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Contents

Introduction

The theory of generation of semigroups of linear contractions, which is the basis of the evolution equation, was developed by Hille and Yosida in 1948. Through this theory the existence and uniqueness for a solution of Cauchy's problem for $u'(t) = Au(t)$ with $u(0) = x \in D(A)$ were proved for an m-dissipative operator A that has a dense domain in a Banach space. The theory of linear semigroups was further deepened by results of Phillips and many other mathematicians, and the theory of linear semigroups has now secured its position as an important area in the field of analysis. In 1953 Kato extented the theory of Hille-Yosida to the case where A depends on time t. Afterward, the theory of evolution equation of parabolic type was brought forth by Kato and Tanabe, and the theory of linear evolution equations has made marked progress. Moreover, during the first half of the 1960s the semilinear evolution equation with a nonlinear perturbation term $u'(t) = A(t) + f(t, u(t))$ was studied by Segal, Browder and Kato, who obtained excellent results. Under this historical background, Komura attracted much attention in 1967 when he announced the theory of generation of nonlinear semigroups in a Hilbert space. The theory was immediately extended by Kato to the case of a Banach space with a uniformly convex conjugate space. Afterward, in 1971, Grandall-Liggett obtained the splendid result that (in a general Banach space, an arbitrary m-dissipative operator always generates a semigroup of nonlinear contractions.) This, together with the work of Komura, has become the basis for the study of nonlinear evolution equations. We shall explore how this theory unfolds for some basic nonlinear models. Now, we expose some motivation of nonlinear evolution problems.

Model 1: Nonlinear Mechanics: A nonlinear classical mechanic is

$$\begin{cases} x''(t) + \alpha x(t)|x(t)| = 0, \quad t > 0 \\ x(0) = x_0, x'(0) = x_1. \end{cases}$$

where $\alpha > 0$. This problem can be converted into a system of first-order ordinary differential equation by way of the change of variable $y = x$, $z = x'$ so that

$$\begin{cases} y' = z, \\ z' = -\alpha y|y| \end{cases}$$

The problem can be written in the equivalent matrix form

$$\begin{bmatrix} y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\alpha|y(t)| & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}, \ t > 0.$$

with the initial condition

$$\begin{bmatrix} y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}.$$

Specifically, the problem can be written compactly in the vector form

$$X'(t) = AX(t), \quad t > 0,$$

with initial condition $X(0) = X_0$.
Where $X : [0, \infty) \to \mathbb{R}^2$ is unknown function, A is an 2×2 matrix, and $X_0 \in \mathbb{R}^2$.

Model 2: (Nonlinear Waves): The model of wave Nonlinear is

$$\begin{cases} \frac{\partial^2 z}{\partial t^2} + \alpha \frac{\partial z}{\partial t} + \beta(\frac{\partial z}{\partial t})^3 - c^2 \frac{\partial^2 z}{\partial x^2} = \int_0^t \varphi(t-s)\sin(z(x,s))ds, \\ z(x,0) = z_0(x), \quad \frac{\partial z}{\partial t}(x,0) = z_1(x), \qquad 0 < x < b, \quad t > 0 \\ z(0,t) = z(b,t) = 0, \qquad t > 0. \end{cases}$$

Where $z_0 \in H^2(0,b) \cap H_0^1$, $z_1 \in H_1^0$, and $\varphi : [0,b] \to \mathbb{R}$ is continuous. The second-order PDE can be written as the following equivalent system of first-order PDE

$$\frac{\partial}{\partial t} \begin{bmatrix} y \\ z \end{bmatrix} (x,t) = \begin{bmatrix} 0 & 1 \\ -c^2 \frac{\partial^2}{\partial x^2} & \alpha + \beta(z)^2 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} (x,t) + \begin{bmatrix} 0 \\ \int_0^t \varphi(t-s)\sin(y)ds \end{bmatrix}, 0 < x < b, t > 0.$$

It natural to view this system in $K = H_0^1(0,b) \times H_0^(0,b)$.
Doing so requires that we define $A : D(A) \subset K \to K$ by

$$A \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ -c^2 \frac{\partial^2 v_1}{\partial x^2} + \alpha v_2 + \beta(v_2)^3 \end{bmatrix},$$

with $D(A) = (H^2(0,b) \cap H_0^1(0,b)) \times H_0^1(0,b)$.

