frac{8}{11\sqrt{\pi}} + \frac{1}{5} < 1,\end{aligned}$$

are satisfied with $T = m = 1, a = 2, b = -1$ and $\nu = \dfrac{1}{2}$.
It follows from Theorem 4.3.4 that the problem (4.33)–(4.35) has at least one solution on J.

Example 2. Consider the following impulsive problem

$$^{c}D^{\frac{1}{2}}_{t_k} y_n(t) = \frac{2 + \|y_n(t)\| + \|^{c}D^{\frac{1}{2}}_{t_k} y_n(t)\|}{108e^{t+3}(1 + \|y_n(t)\| + \|^{c}D^{\frac{1}{2}}_{t_k} y_n(t)\|)}, \text{ for each, } t \in J_0 \cup J_1. \tag{4.36}$$

$$\Delta y_n|_{t=\frac{1}{3}} = \frac{\|y_n(\frac{1}{3}^-)\|}{6 + \|y_n(\frac{1}{3}^-)\|}, \tag{4.37}$$

$$y_n(0) = -y_n(1), \tag{4.38}$$

where $J_0 = \left[0, \frac{1}{3}\right]$, $J_1 = \left(\frac{1}{3}, 1\right]$, $t_0 = 0$, and $t_1 = \frac{1}{3}$.
Set

$$E = l^1 = \{y = (y_1, y_2, ..., y_n, ...), \sum_{n=1}^{\infty} |y_n| < \infty\},$$

$$f = (f_1, f_2, ..., f_n, ...),$$

such that

$$f(t,u,v) = \frac{2 + \|u\| + \|v\|}{108e^{t+3}(1 + \|u\| + \|v\|)}, \quad t \in [0,1], \ u, v \in E.$$

Clearly, the function f is jointly continuous.
E is a Banach space with the norm $||y|| = \sum_{n=1}^{\infty} |y_n|$.
For any $u, \bar{u}, v, \bar{v} \in E$ and $t \in [0,1]$:

$$||f(t,u,v) - f(t,\bar{u},\bar{v})|| \leq \frac{1}{108e^3}(||u - \bar{u}|| + ||v - \bar{v}||).$$

Hence condition (P2) is satisfied with $K = L = \frac{1}{108e^3}$.
And let

$$I_1(u) = \frac{\|u\|}{6 + \|u\|}, \quad u \in E.$$

Let $u, v \in E$. Then we have

$$\|I_1(u) - I_1(v)\| = \Big\|\frac{u}{6+u} - \frac{v}{6+v}\Big\| \leq \frac{1}{6}\|u - v\|.$$

Hence the condition (P3) is satisfied with $\widetilde{l} = \frac{1}{6}$.
The condition

$$\begin{aligned}\left(\frac{|b|}{|a+b|} + 1\right)\left(m\widetilde{l} + \frac{(m+1)KT^{\nu}}{(1-L)\Gamma(\nu+1)}\right) &= \frac{3}{2}\left(\frac{1}{6} + \frac{\frac{2}{12}}{(1-\frac{1}{12})\Gamma(\frac{3}{2})}\right) \\ &= \frac{6}{11\sqrt{\pi}} + \frac{1}{4} < 1,\end{aligned}$$

is satisfied with $T = m = 1, a = 1, b = 1$ and $\nu = \frac{1}{2}$.
Also, we have

$$m\widetilde{l} = \frac{1}{6} < 1.$$

It follows from Theorem 4.3.5 that the problem $(4.36) - (4.38)$ has at least one solution on

Chapitre 5

Existence and Stability Results for Nonlinear Implicit Fractional Differential Equations with Finite Delay and Impulses

5.1 Introduction and Motivations

In this chapter, we establish, in Section 5.2, existence, uniqueness and stability results to the following nonlinear implicit fractional differential equation with finite delay and impulses

$$^{c}D^{\alpha}_{t_k}y(t) = f(t, y_t, ^{c}D^{\alpha}_{t_k}y(t)), \text{ for each , } t \in (t_k, t_{k+1}],\ k = 0, \ldots, m,\ 0 < \alpha \leq 1,$$

$$\Delta y|_{t_k} = I_k(y_{t_k^-}),\ k = 1, \ldots, m,$$

$$y(t) = \varphi(t),\ t \in [-r, 0],\ r > 0$$

where $^{c}D^{\alpha}_{t_k}$ is the Caputo fractional derivative, $f : J \times PC([-r, 0], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ is a given function, $I_k : PC([-r, 0], \mathbb{R}) \to \mathbb{R}$, and $\varphi \in PC([-r, 0], \mathbb{R}), 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$.
For each function y_t defined on $[-r, T]$ and for any $t \in J$, we denote by y_t the element of $PC\left([-r, 0], \mathbb{R}\right)$ defined by :

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0],$$

$y_t(.)$ represent the history of the state from time $t - r$ up to time t.
Here $\Delta y|_{t_k} = y(t_k^+) - y(t_k^-)$, where $y(t_k^+) = \lim_{h \to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h \to 0^-} y(t_k + h)$ represent the right and left limits of y_t at $t = t_k$, respectively.

An extension of this problem is given in Section 5.3. More precisely, we shall present a result of existence and uniqueness for the implicit fractional differential equation with finite delay and impulses in Banach space

$$^cD^{\nu}_{t_k}y(t) = f(t, y_t, {}^cD^{\nu}_{t_k}y(t)), \text{ for each }, \ t \in (t_k, t_{k+1}], \ k = 0, \dots, m, \ 0 < \nu \leq 1,$$

$$\Delta y|_{t=t_k} = I_k(y_{t_k^-}), \ k = 1, \dots, m,$$

$$y(t) = \varphi(t), \ t \in [-r, 0], \ r > 0$$

where $^cD^{\nu}_{t_k}$ is the Caputo fractional derivative, $(E, ||\cdot||)$ is a real Banach space, $f : J \times PC([-r, 0], E) \times E \to E$ is a given function, $I_k : PC([-r, 0], E) \to E$, and $\varphi \in PC([-r, 0], E), 0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$.

Impulsive fractional differential equations are a very important class of fractional differential equations because many phenomena from physics, chemistry, engineering, biology, etc... can be represented by the impulsive fractional differential equations.
On the other hand, the theory of impulsive differential equations describes the process subject to abrupt change in their states at times. Impulsive differential equations have received much attention, we refer the reader to books [30, 31, 55, 79, 103, 121, 133], and the papers [66, 84, 85, 138], the references therein.

In [61], Benchohra and Slimani considered the existence and uniqueness of solutions for the initial value problems with impulses,

$$\begin{aligned} {}^cD^{\alpha}y(t) &= f(t, y(t)), \quad t \in J = [0, T], \ t \neq t_k, \ 0 < \alpha \leq 1, \\ \Delta y\big|_{t=t_k} &= I_k(y(t_k^-)), \\ y(0) &= y_0, \end{aligned}$$

where $k = 1, \dots, m$, $^cD^{\alpha}$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \to \mathbb{R}$ is a given function, $I_k : \mathbb{R} \to \mathbb{R}$, and $y_0 \in \mathbb{R}$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k$.

In [60], Benchohra and Seba, using Mönch's fixed point theorem combined with the technique of measures of noncompactness, considered the existence and uniqueness of solutions for the initial value problems with impulses,

$$\begin{aligned} {}^cD^{\alpha}y(t) &= f(t, y(t)), \quad t \in J = [0, T], \ t \neq t_k, \ 0 < \alpha \leq 1, \\ \Delta y\big|_{t=t_k} &= I_k(y(t_k^-)), \\ y(0) &= y_0, \end{aligned}$$

where $k = 1, \dots, m$, $^cD^{\alpha}$ is the Caputo fractional derivative, $f : J \times E \to E$ is a given function, $I_k : E \to E$, $y_0 \in E$, E is a Banach space, and $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$.

In [5], Agarwal *et al.* studied the existence and uniqueness of solutions for the initial value problems, for fractional order differential equations with impulses

$$\begin{aligned} {}^cD^\alpha y(t) = f(t, y(t)), \quad t \in J = [0,T],\ t \neq t_k,\ k = 1,\dots,m,\ 1 < \alpha \leq 2, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1,\dots,m, \\ \Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1,\dots,m, \\ y(0) = y_0, \quad y'(0) = y_1, \end{aligned}$$

where $k = 1,\dots,m$, ${}^cD^\alpha$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \to \mathbb{R}$ is a given function, $I_k : \mathbb{R} \to \mathbb{R}$, $y_0 \in \mathbb{R}$ and $y_1 \in \mathbb{R}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $\Delta y'|_{t=t_k} = y'(t_k^+) - y'(t_k^-)$.

In [53], Benchohra *et al.* discussed the existence of solutions for the initial value problems, for fractional order differential inclusions,

$$\begin{aligned} {}^cD^\alpha y(t) \in F(t, y(t)), \quad t \in J = [0,T],\ t \neq t_k,\ k = 1,\dots,m,\ 1 < \alpha \leq 2, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1,\dots,m, \\ \Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1,\dots,m, \\ y(0) = y_0, \quad y'(0) = y_1, \end{aligned}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative, $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$), I_k and $\overline{I}_k : \mathbb{R} \to \mathbb{R}$, $k = 1,\dots,m$, and $y_0, y_1 \in \mathbb{R}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $\Delta y'|_{t=t_k} = y'(t_k^+) - y'(t_k^-)$.

5.2 Existence Results for the NIFDE with Finite Delay and Impulses

[1]

5.2.1 Introduction

In this Section, we establish, existence, uniqueness and stability results of solutions for the following problem for nonlinear implicit fractional differential equations with finite delay and impulses :

$$ {}^cD^\alpha_{t_k} y(t) = f(t, y_t, {}^c D^\alpha_{t_k} y(t)), \text{ for each } , \ t \in (t_k, t_{k+1}],\ k = 0,\dots,m,\ 0 < \alpha \leq 1, \quad (5.1)$$

1. W. Albarakati, M. Benchohra and S. Bouriah, Existence and Stability Results for Nonlinear Implicit Fractional Differential Equations with Delay and Impulses, *Differential Equations and Applications* Vol. 8 No. 2 (2016), 273-293.

$$\Delta y|_{t_k} = I_k(y_{t_k^-}),\ k = 1, \ldots, m, \tag{5.2}$$

$$y(t) = \varphi(t),\ t \in [-r, 0],\ r > 0 \tag{5.3}$$

where ${}^cD^{\alpha}_{t_k}$ is the Caputo fractional derivative, $f : J \times PC([-r, 0], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ is a given function, $I_k : PC([-r, 0], \mathbb{R}) \to \mathbb{R}$, and $\varphi \in PC([-r, 0], \mathbb{R}), 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$.
For each function y_t defined on $[-r, T]$ and for any $t \in J$, we denote by y_t the element of $PC\left([-r, 0], \mathbb{R}\right)$ defined by :

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0],$$

$y_t(.)$ represent the history of the state from time $t - r$ up to time t.
Here $\Delta y|_{t_k} = y(t_k^+) - y(t_k^-)$, where $y(t_k^+) = \lim_{h \to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h \to 0^-} y(t_k + h)$ represent the right and left limits of y_t at $t = t_k$, respectively.
The arguments are based upon the Banach contraction principle, and Schaefer's fixed point theorem. We present two examples to show the applicability of our results.

5.2.2 Existence of Solutions

Denote by $C(J, \mathbb{R})$ the Banach space of continuous functions from J into $\mathbb{R}$, with the usual supremum norm

$$\|y\|_{\infty} = sup\{|y(t)|, \quad t \in J\}.$$

Let $J_0 = [t_0, t_1]$ and $J_k = (t_k, t_{k+1}]$ where $k = 1, \ldots, m$.
Consider the set of functions

$$\begin{aligned} PC([-r,0],\mathbb{R}) \ &= \{y : [-r, 0] \to \mathbb{R} : y \in C((\tau_k, \tau_{k+1}], \mathbb{R}),\ k = 0, \ldots, m \text{ and there exist} \\ y(\tau_k^-) \text{ and } y(\tau_k^+),\ k = 1, \ldots, m' &\text{ with } y(\tau_k^-) = y(\tau_k) \text{ and } \tau_k = t_k - t, \ \text{ for each } t \in J_k\}. \end{aligned}$$

$PC([-r, 0], \mathbb{R})$ is a Banach space with the norm

$$\|y\|_{PC} = \sup_{t \in [-r,0]} |y(t)|.$$

$$PC([0, T], \mathbb{R}) = \{y : [0, T] \to \mathbb{R} : y \in C((t_k, t_{k+1}], \mathbb{R}),\ k = 1, ..., m, \text{ and there exist } y(t_k^-) \text{ and } y(t_k^+), k = 1, ..., m \text{ with } y(t_k^-) = y(t_k)\}.$$

$PC([0, T], \mathbb{R})$ is a Banach space with the norm

$$\|y\|_{C} = \sup_{t \in [0,T]} |y(t)|.$$

$$\Omega = \{y : [-r, T] \to \mathbb{R} : y|_{[-r,0]} \in PC([-r, 0], \mathbb{R}) \text{ and } y|_{[0,T]} \in PC([0, T], \mathbb{R})\}.$$

Ω is a Banach space with the norm

$$\|y\|_{\Omega} = \sup_{t \in [-r,T]} |y(t)|.$$

Definition 5.2.1 A function $y \in \Omega$ whose α-derivative exists on J_k is said to be a solution of (5.1)–(5.3) if y satisfies the equation ${}^cD^{\alpha}_{t_k}y(t) = f(t, y_t, {}^c D^{\alpha}_{t_k}y(t))$ on J_k, and satisfy the conditions

$$\Delta y|_{t=t_k} = I_k(y_{t_k^-}),\ k = 1, \dots, m,$$
$$y(t) = \varphi(t),\ t \in [-r, 0].$$

To prove the existence of solutions to (5.1)–(5.3), we need the following auxiliary Lemma.

Lemma 5.2.2 *Let $0 < \alpha \leq 1$ and let $\sigma : J \to \mathbb{R}$ be continuous. A function y is a solution of the fractional integral equation*

$$y(t) = \begin{cases} \varphi(0) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1}\sigma(s)ds & \text{if } t \in [0, t_1] \\[2ex] \varphi(0) + \displaystyle\sum_{i=1}^{k} I_i(y_{t_i^-}) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1}\sigma(s)ds \\ + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_k}^{t} (t-s)^{\alpha-1}\sigma(s)ds, & \text{if } t \in (t_k, t_{k+1}], \\[2ex] \varphi(t),\ t \in [-r, 0], \end{cases} \tag{5.4}$$

where $k = 1, \dots, m$, if and only if y is a solution of the following fractional problem

$${}^cD^{\alpha}_{t_k}y(t) = \sigma(t), \quad t \in J_k, \tag{5.5}$$
$$\Delta y|_{t=t_k} = I_k(y_{t_k^-}), \quad k = 1, \dots, m, \tag{5.6}$$
$$y(t) = \varphi(t),\ t \in [-r, 0]. \tag{5.7}$$

Proof. Assume y satisfies (5.5)-(5.7). If $t \in [0, t_1]$ then

$${}^cD^{\alpha}y(t) = \sigma(t).$$

Lemma 1.3.7 implies

$$y(t) = \varphi(0) + I^{\alpha}\sigma(t) = \varphi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\sigma(s)ds.$$

If $t \in (t_1, t_2]$ then Lemma 1.3.7 implies

$$y(t) \ = \ y(t_1^+) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1}\sigma(s)ds$$

$$
\begin{aligned}
&= \Delta y|_{t=t_1} + y(t_1^-) + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(t-s)^{\alpha-1}\sigma(s)ds \\
&= I_1(y_{t_1^-}) + \left[\varphi(0) + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\sigma(s)ds\right] \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(t-s)^{\alpha-1}\sigma(s)ds. \\
&= \varphi(0) + I_1(y_{t_1^-}) + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\sigma(s)ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(t-s)^{\alpha-1}\sigma(s)ds.
\end{aligned}
$$

If $t \in (t_2, t_3]$, then from Lemma 1.3.7, we get

$$
\begin{aligned}
y(t) &= y(t_2^+) + \frac{1}{\Gamma(\alpha)}\int_{t_2}^{t}(t-s)^{\alpha-1}\sigma(s)ds \\
&= \Delta y|_{t=t_2} + y(t_2^-) + \frac{1}{\Gamma(\alpha)}\int_{t_2}^{t}(t-s)^{\alpha-1}\sigma(s)ds \\
&= I_2(y_{t_2^-}) + \Bigg[\varphi(0) + I_1(y_{t_1^-}) + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\sigma(s)ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\sigma(s)ds\Bigg] + \frac{1}{\Gamma(\alpha)}\int_{t_2}^{t}(t-s)^{\alpha-1}\sigma(s)ds. \\
&= \varphi(0) + \left[I_1(y_{t_1^-}) + I_2(y_{t_2^-})\right] + \Bigg[\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\sigma(s)ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\sigma(s)ds\Bigg] + \frac{1}{\Gamma(\alpha)}\int_{t_2}^{t}(t-s)^{\alpha-1}\sigma(s)ds.
\end{aligned}
$$

Repeating the process in this ways, the solution $y(t)$ for $t \in (t_k, t_{k+1}]$ where $k = 1, \ldots, m$, can be written as

$$
\begin{aligned}
y(t) &= \varphi(0) + \sum_{i=1}^{k} I_i(y_{t_i^-}) + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\sigma(s)ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\sigma(s)ds.
\end{aligned}
$$

Conversely, assume that y satisfies the impulsive fractional integral equation (5.4). If $t \in [0, t_1]$ then $y(0) = \varphi(0)$ and using the fact that ${}^cD^\alpha$ is the left inverse of I^α we get

$$
{}^cD^\alpha y(t) = \sigma(t), \quad \text{for each } t \in [0, t_1].
$$

If $t \in (t_k, t_{k+1}]$, $k = 1, \ldots, m$ and using the fact that ${}^cD^\alpha C = 0$, where C is a constant, we get

$$
{}^cD^\alpha y(t) = \sigma(t), \text{for each } t \in (t_k, t_{k+1}].
$$

Also, we can easily show that

$$\Delta y|_{t=t_k} = I_k(y_{t_k^-}), \quad k = 1, \dots, m.$$

□

We are now in a position to state and prove our existence result for the problem (5.1)–(5.3) based on Banach's fixed point.

Theorem 5.2.3 *Assume*

(H1) The function $f : J \times PC([-r,0],\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H2) There exist constants $K > 0$ and $0 < L < 1$ such that

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \leq K\|u-\bar{u}\|_{PC} + L|v-\bar{v}|$$

for any $u,\bar{u} \in PC([-r,0],\mathbb{R})$, $v,\bar{v} \in \mathbb{R}$ and $t \in J$.

(H3) There exists a constant $\widetilde{l} > 0$ such that

$$|I_k(u) - I_k(\overline{u})| \leq \widetilde{l}\|u-\overline{u}\|_{PC},$$

for each $u,\overline{u} \in PC([-r,0],\mathbb{R})$ and $k = 1,\dots,m$.

If

$$m\widetilde{l} + \frac{(m+1)KT^\alpha}{(1-L)\Gamma(\alpha+1)} < 1, \tag{5.8}$$

then there exists a unique solution for the problem (5.1)-(5.3) *on J.*

Proof. Transform the problem (5.1)-(5.3) into a fixed point problem. Consider the operator $N : \Omega \to \Omega$ defined by

$$Ny(t) = \begin{cases} \varphi(0) + \displaystyle\sum_{0<t_k<t} I_k(y_{t_i^-}) + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k-s)^{\alpha-1} g(s)ds \\ +\displaystyle\frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} g(s)ds, & t \in [0,T], \\ \\ \varphi(t), \qquad\qquad t \in [-r,0], \end{cases} \tag{5.9}$$

where $g \in C(J,\mathbb{R})$ be such that

$$g(t) = f(t, y_t, g(t)).$$

Clearly, the fixed points of operator N are solutions of problem (5.1)–(5.3).
Let $u, w \in \Omega$. If $t \in [-r,0]$, then

$$|N(u)(t) - N(w)(t)| = 0.$$

For $t \in J$, we have

$$\begin{aligned}
|N(u)(t) - N(w)(t)| &\leq \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} |g(s) - h(s)| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} |g(s) - h(s)| ds \\
&+ \sum_{0<t_k<t} |I_k(u_{t_k^-}) - I_k(w_{t_k^-})|,
\end{aligned}$$

where $g, h \in C(J, \mathbb{R})$ be such that

$$g(t) = f(t, u_t, g(t)),$$

and

$$h(t) = f(t, w_t, h(t)).$$

By $(H2)$ we have

$$\begin{aligned}
|g(t) - h(t)| &= |f(t, u_t, g(t)) - f(t, w_t, h(t))| \\
&\leq K\|u_t - w_t\|_{PC} + L|g(t) - h(t)|.
\end{aligned}$$

Then

$$|g(t) - h(t)| \leq \frac{K}{1-L} \|u_t - w_t\|_{PC}.$$

Therefore, for each $t \in J$

$$\begin{aligned}
|N(u)(t) - N(w)(t)| &\leq \frac{K}{(1-L)\Gamma(\alpha)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|u_s - w_s\|_{PC} ds \\
&+ \frac{K}{(1-L)\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} \|u_s - w_s\|_{PC} ds \\
&+ \sum_{k=1}^{m} \widetilde{l} \|u_{t_k^-} - w_{t_k^-}\|_{PC}. \\
&\leq \left[m\widetilde{l} + \frac{mKT^\alpha}{(1-L)\Gamma(\alpha+1)} + \frac{KT^\alpha}{(1-L)\Gamma(\alpha+1)} \right] \|u - w\|_\Omega.
\end{aligned}$$

Thus

$$\|N(u) - N(w)\|_\Omega \leq \left[m\widetilde{l} + \frac{(m+1)KT^\alpha}{(1-L)\Gamma(\alpha+1)} \right] \|u - w\|_\Omega.$$

By (5.8), the operator N is a contraction. Hence, by Banach's contraction principle, N has a unique fixed point which is a unique solution of the problem (5.1)–(5.3).

□

Our second result is based on Schaefer's fixed point theorem.

Theorem 5.2.4 *Assume (H1), (H2) and*

(H4) There exist $p, q, r \in C(J, \mathbb{R}_+)$ with $r^* = \sup_{t\in J} r(t) < 1$ such that

$$|f(t,u,w)| \leq p(t) + q(t)\|u\|_{PC} + r(t)|w| \text{ for } t \in J,\ u \in PC([-r,0],\mathbb{R}) \text{ and } w \in \mathbb{R}.$$

(H5) The functions $I_k : PC([-r,0],\mathbb{R}) \to \mathbb{R}$ are continuous and there exist constants $M^*, N^* > 0$ with $mM^* < 1$ such that

$$|I_k(u)| \leq M^*\|u\|_{PC} + N^* \text{ for each } u \in PC([-r,0],\mathbb{R}),\ k = 1,\dots,m.$$

Then, the problem (5.1)-(5.3) *has at least one solution.*

Proof. Let the operator N defined in (5.9). We shall use Schaefer's fixed point theorem to prove that N has a fixed point. The proof will be given in several steps.