Model 3: (Nonlinear Multi-valued Diffusion) The following model studied in [101] describes diffusion within a heterogenous material described by a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$ comprised of two components undergoing phase changes.

$$\begin{cases} \frac{\partial}{\partial t} a(z) - \triangle \alpha_1(z) + \beta(z-w) \ni \int_0^t k_1(t-s) f(s,x)ds \\ \frac{\partial}{\partial t} b(w) - \triangle \alpha_2(z) - \beta(z-w) \ni \int_0^t k_2(t-s) g(s,x)ds \\ 0 \in_1 (z(x,t)), 0 \in_2 (w(x,t)) \quad x \in \partial\Omega, \quad t > 0 \\ w(x,0) = w_0(x), \quad z(x,0) = z_0(x), \quad x \in \Omega. \end{cases}$$

where $\alpha_1, \alpha_2, \beta : \mathbb{R} \to \mathbb{R}$ are maximal operators such that $0 \in \alpha_i(0)$, $(i = 1, 2)$, $0 \in \beta(0)$. The system can be reformulated as the abstract function evolution inclusion

$$\begin{cases} u'(t) \in Au(t) + F(u,t), 0 < t < b \\ u(0) = u_0. \end{cases}$$

Where $A : D(A) \subset H \to H$ is given by

$$A \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} -\triangle\alpha_1(z) + \beta(z-w) \\ -\triangle\alpha_2(z) - \beta(z-w) \end{bmatrix}.$$

with $H = H_0^1(0,b) \times H^0(0,b)$.

Differential equations with impulses were considered for the first time by Milman and Myshkis [67] and then followed by a period of active research which culminated with the monograph by Halanay and Wexler [48]. Many phenomena and evolution processes in the field of physics, chemical technology, population dynamics, and natural sciences may change state abruptly or be subject to short-term perturbations (see for instance [1, 65] and the references therein). These short perturbations may be seen as impulses. Impulsive problems arise also in various applications in communications, chemical technology, mechanics (jump discontinuities in velocity), electrical engineering, medicine, and biology. For instance, in the periodic treatment of some diseases, impulses correspond to the administration of a drug treatment. In environmental sciences, impulses correspond to seasonal changes of the water level of artificial reservoirs. Their models are described by impulsive differential equations and inclusions. Various mathematical results (existence, asymptotic behavior, ...) have been obtained; see [11, 20, 37, 38, 46, 65, 78] and the references therein.

This thesis is devoted to the existence of solutions for different classes of initial values problems for impulsive differential equation and inclusions with fixed and variable moments. This thesis is arranged as follows:

- In the first chapter we introduce notation, definitions, lemmas and fixed point theorems which are used throughout this thesis.

- In the second chapter, we present basic notions of dissipative nonlinear operator in Banach space.

- In the third chapter we are concerned with nonlinear semigroups which have a connection to solution of nonlinear evolution with dissipative operator. We introduce the Nonlinear semigroups and their generators and nonlinear evolution with dissipative operator.

- In the fourth chapter we consider the following class of impulsive evolution inclusions

$$\begin{cases} y'(t) & \in & Ay(t) + F(t, y(t)), \qquad \text{a.e. } t \in [0, b], \\ y(t_k^+) - y(t_k) & = & I_k(y(t_k)), \quad k = 1, \dots, n, \\ y(0) & = & x, \end{cases} \tag{0.1}$$

where $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = b$, $F\colon\ J \times E \to \mathcal{P}(E)$ is a multifunction, and $x \in \overline{D(A)}$. The operator A is the infinitesimal generator of a nonlinear semigroup, $I_k \in C(E, E)\,(k = 1, \dots, m)$, and $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$. The notations $y(t_k^+) = \lim\limits_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim\limits_{h\to 0^+} y(t_k - h)$ stand for the right and the left limits of the function y at $t = t_k$, respectively.

The Cauchy problem () with A a generator of C_0-semigroup is well understood (see, for example [20, 37, 38, 58, 75] and the references therein).

In the case where $F = I_k = 0$, the fondamental existence result for () is due to Crandall [32] and Crandall and Liggett [33], who proved that an accretive operator A generates a nonlinear semigroup. The existence problem on the compact and unbounded interval when the impulses are absent, was studied by Attouch *et al* [10], Barbu [16], Bénilan [21, 22], Bénilan and Brezis [23], Browder [30], Kato [59, 60], and Kobayashi [62].

Very recently by using a combination of compactness arguments and fixed point theory, the existence results and compactness of solution handling Cauchy condition, periodic, anti-periodic, multi-point and nonlocal conditions of nonlinear evolution equations and inclusions without impulses, various mathematical results (existence, asymptotic behavior,. . .) have been obtained; see, for examples [5, 41, 47, 91, 76, 73, 94, 95, 96, 97, 98].

In this chapter we discuss the existence of solution for some class of nonlinear evolution inclusions with impulses effect, in the case where nonlinearity may be either convex or non convex. In the first part of this work, we assume that the multifunction is almost strongly-weakly u.s.c with convex valued. For second part we present the non-convex where the right-hand said is l.s.c with decomposable valued. Our results will be obtained by means of technique of fixe point theorems.

- In the last chapter, we prove some variants of random Perov, Schauder, Krasnoselskii and Leray-Schauder-type fixed point theorems fixed in generalized Banach spaces. The results are used to prove the existence of solution for random differential equations with initial and boundary conditions.

Key words and phrases: Impulsive differential inclusion, nonlinear differential equation , integral solution, compact semigroup, multifunction, fixed point theorem's, random operator.

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Chapter 1

Preliminaries

In this chapter we describe basic results of functional analysis that are needed later. For the details, refer, for example, to [54, 68, 102].

1.1 Element of Functional Analysis

Let E be a real or complex Banach space.

Definition 1.1.1 *(1) $X \subset E$ is called a subspace if $\alpha x + \beta y \in X$ for all $\alpha, \beta \in \mathbb{K}, (\mathbb{K} = \mathbb{R} or \quad \mathbb{C})$ and $x, y \in X$.*

(2) For $S \subset E$, the set consisting of all linear combinations of S is obviously a linear set. We say this linear set is generated by S.

Definition 1.1.2 **(1)** *$X \subset E$ is called a convex set if $\alpha x + (1-\alpha)y \in X$ for all α with $0 < \alpha < 1$ and all elements $x, y \in X$.*

(2) *$\alpha_1 x_1 + \ldots + \alpha_n x_n$ with $x_i \in E \quad (i = 1, \ldots, n)$, $\alpha_i \geq 0$, and $\alpha_1 + \ldots + \alpha_n = 1$, is called a convex combination of $x_1, \ldots, x_n$.*
For $S \subset E$, the set consisting of all convex combinations of elements of S is obviously a convex set. we denote this convex set by $co(S)$, and we say that the convex set is generated by S. The closure $\overline{co}(S)$ of $co(S)$ also forms a convex set. We denote this closed convex set by $\overline{co}S$.

Definition 1.1.3 *Let E and F be Banach spaces.*

(1) *An operator $A : D(A) \subset X \to R(A) \subset F$ such that*

$$\|Ax - Ay\| \leq \|x - y\|, \quad x, y \in D(A)$$

is called a contraction operator.
Where $D(A)$, $R(A)$ we denote domain and range of A, respectively

(2) *An operator $A : D(A) \subset E \to R(A) \subset F$, is called a strict contraction if there exists a constant α with $0 \leq \alpha < 1$such that*

$$\|Ax - Ay\| \leq \alpha \|x - y\|, \quad x, y \in D(A)$$

We have the following fixed point theorem.