Step 1 : *N is continuous.* Let $\{u_n\}$ be a sequence such that $u_n \to u$ in Ω. If $t \in [-r,0]$, then

$$|N(u_n)(t) - N(u)(t)| = 0.$$

For $t \in J$, we have

$$\begin{aligned} |N(u_n)(t) - N(u)(t)| &\leq \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} |g_n(s) - g(s)| ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |g_n(s) - g(s)| ds \\ &+ \sum_{0<t_k<t} |I_k(u_{nt_k^-}) - I_k(u_{t_k^-})|, \end{aligned} \tag{5.10}$$

where $g_n, g \in C(J,\mathbb{R})$ such that

$$g_n(t) = f(t, u_{nt}, g_n(t)),$$

and

$$g(t) = f(t, u_t, g(t)).$$

By $(H2)$, we have

$$\begin{aligned} |g_n(t) - g(t)| &= |f(t, u_{nt}, g_n(t)) - f(t, u_t, g(t))| \\ &\leq K\|u_{nt} - u_t\|_{PC} + L|g_n(t) - g(t)|. \end{aligned}$$

Then

$$|g_n(t) - g(t)| \leq \frac{K}{1-L}\|u_{nt} - u_t\|_{PC}.$$

Since $u_n \to u$, then we get $g_n(t) \to g(t)$ as $n \to \infty$ for each $t \in J$. And let $\eta > 0$ be such that, for each $t \in J$, we have $|g_n(t)| \leq \eta$ and $|g(t)| \leq \eta$. Then, we have

$$\begin{array}{rcl} (t-s)^{\alpha-1}|g_n(s)-g(s)| & \leq & (t-s)^{\alpha-1}[|g_n(s)|+|g(s)|] \\ & \leq & 2\eta(t-s)^{\alpha-1}, \end{array}$$

and

$$\begin{array}{rcl} (t_k-s)^{\alpha-1}|g_n(s)-g(s)| & \leq & (t_k-s)^{\alpha-1}[|g_n(s)|+|g(s)|] \\ & \leq & 2\eta(t_k-s)^{\alpha-1}. \end{array}$$

For each $t \in J$, the functions $s \to 2\eta(t-s)^{\alpha-1}$ and $s \to 2\eta(t_k-s)^{\alpha-1}$ are integrable on $[0,t]$, then the Lebesgue Dominated Convergence Theorem and (5.10) imply that

$$|N(u_n)(t) - N(u)(t)| \to 0 \ as\ n \to \infty,$$

and hence

$$\|N(u_n) - N(u)\|_\Omega \to 0 \text{ as } n \to \infty.$$

Consequently, N is continuous.

Step 2 : *N maps bounded sets into bounded sets in Ω.* Indeed, it is enough to show that for any $\eta^* > 0$, there exists a positive constant ℓ such that for each $u \in B_{\eta^*} = \{u \in \Omega : ||u||_\Omega \leq \eta^*\}$, we have $\|N(u)\|_\Omega \leq \ell$. We have for each $t \in J$,

$$\begin{array}{rcl} N(u)(t) & = & \varphi(0) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k-s)^{\alpha-1} g(s)ds \\ & + & \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_k}^{t} (t-s)^{\alpha-1} g(s)ds + \sum_{0<t_k<t} I_k(u_{t_k^-}), \end{array} \tag{5.11}$$

where $g \in C(J,\mathbb{R})$ be such that

$$g(t) = f(t, u_t, g(t)).$$

By (H4), we have for each $t \in J$,

$$\begin{array}{rcl} |g(t)| & = & |f(t,u_t,g(t))| \\ & \leq & p(t) + q(t)\|u_t\|_{PC} + r(t)|g(t)| \\ & \leq & p(t) + q(t)\|u\|_\Omega + r(t)|g(t)| \\ & \leq & p(t) + q(t)\eta^* + r(t)|g(t)| \\ & \leq & p^* + q^*\eta^* + r^*|g(t)|, \end{array}$$

where $p^* = \sup\limits_{t\in J} p(t)$, and $q^* = \sup\limits_{t\in J} q(t)$.
Then

$$|g(t)| \leq \frac{p^* + q^*\eta^*}{1-r^*} := M.$$

Thus (5.11) implies

$$\begin{aligned}|N(u)(t)| &\leq |\varphi(0)| + \frac{mMT^\alpha}{\Gamma(\alpha+1)} + \frac{MT^\alpha}{\Gamma(\alpha+1)} + m(M^*\|u_{t_k^-}\|_{PC} + N^*)\\ &\leq |\varphi(0)| + \frac{(m+1)MT^\alpha}{\Gamma(\alpha+1)} + m(M^*\|u\|_\Omega + N^*)\\ &\leq |\varphi(0)| + \frac{(m+1)MT^\alpha}{\Gamma(\alpha+1)} + m(M^*\eta^* + N^*) := R.\end{aligned}$$

And if $t \in [-r, 0]$, then

$$|N(u)(t)| \leq \|\varphi\|_{PC},$$

thus

$$\|N(u)\|_\Omega \leq \max\ \{R, \|\varphi\|_{PC}\} := \ell.$$

Step 3 : *N maps bounded sets into equicontinuous sets of Ω.*

Let $t_1, t_2 \in (0, T]$, $t_1 < t_2$, B_{η^*} be a bounded set of Ω as in Step 2, and let $u \in B_{\eta^*}$. Then

$$\begin{aligned}&|N(u)(t_2) - N(u)(t_1)|\\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1} |(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}||g(s)|ds\\ &\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} |(t_2-s)^{\alpha-1}||g(s)|ds + \sum_{0<t_k<t_2-t_1} |I_k(u_{t_k^-})|\\ &\leq \frac{M}{\Gamma(\alpha+1)}[2(t_2-t_1)^\alpha + (t_2^\alpha - t_1^\alpha)] + (t_2-t_1)(M^*\|u_{t_k^-}\|_{PC} + N^*)\\ &\leq \frac{M}{\Gamma(\alpha+1)}[2(t_2-t_1)^\alpha + (t_2^\alpha - t_1^\alpha)] + (t_2-t_1)(M^*\|u\|_\Omega + N^*)\\ &\leq \frac{M}{\Gamma(\alpha+1)}[2(t_2-t_1)^\alpha + (t_2^\alpha - t_1^\alpha)] + (t_2-t_1)(M^*\eta^* + N^*).\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Ascoli-Arzela theorem, we can conclude that $N : \Omega \to \Omega$ is completely continuous.

Step 4 : *A priori bounds.* Now it remains to show that the set

$$E = \{u \in \Omega : u = \lambda N(u) \text{ for some } \ 0 < \lambda < 1\}$$

is bounded. Let $u \in E$, then $u = \lambda N(u)$ for some $0 < \lambda < 1$. Thus, for each $t \in J$ we have

$$\begin{aligned}u(t) &= \lambda\varphi(0) + \frac{\lambda}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k} (t_k-s)^{\alpha-1}g(s)ds\\ &+ \frac{\lambda}{\Gamma(\alpha)}\int_{t_k}^{t} (t-s)^{\alpha-1}g(s)ds + \lambda\sum_{0<t_k<t} I_k(u_{t_k^-}).\end{aligned} \tag{5.12}$$

And, by (H4), we have for each $t \in J$,

$$\begin{aligned} |g(t)| &= |f(t, u_t, g(t))| \\ &\leq p(t) + q(t)\|u_t\|_{PC} + r(t)|g(t)| \\ &\leq p^* + q^*\|u_t\|_{PC} + r^*|g(t)|. \end{aligned}$$

Thus

$$|g(t)| \leq \frac{1}{1-r^*}(p^* + q^*\|u_t\|_{PC}).$$

This implies, by (5.12) and (H5), that for each $t \in J$ we have

$$\begin{aligned} |u(t)| &\leq |\varphi(0)| + \frac{1}{(1-r^*)\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}(p^* + q^*\|u_s\|_{PC})ds \\ &+ \frac{1}{(1-r^*)\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}(p^* + q^*\|u_s\|_{PC})ds \\ &+ m(M^*\|u_{t_k^-}\|_{PC} + N^*). \end{aligned}$$

Consider the function ζ defined by

$$\zeta(t) = \sup\{|u(s)| : -r \leq s \leq t\}, 0 \leq t \leq T,$$

then, there exists $t^* \in [-r, T]$ such that $\zeta(t) = |u(t^*)|$. If $t^* \in [0, T]$,then by the previous inequality, we have for $t \in J$

$$\begin{aligned} \zeta(t) &\leq |\varphi(0)| + \frac{1}{(1-r^*)\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}(p^* + q^*\zeta(s))ds \\ &+ \frac{1}{(1-r^*)\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}(p^* + q^*\zeta(s))ds \\ &+ mM^*\zeta(t) + mN^*. \end{aligned}$$

Thus

$$\begin{aligned} \zeta(t) &\leq \frac{|\varphi(0)| + mN^*}{1-mM^*} + \frac{1}{(1-mM^*)(1-r^*)\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}(p^* + q^*\zeta(s))ds \\ &+ \frac{1}{(1-mM^*)(1-r^*)\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}(p^* + q^*\zeta(s))ds \\ &\leq \frac{|\varphi(0)| + mN^*}{1-mM^*} + \frac{(m+1)p^*T^\alpha}{(1-mM^*)(1-r^*)\Gamma(\alpha+1)} \\ &+ \frac{(m+1)q^*}{(1-mM^*)(1-r^*)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\zeta(s)ds. \end{aligned}$$

Applying Lemma 1.3.8, we get

$$\begin{aligned}\zeta(t) &\leq \left[\frac{|\varphi(0)|+mN^*}{1-mM^*}+\frac{(m+1)p^*T^\alpha}{(1-mM^*)(1-r^*)\Gamma(\alpha+1)}\right]\\ &\times \left[1+\frac{\delta(m+1)q^*T^\alpha}{(1-mM^*)(1-r^*)\Gamma(\alpha+1)}\right]:=A,\end{aligned}$$

where $\delta=\delta(\alpha)$ a constant. If $t^*\in[-r,0]$, then $\zeta(t)=\|\varphi\|_{PC}$, thus for any $t\in J$, $\|u\|_\Omega\leq\zeta(t)$, we have

$$\|u\|_\Omega\leq\max\{\|\varphi\|_{PC},A\}$$

This shows that the set E is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that N has a fixed point which is a solution of the problem $(5.1)-(5.3)$. □

5.2.3 Ulam-Hyers Rassias stability

Here, we adopt the concepts in Wang et it [139] and introduce Ulam's type stability concepts for the problem (5.1)-(5.2).
Let $z\in\Omega,\epsilon>0,\psi>0$ and $\omega\in PC(J,\mathbb{R}_+)$ is nondecreasing. We consider the set of inequalities

$$\begin{cases}|{}^cD^\alpha z(t)-f(t,z_t,{}^cD^\alpha z(t))|\leq\epsilon,\ t\in(t_k,t_{k+1}],k=1,...,m\\ |\Delta z|_{t_k}-I_k(z_{t_k^-})|\leq\epsilon,\ k=1,...,m;\end{cases}\tag{5.13}$$

the set of inequalities

$$\begin{cases}|{}^cD^\alpha z(t)-f(t,z_t,{}^cD^\alpha z(t))|\leq\omega(t),\ t\in(t_k,t_{k+1}],k=1,...,m\\ |\Delta z|_{t_k}-I_k(z_{t_k^-})|\leq\psi,\ k=1,...,m;\end{cases}\tag{5.14}$$

and the set of inequalities

$$\begin{cases}|{}^cD^\alpha z(t)-f(t,z_t,{}^cD^\alpha z(t))|\leq\epsilon\omega(t),\ t\in(t_k,t_{k+1}],k=1,...,m\\ |\Delta z|_{t_k}-I_k(z_{t_k^-})|\leq\epsilon\psi,\ k=1,...,m.\end{cases}\tag{5.15}$$

Definition 5.2.5 *The problem* (5.1)-(5.2) *is Ulam-Hyers stable if there exists a real number $c_{f,m}>0$ such that for each $\epsilon>0$ and for each solution $z\in\Omega$ of the inequality (5.13) there exists a solution $y\in\Omega$ of the problem* (5.1)-(5.2) *with*

$$|z(t)-y(t)|\leq c_{f,m}\epsilon,\ t\in J.$$

Definition 5.2.6 *The problem* (5.1)-(5.2) *is generalized Ulam-Hyers stable if there exists $\theta_{f,m}\in C(\mathbb{R}_+,\mathbb{R}_+),\theta_{f,m}(0)=0$ such that for each solution $z\in\Omega$ of the inequality (5.13) there exists a solution $y\in\Omega$ of the problem* (5.1)-(5.2) *with*

$$|z(t)-y(t)|\leq\theta_{f,m}(\epsilon),\ t\in J.$$

Definition 5.2.7 *The problem* (5.1)-(5.2) *is Ulam-Hyers-Rassias stable with respect to (ω,ψ) if there exists $c_{f,m,\omega}>0$ such that for each $\epsilon>0$ and for each solution $z\in\Omega$ of the inequality (5.15) there exists a solution $y\in\Omega$ of the problem* (5.1)-(5.2) *with*

$$|z(t)-y(t)|\leq c_{f,m,\omega}\epsilon(\omega(t)+\psi),\ t\in J.$$

Definition 5.2.8 *The problem* (5.1)-(5.2) *is generalized Ulam-Hyers-Rassias stable with respect to (ω,ψ) if there exists $c_{f,m,\omega}>0$ such that for each solution $z\in\Omega$ of the inequality (5.14) there exists a solution $y\in\Omega$ of the problem* (5.1)-(5.2) *with*

$$|z(t)-y(t)|\leq c_{f,m,\omega}(\omega(t)+\psi),\ t\in J.$$

Remark 5.2.9 *It is clear that : (i) Definition 5.2.5 implies Definition 5.2.6 ; (ii) Definition 5.2.7 implies Definition 5.2.8 ; (iii) Definition 5.2.7 for $\omega(t)=\psi=1$ implies Definition 5.2.5.*

Remark 5.2.10 *A function $z\in\Omega$ is a solution of the inequality (5.15) if and only if there is $\sigma\in PC(J,\mathbb{R})$ and a sequence $\sigma_k, k=1,...,m$ (which depend on z) such that*

i) $|\sigma(t)|\leq\epsilon\omega(t),\ t\in(t_k,t_{k+1}],k=1,...,m$ *and* $|\sigma_k|\leq\epsilon\psi,k=1,...,m;$

ii) ${}^cD^\alpha z(t)=f(t,z_t,{}^cD^\alpha z(t))+\sigma(t),\ t\in(t_k,t_{k+1}],k=1,...,m\,;$

iii) $\Delta z|_{t_k}=I_k(z_{t_k^-})+\sigma_k,\ k=1,...,m.$

One can have similar remarks for inequalities 5.14 and 5.13.

Now, we state the following Ulam-Hyers-Rassias stable result.

Theorem 5.2.11 *Assume (H1)-(H3), (5.8) and*

(H6) there exists a nondecreasing function $\omega\in PC\left(J,\mathbb{R}_+\right)$ and there exists $\lambda_\omega>0$ such that for any $t\in J$:

$$I^\alpha\omega\left(t\right)\leq\lambda_\omega\omega\left(t\right)$$

are satisfied, then, the problem (5.1)-(5.2) *is Ulam-Hyers-Rassias stable with respect to (ω,ψ).*

proof. Let $z\in\Omega$ be a solution of the inequality (5.15). Denote by y the unique solution of the following problem

$$\begin{cases}{}^cD^\alpha_{t_k}y(t)=f(t,y_t,{}^cD^\alpha_{t_k}y(t)),\ t\in(t_k,t_{k+1}],k=1,...,m\ ;\\ \Delta y|_{t=t_k}=I_k(y_{t_k^-}),\ k=1,...,m\ ;\\ y(t)=z(t)=\varphi(t),\ t\in[-r,0].\end{cases}$$

Using Lemma 5.2.2, we obtain for each $t\in(t_k,t_{k+1}]$

$$y(t)\ =\ \varphi(0)+\sum_{i=1}^k I_i(y_{t_i^-})+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^k\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}g(s)ds$$

$$+ \quad \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}g(s)ds,$$

where $g \in C(J,\mathbb{R})$ be such that

$$g(t) = f(t, y_t, g(t)).$$

Since z solution of the inequality (5.15) and by Remark 5.2.10, we have

$$\begin{cases} {}^cD^{\alpha}_{t_k}z(t) = f(t, z_t, {}^cD^{\alpha}_{t_k}z(t)) + \sigma(t), \ t \in (t_k, t_{k+1}], k = 1, ..., m \ ; \\ \Delta z|_{t=t_k} = I_k(z_{t_k^-}) + \sigma_k, \ k = 1, ..., m \ . \end{cases} \tag{5.16}$$

Clearly, the solution of (5.16) is given by

$$\begin{aligned} z(t) &= \varphi(0) + \sum_{i=1}^{k} I_i(z_{t_i^-}) + \sum_{i=1}^{k}\sigma_i + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}h(s)ds \\ &+ \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\sigma(s)ds + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}h(s)ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\sigma(s)ds, \ t \in (t_k, t_{k+1}], \end{aligned}$$

where $h \in C(J,\mathbb{R})$ be such that

$$h(t) = f(t, z_t, h(t)).$$

Hence for each $t \in (t_k, t_{k+1}]$, it follows that

$$\begin{aligned} |z(t) - y(t)| &\leq \sum_{i=1}^{k}|\sigma_i| + \sum_{i=1}^{k}|I_i(z_{t_i^-}) - I_i(y_{t_i^-})| \\ &+ \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}|h(s)-g(s)|ds \\ &+ \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}|\sigma(s)|ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|h(s)-g(s)|ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|\sigma(s)|. \end{aligned}$$

Thus

$$|z(t) - y(t)| \ \leq \ m\epsilon\psi + (m+1)\epsilon\lambda_\omega\omega(t) + \sum_{i=1}^{k}\widetilde{l}\|z_{t_i^-} - y_{t_i^-}\|_{PC}$$

$$\begin{aligned}&+\quad \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}|h(s)-g(s)|ds\\&+\quad \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|h(s)-g(s)|ds.\end{aligned}$$

By (H2), we have

$$\begin{aligned}|h(t)-g(t)| &= |f(t,z_t,h(t))-f(t,y_t,g(t))|\\&\leq K\|z_t-y_t\|_{PC}+L|g(t)-h(t)|.\end{aligned}$$

Then

$$|h(t)-g(t)|\leq \frac{K}{1-L}\|z_t-y_t\|_{PC}.$$

Therefore, for each $t\in J$

$$\begin{aligned}|z(t)-y(t)| &\leq m\epsilon\psi+(m+1)\epsilon\lambda_\omega\omega(t)+\sum_{i=1}^{k}\widetilde{l}\|z_{t_i^-}-y_{t_i^-}\|_{PC}\\&+\quad \frac{K}{(1-L)\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\|z_s-y_s\|_{PC}ds\\&+\quad \frac{K}{(1-L)\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\|z_s-y_s\|_{PC}ds.\end{aligned}$$

Thus

$$\begin{aligned}|z(t)-y(t)| &\leq \sum_{0<t_i<t}\widetilde{l}\|z_{t_i^-}-y_{t_i^-}\|_{PC}+\epsilon(\psi+\omega(t))(m+(m+1)\lambda_\omega)\\&+\quad \frac{K(m+1)}{(1-L)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\|z_s-y_s\|_{PC}ds.\end{aligned}$$

We consider the function ζ_1 defined by

$$\zeta_1(t)=\sup\{|z(s)-y(s)|:-r\leq s\leq t\},0\leq t\leq T,$$

then, there exists $t^*\in[-r,T]$ such that $\zeta_1(t)=|z(t^*)-y(t^*)|$.
If $t^*\in[-r,0]$, then $\zeta_1(t)=0$.
If $t^*\in[0,T]$, then by the previous inequality, we have

$$\begin{aligned}\zeta_1(t) &\leq \sum_{0<t_i<t}\widetilde{l}\zeta_1(t_i^-)+\epsilon(\psi+\omega(t))(m+(m+1)\lambda_\omega)\\&+\quad \frac{K(m+1)}{(1-L)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\zeta_1(s)ds.\end{aligned}$$

Applying Lemma 1.3.9, we get

$$\begin{aligned}\zeta_1(t) &\leq \epsilon(\psi+\omega(t))(m+(m+1)\lambda_\omega)\\ &\times \left[\prod_{0<t_i<t}(1+\widetilde{l})\exp\left(\int_0^t \frac{K(m+1)}{(1-L)\Gamma(\alpha)}(t-s)^{\alpha-1}ds\right)\right]\\ &\leq c_\omega\epsilon(\psi+\omega(t)),\end{aligned}$$

where

$$\begin{aligned}c_\omega &= (m+(m+1)\lambda_\omega)\left[\prod_{i=1}^{m}(1+\widetilde{l})\exp\left(\frac{K(m+1)T^\alpha}{(1-L)\Gamma(\alpha+1)}\right)\right]\\ &= (m+(m+1)\lambda_\omega)\left[(1+\widetilde{l})\exp\left(\frac{K(m+1)T^\alpha}{(1-L)\Gamma(\alpha+1)}\right)\right]^m.\end{aligned}$$

Thus, the problem (5.1)-(5.2) is Ulam-Hyers-Rassias stable with respect to (ω,ψ).

□

Next, we present the following Ulam-Hyers stable result.