Theorem 1.1.1 *[38] Let X be a closed subset of E. If $A : X \to X$ is a strict contraction operator, then A has a unique fixed point.*

Definition 1.1.4 *Let X be a Banach space*

(1) *We say that X is strictly convex if $\|x + y\| \neq \|x\| + \|y\|$ for all linearly independent elements x and y of E.*

(2) *If $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|x_n + y_n\| = 2$, where $x_n, y_n \in E$, imply that $\lim_{n\to\infty} \|x_n - y_n\| = 0$, then we say that E is uniformly convex.*

As we can easily see, a necessary and sufficient condition for X to be strictly convex is that the points on the segment connecting two distinct points on the surface of the unit sphere in X are contained in the interior of the unit sphere $\{x \in X, \|x\| \leq 1\}$ except for the end points.

Theorem 1.1.2 *A uniformly convex Banach space X is strictly convex.*

Proof. Let X be uniformly convex and suppose

$$\|x + y\| = \|x\| + \|y\|, \quad \text{where} \quad x \neq 0 \quad \text{and} \quad y \neq 0.$$

We can assume $\|x\| \leq \|y\|$. We have

$$\begin{aligned} \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| &= (\|x\|\|y\|)^{-1} \| \|y\|(x + y) - (\|y\| - \|x\|)y\| \\ &\geq (\|x\|\|y\|)^{-1} [\|y\|(\|x\| + \|y\|) - (\|y\| - \|x\|)\|y\|] \\ &= 2 \end{aligned}$$

then $\left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| = 2$, and (since X is uniformly convex) $\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| = 0$. Hence, we obtain $\|y\|x = \|x\|y$, and x and y are not linearly independent. Therefore, X is strictly convex.

□

Example 1.1.1 *In any Hilbert space H the parallelogram law*

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad x, y \in H$$

holds. From this, we see that a Hilbert space is a uniformly convex Banach space.

The set of all continuous linear functionals defined on X forms a Banach space. We denote this Banach space by X^* and call it the deal space of X.We denote deal space $(X^*)^*$ of X^* by X^{**}.

Definition 1.1.5 *If $X \simeq X^{**}$, then X is said to be reflexive.*

Definition 1.1.6 *We say that a sequence $(x_n)_{n\in\mathbb{N}}$ in X converges weakly to x, and express by $w-\lim_{n\in\infty} x_n = x$, if $\lim_{n\to\infty} x^*(x_n) = x^*(x)$ holds for arbitrary $x^* \in X^*$.*

Now, we describe theorems of the Radon-Nikodym type.

Definition 1.1.7 *Let $[a, b]$ be a bounded closed interval and let $f : [a, b] \to X$. f is called absolutely continuous on $[a, b]$ if the following holds:*
For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $[a_i, b_i) \subset [a, b]$, $i = 1, 2, ..., n$ are mutually disjoint and $\sum_{i=1}^{n}(b_i - a_i) < \delta$, then $\sum_{i=1}^{n} \|f(b_i) - f(a_i)\| < \varepsilon$.

Theorem 1.1.3 *[68] (i) Let $f : [a, b] \to X$ be a Bochner integrable on $[a, b]$, and set*

$$g(s) = \int_a^s f(t)dt \quad , \qquad a \leq s \leq b.$$

Then, g is absolutely continuous on $[a, b]$, strongly differentiable at a.e s, and the strong derivative $g'(s) = f(s)$ a.e s.
(ii) If $g : [a, b] \to X$ is absolutely continuous and weakly differentiable at a.e. s, then the weak derivative f of g is Bochner integrable on $[a, b]$ and

$$g(s) = g(a) + \int_a^s f(t)dt \quad , \quad a \leq s \leq b$$

holds. Therefore, g is strongly differentiable at a.e. s and $g'(s) = f(s)$ a.e. s.