Theorem 5.2.12 *Assume that (H1)-(H3) and* (5.8) *are satisfied, then, the problem* (5.1)-(5.2) *is Ulam-Hyers stable*

proof. Let $z\in\Omega$ be a solution of the inequality (5.13). Denote by y the unique solution of the problem

$$\begin{cases} {}^cD^\alpha_{t_k}y(t)=f(t,y_t,{}^cD^\alpha_{t_k}y(t)),\ t\in(t_k,t_{k+1}], k=1,...,m\ ;\\ \Delta y|_{t=t_k}=I_k(y_{t_k^-}),\ k=1,...,m\ ;\\ y(t)=z(t)=\varphi(t),\ t\in[-r,0].\end{cases}$$

From the proof of Theorem 5.2.11, we get the inequality

$$\begin{aligned}\zeta_1(t) &\leq \sum_{0<t_i<t}\widetilde{l}\zeta_1(t_i^-)+m\epsilon+\frac{T^\alpha\epsilon(m+1)}{\Gamma(\alpha+1)}\\ &+ \frac{K(m+1)}{(1-L)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\zeta_1(s)ds.\end{aligned}$$

Applying Lemma 1.3.9, we get

$$\begin{aligned}\zeta_1(t) &\leq \epsilon\left(\frac{m\Gamma(\alpha+1)+T^\alpha(m+1)}{\Gamma(\alpha+1)}\right)\\ &\times \left[\prod_{0<t_i<t}(1+\widetilde{l})\exp\left(\int_0^t \frac{K(m+1)}{(1-L)\Gamma(\alpha)}(t-s)^{\alpha-1}ds\right)\right]\end{aligned}$$

$$\leq \; c_\omega \epsilon,$$

where

$$\begin{aligned} c_\omega &= \left(\frac{m\Gamma(\alpha+1)+T^\alpha(m+1)}{\Gamma(\alpha+1)}\right)\left[\prod_{i=1}^{m}(1+\widetilde{l})\exp\left(\frac{K(m+1)T^\alpha}{(1-L)\Gamma(\alpha+1)}\right)\right] \\ &= \left(\frac{m\Gamma(\alpha+1)+T^\alpha(m+1)}{\Gamma(\alpha+1)}\right)\left[(1+\widetilde{l})\exp\left(\frac{K(m+1)T^\alpha}{(1-L)\Gamma(\alpha+1)}\right)\right]^m. \end{aligned}$$

□

Moreover, if we set $\theta(\epsilon) = c_\omega\epsilon; \theta(0) = 0$, then, the problem (5.1)-(5.2) is generalized Ulam-Hyers stable.

5.2.4 Examples

Example 1. Consider the following impulsive problem

$$^cD^{\frac{1}{2}}_{t_k}y(t) = \frac{e^{-t}}{(11+e^t)}\left[\frac{y_t}{1+y_t} - \frac{^cD^{\frac{1}{2}}_{t_k}y(t)}{1+^c D^{\frac{1}{2}}_{t_k}y(t)}\right], \text{ for each, } t \in J_0 \cup J_1. \tag{5.17}$$

$$\Delta y|_{t=\frac{1}{2}} = \frac{y(\frac{1}{2}^-)}{10+y(\frac{1}{2}^-)}. \tag{5.18}$$

$$y(t) = \varphi(t),\; t \in [-r,0], r > 0, \tag{5.19}$$

where $\varphi \in PC([-r,0],\mathbb{R})$, $J_0 = \left[0,\frac{1}{2}\right]$, $J_1 = \left(\frac{1}{2},1\right]$, $t_0 = 0$, and $t_1 = \frac{1}{2}$.
Set

$$f(t,u,v) = \frac{e^{-t}}{(11+e^t)}\left[\frac{u}{1+u} - \frac{v}{1+v}\right],\; t \in [0,1],\; u \in PC([-r,0],\mathbb{R}) \text{ and } v \in \mathbb{R}.$$

Clearly, the function f is jointly continuous.
For each $u, \bar{u} \in PC([-r,0],\mathbb{R}), v, \bar{v} \in \mathbb{R}$ and $t \in [0,1]$:

$$\begin{aligned} |f(t,u,v) - f(t,\bar{u},\bar{v})| &\leq \frac{e^{-t}}{(11+e^t)}(\|u-\bar{u}\|_{PC} + |v-\bar{v}|) \\ &\leq \frac{1}{12}\|u-\bar{u}\|_{PC} + \frac{1}{12}|v-\bar{v}|. \end{aligned}$$

Hence condition (H2) is satisfied with $K = L = \dfrac{1}{12}$.
And let

$$I_1(u) = \frac{u}{10+u}, \quad u \in PC([-r,0],\mathbb{R}).$$

Let $u, v \in PC([-r, 0], \mathbb{R})$. Then we have

$$|I_1(u) - I_1(v)| = \Big|\frac{u}{10+u} - \frac{v}{10+v}\Big| \leq \frac{1}{10}\|u - v\|_{PC}.$$

Thus condition

$$\begin{aligned} m\widetilde{l} + \frac{(m+1)KT^\alpha}{(1-L)\Gamma(\alpha+1)} &= \left[\frac{1}{10} + \frac{\frac{2}{12}}{(1-\frac{1}{12})\Gamma(\frac{3}{2})}\right] \\ &= \frac{4}{11\sqrt{\pi}} + \frac{1}{10} < 1, \end{aligned}$$

is satisfied with $T = 1, m = 1$ and $\widetilde{l} = \frac{1}{10}$. It follows from Theorem 5.2.3 that the problem (5.17)–(5.19) has a unique solution on J.

Set for any $t \in [0,1]$, $\omega(t) = t, \psi = 1$.
Since

$$I^{\frac{1}{2}}\omega(t) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^t (t-s)^{\frac{1}{2}-1} s ds \leq \frac{2t}{\sqrt{\pi}},$$

then, condition (H6) is satisfied with $\lambda_\omega = \frac{2}{\sqrt{\pi}}$. From which it follows that the problem (5.17)-(5.18) is Ulam-Hyers-Rassias stable with respect to (ω, ψ).

Example 2. Consider the following impulsive problem

$$^cD^{\frac{1}{2}}_{t_k} y(t) = \frac{2 + |y_t| + |^cD^{\frac{1}{2}}_{t_k} y(t)|}{108e^{t+3}(1 + |y_t| + |^cD^{\frac{1}{2}}_{t_k} y(t)|)}, \text{ for each, } t \in J_0 \cup J_1. \tag{5.20}$$

$$\Delta y|_{t=\frac{1}{3}} = \frac{|y(\frac{1}{3}^-)|}{6 + |y(\frac{1}{3}^-)|}, \tag{5.21}$$

$$y(t) = \varphi(t), \ t \in [-r, 0], r > 0, \tag{5.22}$$

where $\varphi \in PC([-r,0],\mathbb{R})$ $J_0 = \left[0, \frac{1}{3}\right]$, $J_1 = \left(\frac{1}{3}, 1\right]$, $t_0 = 0$, and $t_1 = \frac{1}{3}$.
Set

$$f(t, u, v) = \frac{2 + |u| + |v|}{108e^{t+3}(1 + |u| + |v|)}, \quad t \in [0,1], \ u \in PC([-r,0],\mathbb{R}), v \in \mathbb{R}.$$

Clearly, the function f is jointly continuous.
For any $u, \bar{u} \in PC([-r,0],\mathbb{R}), v, \bar{v} \in \mathbb{R}$ and $t \in [0,1]$:

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \leq \frac{1}{108e^3}(\|u - \bar{u}\|_{PC} + |v - \bar{v}|).$$

Hence condition (H2) is satisfied with $K = L = \dfrac{1}{108e^3}$.
We have, for each $t \in [0,1]$,

$$|f(t,u,v)| \leq \frac{1}{108e^{t+3}}(2 + \|u\|_{PC} + |v|).$$

Thus condition (H4) is satisfied with $p(t) = \dfrac{1}{54e^{t+3}}$ and $q(t) = r(t) = \dfrac{1}{108e^{t+3}}$. And let

$$I_1(u) = \frac{|u|}{6+|u|}, \quad u \in PC\left([-r,0],\mathbb{R}\right).$$

We have, for each $u \in PC\left([-r,0],\mathbb{R}\right)$,

$$|I_1(u)| \leq \frac{1}{6}\|u\|_{PC} + 1$$

Thus condition (H5) is satisfied with $M^* = \dfrac{1}{6}$ and $N^* = 1$.
It follows from Theorem 5.2.4 that the problem (5.20)–(5.22) has at least one solution on J.

5.3 Existence Results for the NIFDE with Finite Delay and Impulses in Banach Space

5.3.1 Introduction

The purpose of this Section, is to establish existence and uniqueness results to the following implicit fractional differential equations with finite delay and impulses :

$$^cD^{\nu}_{t_k}y(t) = f(t, y_t, {}^cD^{\nu}_{t_k}y(t)), \text{ for each }, \ t \in (t_k, t_{k+1}], \ k = 0, \ldots, m, \ 0 < \nu \leq 1, \tag{5.23}$$

$$\Delta y|_{t=t_k} = I_k(y_{t_k^-}), \ k = 1, \ldots, m, \tag{5.24}$$

$$y(t) = \varphi(t), \ t \in [-r,0], \ r > 0 \tag{5.25}$$

where $^cD^{\nu}_{t_k}$ is the Caputo fractional derivative, $(E, ||\cdot||)$ is a real Banach space, $f : J \times PC([-r,0],E) \times E \to E$ is a given function, $I_k : PC([-r,0],E) \to E$, and $\varphi \in PC([-r,0],E), 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$.
For each function y_t defined on $[-r,T]$ and for any $t \in J$, we denote by y_t the element of $PC([-r,0],E)$ defined by :

$$y_t(\theta) = y(t+\theta), \quad \theta \in [-r,0],$$

$y_t(\cdot)$ represent the history of the state from time $t-r$ up to time t.
Here $\Delta y|_{t_k} = y(t_k^+) - y(t_k^-)$, where $y(t_k^+) = \lim_{h\to 0^+} y(t_k+h)$ and $y(t_k^-) = \lim_{h\to 0^-} y(t_k+h)$ represent the right and left limits of y_t at $t = t_k$, respectively.
At last, two examples are given to demonstrate the application of our main results.

5.3.2 Existence of Solutions

Let $(E; \|\cdot\|)$ be a valued-Banach space, and $t \in J$. We denote by $C(J,E)$ the space of E valued continuous functions on J with the usual supremum norm

$$\|y\|_\infty = \sup\{||y(t)|| : t \in J\}$$

for any $y \in C(J,E)$.
Let $J_0 = [t_0, t_1]$ and $J_k = (t_k, t_{k+1}]$ where $k = 1, \ldots, m$.
Consider the set of functions

$$PC([-r,0],E) \;\; = \{y : [-r,0] \to E : y \in C((\tau_k, \tau_{k+1}], E),\ k = 0, \ldots, m \text{ and there exist}$$

$$y(\tau_k^-) \text{ and } y(\tau_k^+),\ k = 1, \ldots, m^{'} \text{ with } y(\tau_k^-) = y(\tau_k) \text{ and } \tau_k = t_k - t, \text{ for each } t \in J_k\}.$$

$PC([-r,0],E)$ is a Banach space with the norm

$$\|y\|_{PC} = \sup_{t\in[-r,0]} \|y(t)\|.$$

$$PC([0,T],E) = \{y : [0,T] \to E : y \in C((t_k, t_{k+1}], E),\ k = 1, ..., m, \text{ and there exist } y(t_k^-) \text{ and } y(t_k^+), k = 1, ..., m \text{ with } y(t_k^-) = y(t_k)\}.$$

$PC([0,T],E)$ is a Banach space with the norm

$$\|y\|_C = \sup_{t\in[0,T]} \|y(t)\|.$$

$$\Omega = \{y : [-r,T] \to E : y|_{[-r,0]} \in PC([-r,0],E) \text{ and } y|_{[0,T]} \in PC([0,T],E)\}.$$

Ω is a Banach space with the norm

$$\|y\|_\Omega = \sup_{t\in[-r,T]} \|y(t)\|.$$

Let us defining what we mean by a solution of problem (5.23)-(5.25).

Definition 5.3.1 A function $y \in \Omega$ whose ν-derivative exists on J_k is said to be a solution of (5.23)-(5.25) if y satisfies the equation ${}^cD^\nu_{t_k} y(t) = f(t, y_t, {}^c D^\nu_{t_k} y(t))$ on J_k, and satisfy the conditions

$$\Delta y|_{t=t_k} = I_k(y_{t_k^-}),\ k = 1, \ldots, m,$$
$$y(t) = \varphi(t),\ t \in [-r,0].$$

To prove the existence of solutions to (5.23)-(5.25), we need the following auxiliary Lemma.

Lemma 5.3.2 *Let $0 < \nu \leq 1$ and let $\sigma : J \to E$ be continuous. A function y is a solution of the fractional integral equation*

$$y(t) = \begin{cases} \varphi(0) + \dfrac{1}{\Gamma(\nu)} \displaystyle\int_0^t (t-s)^{\nu-1}\sigma(s)ds & \text{if } t \in [0, t_1] \\ \\ \varphi(0) + \displaystyle\sum_{i=1}^{k} I_i(y_{t_i^-}) + \dfrac{1}{\Gamma(\nu)} \displaystyle\sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\nu-1}\sigma(s)ds \\ + \dfrac{1}{\Gamma(\nu)} \displaystyle\int_{t_k}^{t} (t-s)^{\nu-1}\sigma(s)ds, & \text{if } t \in (t_k, t_{k+1}], \\ \\ \varphi(t), \ t \in [-r, 0], \end{cases} \tag{5.26}$$

where $k = 1, \ldots, m$, if and only if y is a solution of the following fractional problem

$$\begin{aligned} {}^cD^\nu y(t) &= \sigma(t), \quad t \in J_k, \\ \Delta y|_{t=t_k} &= I_k(y_{t_k^-}), \quad k = 1, \ldots, m, \\ y(t) &= \varphi(t), \ t \in [-r, 0]. \end{aligned}$$

This lemma was already proved in the previous section.

First we list the following hypotheses :
(P1) The function $f : J \times PC([-r, 0], E) \times E \to E$ is continuous.
(P2) There exist constants $K > 0$ and $0 < L < 1$ such that

$$||f(t, u, v) - f(t, \bar{u}, \bar{v})|| \leq K||u - \bar{u}||_{PC} + L||v - \bar{v}||$$

for any $u, \bar{u} \in PC([-r, 0], E)$, $v, \bar{v} \in E$ and $t \in J$.
(P3) There exists a constant $\widetilde{l} > 0$ such that

$$\|I_k(u) - I_k(\overline{u})\| \leq \widetilde{l}\|u - \overline{u}\|_{PC},$$

for each $u, \overline{u} \in PC([-r, 0], E)$ and $k = 1, \ldots, m$.

Remark 5.3.3 *[25] Conditions $(P2)$ and $(P3)$ are respectively equivalent to the inequalities*

$$\alpha\Big(f(t, B_1, B_2)\Big) \leq K\alpha(B_1) + L\alpha(B_2)$$

$$\alpha\Big(I_k(B_1)\Big) \leq \widetilde{l}\alpha(B_1),$$

for any bounded sets $B_1 \subseteq PC([-r, 0], E), B_2 \subseteq E$, for each $t \in J$ and $k = 1, \ldots, m$.

Theorem 5.3.4 *Assume (P1)–(P3) hold.*
If

$$m\widetilde{l} + \frac{(m+1)KT^{\nu}}{(1-L)\Gamma(\nu+1)} < 1 \tag{5.27}$$

then the IVP (5.23)-(5.25) has at least one solution on J.

This theorem will be proved in two ways : the first is based on Darbo's fixed point theorem combined with the technique of measures of noncompactness and the second on *Mönch's* fixed point theorem.
Proof 1.

Transform the problem (5.23)-(5.25) into a fixed point problem. Consider the operator $N : \Omega \to \Omega$ defined by

$$Ny(t) = \begin{cases} \varphi(0) + \sum\limits_{0<t_k<t} I_k(y_{t_k^-}) + \dfrac{1}{\Gamma(\nu)} \sum\limits_{0<t_k<t} \displaystyle\int_{t_{k-1}}^{t_k} (t_k - s)^{\nu-1} g(s)ds \\ +\dfrac{1}{\Gamma(\nu)} \displaystyle\int_{t_k}^{t} (t-s)^{\nu-1} g(s)ds, & t \in [0,T], \\ \\ \varphi(t), & t \in [-r, 0], \end{cases} \tag{5.28}$$

where $g \in C(J, E)$ be such that

$$g(t) = f(t, y_t, g(t)).$$

Clearly, the fixed points of operator N are solutions of problem (5.23)–(5.25).
We shall show that N satisfies the assumption of *Darbo's* fixed point Theorem. The proof will be given in several claims.

Claim 1 : *N is continuous.*
Let $\{u_n\}$ be a sequence such that $u_n \to u$ in Ω. If $t \in [-r, 0]$, then

$$\|N(u_n)(t) - N(u)(t)\| = 0.$$

For $t \in J$, we have

$$\begin{aligned} \|N(u_n)(t) - N(u)(t)\| \;&\leq\; \frac{1}{\Gamma(\nu)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\nu-1} \|g_n(s) - g(s)\| ds \\ &+\; \frac{1}{\Gamma(\nu)} \int_{t_k}^{t} (t-s)^{\nu-1} \|g_n(s) - g(s)\| ds \\ &+\; \sum_{0<t_k<t} \|I_k(u_{nt_k^-}) - I_k(u_{t_k^-})\|, \end{aligned} \tag{5.29}$$

where $g_n, g \in C(J, E)$ such that

$$g_n(t) = f(t, u_{nt}, g_n(t)),$$

and

$$g(t) = f(t, u_t, g(t)).$$

By $(P2)$, we have

$$\begin{array}{rcl} \|g_n(t) - g(t)\| & = & \|f(t, u_{nt}, g_n(t)) - f(t, u_t, g(t))\| \\ & \leq & K\|u_{nt} - u_t\|_{PC} + L\|g_n(t) - g(t)\|. \end{array}$$

Then

$$\|g_n(t) - g(t)\| \leq \frac{K}{1-L}\|u_{nt} - u_t\|_{PC}.$$

Since $u_n \to u$, then we get $g_n(t) \to g(t)$ as $n \to \infty$ for each $t \in J$. And let $\eta > 0$ be such that, for each $t \in J$, we have $\|g_n(t)\| \leq \eta$ and $\|g(t)\| \leq \eta$. Then, we have

$$\begin{array}{rcl} (t-s)^{\nu-1}\|g_n(s) - g(s)\| & \leq & (t-s)^{\nu-1}[\|g_n(s)\| + \|g(s)\|] \\ & \leq & 2\eta(t-s)^{\nu-1}, \end{array}$$

and

$$\begin{array}{rcl} (t_k-s)^{\nu-1}\|g_n(s) - g(s)\| & \leq & (t_k-s)^{\nu-1}[\|g_n(s)\| + \|g(s)\|] \\ & \leq & 2\eta(t_k-s)^{\nu-1}. \end{array}$$

For each $t \in J$, the functions $s \to 2\eta(t-s)^{\nu-1}$ and $s \to 2\eta(t_k-s)^{\nu-1}$ are integrable on $[0, t]$, then the Lebesgue Dominated Convergence Theorem and (5.29) imply that

$$\|N(u_n)(t) - N(u)(t)\| \to 0 \text{ } as \text{ } n \to \infty,$$

and hence

$$\|N(u_n) - N(u)\|_\Omega \to 0 \text{ as } n \to \infty.$$

Consequently, N is continuous.
Let the constant R such that :

$$R \geq \max\left\{\frac{(\|\varphi(0)\| + mc_1)\,\Gamma(\nu+1)(1-L) + (m+1)T^\nu f^*}{\Gamma(\nu+1)(1-L) - \left[(m+1)T^\nu K + m\widetilde{l}\Gamma(\nu+1)(1-L)\right]}, \|\varphi\|_{PC}\right\} \quad (5.30)$$

where

$$c_1 = \max_{1\leq k\leq m}\{\sup\{\|I_k(v)\|, v \in PC([-r, 0], E)\}\},$$

and

$$f^* = \sup_{t\in J}\|f(t, 0, 0)\|.$$

Define

$$D_R = \{u \in \Omega : \|u\|_\Omega \leq R\}.$$

It is clear that D_R is a bounded, closed and convex subset of Ω.

Claim 2 : $N(D_R) \subset D_R$.

Let $u \in D_R$ we show that $Nu \in D_R$.
If $t \in [-r, 0]$ then,

$$\|N(u)(t)\| \leq \|\varphi\|_{PC} \leq R.$$

And if $t \in J$, we have

$$\begin{aligned} \|N(u)(t)\| &\leq \|\varphi(0)\| + \frac{1}{\Gamma(\nu)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\nu-1} \|g(s)\| ds \\ &+ \frac{1}{\Gamma(\nu)} \int_{t_k}^{t} (t-s)^{\nu-1} \|g(s)\| ds + \sum_{0<t_k<t} \|I_k(u_{t_k^-})\|. \end{aligned} \tag{5.31}$$

By (P2) we have for each $t \in J$,

$$\begin{aligned} ||g(t)|| &\leq \|f(t, u_t, g(t)) - f(t, 0, 0)\| + \|f(t, 0, 0)\| \\ &\leq K\|u_t\|_{PC} + L\|g(t)\| + f^* \\ &\leq K\|u\|_\Omega + L||g(t)|| + f^* \\ &\leq KR + L||g(t)|| + f^*. \end{aligned}$$

Then

$$||g(t)|| \leq \frac{f^* + KR}{1 - L} := M.$$

Thus, $(5.30), (5.31)$ and (P3) imply that

$$\begin{aligned} ||Nu(t)|| &\leq \|\varphi(0)\| + \frac{mMT^\nu}{\Gamma(\nu+1)} + \frac{MT^\nu}{\Gamma(\nu+1)} + \sum_{k=1}^{m} \|I_k(u_{t_k^-}) - I_k(0)\| + \sum_{k=1}^{m} \|I_k(0)\| \\ &\leq \|\varphi(0)\| + \frac{(m+1)MT^\nu}{\Gamma(\nu+1)} + m\widetilde{l}\|u_{t_k^-}\|_{PC} + mc_1 \\ &\leq \|\varphi(0)\| + \frac{(m+1)MT^\nu}{\Gamma(\nu+1)} + m\widetilde{l}\|u\|_\Omega + mc_1 \\ &\leq \|\varphi(0)\| + \frac{(m+1)MT^\nu}{\Gamma(\nu+1)} + m\widetilde{l}R + mc_1 \\ &\leq R, \end{aligned}$$

from which it follows that for each $t \in [-r, T]$, we have $\|Nu(t)\| \leq R$.
Which implies that $\|Nu\|_\Omega \leq R$
Consequently,

$$N(D_R) \subset D_R.$$

Claim 3 : *$N(D_R)$ is bounded and equicontinuous.*

By Claim 2 we have $N(D_R) = \{N(u) : u \in D_R\} \subset D_R$. Thus, for each $u \in D_R$ we have $\|N(u)\|_\Omega \leq R$ which means that $N(D_R)$ is bounded. Let $t_1, t_2 \in (0, T],\ t_1 < t_2$, and let $u \in D_R$. Then

$$\begin{aligned}
&\|N(u)(t_2) - N(u)(t_1)\| \\
&\leq \frac{1}{\Gamma(\nu)} \int_0^{t_1} |(t_2 - s)^{\nu-1} - (t_1 - s)^{\nu-1}| \|g(s)\| ds \\
&\quad + \frac{1}{\Gamma(\nu)} \int_{t_1}^{t_2} |(t_2 - s)^{\nu-1}| \|g(s)\| ds + \sum_{0<t_k<t_2-t_1} \|I_k(u_{t_k^-}) - I_k(0)\| + \sum_{0<t_k<t_2-t_1} \|I_k(0)\| \\
&\leq \frac{M}{\Gamma(\nu+1)}[2(t_2 - t_1)^\nu + (t_2^\nu - t_1^\nu)] + (t_2 - t_1)(\widetilde{l}\|u_{t_k^-}\|_{PC} + c_1) \\
&\leq \frac{M}{\Gamma(\nu+1)}[2(t_2 - t_1)^\nu + (t_2^\nu - t_1^\nu)] + (t_2 - t_1)(\widetilde{l}\|u\|_\Omega + c_1) \\
&\leq \frac{M}{\Gamma(\nu+1)}[2(t_2 - t_1)^\nu + (t_2^\nu - t_1^\nu)] + (t_2 - t_1)(\widetilde{l}R + c_1).
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero.

Claim 4 : *The operator $N : D_R \to D_R$ is a strict set contraction.*

Let $V \subset D_R$.
If $t \in [-r, 0]$,then

$$\begin{aligned}
\alpha(N(V)(t)) &= \alpha(N(y)(t), y \in V) \\
&= \alpha(\varphi(t), y \in V) \\
&= 0.
\end{aligned}$$

And if $t \in J$, we have

$$\begin{aligned}
\alpha(N(V)(t)) &= \alpha((Ny)(t), y \in V) \\
&\leq \sum_{0<t_k<t} \Big\{\alpha(I_k(y_{t_k^-})), y \in V\Big\} + \frac{1}{\Gamma(\nu)} \sum_{0<t_k<t} \Big\{ \int_{t_{k-1}}^{t_k} (t_k - s)^{\nu-1} \alpha(g(s)) ds, y \in V\Big\} \\
&+ \frac{1}{\Gamma(\nu)} \Big\{ \int_{t_k}^{t} (t - s)^{\nu-1} \alpha(g(s)) ds, y \in V\Big\}.
\end{aligned}$$

Then Remark 5.3.3 and Lemma 1.4.4 imply that, for each $s \in J$,

$$\begin{aligned}
\alpha(\{g(s), y \in V\}) &= \alpha(\{f(s, y(s), g(s)), y \in V\}) \\
&\leq K\alpha(\{y(s), y \in V\}) + L\alpha(\{g(s), y \in V\}).
\end{aligned}$$

Thus

$$\alpha\left(\{g(s), y \in V\}\right) \;\leq\; \frac{K}{1-L}\alpha\{y(s), y \in V\}.$$

On the other hand, for each $t \in J$ and $k = 1, \dots, m$, we have

$$\sum_{0<t_k<t} \alpha\left(\left\{I_k(y_{t_k^-}), y \in V\right\}\right) \;\leq\; m\widetilde{l}\alpha(\{y(t), y \in V\}).$$

Then

$$\begin{aligned}
\alpha(N(V)(t)) \;&\leq\; m\widetilde{l}\alpha(\{y(t), y \in V\}) + \frac{mK}{(1-L)\Gamma(\nu)}\left\{\int_0^t (t-s)^{\nu-1}\{\alpha(y(s))\}ds, y \in V\right\}\\
&+\; \frac{K}{(1-L)\Gamma(\nu)}\left\{\int_0^t (t-s)^{\nu-1}\{\alpha(y(s))\}ds, y \in V\right\}\\
&\leq\; m\widetilde{l}\alpha_c(V) + \left[\frac{mKT^\nu}{(1-L)\Gamma(\nu+1)} + \frac{KT^\nu}{(1-L)\Gamma(\nu+1)}\right]\alpha_c(V)\\
&=\; \left[m\widetilde{l} + \frac{(m+1)KT^\nu}{(1-L)\Gamma(\nu+1)}\right]\alpha_c(V).
\end{aligned}$$

Therefore

$$\alpha_c(NV) \leq \left[m\widetilde{l} + \frac{(m+1)KT^\nu}{(1-L)\Gamma(\nu+1)}\right]\alpha_c(V).$$

So, by (5.27), the operator N is a set contraction. As a consequence of Theorem 1.5.3, we deduce that N has a fixed point which is solution to the problem (5.23)-(5.25). $\square$

Proof 2. Consider the operator N defined in (5.28). We shall show that N satisfies the assumption of *Mönch's* fixed point theorem. We know that $N : D_R \to D_R$ is bounded and continuous, we need to prove that the implication

$$V = \overline{conv}N(V) \quad \text{or} \quad V = N(V) \cup \{0\} \Rightarrow \alpha(V) = 0,$$

holds for every subset V of D_R. Now let V be a subset of D_R such that $V \subset \overline{conv}(N(V) \cup \{0\})$. V is bounded and equicontinuous and therefore the function $t \to v(t) = \alpha(V(t))$ is continuous on $[-r, T]$. By Remark 5.3.3, Lemma 1.4.8 and the properties of the measure α we have for each $t \in J$

$$\begin{aligned}
v(t) \;&\leq\; \alpha(N(V)(t) \cup \{0\})\\
&\leq\; \alpha(N(V)(t))\\
&\leq\; \alpha\{(Ny)(t), y \in V\}\\
&\leq\; m\widetilde{l}\alpha(\{y(t), y \in V\}) + \frac{(m+1)K}{(1-L)\Gamma(\nu)}\left\{\int_0^t (t-s)^{\nu-1}\{\alpha(y(s))\}ds, y \in V\right\}
\end{aligned}$$

$$= m\widetilde{l}v(t) + \frac{(m+1)K}{(1-L)\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}v(s)ds.$$

Then

$$v(t) \leq \frac{(m+1)K}{(1-m\widetilde{l})(1-L)\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}v(s)ds.$$

Lemma 1.3.8 implies that $v(t) = 0$ for each $t \in J$.
For $t \in [-r, 0]$ we have $v(t) = \alpha(\varphi(t)) = 0$, then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzelà theorem, V is relatively compact in D_R. Applying now Theorem 1.5.5 we conclude that N has a fixed point $y \in D_R$. Hence N has a fixed point which is solution to the problem (5.23)-(5.25).

□

5.3.3 Examples

Example 1. Consider the following infinite system

$$^cD^{\frac{1}{2}}_{t_k}y_n(t) = \frac{e^{-t}}{(11+e^t)}\left[\frac{y_{tn}}{1+y_{tn}} - \frac{^cD^{\frac{1}{2}}_{t_k}y_n(t)}{1+^c D^{\frac{1}{2}}_{t_k}y_n(t)}\right], \text{ for each, } t \in J_0 \cup J_1. \tag{5.32}$$

$$\Delta y_n|_{t=\frac{1}{2}} = \frac{y_n(\frac{1}{2}^-)}{10+y_n(\frac{1}{2}^-)}. \tag{5.33}$$

$$y_n(t) = \varphi(t),\ t \in [-r, 0], r > 0, \tag{5.34}$$

where $\varphi \in PC([-r, 0], E)$, $J_0 = \left[0, \frac{1}{2}\right]$, $J_1 = \left(\frac{1}{2}, 1\right]$, $t_0 = 0$, and $t_1 = \frac{1}{2}$.
Set

$$E = l^1 = \{y = (y_1, y_2, ..., y_n, ...), \sum_{n=1}^{\infty}|y_n| < \infty\},$$

and

$$f(t, u, v) = \frac{e^{-t}}{(11+e^t)}\left[\frac{u}{1+u} - \frac{v}{1+v}\right],\ t \in [0, 1],\ u \in PC([-r, 0], E) \text{ and } v \in E.$$

Clearly, the function f is jointly continuous.
E is a Banach space with the norm $||y|| = \sum_{n=1}^{\infty}|y_n|$.
For any $u, \bar{u} \in PC([-r, 0], E)$, $v, \bar{v} \in E$ and $t \in [0, 1]$:

$$||f(t, u, v) - f(t, \bar{u}, \bar{v})|| \leq \frac{1}{12}(||u - \bar{u}||_{PC} + ||v - \bar{v}||).$$

Hence condition (P2) is satisfied with $K = L = \dfrac{1}{12}$.
And let
$$I_1(u) = \frac{u}{10+u}, \quad u \in PC([-r,0],E).$$
Let $u, v \in PC([-r,0],E)$. Then we have
$$\|I_1(u) - I_1(v)\| = \Big\|\frac{u}{10+u} - \frac{v}{10+v}\Big\| \leq \frac{1}{10}\|u-v\|_{PC}.$$
Hence the condition (P3) is satisfied with $\widetilde{l} = \dfrac{1}{10}$.
And the conditions
$$\begin{aligned} m\widetilde{l} + \frac{(m+1)KT^{\nu}}{(1-L)\Gamma(\nu+1)} &= \left[\frac{1}{10} + \frac{\frac{2}{12}}{(1-\frac{1}{12})\Gamma(\frac{3}{2})}\right] \\ &= \frac{4}{11\sqrt{\pi}} + \frac{1}{10} < 1, \end{aligned}$$
are satisfied with $T = m = 1$ and $\nu = \dfrac{1}{2}$. It follows from Theorem 5.3.4 that the problem (5.32)–(5.34) has a at least one solution on J.

Example 2. Consider the following impulsive problem
$$^cD^{\frac{1}{2}}_{t_k} y_n(t) = \frac{2 + \|y_{nt}\| + \|^cD^{\frac{1}{2}}_{t_k} y_n(t)\|}{108e^{t+3}(1 + \|y_{nt}\| + \|^cD^{\frac{1}{2}}_{t_k} y_n(t)\|)}, \text{ for each, } t \in J_0 \cup J_1. \tag{5.35}$$
$$\Delta y_n|_{t=\frac{1}{3}} = \frac{\|y_n(\frac{1}{3}^-)\|}{6 + \|y_n(\frac{1}{3}^-)\|}, \tag{5.36}$$
$$y_n(t) = \varphi(t),\ t \in [-r,0], r > 0, \tag{5.37}$$
where $\varphi \in PC([-r,0],E)$, $J_0 = \left[0, \frac{1}{3}\right]$, $J_1 = \left(\frac{1}{3}, 1\right]$, $t_0 = 0$, and $t_1 = \frac{1}{3}$.
Set
$$E = l^1 = \{y = (y_1, y_2, ..., y_n, ...), \sum_{n=1}^{\infty} |y_n| < \infty\},$$
and
$$f(t,u,v) = \frac{2 + \|u\| + \|v\|}{108e^{t+3}(1 + \|u\| + \|v\|)}, \quad t \in [0,1],\ u \in PC([-r,0],E), v \in E.$$
Clearly, the function f is jointly continuous.
E is a Banach space with the norm $||y|| = \sum_{n=1}^{\infty} |y_n|$.
For any $u, \bar{u} \in PC\left([-r,0],E\right), v, \bar{v} \in E$ and $t \in [0,1]$:
$$||f(t,u,v) - f(t,\bar{u},\bar{v})|| \leq \frac{1}{108e^3}(||u-\bar{u}||_{PC} + ||v-\bar{v}||).$$

Hence condition (P2) is satisfied with $K = L = \dfrac{1}{108e^3}$.
And let

$$I_1(u) = \frac{\|u\|}{6 + \|u\|}, \quad u \in PC([-r, 0], E).$$

Let $u, v \in PC([-r, 0], E)$. Then we have

$$\|I_1(u) - I_1(v)\| = \Big\| \frac{u}{6+u} - \frac{v}{6+v} \Big\| \leq \frac{1}{6} \|u - v\|_{PC}.$$

Hence the condition (P3) is satisfied with $\widetilde{l} = \dfrac{1}{6}$.
The condition

$$\begin{aligned} m\widetilde{l} + \frac{(m+1)KT^{\nu}}{(1-L)\Gamma(\nu+1)} &= \left[\frac{1}{6} + \frac{\frac{2}{12}}{(1 - \frac{1}{12})\Gamma(\frac{3}{2})} \right] \\ &= \frac{4}{11\sqrt{\pi}} + \frac{1}{6} < 1, \end{aligned}$$

is satisfied with $T = m = 1$ and $\nu = \dfrac{1}{2}$.
Also, we have

$$m\widetilde{l} = \frac{1}{6} < 1.$$

It follows from Theorem 5.3.4 that the problem (5.35)-(5.37) has at least one solution on J.

Chapitre 6

Existence and Stability Results for Neutral Functional Differential Equations of Fractional Order with Finite Delay and Impulses

6.1 Introduction and Motivations

In this Chapter, we establish, in Section 6.2, existence, uniqueness and stability results to the following nonlinear implicit neutral fractional-order differential equation with finite delay

$$^cD^{\alpha}[y(t) - g(t, y_t)] = f(t, y_t, {}^cD^{\alpha}y(t)),\ t \in J = [0, T],\ T > 0\ , 0 < \alpha \leq 1$$

$$y(t) = \varphi(t),\ t \in [-r, 0],\ r > 0$$

where $f : J \times C([-r, 0], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ and $g : J \times C([-r, 0], \mathbb{R}) \to \mathbb{R}$ are two given functions such that $g(0, \varphi) = 0$ and $\varphi \in C([-r, 0], \mathbb{R})$.

For each function y_t defined on $[-r, T]$ and for any $t \in J$, we denote by y_t the element of $C([-r, 0], \mathbb{R})$ defined by :

$$y_t(\theta) = y(t + \theta),\quad \theta \in [-r, 0],$$

$y_t(\cdot)$ represent the evolution history of system state from time $t - r$ to time t.

An extension of this problem is given in Section 6.3. More precisely, we shall present a result of existence and uniqueness for the nonlinear implicit neutral fractional-order differential equation with finite delay and impulses

$$^cD^{\alpha}_{t_k}[y(t) - \phi(t, y_t)] = f(t, y_t, {}^cD^{\alpha}_{t_k}y(t)),\ \text{for each } t \in (t_k, t_{k+1}],\ k = 0, \ldots, m,\ 0 < \alpha \leq 1,$$

$$\Delta y|_{t_k} = I_k(y_{t_k^-}),\ k = 1, \dots, m,$$
$$y(t) = \varphi(t),\ t \in [-r, 0],\ r > 0,$$

where ${}^cD^\alpha_{t_k}$ is the Caputo fractional derivative, $f : J \times PC([-r, 0], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$, $\phi : J \times PC([-r, 0], \mathbb{R}) \to \mathbb{R}$ are given functions with $\phi(0, \varphi) = 0, I_k : PC([-r, 0], \mathbb{R}) \to \mathbb{R}$ and $\varphi \in PC([-r, 0], \mathbb{R}), 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, and $PC([-r, 0], \mathbb{R})$ is a space to be specified later.
And $\Delta y|_{t_k} = y(t_k^+) - y(t_k^-)$, where $y(t_k^+) = \lim_{h \to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h \to 0^-} y(t_k + h)$ represent the right and left limits of y_t at $t = t_k$, respectively.

6.2 Existence and Stability Results for Neutral NIFDE with Finite Delay

[1]

6.2.1 Introduction

In this Section, we establish, existence, uniqueness and stability results to the following nonlinear implicit neutral fractional-order differential equation with finite delay

$$ {}^cD^\alpha[y(t) - g(t, y_t)] = f(t, y_t, {}^cD^\alpha y(t)),\ t \in J = [0, T],\ T > 0\ , 0 < \alpha \leq 1 \tag{6.1}$$

$$y(t) = \varphi(t),\ t \in [-r, 0],\ r > 0 \tag{6.2}$$

where $f : J \times C([-r, 0], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ and $g : J \times C([-r, 0], \mathbb{R}) \to \mathbb{R}$ are two given functions such that $g(0, \varphi) = 0$ and $\varphi \in C([-r, 0], \mathbb{R})$.
The arguments are based upon the Banach contraction principle. Two examples are given to show the applicability of our results.

6.2.2 Existence of Solutions

By $C([-r, 0], \mathbb{R})$, $C([0, T], \mathbb{R})$ we denote the Banach spaces of all continuous functions from $[-r, 0]$ into $\mathbb{R}$ (resp from $[0, T]$ into $\mathbb{R}$) with the norms :
$\|y\|_C = \{\sup |y(t)|, t \in [-r, 0]\}$ and $\|y\|_\infty = \{\sup |y(t)|, t \in [0, T]\}$ (respectively).
Set

$$\Omega = \{y : [-r, T] \to \mathbb{R} : y|_{[-r,0]} \in C([-r, 0], \mathbb{R}) \text{ and } y|_{[0,T]} \in C([0, T], \mathbb{R})\}.$$

Ω is a Banach space with the norm

$$\|y\|_\Omega = \sup_{t \in [-r, T]} |y(t)|.$$

1. M. Benchohra and S. Bouriah, Existence and Stability Results for Neutral Functional Differential Equations of Fractional Order with Finite Delay, *Dynamics of Continuous, Discrete and Impulsive Systems.*

Definition 6.2.1 *A function $y \in \Omega$ is called solution of the problem* (6.1)-(6.2) *if it satisfies the equation* (6.1) *on J and the condition* (6.2) *on* $[-r, 0]$.

Lemma 6.2.2 *Let $0 < \alpha \leq 1$ and $h : [0, T] \to \mathbb{R}$ a continuous function. Then, the linear problem*

$$^{c}D^{\alpha}[y(t) - g(t, y_t)] = h(t), \ t \in J$$
$$y(t) = \varphi(t), \ t \in [-r, 0]$$

has a unique solution which is given by

$$y(t) = \begin{cases} \varphi(0) + g(t, y_t) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} h(s) ds, \ t \in J \\ \varphi(t), \quad t \in [-r, 0]. \end{cases}$$

Lemma 6.2.3 *Let $f : J \times C([-r, 0], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ a continuous function, then the problem* (6.1)-(6.2) *is equivalent to the problem*

$$y(t) = \begin{cases} \varphi(0) + I^{\alpha} K_y(t), \ t \in J \\ \varphi(t), \ t \in [-r, 0] \end{cases} \tag{6.3}$$

where $K_y \in C(J, \mathbb{R})$ satisfies the functional equation

$$K_y(t) = f(t, y_t, K_y(t)) +^{c} D^{\alpha} g(t, y_t).$$

Proof. Let y solution of the problem (6.3), show that y is solution of (6.1)-(6.2).
We have

$$y(t) = \begin{cases} \varphi(0) + I^{\alpha} K_y(t), \ t \in J \\ \varphi(t), \ t \in [-r, 0] \end{cases}$$

for $t \in [-r, 0]$, we have $y(t) = \varphi(t)$, so the condition (6.2) is satisfied.
On the other hand, for $t \in J$, we have

$$^{c}D^{\alpha} y(t) = K_y(t) = f(t, y_t, K_y(t)) +^{c} D^{\alpha} g(t, y_t).$$

So

$$^{c}D^{\alpha}\left[y(t) - g(t, y_t)\right] = f(t, y_t,^{c} D^{\alpha} y(t)).$$

Then, y is well solution of the problem (6.1)-(6.2).

□

Lemma 6.2.4 *Under assumptions :*

(H1) $f : J \times C\left([-r, 0], \mathbb{R}\right) \times \mathbb{R} \to \mathbb{R}$ *is a continuous function.*

(H2) there exist $K > 0$ and $0 < \overline{K} < 1$ such that :

$$|f(t, u, v) - f(t, \bar{u}, \bar{v})| \leq K \|u - \bar{u}\|_C + \overline{K} |v - \bar{v}|$$

for any $u, \bar{u} \in C\left([-r, 0], \mathbb{R}\right), v, \bar{v} \in \mathbb{R}$ *and* $t \in J$.

(H3) there exists $L > 0$ such that :

$$|g(t,u) - g(t,v)| \leq L \|u - v\|_C$$

for any $u, v \in C([-r,0], \mathbb{R})$ and $t \in J$.
And if

$$\frac{KT^\alpha}{\left(1-\overline{K}\right)\Gamma(\alpha+1)} + \frac{L}{\left(1-\overline{K}\right)} < 1, \tag{6.4}$$

then, the problem (6.1)-(6.2) *has a unique solution.*

Proof. Let the operator $N : \Omega \to \Omega$ defined by

$$Ny(t) = \begin{cases} \varphi(0) + I^\alpha K_y(t), & t \in J \\ \varphi(t), & t \in [-r,0]. \end{cases} \tag{6.5}$$

By Lemma 6.2.3, it is clear that the fixed points of N are the solutions of the problem (6.1)-(6.2).
Let $y, \tilde{y} \in \Omega$. If $t \in [-r,0]$, then

$$|Ny(t) - N\tilde{y}(t)| = 0.$$

For $t \in J$, we have

$$|Ny(t) - N\tilde{y}(t)| = |I^\alpha K_y(t) - I^\alpha K_{\tilde{y}}(t)| \leq I^\alpha |K_y(t) - K_{\tilde{y}}(t)|. \tag{6.6}$$

For any $t \in J$

$$\begin{aligned} |K_y(t) - K_{\tilde{y}}(t)| &\leq |f(t, y_t, K_y(t)) - f(t, \tilde{y}_t, K_{\tilde{y}}(t))| \\ &\quad + {}^cD^\alpha |g(t,y_t) - g(t,\tilde{y}_t)| \\ &\leq K \|y_t - \tilde{y}_t\|_C + \overline{K} |K_y(t) - K_{\tilde{y}}(t)| \\ &\quad + {}^cD^\alpha |g(t,y_t) - g(t,\tilde{y}_t)|. \end{aligned}$$

Thus

$$|K_y(t) - K_{\tilde{y}}(t)| \leq \frac{K}{1-\overline{K}} \|y_t - \tilde{y}_t\|_C + \left(\frac{1}{1-\overline{K}}\right) {}^cD^\alpha |g(t,y_t) - g(t,\tilde{y}_t)|. \tag{6.7}$$

By replacing (6.7) in the inequality (6.6), we find

$$\begin{aligned} |Ny(t) - N\tilde{y}(t)| &\leq \frac{K}{\left(1-\overline{K}\right)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|y_s - \tilde{y}_s\|_C \, ds \\ &\quad + \frac{1}{1-\overline{K}} I^\alpha \, {}^cD^\alpha |g(t,y_t) - g(t,\tilde{y}_t)| \\ &\leq \frac{KT^\alpha}{\left(1-\overline{K}\right)\Gamma(\alpha+1)} \|y - \tilde{y}\|_\Omega \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{1-\overline{K}}\left(|g(t,y_t)-g(t,\tilde{y}_t)|+|g\left(0,y_0\right)-g\left(0,\tilde{y}_0\right)|\right)\\
\leq\ & \frac{KT^{\alpha}}{\left(1-\overline{K}\right)\Gamma\left(\alpha+1\right)}\left\|y-\tilde{y}\right\|_{\Omega}+\frac{L}{1-\overline{K}}\left\|y_t-\tilde{y}_t\right\|_{C}\\
\leq\ & \left[\frac{KT^{\alpha}}{\left(1-\overline{K}\right)\Gamma\left(\alpha+1\right)}+\frac{L}{1-\overline{K}}\right]\left\|y-\tilde{y}\right\|_{\Omega},
\end{aligned}
$$

then

$$
\left\|Ny-N\tilde{y}\right\|_{\Omega}\leq\left[\frac{KT^{\alpha}}{\left(1-\overline{K}\right)\Gamma\left(\alpha+1\right)}+\frac{L}{\left(1-\overline{K}\right)}\right]\left\|y-\tilde{y}\right\|_{\Omega}.
$$

From (6.4), it follows that N admits a unique fixed point which is solution of the problem (6.1)-(6.2).