Theorem 1.1.4 *[68] Let X be a reflexive Banach space. A necessary and sufficient condition for $g : [a, b] \to X$ to be absolutely continuous on $[a, b]$ is that there exists a Bochner integrable function f on $[a, b]$ such that*

$$g(s) = g(a) + \int_a^s f(t)dt, \quad a \leq s \leq b.$$

In this case, g is strongly differentiable at a.e. s and $g'(s) = f(s)$, a.e. s.

In what follows, $L^1([0, b], X)$ denotes the Banach space of functions $y : [0, b] \to X$ while are Bochner integrable with norm

$$\|y\|_{L^1} = \int_0^b \|y(t)\|dt.$$

Definition 1.1.8 *Let (Ω, Σ, μ) be a finite measure space. A subset $\mathcal{C}$ in $L^1(\Omega, \Sigma, \mu)$ is called uniformly integrable if, for each $\varepsilon > 0$ there exist $\delta(\varepsilon) > 0$ such that, for each measurable subset $\mathcal{R} \subset \Sigma$ whose $\mu(\mathcal{R}) < \delta(\varepsilon)$, we have*

$$\int_{\mathcal{R}} \|f(s)\| d\mu(s) < \varepsilon.$$

Remark 1.1.1 *Let $\mathcal{C} \subset L^1(\Omega, \Sigma, \mu)$, then:*
(i) if $\mu(\Omega) < \infty$ and $\mathcal{C}$ is bounded in $L^p(\Omega, \Sigma, \mu)$ where $p > 1$, then $\mathcal{C}$ is uniformly integrable.
(ii) if there exist $p \in L^1(\Omega, \Sigma, \mathbb{R}_+)$ such that

$$\|f(w)\| \leq p(w), \quad \textit{for each} \quad f \in \mathcal{C} \quad \textit{and a.e.} \quad w \in \Omega,$$

then $\mathcal{C}$ is uniformly integrable.

Let $K \subset X$. We define $\mathcal{K}$ by

$$\mathcal{K} = \{f \in L^1(\Omega, \Sigma, \mu) : \quad f(w) \in K \quad \textit{a.e.} \quad w \in \Omega\}.$$

Theorem 1.1.5 *[36]*
Let (Ω, Σ, μ) be a finite measure space, X be a Banach space, and $\mathcal{K}$ be a bounded uniformly integrable subset of $L^1(\Omega, \Sigma, \mu)$. suppose that given $\varepsilon > 0$ there exists a measurable set Ω_ε and a weakly compact set $K_\varepsilon \subset X$ such that $\mu(\Omega \backslash \Omega_\varepsilon) < \varepsilon$ and for each $f \in \mathcal{K}$, $f(w) \in K_\varepsilon$ for almost all $w \in \Omega_\varepsilon$. Then $\mathcal{K}$ is a relatively weakly compact subset of $L^1(\Omega, \Sigma, \mu)$.

1.2 Multi-valued maps analysis

In this section we include several basic notions and results referring to multivalued mappings, i.e., to maps whose values are sets (multiple-valued or called also set-valued maps) which will be used throughout this thesis. For the details, refer, for example, to Aubin and Frankowska [13], Andres-Górniewicz [6], Djebali *et al.* [38], Górniewicz [45], Hu-Papageorgiou [55, 56], Kamenskii *et al* [58], Kisielewicz [61], Tolstonogov [93], Aubin and Cellina [12] and Deimling [35].

1.2.1 Definitions and properties

Let X, Y, Z be three spaces.