□

6.2.3 Ulam-Hyers Stability Results

for the implicit fractional-order differential equation (6.1), we adopt the definition in Rus [125] for : Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability.

Definition 6.2.5 *The equation* (6.1) *is Ulam-Hyers stable if there exists a real number* $c_f>0$ *such that for each* $\epsilon>0$ *and for each solution* $z\in C^1\left(J,\mathbb{R}\right)$ *of the inequality*

$$
|^{c}D^{\alpha}z(t)-f(t,z_t,^{c}D^{\alpha}z(t))-^{c}D^{\alpha}g(t,z_t)|\leq\epsilon,\ \ t\in J,
$$

there exists a solution $y\in C^1\left(J,\mathbb{R}\right)$ *of equation* (6.1) *with*

$$
|z(t)-y(t)|\leq c_f\epsilon,\ \ t\in J.
$$

Definition 6.2.6 *The equation* (6.1) *is generalized Ulam-Hyers stable if there exists* $\psi_f\in C\left(\mathbb{R}_+,\mathbb{R}_+\right),\ \psi_f\left(0\right)=0$, *such that for each solution* $z\in C^1\left(J,\mathbb{R}\right)$ *of the inequality*

$$
|^{c}D^{\alpha}z(t)-f(t,z_t,^{c}D^{\alpha}z(t))-^{c}D^{\alpha}g(t,z_t)|\leq\epsilon,\ \ t\in J,
$$

there exists a solution $y\in C^1\left(J,\mathbb{R}\right)$ *of the equation* (6.1) *with*

$$
|z(t)-y(t)|\leq\psi_f\left(\epsilon\right),\ \ t\in J.
$$

Definition 6.2.7 *The equation* (6.1) *is Ulam-Hyers-Rassias stable with respect to* $\phi\in C\left(J,\mathbb{R}_+\right)$ *if there exists a real number* $c_f>0$ *such that for each* $\epsilon>0$ *and for each solution* $z\in C^1\left(J,\mathbb{R}\right)$ *of the inequality*

$$
|^{c}D^{\alpha}z(t)-f(t,z_t,^{c}D^{\alpha}z(t))-^{c}D^{\alpha}g(t,z_t)|\leq\epsilon\phi\left(t\right),\ \ t\in J,
$$

there exists a solution $y\in C^1\left(J,\mathbb{R}\right)$ *of equation* (6.1) *with*

$$
|z(t)-y(t)|\leq c_f\epsilon\phi\left(t\right),\ \ t\in J.
$$

Definition 6.2.8 *The equation* (6.1) *is generalized Ulam-Hyers-Rassias stable with respect to* $\phi \in C(J, \mathbb{R}_+)$ *if there exists a real number* $c_{f,\phi} > 0$ *such that for each solution* $z \in C^1(J, \mathbb{R})$ *of the inequality*

$$|^c D^\alpha z(t) - f(t, z_t, ^c D^\alpha z(t)) -^c D^\alpha g(t, z_t)| \leq \phi(t), \ t \in J,$$

there exists a solution $y \in C^1(J, \mathbb{R})$ *of equation* (6.1) *with*

$$|z(t) - y(t)| \leq c_{f,\phi} \phi(t), \ t \in J.$$

Remark 6.2.9 *A function* $z \in C^1(J, \mathbb{R})$ *is a solution of the inequality*

$$|^c D^\alpha z(t) - f(t, z_t, ^c D^\alpha z(t)) -^c D^\alpha g(t, z_t)| \leq \epsilon, \ t \in J,$$

if and only if there exists a function $h \in C(J, \mathbb{R})$ *(which depends on* y*) such that*

i) $|h(t)| \leq \epsilon$, $\forall t \in J$.

ii) $^c D^\alpha [z(t) - g(t, z_t)] = f(t, z_t, ^c D^\alpha z(t)) + h(t)$, $t \in J$.

Remark 6.2.10 *Clearly,*

i) *Definition 6.2.5* $\Rightarrow$ *Definition 6.2.6*

ii) *Definition 6.2.7* $\Rightarrow$ *Definition 6.2.8.*

Remark 6.2.11 *A solution of the implicit differential equation*

$$|^c D^\alpha z(t) - f(t, z_t, ^c D^\alpha z(t)) -^c D^\alpha g(t, z_t)| \leq \epsilon, \ t \in J,$$

with fractional order is called an fractional $\epsilon-$*solution of the implicit fractional differential equation* (6.1).

Theorem 6.2.12 *Assume that* $(H1)$-$(H3)$, (6.4) *are satisfied. If*

$$\overline{K} + L < 1, \tag{6.8}$$

then the problem (6.1)-(6.2) *is Ulam-Hyers stable.*

Proof. Let $\epsilon > 0$ and $z \in \Omega$ a function which verifies the inequality

$$|^c D^\alpha z(t) - f(t, z_t, ^c D^\alpha z(t)) -^c D^\alpha g(t, z_t)| \leq \epsilon \text{ for each } t \in J$$

this inequality is equivalent to

$$|^c D^\alpha z(t) - K_z(t)| \leq \epsilon \tag{6.9}$$

and let $y \in \Omega$ be the unique solution of the problem

$$\begin{cases} ^c D^\alpha [y(t) - g(t, y_t)] = f(t, y_t, ^c D^\alpha y(t)), \ t \in J \\ z(t) = y(t) = \varphi(t), \ t \in [-r, 0]. \end{cases}$$

By integration of the inequality (6.9) , we obtain

$$|z(t) - I^{\alpha}K_z(t)| \leq \frac{\epsilon T^{\alpha}}{\Gamma(\alpha+1)}.$$

We consider the function γ_1 defined by

$$\gamma_1(t) = \sup\{|z(s) - y(s)| : -r \leq s \leq t\}, 0 \leq t \leq T,$$

then, there exists $t^* \in [-r, T]$ such that $\gamma_1(t) = |z(t^*) - y(t^*)|$.
If $t^* \in [-r, 0]$, then $\gamma_1(t) = 0$.
If $t^* \in [0, T]$, then

$$\begin{aligned}\gamma_1(t) &\leq |z(t) - I^{\alpha}K_z(t)| + I^{\alpha}|K_z(t) - K_y(t)| \\ &\leq \frac{\epsilon T^{\alpha}}{\Gamma(\alpha+1)} + I^{\alpha}|K_z(t) - K_y(t)|. \end{aligned} \tag{6.10}$$

On the other hand, we have

$$\begin{aligned}|K_z(t) - K_y(t)| &\leq |f(t, z_t, K_z(t)) - f(t, y_t, K_y(t)| \\ &\quad + {}^cD^{\alpha}|g(t, z_t) - g(t, y_t)| \\ &\leq K\gamma_1(t) + \overline{K}|K_z(t) - K_y(t)| \\ &\quad + {}^cD^{\alpha}|g(t, z_t) - g(t, y_t)|,\end{aligned}$$

then

$$|K_z(t) - K_y(t)| \leq \frac{K}{1-\overline{K}}\gamma_1(t) + \frac{1}{1-\overline{K}}\ {}^cD^{\alpha}|g(t, z_t) - g(t, y_t)| \tag{6.11}$$

by replacing (6.11) in the inequality (6.10), we get

$$\begin{aligned}\gamma_1(t) &\leq \frac{\epsilon T^{\alpha}}{\Gamma(\alpha+1)} + \frac{K}{\left(1-\overline{K}\right)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\gamma_1(s)ds \\ &\quad + \frac{1}{1-\overline{K}}|g(t, z_t) - g(t, y_t)| \\ &\leq \frac{\epsilon T^{\alpha}}{\Gamma(\alpha+1)} + \frac{K}{\left(1-\overline{K}\right)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\gamma_1(s)ds \\ &\quad + \frac{L}{1-\overline{K}}\gamma_1(t),\end{aligned}$$

then

$$\gamma_1(t) \leq \frac{\epsilon T^{\alpha}\left(1-\overline{K}\right)}{\left[1-\left(\overline{K}+L\right)\right]\Gamma(\alpha+1)} + \frac{K}{\left[1-\left(\overline{K}+L\right)\right]\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\gamma_1(s)ds$$

and by the Gronwall's Lemma, we get

$$\gamma_1(t) \leq \frac{\epsilon T^\alpha\left(1-\overline{K}\right)}{\left[1-\left(\overline{K}+L\right)\right]\Gamma\left(\alpha+1\right)}\left[1+\frac{KT^\alpha\sigma_1}{\left[1-\left(\overline{K}_1+L\right)\right]\Gamma\left(\alpha+1\right)}\right]:=c\epsilon,$$

where $\sigma_1 = \sigma_1(\alpha)$ a constant, which completes the proof of the theorem. Moreover, if we set $\psi(\epsilon) = c\epsilon; \psi(0) = 0$, then the problem (6.1)-(6.2) is generalized Ulam-Hyers stable.

□

Theorem 6.2.13 *Assume that* $(H1)$-$(H3)$, (6.4), (6.8) *and*

(H4) there exists an increasing function $\phi \in C\left(J, \mathbb{R}_+\right)$ *and there exists* $\lambda_\phi > 0$ *such that for any* $t \in J$:

$$I^\alpha \phi\left(t\right) \leq \lambda_\phi \phi\left(t\right)$$

are satisfied. Then, the problem (6.1)-(6.2) *is Ulam-Hyers-Rassias stable.*

Proof. Let $z \in \Omega$ solution of the following inequality

$$\left|{}^cD^\alpha z(t) - f(t, z_t, {}^c D^\alpha z(t)) - {}^c D^\alpha g(t, z_t)\right| \leq \epsilon\phi\left(t\right), \ t \in J, \ \epsilon > 0$$

this inequality is equivalent to

$$\left|{}^cD^\alpha z(t) - K_z(t)\right| \leq \epsilon\phi\left(t\right) \tag{6.12}$$

and let $y \in \Omega$ the unique solution of Cauchy problem

$$\begin{cases} {}^cD^\alpha\left[y(t) - g(t, y_t)\right] = f(t, y_t, {}^c D^\alpha y(t)), \ t \in J \\ z(t) = y(t) = \varphi(t), \ t \in \left[-r, 0\right]. \end{cases}$$

By integration of (6.12), we obtain for any $t \in J$

$$\left|z(t) - I^\alpha K_z(t)\right| \leq \epsilon I^\alpha \phi\left(t\right) \leq \epsilon\lambda_\phi\phi\left(t\right).$$

Using the function γ_1 which is defined in the proof of the theorem 6.2.12, we get :
if $t^* \in [-r, 0]$ then $\gamma_1\left(t\right) = 0$.
If $t^* \in [0, T]$, then we have

$$\begin{aligned} \gamma_1\left(t\right) &\leq \left|z(t) - I^\alpha K_z(t)\right| + I^\alpha\left|K_z(t) - K_y(t)\right| \\ &\leq \epsilon\lambda_\phi\phi\left(t\right) + I^\alpha\left|K_z(t) - K_y(t)\right| \end{aligned} \tag{6.13}$$

from which it follows that

$$\left|K_z(t) - K_y(t)\right| \leq \frac{K}{1-\overline{K}}\gamma_1\left(t\right) + \frac{1}{1-\overline{K}}\ {}^cD^\alpha\left|g(t, z_t) - g(t, y_t)\right|. \tag{6.14}$$

By replacing (6.14) in the inequality (6.13), we obtain

$$\begin{aligned}\gamma_1(t) &\leq \epsilon\lambda_\phi\phi(t) + \frac{K}{\left(1-\overline{K}\right)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\gamma_1(s)\,ds \\ &\quad + \frac{1}{1-\overline{K}}\left|g(t,z_t)-g(t,y_t)\right| \\ &\leq \epsilon\lambda_\phi\phi(t) + \frac{K}{\left(1-\overline{K}\right)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\gamma_1(s)\,ds + \frac{L}{1-\overline{K}}\gamma_1(t),\end{aligned}$$

then

$$\gamma_1(t) \leq \frac{\left(1-\overline{K}\right)\epsilon\lambda_\phi\phi(t)}{1-\left(\overline{K}+L\right)} + \frac{K}{\left[1-\left(\overline{K}+L\right)\right]\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\gamma_1(s)\,ds,$$

by the Gronwall's Lemma, we get

$$\begin{aligned}\gamma_1(t) &\leq \frac{\left(1-\overline{K}\right)\epsilon\lambda_\phi\phi(t)}{1-\left(\overline{K}+L\right)}\left[1+\frac{KT^\alpha\sigma_2}{\left[1-\left(\overline{K}+L\right)\right]\Gamma(\alpha+1)}\right] \\ &\leq \left[\frac{\left(1-\overline{K}\right)\lambda_\phi}{1-\left(\overline{K}+L\right)}\left(1+\frac{KT^\alpha\sigma_2}{\left[1-\left(\overline{K}+L\right)\right]\Gamma(\alpha+1)}\right)\right]\epsilon\phi(t) = c\epsilon\phi(t),\end{aligned}$$

where $\sigma_2 = \sigma_2(\alpha)$ a constant. Then the problem (6.1)-(6.2) is Ulam-Hyers-Rassias stable.

□

6.2.4 Examples

Example 1. Consider the problem of neutral fractional differential equation :

$$^cD^{\frac{1}{2}}\left[y(t) - \frac{te^{-t}\|y_t\|_C}{(9+e^t)(1+\|y_t\|_C)}\right] = \frac{2+\|y_t\|_C + |^cD^{\frac{1}{2}}y(t)|}{12e^{t+9}(1+\|y_t\|_C + |^cD^{\frac{1}{2}}y(t)|)}, \quad t\in[0,1] \quad (6.15)$$

$$y(t) = \varphi(t); \quad t\in[-r,0], \quad r>0 \quad (6.16)$$

where $\varphi \in C([-r,0],\mathbb{R})$.
Set

$$g(t,w) = \frac{te^{-t}w}{(9+e^t)(1+w)}, \quad (t,w)\in[0,1]\times[0,+\infty)$$

and

$$f(t,u,v) = \frac{2+u+v}{12e^{t+9}(1+u+v)}, \quad (t,u,v)\in[0,1]\times[0,+\infty)\times[0,+\infty).$$

Observe that $g(0,w)=0$, for any $w\in[0,+\infty)$.

Clearly, the function f is continuous. Hence, $(H1)$ is satisfied.
We have,

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \le \frac{1}{12e^9}\left(\|u-\bar{u}\|_C + |v-\bar{v}|\right)$$

$$|g(t,u) - g(t,\bar{u})| \le \frac{1}{10}\|u-\bar{u}\|_C$$

for any $u, \bar{u} \in C([-r,0],\mathbb{R}),\ v,\bar{v} \in \mathbb{R}\ and\ t \in [0,1]$.
Hence, conditions $(H2)$ and $(H3)$ are satisfied with $K = \overline{K} = \dfrac{1}{12e^9}$ and $L = \dfrac{1}{10}$.
And condition

$$\frac{KT^\alpha}{(1+\overline{K})\Gamma(\alpha+1)} + \frac{L}{(1-\overline{K})} = \frac{20+12e^9\sqrt{\pi}}{10\sqrt{\pi}(12e^9-1)} < 1,$$

is satisfied with $T = 1$, $\alpha = \dfrac{1}{2}$.
By Lemma 6.2.4, the problem (6.15)-(6.16) admits a unique solution.
Since

$$\overline{K} + L = \frac{10+12e^9}{120e^9} < 1,$$

then, by Theorem 6.2.12, the problem (6.15)-(6.16) is Ulam-Hyers stable.

Example 2. Consider the problem of neutral fractional differential equation :

$$^cD^{\frac{1}{2}}\left[y(t) - \frac{t}{5e^{t+2}\left(1+\|y_t\|_C\right)}\right] = \frac{e^{-t}}{7+e^t}\left[\frac{\|y_t\|_C}{1+\|y_t\|_C} - \frac{|^cD^{\frac{1}{2}}y(t)|}{1+|^cD^{\frac{1}{2}}y(t)|}\right], t \in [0,1] \tag{6.17}$$

$$y(t) = \varphi(t), \quad t \in [-r,0], \quad r > 0, \tag{6.18}$$

where $\varphi \in C([-r,0],\mathbb{R})$.
Set

$$g(t,w) = \frac{t}{5e^{t+2}(1+w)}, \quad (t,w) \in [0,1] \times [0,+\infty),$$

and

$$f(t,u,v) = \frac{e^{-t}}{(7+e^t)}\left(\frac{u}{1+u} - \frac{v}{1+v}\right), \quad (t,u,v) \in [0,1] \times [0,+\infty) \times [0,+\infty).$$

Observe that $g(0,w) = 0$, for any $w \in [0,+\infty)$.
Clearly, the function f is continuous. Hence, $(H1)$ is satisfied.

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \le \frac{1}{8}\|u-\bar{u}\|_C + \frac{1}{8}|v-\bar{v}|$$

$$|g(t,u) - g(t,\bar{u})| \le \frac{1}{5e^2}\|u-\bar{u}\|_C$$

for any $u, \bar{u} \in C([-r,0],\mathbb{R}),\ v, \bar{v} \in \mathbb{R}\ and\ t \in [0,1]$.
Hence, conditions $(H2)$ and $(H3)$ are satisfied with $K = \overline{K} = \frac{1}{8}$ and $L = \frac{1}{5e^2}$.
We have

$$\frac{KT^\alpha}{(1+\overline{K})\Gamma(\alpha+1)} + \frac{L}{(1-\overline{K})} = \frac{10e^2 + 8\sqrt{\pi}}{35e^2\sqrt{\pi}} < 1.$$

By Lemma 6.2.4, the problem (6.17)-(6.18) admits a unique solution.
Since

$$\overline{K} + L = \frac{5e^2 + 8}{40e^2} < 1,$$

then, by Theorem 6.2.12, the problem (6.17)-(6.18) is Ulam-Hyers stable.

6.3 Existence Results for the Neutral IFDE with Finite Delay and Impulses

[2]

6.3.1 Introduction

The purpose of this section, is to establish existence, uniqueness and stability results to the following implicit neutral differential equations of fractional order with finite delay and Impulses

$$^cD^\alpha_{t_k}[y(t) - \phi(t, y_t)] = f(t, y_t, {}^cD^\alpha_{t_k} y(t)),\ \text{for each } t \in (t_k, t_{k+1}],\ k = 0, \dots, m,\ 0 < \alpha \le 1, \tag{6.19}$$

$$\Delta y|_{t_k} = I_k(y_{t_k^-}),\ k = 1, \dots, m, \tag{6.20}$$

$$y(t) = \varphi(t),\ t \in [-r, 0],\ r > 0, \tag{6.21}$$

where $^cD^\alpha_{t_k}$ is the Caputo fractional derivative, $f : J \times PC([-r,0],\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$, $\phi : J \times PC([-r,0],\mathbb{R}) \to \mathbb{R}$ are given functions with $\phi(0,\varphi) = 0, I_k : PC([-r,0],\mathbb{R}) \to \mathbb{R}$ and $\varphi \in PC([-r,0],\mathbb{R}), 0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, and $PC([-r,0],\mathbb{R})$ is a space to be specified later. For each function y defined on $[-r,T]$ and for any $t \in J$, we denote by y_t the element of $PC([-r,0],\mathbb{R})$ defined by :

$$y_t(\theta) = y(t+\theta),\quad \theta \in [-r, 0],$$

that is, $y_t(\cdot)$ represents the history of the state from time $t - r$ up to time t.
And $\Delta y|_{t_k} = y(t_k^+) - y(t_k^-)$, where $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^-} y(t_k + h)$ represent the right and left limits of y_t at $t = t_k$, respectively.
The arguments are based upon the Banach contraction principle, and Schaefer's fixed point theorem. At last, an example is included to show the applicability of our results.

2. M. Benchohra, S. Bouriah and J.Henderson, Existence and Stability Results for Nonlinear Implicit Neutral Fractional Differential Equations with Finite Delay and Impulses, *Comm. Appl. Nonlinear Anal.* **22** (1), (2015), 46-67.

6.3.2 Existence of Solutions

By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\}.$$

Let $J_0 = [t_0, t_1], J_k = (t_k, t_{k+1}]$ where $k = 1, \dots, m$ and $-r = \tau_0 < \tau_1 < \dots < \tau_l < \tau_{l+1} = 0$, with $l \leq m$.
Consider the sets of functions

$$PC([-r,0],\mathbb{R}) = \{y : [-r,0] \to \mathbb{R} : y \in C((\tau_k, \tau_{k+1}],\mathbb{R}),\ k = 1, ..., l,\ \text{and there exist } y(\tau_k^-) \text{ and } y(\tau_k^+), k = 1, ..., l \text{ with } \ y(\tau_k^-) = y(\tau_k) \text{ and } \tau_k = t_k - t, \ \ \text{for each } t \in J_k\}.$$

$PC([-r,0],\mathbb{R})$ is a Banach space with the norm

$$\|y\|_{PC} = \sup_{t\in[-r,0]} |y(t)|.$$

$$PC([0,T],\mathbb{R}) = \{y : [0,T] \to \mathbb{R} : y \in C((t_k, t_{k+1}],\mathbb{R}),\ k = 1, ..., m,\ \text{and there exist } y(t_k^-) \text{ and } y(t_k^+), k = 1, ..., m \text{ with } y(t_k^-) = y(t_k)\}.$$

$PC([0,T],\mathbb{R})$ is a Banach space with the norm

$$\|y\|_C = \sup_{t\in[0,T]} |y(t)|.$$

$$\Omega = \{y : [-r,T] \to \mathbb{R} : y|_{[-r,0]} \in PC([-r,0],\mathbb{R}) \text{ and } y|_{[0,T]} \in PC([0,T],\mathbb{R})\}.$$

Ω is a Banach space with the norm

$$\|y\|_\Omega = \sup_{t\in[-r,T]} |y(t)|.$$

Definition 6.3.1 A function $y \in \Omega$ whose α-derivative exists on J_k is said to be a solution of (6.19)–(6.21) if y satisfies the equation ${}^cD^\alpha_{t_k}(y(t) - \phi(t, y_t)) = f(t, y_t, {}^cD^\alpha_{t_k} y(t))$ on J_k, and satisfy the conditions

$$\Delta y|_{t=t_k} = I_k(y_{t_k^-}),\ k = 1, \dots, m,$$
$$y(t) = \varphi(t),\ t \in [-r, 0].$$

To prove the existence of solutions to (6.19)–(6.21), we need the following auxiliary Lemma.

Lemma 6.3.2 *Let $0 < \alpha \leq 1$ and let $\sigma : J \to \mathbb{R}$ be continuous. A function y is a solution of the fractional integral equation*

$$y(t) = \begin{cases} \varphi(0) + \phi(t, y_t) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1}\sigma(s)ds & \text{if } t \in [0, t_1] \\ \\ \varphi(0) + \phi(t, y_t) + \displaystyle\sum_{i=1}^{k} I_i(y_{t_i^-}) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1}\sigma(s)ds \\ +\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_k}^t (t-s)^{\alpha-1}\sigma(s)ds, & \text{if } t \in (t_k, t_{k+1}], \\ \\ \varphi(t), \ t \in [-r, 0], \end{cases} \tag{6.22}$$

where $k = 1, \dots, m$, if and only if y is a solution of the following fractional problem

$$^cD^\alpha(y(t) - \phi(t, y_t)) = \sigma(t), \quad t \in J_k, \tag{6.23}$$

$$\Delta y|_{t=t_k} = I_k(y_{t_k^-}), \ k = 1, \dots, m, \tag{6.24}$$

$$y(t) = \varphi(t), \ t \in [-r, 0]. \tag{6.25}$$

Proof. Assume y satisfies (6.23)-(6.25). If $t \in [0, t_1]$ then

$$^cD^\alpha(y(t) - \phi(t, y_t)) = \sigma(t).$$

Lemma 1.3.7 implies

$$y(t) - \phi(t, y_t) = \varphi(0) + I^\alpha\sigma(t) = \varphi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\sigma(s)ds.$$

If $t \in (t_1, t_2]$ then Lemma 1.3.7 implies

$$\begin{aligned} y(t) - \phi(t, y_t) &= y(t_1^+) - \phi(t_1, y_{t_1}) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1}\sigma(s)ds \\ &= \Delta y|_{t=t_1} + y(t_1^-) - \phi(t_1, y_{t_1}) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1}\sigma(s)ds \\ &= I_1(y_{t_1^-}) + \left[\varphi(0) + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1}\sigma(s)ds\right] \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1}\sigma(s)ds. \\ &= \varphi(0) + I_1(y_{t_1^-}) + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1}\sigma(s)ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1}\sigma(s)ds. \end{aligned}$$

If $t \in (t_2, t_3]$, then from Lemma 1.3.7, we get

$$\begin{aligned} y(t) - \phi(t, y_t) &= y(t_2^+) - \phi(t_2, y_{t_2}) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1}\sigma(s)ds \\ &= \Delta y|_{t=t_2} + y(t_2^-) - \phi(t_2, y_{t_2}) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1}\sigma(s)ds \\ &= I_2(y_{t_2^-}) + \Bigg[\varphi(0) + I_1(y_{t_1^-}) + \frac{1}{\Gamma(\alpha)} \int_{0}^{t_1} (t_1-s)^{\alpha-1}\sigma(s)ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\sigma(s)ds\Bigg] + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1}\sigma(s)ds. \\ &= \varphi(0) + \Big[I_1(y_{t_1^-}) + I_2(y_{t_2^-})\Big] + \Bigg[\frac{1}{\Gamma(\alpha)} \int_{0}^{t_1} (t_1-s)^{\alpha-1}\sigma(s)ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\sigma(s)ds\Bigg] + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1}\sigma(s)ds. \end{aligned}$$

Repeating the process in this way, the solution $y(t)$, for $t \in (t_k, t_{k+1}]$ where $k = 1, \ldots, m$, can be written as

$$\begin{aligned} y(t) &= \varphi(0) + \phi(t, y_t) + \sum_{i=1}^{k} I_i(y_{t_i^-}) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1}\sigma(s)ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}\sigma(s)ds. \end{aligned}$$

Conversely, assume that y satisfies the impulsive fractional integral equation (6.22). If $t \in [0, t_1]$ then $y(0) = \varphi(0)$ and using the fact that ${}^cD^\alpha$ is the left inverse of I^α we get

$$^cD^\alpha(y(t) - \phi(t, y_t)) = \sigma(t), \quad \text{for each } t \in [0, t_1].$$

If $t \in (t_k, t_{k+1}]$, $k = 1, \ldots, m$, and using the fact that ${}^cD^\alpha C = 0$, where C is a constant, we get

$$^cD^\alpha(y(t) - \phi(t, y_t)) = \sigma(t), \text{for each } t \in (t_k, t_{k+1}].$$

Also, we can easily show that

$$\Delta y|_{t=t_k} = I_k(y_{t_k^-}), \quad k = 1, \ldots, m.$$

□

We are now in a position to state and prove our existence result for the problem (6.19)−(6.21) based on Banach's fixed point.

Theorem 6.3.3 *Assume*

(P1) The function $f : J \times PC([-r, 0], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ *is continuous.*

(P2) There exist constants $K > 0, \overline{L} > 0$ *and* $0 < L < 1$ *such that*

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \leq K\|u-\bar{u}\|_{PC} + L|v-\bar{v}|$$

and

$$|\phi(t,u) - \phi(t,\bar{u})| \leq \overline{L}\|u-\bar{u}\|_{PC}$$

for any $u, \bar{u} \in PC([-r,0],\mathbb{R}),\ v, \bar{v} \in \mathbb{R}$ *and* $t \in J$.

(P3) There exists a constant $\widetilde{l} > 0$ *such that*

$$|I_k(u) - I_k(\overline{u})| \leq \widetilde{l}\|u-\overline{u}\|_{PC},$$

for each $u, \overline{u} \in PC([-r,0],\mathbb{R})$ *and* $k = 1, \ldots, m$.

If

$$m\widetilde{l} + \overline{L} + \frac{(m+1)KT^{\alpha}}{(1-L)\Gamma(\alpha+1)} < 1, \tag{6.26}$$

then there exists a unique solution for the problem (6.19)-(6.21) *on* J.

Proof. Transform the problem (6.19)-(6.21) into a fixed point problem. Consider the operator $N : \Omega \to \Omega$ defined by

$$Ny(t) = \begin{cases} \varphi(0) + \phi(t,y_t) + \displaystyle\sum_{0<t_k<t} I_k(y_{t_k^-}) + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} g(s) ds \\ + \displaystyle\frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} g(s) ds, & t \in [0,T], \\ \varphi(t), & t \in [-r,0], \end{cases} \tag{6.27}$$

where $g \in C(J,\mathbb{R})$ be such that

$$g(t) = f(t, y_t, g(t)).$$

Clearly, the fixed points of operator N are solutions of problem (6.19)–(6.21).
Let $u, w \in \Omega$. If $t \in [-r,0]$, then

$$|N(u)(t) - N(w)(t)| = 0.$$

For $t \in J$, we have

$$\begin{aligned} |N(u)(t) - N(w)(t)| &\leq \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} |g(s) - h(s)| ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |g(s) - h(s)| ds + |\phi(t,u_t) - \phi(t,w_t)| \end{aligned}$$

$$+ \sum_{0<t_k<t} |I_k(u_{t_k^-}) - I_k(w_{t_k^-})|,$$

where $g, h \in C(J, \mathbb{R})$ be such that

$$g(t) = f(t, u_t, g(t)),$$

and

$$h(t) = f(t, w_t, h(t)).$$

By $(P2)$ we have

$$\begin{aligned} |g(t) - h(t)| &= |f(t, u_t, g(t)) - f(t, w_t, h(t))| \\ &\le K\|u_t - w_t\|_{PC} + L|g(t) - h(t)|. \end{aligned}$$

Then

$$|g(t) - h(t)| \le \frac{K}{1-L}\|u_t - w_t\|_{PC}.$$

Therefore, for each $t \in J$

$$\begin{aligned} |N(u)(t) - N(w)(t)| &\le \frac{K}{(1-L)\Gamma(\alpha)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}\|u_s - w_s\|_{PC} ds \\ &+ \frac{K}{(1-L)\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}\|u_s - w_s\|_{PC} ds \\ &+ \sum_{k=1}^{m} \widetilde{l}\|u_{t_k^-} - w_{t_k^-}\|_{PC} + \overline{L}\|u_t - w_t\|_{PC}. \\ &\le \left[m\widetilde{l} + \overline{L} + \frac{mKT^\alpha}{(1-L)\Gamma(\alpha+1)} \right. \\ &\quad \left. + \frac{KT^\alpha}{(1-L)\Gamma(\alpha+1)}\right]\|u - w\|_\Omega. \end{aligned}$$

Thus

$$\|N(u) - N(w)\|_\Omega \le \left[m\widetilde{l} + \overline{L} + \frac{(m+1)KT^\alpha}{(1-L)\Gamma(\alpha+1)}\right]\|u - w\|_\Omega.$$

By (6.26), the operator N is a contraction. Hence, by Banach's contraction principle, N has a unique fixed point which is a unique solution of the problem (6.19)–(6.21). □

Our second result is based on Schaefer's fixed point theorem.

Theorem 6.3.4 *Assume (P1), (P2) and*

(P4) There exist $p, q, r \in C(J, \mathbb{R}_+)$ *with* $r^* = \sup_{t\in J} r(t) < 1$ *such that*

$$|f(t,u,w)| \leq p(t) + q(t)\|u\|_{PC} + r(t)|w| \text{ for } t \in J,\ u \in PC([-r,0],\mathbb{R}) \text{ and } w \in \mathbb{R}.$$

(P5) The functions $I_k : PC([-r,0],\mathbb{R}) \to \mathbb{R}$ *are continuous and there exist constants* $M^*, N^* > 0$ *such that*

$$|I_k(u)| \leq M^*\|u\|_{PC} + N^* \text{ for each } u \in PC([-r,0],\mathbb{R}),\ k = 1, \ldots, m.$$

(P6) The function ϕ *is completely continuous, and for each bounded set* B_{η^*} *in* Ω, *the set* $\{t \to \phi(t, y_t) : y \in B_{\eta^*}\}$ *is equicontinuous in* $PC(J,\mathbb{R})$ *and there exist two constants* $d_1 > 0, d_2 > 0$ *with* $mM^* + d_1 < 1$ *such that*

$$|\phi(t,u)| \leq d_1\|u\|_{PC} + d_2, t \in J, u \in PC([-r,0],\mathbb{R}).$$

Then, the problem (6.19)-(6.21) *has at least one solution.*

Proof. We consider the operator $N_1 : \Omega \to \Omega$ defined by

$$N_1 y(t) = \begin{cases} \varphi(0) + \sum_{0<t_k<t} I_k(y_{t_k^-}) + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} g(s) ds \\ + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} g(s) ds, & t \in [0,T], \\ \\ \varphi(t), & t \in [-r,0], \end{cases}$$

The operator N defined in (6.27) can be written as

$$Ny(t) = \phi(t, y_t) + N_1 y(t), \text{ for each } t \in J.$$

We shall use Schaefer's fixed point theorem to prove that N has a fixed point. So we have to show that N is completely continuous. since ϕ is completely continuous by (P6), we shall show that N_1 is completely continuous.
The proof will be given in several steps.

Step 1 : N_1 *is continuous.* Let $\{u_n\}$ be a sequence such that $u_n \to u$ in Ω. If $t \in [-r, 0]$, then

$$|N_1(u_n)(t) - N_1(u)(t)| = 0.$$

For $t \in J$, we have

$$\begin{aligned}
|N_1(u_n)(t) - N_1(u)(t)| &\leq \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} |g_n(s) - g(s)| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |g_n(s) - g(s)| ds \\
&+ \sum_{0<t_k<t} |I_k(u_{nt_k^-}) - I_k(u_{t_k^-})| \\
&\leq \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} |g_n(s) - g(s)| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |g_n(s) - g(s)| ds \\
&+ \sum_{0<t_k<t} \widetilde{l} \|u_{nt_k^-} - u_{t_k^-}\|_{PC}
\end{aligned}$$

and then

$$\begin{aligned}
|N_1(u_n)(t) - N_1(u)(t)| &\leq \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} |g_n(s) - g(s)| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |g_n(s) - g(s)| ds \\
&+ m\widetilde{l} \|u_n - u\|_{\Omega},
\end{aligned} \tag{6.28}$$

where $g_n, g \in C(J, \mathbb{R})$ such that

$$g_n(t) = f(t, u_{nt}, g_n(t)),$$

and

$$g(t) = f(t, u_t, g(t)).$$

By $(P2)$, we have

$$\begin{aligned}
|g_n(t) - g(t)| &= |f(t, u_{nt}, g_n(t)) - f(t, u_t, g(t))| \\
&\leq K\|u_{nt} - u_t\|_{PC} + L|g_n(t) - g(t)|.
\end{aligned}$$

Then

$$|g_n(t) - g(t)| \leq \frac{K}{1-L} \|u_{nt} - u_t\|_{PC}.$$

Since $u_n \to u$, then we get $g_n(t) \to g(t)$ as $n \to \infty$ for each $t \in J$. And let $\eta > 0$ be such that, for each $t \in J$, we have $|g_n(t)| \leq \eta$ and $|g(t)| \leq \eta$. Then, we have

$$\begin{aligned}
(t-s)^{\alpha-1} |g_n(s) - g(s)| &\leq (t-s)^{\alpha-1} [|g_n(s)| + |g(s)|] \\
&\leq 2\eta (t-s)^{\alpha-1},
\end{aligned}$$

and

$$\begin{aligned} (t_k-s)^{\alpha-1}|g_n(s)-g(s)| &\leq (t_k-s)^{\alpha-1}[|g_n(s)|+|g(s)|] \\ &\leq 2\eta(t_k-s)^{\alpha-1}. \end{aligned}$$

For each $t \in J$, the functions $s \to 2\eta(t-s)^{\alpha-1}$ and $s \to 2\eta(t_k-s)^{\alpha-1}$ are integrable on $[0,t]$, then the Lebesgue Dominated Convergence Theorem and (6.28) imply that

$$|N_1(u_n)(t) - N_1(u)(t)| \to 0 \ as \ n \to \infty,$$

and hence

$$\|N_1(u_n) - N_1(u)\|_\Omega \to 0 \text{ as } n \to \infty.$$

Consequently, N_1 is continuous.

Step 2 : *N_1 maps bounded sets into bounded sets in Ω*. Indeed, it is enough to show that for any $\eta^* > 0$, there exists a positive constant ℓ such that for each $u \in B_{\eta^*} = \{u \in \Omega : ||u||_\Omega \leq \eta^*\}$, we have $\|N_1(u)\|_\Omega \leq \ell$. We have for each $t \in J$,

$$\begin{aligned} N_1(u)(t) &= \varphi(0) + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k-s)^{\alpha-1} g(s)ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} g(s)ds, \\ &+ \sum_{0<t_k<t} I_k(u_{t_k^-}), \end{aligned} \tag{6.29}$$

where $g \in C(J, \mathbb{R})$ is such that

$$g(t) = f(t, u_t, g(t)).$$

By (P4), we have for each $t \in J$,

$$\begin{aligned} |g(t)| &= |f(t, u_t, g(t))| \\ &\leq p(t) + q(t)\|u_t\|_{PC} + r(t)|g(t)| \\ &\leq p(t) + q(t)\|u\|_\Omega + r(t)|g(t)| \\ &\leq p(t) + q(t)\eta^* + r(t)|g(t)| \\ &\leq p^* + q^*\eta^* + r^*|g(t)|, \end{aligned}$$

where $p^* = \sup_{t \in J} p(t)$, and $q^* = \sup_{t \in J} q(t)$.

Then

$$|g(t)| \leq \frac{p^* + q^*\eta^*}{1-r^*} := M.$$

Thus (6.29) implies

$$\begin{aligned}|N_1(u)(t)| &\leq |\varphi(0)| + \frac{mMT^\alpha}{\Gamma(\alpha+1)} + \frac{MT^\alpha}{\Gamma(\alpha+1)} + \sum_{k=1}^{m}\left(M^*\|u_{t_k^-}\|_{PC} + N^*\right)\\ &\leq |\varphi(0)| + \frac{(m+1)MT^\alpha}{\Gamma(\alpha+1)} + m\left(M^*\|u_{t_k^-}\|_\Omega + N^*\right)\\ &\leq |\varphi(0)| + \frac{(m+1)MT^\alpha}{\Gamma(\alpha+1)} + m\left(M^*\eta^* + N^*\right) := R.\end{aligned}$$

And if $t \in [-r, 0]$, then

$$|N_1(u)(t)| \leq \|\varphi\|_{PC},$$

thus

$$\|N_1(u)\|_\Omega \leq \max\ \{R, \|\varphi\|_{PC}\} := \ell.$$

Step 3 : N_1 *maps bounded sets into equicontinuous sets of* Ω.

Let $\tau_1, \tau_2 \in (0, T]$, $\tau_1 < \tau_2$, B_{η^*} be a bounded set of Ω as in Step 2, and let $u \in B_{\eta^*}$. Then

$$\begin{aligned}&|N_1(u)(\tau_2) - N_1(u)(\tau_1)|\\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^{\tau_1} |(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}||g(s)|ds\\ &\quad + \frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2} |(\tau_2 - s)^{\alpha-1}||g(s)|ds + \sum_{0<t_k<\tau_2-\tau_1} |I_k(u_{t_k^-})|\\ &\leq \frac{M}{\Gamma(\alpha+1)}[2(\tau_2-\tau_1)^\alpha + (\tau_2^\alpha - \tau_1^\alpha)] + (\tau_2 - \tau_1)\left(M^*\|u_{t_k^-}\|_\Omega + N^*\right)\\ &\leq \frac{M}{\Gamma(\alpha+1)}[2(\tau_2-\tau_1)^\alpha + (\tau_2^\alpha - \tau_1^\alpha)] + (\tau_2 - \tau_1)\left(M^*\eta^* + N^*\right).\end{aligned}$$

As $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Ascoli-Arzela theorem, we can conclude that $N_1 : \Omega \to \Omega$ is completely continuous.

Step 4 : *A priori bounds.* Now it remains to show that the set

$$E = \{u \in \Omega : u = \lambda N(u) \text{ for some } \ 0 < \lambda < 1\}$$

is bounded. Let $u \in E$. Then $u = \lambda N(u)$ for some $0 < \lambda < 1$. Thus, for each $t \in J$ we have

$$\begin{aligned}u(t) &= \lambda\varphi(0) + \lambda\phi(t, y_t) + \frac{\lambda}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k - s)^{\alpha-1}g(s)ds\\ &+ \frac{\lambda}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}g(s)ds + \lambda\sum_{0<t_k<t} I_k(u_{t_k^-}).\end{aligned} \tag{6.30}$$

And, by (P4), we have for each $t \in J$,

$$\begin{aligned} |g(t)| &= |f(t, u_t, g(t))| \\ &\leq p(t) + q(t)\|u_t\|_{PC} + r(t)|g(t)| \\ &\leq p^* + q^*\|u_t\|_{PC} + r^*|g(t)|. \end{aligned}$$

Thus

$$|g(t)| \leq \frac{1}{1-r^*}(p^* + q^*\|u_t\|_{PC}).$$

This implies, by (6.30), (P5) and (P6), that for each $t \in J$ we have

$$\begin{aligned} |u(t)| &\leq |\varphi(0)| + d_1\|u_t\|_{PC} + d_2 \\ &+ \frac{1}{(1-r^*)\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}(p^* + q^*\|u_s\|_{PC})ds \\ &+ \frac{1}{(1-r^*)\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}(p^* + q^*\|u_s\|_{PC})ds \\ &+ m\left(M^*\|u_{t_k^-}\|_{PC} + N^*\right). \end{aligned}$$

Consider the function ν defined by

$$\nu(t) = \sup\{|u(s)| : -r \leq s \leq t\}, 0 \leq t \leq T.$$

Then, there exists $t^* \in [-r, T]$ such that $\nu(t) = |u(t^*)|$. If $t^* \in [0, T]$, then by the previous inequality, we have for $t \in J$

$$\begin{aligned} \nu(t) &\leq |\varphi(0)| + \frac{1}{(1-r^*)\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}(p^* + q^*\nu(s))ds \\ &+ \frac{1}{(1-r^*)\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}(p^* + q^*\nu(s))ds \\ &+ (mM^* + d_1)\nu(t) + (mN^* + d_2). \end{aligned}$$

Thus

$$\begin{aligned} \nu(t) &\leq \frac{1}{(1-(mM^*+d_1))(1-r^*)\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}(p^* + q^*\nu(s))ds \\ &+ \frac{|\varphi(0)| + mN^* + d_2}{1-(mM^*+d_1)} \\ &+ \frac{1}{(1-(mM^*+d_1))(1-r^*)\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}(p^* + q^*\nu(s))ds \end{aligned}$$

$$\begin{aligned}&\leq \frac{|\varphi(0)|+mN^*+d_2}{1-(mM^*+d_1)}+\frac{(m+1)p^*T^\alpha}{(1-(mM^*+d_1))(1-r^*)\Gamma(\alpha+1)}\\&+\frac{(m+1)q^*}{(1-(mM^*+d_1))(1-r^*)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\nu(s)ds.\end{aligned}$$

Applying Lemma 1.3.8, we get

$$\begin{aligned}\nu(t) &\leq \left[\frac{|\varphi(0)|+mN^*+d_2}{1-(mM^*+d_1)}+\frac{(m+1)p^*T^\alpha}{(1-(mM^*+d_1))(1-r^*)\Gamma(\alpha+1)}\right]\\&\times \left[1+\frac{\delta(m+1)q^*T^\alpha}{(1-(mM^*+d_1))(1-r^*)\Gamma(\alpha+1)}\right]:=A,\end{aligned}$$

where $\delta=\delta(\alpha)$ a constant. If $t^*\in[-r,0]$, then $\nu(t)=\|\varphi\|_{PC}$, thus for any $t\in J, \|u\|_\Omega\leq\nu(t)$, we have

$$\|u\|_\Omega\leq\max\{\|\varphi\|_{PC},A\}$$

This shows that the set E is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that N has a fixed point which is a solution of the problem (6.19)−(6.21).