Notations 1.2.1 *Denote by,*

$$\begin{aligned} \mathcal{P}(Y) &= \{A \subseteq Y : A \neq \emptyset\} \\ \mathcal{P}_c l(Y) &= \{A \subseteq Y : A \textit{ closed }\} \end{aligned}$$

$$\begin{aligned}
\mathcal{P}_{cp}(Y) &= \{A \subseteq Y : A \text{ compact } \} \\
\mathcal{P}_{b}(Y) &= \{A \subseteq Y : A \text{ bounded } \}\ Y \text{ be metric space} \\
\mathcal{P}_{cv}(Y) &= \{A \subseteq Y : A \text{ convex } \}\ Y \text{ be metric space} \\
\mathcal{P}_{cv,cp}(Y) &= \{A \subseteq Y : A \text{ convex and compact } \} Y \text{ be metric space} .
\end{aligned}$$

Definition 1.2.1 • *Assume that for every point $x \in X$ a nonempty subset $F(x)$ of Y is given, in this case, we say that F is a multi-valued mapping from X to Y and we write $F : X \to \mathcal{P}(Y)$.*
• *A multi-valued map F is characterized by its graph $Gr(F)$, the subset of the product space $X \times Y$ defined by*

$$Gr(F) := \{(x, y) \in X \times Y : y \in F(x)\}.$$

• *Let A be a subset of X. The set $F(A) = \bigcup_{x \in A} F(x)$ is called the image of A under F.*
• *Let $F : X \to \mathcal{P}(Y)$ and $G : Y \to \mathcal{P}(Z)$ two multi-valued maps, the composition $G \circ F : X \to \mathcal{P}(Z)$ of F and G is defined by:*

$$(G \circ F)(x) = \bigcup \{G(y) : y \in F(x)\} \quad \text{for every} \quad x \in X.$$

• *For any subset $B \subset Y$ we define the small counter image $F_+^{-1}(B)$ and the large counter image $F_-^{-1}(B)$ of B under F as follows:*

$$F_+^{-1}(B) = \{x \in X : F(x) \subset B\},$$

$$F_-^{-1}(B) = \{x \in X : F(x) \cap B \neq \emptyset\},$$

• *Let given the maps $F, G : X \to \mathcal{P}(Y)$. Define $(F \times G)$, $F \cap G : X \to \mathcal{P}(Y)$ and $F \cup G : X \to \mathcal{P}(Y)$ as follows:*

$$(F \times G)(x) = F(x) \times G(x),$$

$$(F \cap G)(x) = F(x) \cap G(x),$$

$$(F \cup G)(x) = F(x) \cup G(x),$$

for every $x \in X$ and $F(x) \cap G(x) \neq \emptyset$.

Proposition 1.2.1 *Let $F : X \to \mathcal{P}(Y)$ be a multi-valued map and A, B two subsets of X and Y respectively , then we have the following properties:*
$A \subset F_-^{-1}(F(A))$ and $F(F_-^{-1}(B)) \subset B$.

Proposition 1.2.2 **(a)** *let $F, G : X \to \mathcal{P}(Y)$ two multi-valued and B be a subset of Y, then*

1. $(F \cup G)_-^{-1}(B) = F_-^{-1}(B) \cup G_-^{-1}(B)$

2. $(F \cup G)_+^{-1}(B) = F_+^{-1}(B) \cup G_+^{-1}(B)$
3. $(F \cap G)_-^{-1}(B) \supset F_-^{-1}(B) \cap G_-^{-1}(B)$
4. $(F \cap G)_+^{-1}(B) \subset F_+^{-1}(B) \cap G^{-1}(B)$.

Example 1.2.1 1) *Let* $F : [0,1] \to \mathcal{P}(\mathbb{R})$ *be defined*

$$F(x) = \begin{cases} [0,1], & \text{for} \quad x \neq \frac{1}{2} \\ \{0\}, & \text{for} \quad x = \frac{1}{2}. \end{cases}$$