□

6.3.3 Ulam-Hyers Stability Results

Here, we adopt the concepts in Wang *et al.* [139] and introduce Ulam's type stability concepts for the problem (6.19)-(6.20).
Let $z\in PC(J,\mathbb{R}), \epsilon>0, \psi>0$, and $\omega\in PC(J,\mathbb{R}_+)$ be nondecreasing. We consider the set of inequalities

$$\begin{cases}|^cD^\alpha(z(t)-\phi(t,z_t))-f(t,z_t,{}^cD^\alpha z(t))|\leq\epsilon,\ t\in(t_k,t_{k+1}], k=1,...,m\\|\Delta z|_{t=t_k}-I_k(z_{t_k^-})|\leq\epsilon,\ k=1,...,m;\end{cases}\tag{6.31}$$

the set of inequalities

$$\begin{cases}|^cD^\alpha(z(t)-\phi(t,z_t))-f(t,z_t,{}^cD^\alpha z(t))|\leq\omega(t),\ t\in(t_k,t_{k+1}], k=1,...,m\\|\Delta z|_{t=t_k}-I_k(z_{t_k^-})|\leq\psi,\ k=1,...,m;\end{cases}\tag{6.32}$$

and the set of inequalities

$$\begin{cases}|^cD^\alpha(z(t)-\phi(t,z_t))-f(t,z_t,{}^cD^\alpha z(t))|\leq\epsilon\omega(t),\ t\in(t_k,t_{k+1}], k=1,...,m\\|\Delta z|_{t=t_k}-I_k(z_{t_k^-})|\leq\epsilon\psi,\ k=1,...,m;\end{cases}\tag{6.33}$$

Definition 6.3.5 *The problem* (6.19)-(6.20) *is Ulam-Hyers stable if there exists a real number $c_{f,m}>0$ such that for each $\epsilon>0$ and for each solution $z\in PC(J,\mathbb{R})$ of the inequality (6.31) there exists a solution $y\in PC([-r,0],\mathbb{R})$ of the problem* (6.19)-(6.20) *with*

$$|z(t)-y(t)|\leq c_{f,m}\epsilon,\ t\in J.$$

Definition 6.3.6 *The problem* (6.19)-(6.20) *is generalized Ulam-Hyers stable if there exists* $\theta_{f,m} \in C(\mathbb{R}_+, \mathbb{R}_+), \theta_{f,m}(0) = 0$ *such that for each solution* $z \in PC(J, \mathbb{R})$ *of the inequality (6.31) there exists a solution* $y \in PC([-r, 0], \mathbb{R})$ *of the problem* (6.19)-(6.20) *with*

$$|z(t) - y(t)| \leq \theta_{f,m}(\epsilon), \ t \in J.$$

Definition 6.3.7 *The problem* (6.19)-(6.20) *is Ulam-Hyers-Rassias stable with respect to* (ω, ψ) *if there exists* $c_{f,m,\omega} > 0$ *such that for each* $\epsilon > 0$ *and for each solution* $z \in PC(J, \mathbb{R})$ *of the inequality (6.33) there exists a solution* $y \in PC([-r, 0], \mathbb{R})$ *of the problem* (6.19)-(6.20) *with*

$$|z(t) - y(t)| \leq c_{f,m,\omega}\epsilon(\omega(t) + \psi), \ t \in J.$$

Definition 6.3.8 *The problem* (6.19)-(6.20) *is generalized Ulam-Hyers-Rassias stable with respect to* (ω, ψ) *if there exists* $c_{f,m,\omega} > 0$ *such that for each solution* $z \in PC(J, \mathbb{R})$ *of the inequality (6.32) there exists a solution* $y \in PC([-r, 0], \mathbb{R})$ *of the problem* (6.19)-(6.20) *with*

$$|z(t) - y(t)| \leq c_{f,m,\omega}(\omega(t) + \psi), \ t \in J.$$

Remark 6.3.9 *It is clear that : (i) Definition 6.3.5 implies Definition 6.3.6 ; (ii) Definition 6.3.7 implies Definition 6.3.8 ; (iii) Definition 6.3.7 for* $\omega(t) = \psi = 1$ *implies Definition 6.3.5.*

Remark 6.3.10 *A function* $z \in PC(J, \mathbb{R})$ *is a solution of the inequality (6.33) if and only if there is* $\sigma \in PC(J, \mathbb{R})$ *and a sequence* $\sigma_k, k = 1, ..., m$ *(which depend on z) such that*

i) $|\sigma(t)| \leq \epsilon\omega(t), \ t \in (t_k, t_{k+1}], k = 1, ..., m$ *and* $|\sigma_k| \leq \epsilon\psi, k = 1, ..., m;$

ii) ${}^cD^\alpha(z(t) - \phi(t, z_t)) = f(t, z_t, {}^cD^\alpha z(t)) + \sigma(t), \ t \in (t_k, t_{k+1}], k = 1, ..., m\,;$

iii) $\Delta z|_{t_k} = I_k(z_{t_k^-}) + \sigma_k, \ k = 1, ..., m.$

One can state remarks for inequalities 6.32 and 6.31.

Theorem 6.3.11 *Assume (P1)-(P3), (6.26) and*

(P7) there exists a nondecreasing function $\omega \in PC\left(J, \mathbb{R}_+\right)$ *and there exists* $\lambda_\omega > 0$ *such that for any* $t \in J$:

$$I^\alpha \omega\left(t\right) \leq \lambda_\omega \omega\left(t\right)$$

are satisfied, and if $\overline{L} < 1$, *then the problem* (6.19)-(6.20) *is Ulam-Hyers-Rassias stable with respect to* (ω, ψ).

proof. Let $z \in \Omega$ be a solution of the inequality (6.33). Denote by y the unique solution of the following problem

$$\begin{cases} {}^cD^{\alpha}_{t_k}[y(t) - \phi(t, y_t)] = f(t, y_t, {}^c D^{\alpha}_{t_k} y(t)), \ t \in (t_k, t_{k+1}], k = 1, ..., m \ ; \\ \Delta y|_{t=t_k} = I_k(y_{t_k^-}), \ k = 1, ..., m \ ; \\ y(t) = z(t) = \varphi(t), \ t \in [-r, 0]. \end{cases}$$

Using Lemma 6.3.2, we obtain for each $t \in (t_k, t_{k+1}]$

$$\begin{aligned} y(t) &= \varphi(0) + \phi(t, y_t) + \sum_{i=1}^{k} I_i(y_{t_i^-}) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} g(s) ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} g(s) ds, \ t \in (t_k, t_{k+1}], \end{aligned}$$

where $g \in C(J, \mathbb{R})$ be such that

$$g(t) = f(t, y_t, g(t)).$$

Since z solution of the inequality (6.33) and by Remark 6.3.10, we have

$$\begin{cases} {}^cD^{\alpha}_{t_k}[z(t) - \phi(t, z_t)] = f(t, z_t, {}^c D^{\alpha}_{t_k} z(t)) + \sigma(t), \ t \in (t_k, t_{k+1}], k = 1, ..., m \ ; \\ \Delta z|_{t=t_k} = I_k(z_{t_k^-}) + \sigma_k, \ k = 1, ..., m \ . \end{cases} \tag{6.34}$$

Clearly, the solution of (6.34) is given by

$$\begin{aligned} z(t) &= \varphi(0) + \phi(t, z_t) + \sum_{i=1}^{k} I_i(z_{t_i^-}) + \sum_{i=1}^{k} \sigma_i + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} h(s) ds \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} \sigma(s) ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} h(s) ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \sigma(s) ds, \ t \in (t_k, t_{k+1}], \end{aligned}$$

where $h \in C(J, \mathbb{R})$ be such that

$$h(t) = f(t, z_t, h(t)).$$

Hence for each $t \in (t_k, t_{k+1}]$, it follows that

$$|z(t) - y(t)| \leq \sum_{i=1}^{k} |\sigma_i| + |\phi(t, z_t) - \phi(t, y_t)| + \sum_{i=1}^{k} |I_i(z_{t_i^-}) - I_i(y_{t_i^-})|$$

$$
\begin{aligned}
&+ \quad \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |\sigma(s)| ds \\
&+ \quad \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |h(s) - g(s)| ds \\
&+ \quad \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} |h(s) - g(s)| ds \\
&+ \quad \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} |\sigma(s)|.
\end{aligned}
$$

Thus

$$
\begin{aligned}
|z(t) - y(t)| \quad &\leq \quad m\epsilon\psi + (m+1)\epsilon\lambda_\omega \omega(t) + \overline{L} \|z_t - y_t\|_{PC} + \sum_{i=1}^{k} \widetilde{l} \|z_{t_i^-} - y_{t_i^-}\|_{PC} \\
&+ \quad \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |h(s) - g(s)| ds \\
&+ \quad \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} |h(s) - g(s)| ds.
\end{aligned}
$$

By (P2), we have

$$
\begin{aligned}
|h(t) - g(t)| \quad &= \quad |f(t, z_t, h(t)) - f(t, y_t, g(t))| \\
&\leq \quad K \|z_t - y_t\|_{PC} + L|g(t) - h(t)|.
\end{aligned}
$$

Then

$$
|h(t) - g(t)| \leq \frac{K}{1 - L} \|z_t - y_t\|_{PC}.
$$

Therefore, for each $t \in J$

$$
\begin{aligned}
|z(t) - y(t)| \quad &\leq \quad m\epsilon\psi + (m+1)\epsilon\lambda_\omega \omega(t) + \overline{L} \|z_t - y_t\|_{PC} + \sum_{i=1}^{k} \widetilde{l} \|z_{t_i^-} - y_{t_i^-}\|_{PC} \\
&+ \quad \frac{K}{(1-L)\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} \|z_s - y_s\|_{PC} ds \\
&+ \quad \frac{K}{(1-L)\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} \|z_s - y_s\|_{PC} ds.
\end{aligned}
$$

Thus

$$
|z(t) - y(t)| \quad \leq \quad \sum_{0 < t_i < t} \widetilde{l} \|z_{t_i^-} - y_{t_i^-}\|_{PC} + \epsilon(\psi + \omega(t))(m + (m+1)\lambda_\omega)
$$

$$+ \quad \overline{L}\|z_t - y_t\|_{PC} + \frac{K(m+1)}{(1-L)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\|z_s - y_s\|_{PC} ds.$$

We consider the function ν_1 defined by

$$\nu_1(t) = \sup\{|z(s) - y(s)| : -r \le s \le t\}, 0 \le t \le T,$$

then, there exists $t^* \in [-r, T]$ such that $\nu_1(t) = |z(t^*) - y(t^*)|$.
If $t^* \in [-r, 0]$, then $\nu_1(t) = 0$.
If $t^* \in [0, T]$, then by the previous inequality, we have

$$\begin{aligned} \nu_1(t) &\le \sum_{0<t_i<t} \frac{\widetilde{l}}{1-\overline{L}} \nu_1(t_i^-) + \frac{\epsilon(\psi + \omega(t))(m + (m+1)\lambda_\omega)}{1-\overline{L}} \\ &+ \frac{K(m+1)}{(1-\overline{L})(1-L)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\nu_1(s) ds. \end{aligned}$$

Applying Lemma 1.3.9, we get

$$\begin{aligned} \nu_1(t) &\le \frac{\epsilon(\psi + \omega(t))(m + (m+1)\lambda_\omega)}{1-\overline{L}} \\ &\times \left[\prod_{0<t_i<t}\left(1 + \frac{\widetilde{l}}{1-\overline{L}}\right) \exp\left(\int_0^t \frac{K(m+1)}{(1-\overline{L})(1-L)\Gamma(\alpha)}(t-s)^{\alpha-1} ds\right)\right] \\ &\le c_\omega \epsilon(\psi + \omega(t)), \end{aligned}$$

where

$$\begin{aligned} c_\omega &= \frac{(m + (m+1)\lambda_\omega)}{1-\overline{L}} \left[\prod_{i=1}^m \left(1 + \frac{\widetilde{l}}{1-\overline{L}}\right) \exp\left(\frac{K(m+1)T^\alpha}{(1-\overline{L})(1-L)\Gamma(\alpha+1)}\right)\right] \\ &= \frac{(m + (m+1)\lambda_\omega)}{1-\overline{L}} \left[\left(1 + \frac{\widetilde{l}}{1-\overline{L}}\right) \exp\left(\frac{K(m+1)T^\alpha}{(1-\overline{L})(1-L)\Gamma(\alpha+1)}\right)\right]^m. \end{aligned}$$

Thus, the problem (6.19)-(6.20) is Ulam-Hyers-Rassias stable with respect to (ω, ψ). □

Next, we present the following Ulam-Hyers stability result.

Theorem 6.3.12 *Assume that (P1)-(P3) and* (6.26) *are satisfied and if* $\overline{L} < 1$, *then the problem* (6.19)-(6.20) *is Ulam-Hyers stable*

proof. Let $z \in \Omega$ be a solution of the inequality (6.31). Denote by y the unique solution of the problem

$$\begin{cases} {}^cD_{t_k}^\alpha[y(t) - \phi(t, y_t)] = f(t, y_t, {}^cD_{t_k}^\alpha y(t)), \ t \in (t_k, t_{k+1}], k = 1, ..., m \ ; \\ \Delta y|_{t=t_k} = I_k(y_{t_k^-}), \ k = 1, ..., m \ ; \\ y(t) = z(t) = \varphi(t), \ t \in [-r, 0]. \end{cases}$$

From the proof of Theorem 6.3.11, we get the inequality

$$\begin{aligned}\nu_1(t) &\leq \sum_{0<t_i<t} \frac{\widetilde{l}}{1-\overline{L}}\nu_1(t_i^-) + \frac{m\epsilon}{1-\overline{L}} + \frac{T^\alpha \epsilon(m+1)}{(1-\overline{L})\Gamma(\alpha+1)} \\ &+ \frac{K(m+1)}{(1-\overline{L})(1-L)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\nu_1(s)ds.\end{aligned}$$

Applying Lemma 1.3.9, we get

$$\begin{aligned}\nu_1(t) &\leq \epsilon\left(\frac{m\Gamma(\alpha+1)+T^\alpha(m+1)}{(1-\overline{L})\Gamma(\alpha+1)}\right) \\ &\times \left[\prod_{0<t_i<t}\left(1+\frac{\widetilde{l}}{1-\overline{L}}\right)\exp\left(\int_0^t \frac{K(m+1)}{(1-\overline{L})(1-L)\Gamma(\alpha)}(t-s)^{\alpha-1}ds\right)\right] \\ &\leq c\epsilon,\end{aligned}$$

where

$$\begin{aligned} c &= \left(\frac{m\Gamma(\alpha+1)+T^\alpha(m+1)}{(1-\overline{L})\Gamma(\alpha+1)}\right)\left[\prod_{i=1}^{m}\left(1+\frac{\widetilde{l}}{1-\overline{L}}\right)\exp\left(\frac{K(m+1)T^\alpha}{(1-\overline{L})(1-L)\Gamma(\alpha+1)}\right)\right] \\ &= \left(\frac{m\Gamma(\alpha+1)+T^\alpha(m+1)}{(1-\overline{L})\Gamma(\alpha+1)}\right)\left[\left(1+\frac{\widetilde{l}}{1-\overline{L}}\right)\exp\left(\frac{K(m+1)T^\alpha}{(1-\overline{L})(1-L)\Gamma(\alpha+1)}\right)\right]^m.\end{aligned}$$

□

Moreover, if we set $\theta(\epsilon) = c\epsilon; \theta(0) = 0$, then, the problem (6.19)-(6.20) is generalized Ulam-Hyers stable.

6.3.4 An Example.

Consider the following impulsive problem, for each $t \in J_0 \cup J_1$,

$$ {}^cD^{\frac{1}{2}}_{t_k}\left[y(t) - \frac{te^{-t}|y_t|}{(9+e^t)(1+|y_t|)}\right] = \frac{e^{-t}}{(11+e^t)}\left[\frac{|y_t|}{1+|y_t|} - \frac{|{}^cD^{\frac{1}{2}}_{t_k}y(t)|}{1+|{}^cD^{\frac{1}{2}}_{t_k}y(t)|}\right], \tag{6.35}$$

$$\Delta y|_{t=\frac{1}{2}} = \frac{|y(\frac{1}{2}^-)|}{10+|y(\frac{1}{2}^-)|}, \tag{6.36}$$

$$y(t) = \varphi(t),\ t\in[-r,0], r>0, \tag{6.37}$$

where $\varphi \in PC([-r,0],\mathbb{R})$, $J_0 = \left[0,\frac{1}{2}\right]$, $J_1 = \left(\frac{1}{2},1\right]$, $t_0 = 0$, and $t_1 = \frac{1}{2}$.
For $t\in[0,1]$, $u \in PC([-r,0],\mathbb{R})$, and $v \in \mathbb{R}$, set

$$f(t,u,v) = \frac{e^{-t}}{(11+e^t)}\left[\frac{|u|}{1+|u|} - \frac{|v|}{1+|v|}\right]$$

and

$$\phi(t,u)=\frac{te^{-t}|u|}{(9+e^{t})(1+|u|)}.$$

Notice that $\phi(0,\varphi)=0$, for any $\varphi\in PC([-r,0],\mathbb{R})$.
Clearly, the function f is jointly continuous.
For each $u,\bar{u}\in PC([-r,0],\mathbb{R}), v,\bar{v}\in\mathbb{R}$ and $t\in[0,1]$, we have

$$\begin{aligned}|f(t,u,v)-f(t,\bar{u},\bar{v})| &\leq \frac{e^{-t}}{(11+e^{t})}(\|u-\bar{u}\|_{PC}+|v-\bar{v}|)\\ &\leq \frac{1}{12}\|u-\bar{u}\|_{PC}+\frac{1}{12}|v-\bar{v}|\end{aligned}$$

and

$$|\phi(t,u)-\phi(t,\bar{u})|\leq\frac{1}{10}\|u-\bar{u}\|_{PC}.$$

Hence condition (P2) is satisfied with $K=L=\dfrac{1}{12},\overline{L}=\dfrac{1}{10}$.
Let

$$I_1(u)=\frac{|u|}{10+|u|},\quad u\in PC([-r,0],\mathbb{R}).$$

For each $u,v\in PC([-r,0],\mathbb{R})$, we have

$$|I_1(u)-I_1(v)|=\left|\frac{|u|}{10+|u|}-\frac{|v|}{10+|v|}\right|\leq\frac{1}{10}\|u-v\|_{PC}.$$

Thus condition

$$\begin{aligned}m\widetilde{l}+\overline{L}+\frac{(m+1)KT^{\alpha}}{(1-L)\Gamma(\alpha+1)} &= \frac{2}{10}+\frac{\frac{1}{6}}{(1-\frac{1}{12})\Gamma(\frac{3}{2})}\\ &= \frac{4}{11\sqrt{\pi}}+\frac{2}{10}<1,\end{aligned}$$

is satisfied with $T=1$,$m=1$ and $\widetilde{l}=\dfrac{1}{10}$. It follows from Theorem 6.3.3 that the problem (6.35)–(6.37) has a unique solution on J.

Set for any $t\in[0,1]$, $\omega(t)=t$ and $\psi=1$. Since

$$I^{\frac{1}{2}}\omega(t)=\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^t(t-s)^{\frac{1}{2}-1}sds\leq\frac{2t}{\sqrt{\pi}},$$

then, condition (P7) is satisfied with $\lambda_{\omega}=\dfrac{2}{\sqrt{\pi}}$, and since $\overline{L}<1$, it follows that the problem (6.35)-(6.36) is Ulam-Hyers-Rassias stable with respect to (ω,ψ).

Conclusion and Perspective

In this thesis, we have considered the following nonlinear implicit fractional differential equation

$$^{c}D^{\alpha}y(t) = f(t, y(t), {}^{c}D^{\alpha}y(t)), \text{ for each } t \in J,\ 0 < \alpha \leq 1, \tag{6.38}$$

subjected to boundary conditions, non-local conditions, delay and impulse. Here $^{c}D^{\alpha}$ is the ***Caputo*** fractional derivative.
The problem of stability was discussed. We discussed and established the existence, the uniqueness and the stability of the solution for implicit neutral fractional differential equation with finite delay and impulses.
We plan to study the same question such as existence, uniqueness and stability to the equation (6.38) in the case where the derivative is of type ***Hadamard***.

Bibliographie

[1] R.P. Agarwal and B. Ahmad, Existence theory for anti-periodic boundary value problems of fractional differential equations and inclusions, *Comput. Math. Appl.* **62** (2011), 1200-1214.

[2] S. Abbas and M. Benchohra, On the generalized Ulam-Hyers-Rassias stability for Darboux problem for partial fractional implicit differential equations. *Applied Mathematics E-Notes*, **14** (2014), 20-28.

[3] S. Abbas, M. Benchohra and G M. N'Guérékata, *Topics in Fractional Differential Equations*, Springer-Verlag, New York, 2012.

[4] R. P. Agarwal, S. Arshad, D. O'Regan, V. Lupulescu, Fuzzy fractional integral equations under compactness type condition. *Fract. Calc. Appl. Anal.* **15** (2012), 572-590.

[5] R.P. Agarwal, M. Benchohra and B.A. Slimani, Existence results for differential equations with fractional order impulses, *Mem. Differential Equations Math. Phys.* **44** (1) (2008), 1-21.

[6] S. Abbas, M. Benchohra and A N. Vityuk, On fractional order derivatives and Darboux problem for implicit differential equations. *Frac. Calc. Appl. Anal.* **15**, (2012), 168-182.

[7] R.P Agarwal, M. Benchohra and S. Hamani, Boundary value problems for fractional differential equations, *Adv. Stud. Contemp. Math.* **16** (2) (2008), 181-196.

[8] S.A. Abd-Salam, A.M.A. El-Sayed, On the stability of a fractional-order differential equation with nonlocal initial condition, *Electron. J. Qual. Theory Differ. Equat.* 29 (2008), 1-8.

[9] R. P. Agarwal, M. Meehan and D. O'Regan, *Fixed Point Theory and Applications*, Cambridge University Press, Cambridge, 2001.

[10] B. Ahmad, J.R. Graef, Coupled systems of nonlinear fractional differential equations with nonlocal boundary conditions. *Panamer. Math. J.* **19** (2009), 29–39.

[11] B. Ahmad and J.J. Nieto, Existence of solutions for nonlocal boundary value problems of higher-order nonlinear fractional differential equations. *Abstrac. Appl. Anal.* **2009** (2009), Article ID 494720, 9 pages.

[12] B. Ahmad and J.J. Nieto, Existence of solutions for impulsive anti-periodic boundary value problems of fractional order, *Taiwaness J. Math.*, **15** (3) (2011), 981-993.

[13] B. Ahmad and J.J. Nieto, Anti-periodic fractional boundary value problems with nonlinear term depending on lower order derivative, *Fract. Calc. Appl. Anal.*, **15** (2012), 451-462.

[14] B. Ahmad and J.J. Nieto, Anti-periodic fractional boundary value problems, *Computers and Mathematics with Applications*, **62** (3) (2011), 1150-1156.

[15] B. Ahmad, J.J. Nieto, A. Alsaedi and N. Mohamad, On a New Class of Anti-periodic Fractional Boundary Value Problems, *Abst. and Applied Analysis*, (2013), 7 p.