2) *Let* $f : X :\to Y$ *be a continuous map. Then we can consider its inverse as a multi-valued map* $F_f : Y \to \mathcal{P}(X)$ *defined by*

$$F_f = f^{-1}(y) \qquad \text{for} \quad y \in Y.$$

3) ***Control problems.*** *Assume we have to solve the following problem,*

$$\begin{cases} y'(t) = f(t, y(t), u(t)), \\ y(t_0) = y_0. \end{cases} \tag{1.1}$$

controlled by parameters $u(t)$*, where* $f : [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$*.*
In order to solve we define a multi-valued map $F : [t_0, T] \times \mathbb{R}^n \to (\mathbb{R}^n)$ *as follows*

$$F(t, y) = \{f(t, y, u)\}_{u \in U}.$$

Then solutions of are solutions of the following differential inclusion:

$$\begin{cases} y'(t) \in F(t, y(t)), \\ y(t_0) = y_0. \end{cases} \tag{1.2}$$

So any control problem () can be translated, in view of multi-valued maps onto problem ().

Definition 1.2.2 *Let* (X, d) *and* (Y, δ) *be two metric spaces and* $F : X \to \mathcal{P}(Y)$ *be a multi-valued with nonempty and closed values. A single-valued map* $f : X \to Y$ *is said to be a selection of* F *and write* $f \subset F$ *whenever* $f(x) \in F(x)$ *for every* $x \in X$*.*

1.2.2 Hausdorff metric space

The Hausdorff metric is defined on a metric space and is used to quantify the distance between subsets of the given metric space. Let (X, d) be a metric space. In what follows, given $x \in X$ and $A \in \mathcal{P}(X)$, the distance of x from A is defined by

$$d(x, A) = \inf\{d(x, a) : a \in A\}.$$

Similarly, for $y \in X$ and $B \in \mathcal{P}(X)$

$$d(B, y) = \inf\{d(b, y) : b \in B\}.$$

As usual, $d(x, \emptyset) = d(\emptyset, y) = +\infty$.

Definition 1.2.3 *Let $A, B \in \mathcal{P}(X)$, we define the Hausdorff distance between A and B by*

$$H(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\}.$$

From the definition we can easily prove the following properties:

1) $H(A,A) = 0, \quad \text{for all} \quad A \in \mathcal{P}(X)$

2) $H(A,B) = H(B,A), \quad \text{for all} \quad A, B \in \mathcal{P}(X)$

3) $H(A,B) \leq H(A,C) + H(C,B) \quad \text{for all} \quad A, B, C \in \mathcal{P}(X).$

Lemma 1.2.1 *Given $\varepsilon > 0$, let*

$$A_\varepsilon = \{x \in X : d(x,A) < \varepsilon\} \quad \textit{and} \quad B_\varepsilon = \{x \in X : d(B,x) < \varepsilon\}.$$

Then from the above definition we have

$$H(A,B) = \inf\{\varepsilon > 0 : B \subset A_\varepsilon, \quad A \subset B_\varepsilon\}.$$

Proof Let $\varepsilon > 0$, such that $A \subset B_\varepsilon$, then

$$d(a,B) \leq \varepsilon, \quad \text{for all} \qquad a \in A$$

this implies

$$\sup_{a\in A} d(a,B) \leq \inf\{\varepsilon > 0 : A \subset B_\varepsilon\}.$$

We suppose that $\sup_{a\in A} d(a,B) < \inf\{\varepsilon > 0, A \subset B_\varepsilon, \}$ this implies that

$$d(a,B) < \inf\{\varepsilon > 0, A \subset B_\varepsilon\} \quad \text{for all} \quad a \in A,$$

then is there exists $\xi > 0$, such that

$$d(a,B) < \xi < \inf\{\varepsilon > 0, A \subset B_\varepsilon\} \quad \text{for all} \quad a \in A,$$

therefore $A \subset B_\xi$, then $\inf\{\varepsilon > 0, A \subset B_\varepsilon\} \leq \xi$ this is contradiction, then

$$\sup_{a\in A} d(a,B) = \inf\{\varepsilon > 0, A \subset B_\varepsilon\},$$

and this completes the proof.

Hence $H(.,.)$ is an extended pseudometric on $\mathcal{P}(X)$. We have $d(a,A) = 0$ if and only if $a \in \overline{A}$ therefore we can prove that

$$H(A,B) = 0 \quad \text{if and only if} \quad \overline{A} = \overline{B}.$$

Then $(\mathcal{P}_{cl}(X), H)$ becomes a metric space.