[16] B. Ahmad and S. Sivasundaram, Existence results for nonlinear impulsive hybrid boundary value problems involving fractional differential equations. *Nonlinear Anal. Hybrid Syst.* **3** (3) (2009), 251–258.

[17] B. Ahmad, and S. Sivasundaram, Theory of fractional differential equations with three-point boundary conditions, *Commun. Appl. Anal.* **12** (2008), 479-484.

[18] K. K. Akhmerov, M. I. Kamenskii, A. S. Potapov, A. E. Rodkina and B. N. Sadovskii, *Measures of noncompactness and condensing operators*, Birkhäuser Verlag, Basel, Bostn, Berlin, 1992.

[19] W. Albarakati, M. Benchohra and **S. Bouriah**, Existence and stability results for nonlinear implicit fractional differential equations with delay and impulses, *Differential Equations and Applications* Vol. 8 No. 2 (2016), 273-293.

[20] J. C. Alvàrez, Measure of noncompactness and fixed points of nonexpansive condensing mappings in locally convex spaces, *Rev. Real. Acad. Cienc. Exact. Fis. Natur. Madrid* **79** (1985), 53-66.

[21] C. Alsina and R. Ger, On some inequalities and stability results related to the exponential function. *J. Inequal. Appl.* **2**, (1998), 373-380.

[22] G. A. Anastassiou, *Advances on Fractional Inequalities*, Springer New York Dordrecht Heidelberg London, 2011.

[23] J. Andres and L. Górniewicz, *Topological Fixed Point Principles for Boundary Value Problems,* Kluwer Academin Publishers, Dordrecht (2003).

[24] T. Aoki, On the stability of the linear transformation in Banach spaces. *J. Math. Soc. Japan* **2**, (1950), 64-66.

[25] J. Appell, Implicit Functions, Nonlinear Integral Equations, and the Measure of Noncompactness of the superposition Operator. *J. Math. Anal. Appl.* **83**, (1981), 251-263.

[26] A. Babakhani and V. Daftardar-Gejji, Existence of positive solutions for multi-term non-autonomous fractional differential equations with polynomial coefficients. *Electron. J. Differential Equations* **2006**, No. 129, 12 pp.

[27] A. Babakhani and V. Daftardar-Gejji, Existence of positive solutions for N-term non-autonomous fractional differential equations, *Positivity* **9** (2) (2005), 193-206.

[28] Z. Bai, On positive solutions of a nonlocal fractional boundary value problem, *Nonlinear Anal.* **72** (2) (2010), 916-924, 2010.

[29] D.D Bainov and S.G. Hristova, Integral inequalities of Gronwall type for piecewise continuous functions ,*J. Appl. Math. Stoc. Anal.*, 10(1997), 89-94.

[30] D. D. Bainov and P. S. Simeonov, *Systems with Impulsive effect,* Horwood, Chichister, 1989.

[31] D. D. Bainov and P. S. Simeonov, *Impulsive differential equations : periodic solutions and applications,* Pitman Monographs and Surveys in Pure and Applied Mathematics, **66**, Longman Scientific & Technical and John Wiley & Sons, Inc., New York, 1993.

[32] D. Baleanu, K. Diethelm, E. Scalas, and J.J. Trujillo, *Fractional Calculus Models and Numerical Methods*, World Scientific Publishing, New York, 2012.

[33] D. Baleanu, Z.B. Güvenç and J.A.T. Machado, *New Trends in Nanotechnology and Fractional Calculus Applications*, Springer, New York, 2010.

[34] D. Baleanu, J.A.T. Machado and A.C.-J. Luo *Fractional Dynamics and Control*, Springer, 2012.

[35] J. Banaŝ and K. Goebel, *Measures of Noncompactness in Banach Spaces*, Marcel Dekker, New York, 1980.

[36] J. Banaŝ and L. Olszowy, Measures of noncompactness related to monotonicity, *Comment. Math. (Prace Mat.)* **41** (2001), 13-23.

[37] Banaś, J, Zając, T : A new approach to the theory of functional integral equations of fractional order. J. Math. Anal. Appl. **375**, 375-387 (2011)

[38] M. Belmekki and M. Benchohra, Existence results for fractional order semilinear functional differential equations, *Proc. A. Razmadze Math. Inst.* **146** (2008), 9-20.

[39] M. Benchohra and F. Berhoun, Impulsive fractional differential equations with variable times, *Comput. Math. Appl.* **59** (2010), 1245-1252.

[40] M. Benchohra and F. Berhoun, Impulsive fractional differential equations with state dependent delay, *Commun. Appl. Anal.* **14** (2) (2010), 213-224.

[41] M. Benchohra, F. Berhoun, N. Hamidi and J.J. Nieto, Fractional differential inclusions with anti-periodic boundary conditions, *Nonlinear Anal. Forum* **19** (2014), 27-35.

[42] M. Benchohra, and **S. Bouriah**, Existence and stability results for nonlinear implicit fractional differential equations with impulses, *Mem. Differential Equations Math. Phys.*

[43] M. Benchohra, **S. Bouriah** and M.Darwish, Nonlinear boundary value problem for implicit differential equations of fractional order in Banach spaces, *Fixed Point Theory.*

[44] M. Benchohra, **S. Bouriah** and J.Henderson, Existence and stability results for nonlinear implicit neutral fractional differential equations with finite delay and impulses, *Comm. Appl. Nonlinear Anal.* **22** (1)(2015), 46-67.

[45] M. Benchohra and **S. Bouriah**, Existence and Stability Results for Nonlinear Boundary Value Problem for Implicit Differential Equations of Fractional Order, *Moroccan Journal Pure. Appl. Anal.* **1** (1), (2015),22-36.

[46] M. Benchohra and **S. Bouriah**, Existence and Stability Results for Neutral Functional Differential Equation of Fractional Order with Finite Delay, *Dynamics of Continuous, Discrete and Impulsive Systems.*

[47] M. Benchohra, N. Hamidi and J. Henderson, Fractional differential equations with anti-periodic boundary conditions *Numer. Funct. Anal. Optim.* **34** (4) (2013), 404-414.

[48] M. Benchohra, F. Berhoun and G M. N'Guérékata, Bounded solutions for fractional order differential equations on the half-line, *Bull. Math. Anal. Appl.* **146** (4) (2012), 62-71.

[49] M. Benchohra, A. Cabada and D. Seba, An existence result for non-linear fractional differential equations on Banach spaces, *Boundary Value Problems.* **2009** (2009), Article ID 628916, 11 pages.

[50] M. Benchohra, J.R. Graef and S. Hamani, Existence results for boundary value problems with nonlinear fractional differential equations, *Appl. Anal.* **87** (7) (2008), 851-863.

[51] M. Benchohra, and S. Hamani, Boundary Value Problems for Differential Inclusions with Fractional Order, *Discus. Mathem. Diff. Incl., Contr. and Optim.* **28** (2008), 147-164.

[52] M. Benchohra, S. Hamani and S.K. Ntouyas, Boundary value problems for differential equations with fractional order, *Surv. Math. Appl.* **3** (2008), 1-12.

[53] M. Benchohra, S. Hamani, J. J. Nieto and B. A. Slimani, Existence of solutions to differential inclusions with fractional order and impulses, *Electron. J. Differential Equations* Vol. 2010 (2010), No. 80, 1-18.

[54] M. Benchohra and B. Hedia, Positive solutions for boundary value problems with fractional order, *Inter. Journ. of Adv. Mathem. Scienc.,* **1** (1), (2013), 12-22.

[55] M. Benchohra, J. Henderson and S. K. Ntouyas, *Impulsive Differential Equations and Inclusions*, Hindawi Publishing Corporation, **2**, New York, 2006.

[56] M. Benchohra, J. Henderson, S.K. Ntouyas and A. Ouahab, Existence results for fractional order functional differential equations with infinite delay, *J. Math. Anal. Appl.* **338** (2) (2008), 1340-1350.

[57] M. Benchohra and J. E. Lazreg, Nonlinear fractional implicit differential equations. *Commun. Appl. Anal.* **17** (2013), 471-482.

[58] M. Benchohra and J. E. Lazreg, Existence and uniqueness results for nonlinear implicit fractional differential equations with boundary conditions. *Romanian J. Math. Comput. Sc.* **4** (1) (2014), 60-72.

[59] M. Benchohra and J. E. Lazreg, Existence results for nonlinear implicit fractional differential equations. *Surv. Math. Appl.* **9** (2014), 79-92.

[60] M. Benchohra and D. Seba, Impulsive fractional differential equations in Banach Spaces *Electron. J. Qual. Theory Differ. Equ.*. Spec. Ed. I (2009), No. 8, pp. 1-14.

[61] M. Benchohra and B.A. Slimani, Existence and uniqueness of solutions to impulsive fractional diferential equations, *Electron. J. Differential Equations*, **2009** (10) (2009), 1-11.

[62] T.A. Burton and C. Kirk, A fixed point theorem of Krasnoselskii-Schaefer type. *Math. Nachr.* **189** (1989), 23-31.

[63] L. Byszewski, Theorems about existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem, *J. Math. Anal. Appl.* **162** (1991), 494-505.

[64] L. Byszewski, Existence and uniqueness of mild and classical solutions of semilinear functional-differential evolution nonlocal Cauchy problem. *Selected problems of mathematics*, 25–33, 50th Anniv. Cracow Univ. Technol. Anniv. Issue, 6, Cracow Univ. Technol., Krakow, 1995

[65] L. Byszewski and V. Lakshmikantham, Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space, *Appl. Anal.* **40** (1991), 11-19.

[66] Y. Chang, A. Anguraj and P. Karthikeyan, Existence results for initial value problems with integral condition for impulsive fractional differential equations. *J. Fract. Calc. Appl.* **2** (7) (2012), 1-10.

[67] A. Chen and Y. Chen, Existence of Solutions to Anti-Periodic Boundary value problem for nonlinear Fractional Differential Equations with Impulses. *Adv. Difference. Equat.* (2011), 17 pages.

[68] J. Chen, F. Liu, I. Turner and V. Anh, The fundamental and numerical solutions of the Riesz space-fractional reaction-dispersion equation, *The ANZIAM Journal* **50** (1) (2008), 45-57

[69] Y. J. Cho , Th. M. Rassias and R. Saadati, *Stability of Functional Equations in Random Normed Spaces*, Science-Business Media, Springer 52, 2013.

[70] G. Darbo, Punti uniti in transformazioni a condominio non compatto, *Rend Sem. Mat. Univ. Padova* **24** (1955), 84-92.

[71] K. Deng, Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions, *J. Math. Anal. Appl.* **179** (1993), 630-637.

[72] K. Diethelm, *The Analysis of Fractional Differential Equations*, Lecture Notes in Mathematics, 2010.

[73] A. El-Sayed, F. Gaafar, Stability of a nonlinear non-autonomous fractional order systems with different delays and non-local conditions. *Adv. Difference Equations* **47** (1) (2011), 12 pp.

[74] N. Engheta, Fractional Curl Operator in Electromagnetics. *Microwave Opt. Tech. Lett.* **17** (1) (1998), 86-91.

[75] M. Faryad and Q. A. Naqvi, Fractional Rectangular Waveguide. *Progress In Electromagnetics Research* PIER **75** (2007), 383-396.

[76] P. Gavruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.* **184** (1994), 431-436.

[77] K. Goebel, *Concise Course on Fixed Point Theorems*, Yokohama Publishers, Japan, 2002.

[78] L. Górniewicz, *Topological Fixed Point Theory of Multivalued Mappings*, Mathematics and its Applications, 495, Kluwer Academic Publishers, Dordrecht, 1999.

[79] J. R. Graef, J. Henderson and A. Ouahab, *Impulsive Differential Inclusions. A Fixed Point Approch*, De Gruyter, Berlin/Boston, 2013.

[80] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[81] D.J. Guo, V. Lakshmikantham, X. Liu, *Nonlinear Integral Equations in Abstract Spaces,* Kluwer Academic Publishers, Dordrecht, 1996.

[82] Z. Guo, M. Liu, Existence and uniqueness of solutions for fractional order integrodifferential equations with nonlocal initial conditions. *Panamer. Math. J.* **21** (2011), 51-61.

[83] J. Hale and S.M. Verduyn Lunel, *Introduction to Functional Differential Equations*, , Applied Mathematicals Sciences, 99, Springer-Verlag, New York, 1993.

[84] J. Henderson and A. Ouahab, Impulsive differential inclusions with fractional order, *Comput. Math. Appl.* **59** (2010), 1191-1226.

[85] J. Henderson and A. Ouahab, A Filippov's theorem, some existence results and the compactness of solution sets of impulsive fractional order differential inclusions, submitted.

[86] R. Hermann, *Fractional Calculus : An Introduction For Physicists*, World Scientific Publishing Co. Pte. Ltd. 2011.

[87] N. Heymans and I. Podlubny, Pysical interpretation of initial conditions for fractional differential equations with Riemnn-Liouville fractional derivatives, *Rheol Acta* **45** (2006), 765-771.

[88] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[89] Ch. Horvath, Measure of Non-compactness and multivalued mappings in complete metric topological spaces, *J. Math. Anal. Appl.* **108** (1985), 403-408.

[90] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad.Sci. U.S.A. **27** (1941), 222-224.

[91] M. Hu and L. Wang, Existence of solutions for a nonlinear fractional differential equation with integral boundary condition, *Int. J. Math. Comp. Sc.*, **7** (1) (2011).

[92] Rabha W. Ibrahim, Stability for Univalent Solutions of Complex Fractional Differential Equations, *Proceedings of the Pakistan Academy of Sciences*,**49** (3) (2012), 227-232.

[93] K.W. Jun and H.M. Kim, On the stability of an n-dimensional quadratic and additive functional equation, *Math. Inequal. Appl.* **19** (9) (2006), 854-858.

[94] S.M. Jung, On the Hyers-Ulam stability of the functional equations that have the quadratic property, *J. Math. Anal. Appl.* **222** (1998), 126-137.

[95] S.M. Jung, Hyers-Ulam stability of linear differential equations of first order, II *Appl. Math. Lett.* **19** (2006), 854-858.

[96] S.M. Jung, K.S. Lee, Hyers-Ulam stability of first order linear partial differential equations with constant coefficients *Math. Inequal. Appl.* **10** (2007), 261-266.

[97] K. Karthikeyan and J.J. Trujillo, Existence and uniqueness results for fractional integrodifferential equations with boundary value conditions, *Commun. Nonlinear Sci. Numer. Simulat.* **17** (2012) 4037-4043.

[98] G.H. Kim, On the stability of functional equations with square-symmetric operation, *Math. Inequal. Appl* **17** (4) (2001), 257-266.

[99] A.A. Kilbas and S. A. Marzan, Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiable functions, *Diff. Equat.* **41** (2005), 84-89.

[100] A.A. Kilbas, Hari M. Srivastava, and Juan J. Trujillo, *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[101] W.A. Kirk and B. Sims, *Handbook of Metric Fixed Point Theory*, Springer-Science + Business Media, B.V, Dordrecht, 2001.

[102] K. Kuratowski, Sur les espaces complets, *Fund. Math.* **15** (1930), 301-309.

[103] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov ; *Theory of Impulsive Differntial Equations*, Worlds Scientific,Singapore, 1989.

[104] V. Lakshmikantham, S. Leela and J. Vasundhara, *Theory of Fractional Dynamic Systems*, Cambridge Academic Publishers, Cambridge, 2009.

[105] V. Lakshmikantham, and J. Vasundhara Devi. Theory of fractional differential equations in a Banach space. *Eur. J. Pure Appl. Math.* **1** (2008), 38-45.

[106] V. Lakshmikantham and A.S. Vatsala, Basic theory of fractional differential equations, *Nonlinear Anal.* **69** (2008), 2677-2682.

[107] V. Lakshmikantham and A.S. Vatsala, General uniqueness and monotone iterative technique for fractional differential equations *Appl. Math. Lett.*, **21** (2008) 828-834.

[108] V. Lakshmikantham and A.S. Vatsala, Theory of fractional differential inequalities and applications, *Commun. Appl. Anal.*, **11**, No. 3-4 (2007), 395-402.

[109] Q. Liu, F. Liu, I. Turner and V. Anh, Numerical simulation for the 3D seep age flow with fractional derivatives in porous media, *IMA Journal of Applied Mathematics* **74** (2) (2009), 201-229.

[110] Lizhen Chen, Zhenbin Fan, On mild solutions to fractional differential equations with nonlocal conditions, *Electron. J. Qual. Theory Differ. Equ.* **53** (2011), 1-13.

[111] A. J. Luo and V. Afraimovich, *Long-range Interactions, Stochasticity and Fractional Dynamics*, Springer, New York, Dordrecht, Heidelberg, London, 2010.

[112] C. Li, F. Zeng, and F. Liu, Spectral approximations to the fractional integral and derivative, *Frac. Calc. Appl. Anal.* 15 (3) (2012), 383-406.

[113] V.D. Milman and A.A. Myshkis, On the stability of motion in the presence of impulses, *Sib. Math. J.* (in Russian), **1** (1960), 233-237.

[114] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.

[115] H. Mönch, Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces, *Nonlinear Anal.* **4** (1980), 985-999.

[116] Q. A. Naqvi and M. Abbas, Complex and Higher Order Fractional Curl Operator in Electromagnetics, *Optics Communications* **241**, (2004), 349-355.

[117] G M. N'Guérékata, A Cauchy problem for some fractional abstract differential equation with non local conditions, *Nonlinear Anal.* **70** (5) (2009), 1873-1876.

[118] G M. N'Guérékata, Corrigendum : A Cauchy problem for some fractional differential equations, *Commun. Math. Anal.* **7** (2009), 11-11.

[119] K.B. Oldham and J. Spanier, *The Fractional Calculus : theory and application of differentiation and integration to arbitrary order*, Academic Press, New York, London, 1974.

[120] M. D. Ortigueira, *Fractional Calculus for Scientists and Engineers*. Lecture Notes in Electrical Engineering, 84. Springer, Dordrecht, 2011.

[121] N.A. Perestyuk, V.A. Plotnikov, A.M. Samoilenko and N.V. Skripnik, *Differential Equation with Impulse Effects, Multivalued Right-hand Sides with Discontinuities* Walter de Gruyter, Berlin/Boston, 2011.

[122] Ivo Petras, *Fractional-Order Nonlinear Systems : Modeling, Analysis and Simulation* Springer Heidelberg Dordrecht London New York, 2011.

[123] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[124] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.* **72** , (1978), 297-300.

[125] J. M. Rassias, *Functional Equations, Difference Inequalities and Ulam Stability Notions (F.U.N.)*, Nova Science Publishers, Inc. New York, 2010.

[126] Th. M. Rassias and J. Brzdek, *Functional Equations in Mathematical Analysis*, Springer 86, New York Dordrecht Heidelberg London 2012.

[127] S. S. Ray, A new approach for the application of Adomian decomposition method for the solution of fractional space diffusion equation with insulated ends, *Appl. Math. Comput.* **202** (2) (2008), 544-549.

[128] B. Ross, *Fractional Calculus and Its Applications*, Proceedings of the International Conference, New Haven, Springer-Verlag, New York (1974)

[129] I. A. Rus, Ulam stabilities of ordinary differential equations in a Banach space, *Carpathian J. Math.* **26** (2010), 103-107.

[130] X. Su and L. Liu, Existence of solution for boundary value problem of nonlinear fractional differential equation, *Appl. Math.* **22** (3) (2007) 291-298.

[131] P. Sahoo, T. Barman and J. P. Davim, *Fractal Analysis in Machining*, Springer, New York, Dordrecht, Heidelberg, London, 2011.

[132] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[133] A.M. Samoilenko and N.A. Perestyuk, *Impulsive Differential Equations* World Scientific, Singapore, 1995.

[134] V.E. Tarasov, *Fractional Dynamics : Application of Fractional Calculus to Dynamics of particles, Fields and Media*, Springer, Heidelberg ; Higher Education Press, Beijing, 2010.

[135] J. M. Ayerbe Toledano, T. Dominguez Benavides and G. Lopez Acedo, *Measures of Noncompactness in Metric Fixed Point Theory*, BirkhÄauser, Basel, 1997.

[136] S. M. Ulam, *Problems in Modern Mathematics*, Chapter 6, JohnWiley and Sons, New York, USA,1940).

[137] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience, New York,1960.

[138] G. Wang, B. Ahmad, L. Zhang and J. J Nieto, Comments on the concept of existence of solution for impulsive fractional differential equations , *Electron. Commun. Nonlinear Sci. Numer. Simulat.* **19** (2014), 401-403.

[139] J. Wang, M. Fečkan and Y. Zhou, Ulam's type stability of impulsive ordinary differential equations, *J. Math. Anal. Appl.* **395** (2012), 258-264.

[140] G. Wang, B. Ahmad and L. Zhang, Impulsive anti-periodic boundary value problem for nonlinear differential equations of fractional order. *Nonlinear Anal.* **74** (2011), 792-804.

[141] G. Wang, B. Ahmad and L. Zhang, Some existence results for impulsive nonlinear fractional differential equations with mixed boundary conditions, *Comput. Math. Appl.* **62** (2011), 1389-1397.

[142] F. Wang and Z. Liu, Anti-periodic fractional boundary value problems for nonlinear differential equations of fractional order, *Adv. in Difference Equat.*, (2012), 12 pages.

[143] J. Wang, L. Lv and Y. Zhou, Ulam stability and data dependence for fractional differential equations with Caputo derivative, *Electron. J. Qual. Theory Differ. Equat.* **63** (2011), 1-10.

[144] J. Wang and Y. Zhang, Existence and stability of solutions to nonlinear impulsive differential equations in β-normed spaces, *Electron. J. Differential Equations*,(83) (2014), 1-10.

[145] H. Ye, J. Gao, and Y. Ding, A generalized Gronwall inequality and its application to a fractional differential equation, *J. Math. Anal. Appl.* **328** (2007), 1075-1081.

[146] K. Yosida, *Fonctional Analysis,* 6th edn. Springer-Verlag, Berlin, 1980.

[147] S. Zhai, X. Feng, and Z. Weng, New high-order compact adi algorithms for 3D nonlinear time-fractional convection-diffusion equation, *Math. Prob. Eng* **2013** (2013), 11 pages.

[148] S. Zhang, Positive solutions for boundary-value problems of nonlinear fractional diffrential equations, *Electron. J. Differential Equations* (36) (2006), 1-12.

[149] X. Zhang, J. Liu, L. Wei and C. Ma, Finite element method for Grwünwald-Letnikov time-fractional partial differential equation, *Appl. Anal.* **92** (10) (2013), 1-12.

[150] L. Zhang and G. Wang, Existence of solutions for nonlinear fractional differential equations with impulses and anti-Periodic boundary conditions, *Electron. J. Qual. Theory Differ. Equ.*(7) (2011), 1-11.