Theorem 1.2.1 *If* (X, d) *is a complete metric space, then the metric space* $(\mathcal{P}_{cl}(X), H)$ *is complete.*

Definition 1.2.4 G *is called upper semi-continuous (u.s.c. for short) on* X *if for each* $x_0 \in X$*, the set* $G(x_0)$ *is a nonempty, closed subset of* X*, and if for each open set* N *of* Y *containing* $G(x_0)$*, there exists an open neighborhood* M *of* x_0 *such that* $G(M) \subseteq N$.

Definition 1.2.5 *A multifunction* $F : J \times \overline{D(A)} \to \mathcal{P}(X)$ *is said to be almost strongly weakly u.s.c. if for each* $\epsilon > 0$*, there exists a Lebesgue measurable subset* $J_\epsilon \subset J$ *whose Lebesgue measure* $\mu(J_\epsilon) \leq \epsilon$ *and such that* $F : J \backslash J_\epsilon \times \overline{D(A)} \to \mathcal{P}(X)$ *is u.s.c., with* $J \backslash J_\epsilon \times \overline{D(A)}-$*endowed with the strong topology and* $X-$*endowed with the weak topology.*

1.2.3 Measurable selections

Definition 1.2.6 $F : [0, b] \times X \to \mathcal{P}(Y)$ *is said:*
(1) *Integrable if it has a summable selection* $f \in L^1([0, b] \times X, Y)$
(2) *Integrably bounded, if there exists* $q \in L^1([0, b], \mathbb{R}^+)$ *such that*

$$\|F(t, z)\|_{\wp} \leq q(t), for \; a.e \quad t \in [0, b] \quad and \; every \quad z \in X,$$

were

$$\|F(t, z)\|_{\wp} := \sup\{\|u\| : u \in F(t, z)\}.$$

Definition 1.2.7 *A multi-valued map* $F : [0, b] \to \mathcal{P}(Y)$ *is said measurable provide for every open* $\mathcal{U} \subset Y$*, the set* $F_-^{-1}(\mathcal{U})$ *is Lebesgue measurable.*

A characterization of the measurability of the multi-valued is given by the following lemma.

Lemma 1.2.2 *The multi-valued map* $F : [0, b] \to \mathcal{P}(Y)$ *with nonempty closed values is measurable if and only if for each* $x \in Y$*, the function* $\psi : [0, b] \to [0, +\infty)$ *defined by*

$$\psi(t) = d(x, F(t)) = \inf\{\|x - y\| : y \in F(t)\}, t \in [0, b],$$

is Lebesgue measurable.

The following lemmas are needed in this thesis, The first one is celebrated by Kuratowski-Ryll-Nardzewski selection theorem.

Lemma 1.2.3 *Let* Y *be a separable metric space and* $F : [0, b] \to \mathcal{P}(Y)$ *a measurable multi-valued map with nonempty closed values. Then* F *has a measurable selection.*

Lemma 1.2.4 *Let X, Y be two real Banach spaces, $F : X \to \mathcal{P}(Y)$ with nonempty weakly compact values be strongly weakly u.s.c multi-valued, then for each compact subset $C \subset X$, the set $\bigcup_{x\in C} F(x)$ is weakly compact. In particular, for each compact subset C of X , there exists $M > 0$ such that $\|y\| \leq M$ for each $x \in C$ and each $y \in F(x)$.*

Proof. Let C be a weakly compact subset in X and let $\{D_i; i \in I\}$ be an arbitrary weakly open covering of $\bigcup_{x\in C} F(x)$. Since F is weakly compact valued, for each $x \in C$ there exists $\eta \in \mathbb{N}$ such that

$$F(x) \subseteq \bigcup_{1\leq k\leq \eta} D_{i_k}.$$

But F is strongly-weakly u.s.c. and therefore there exists a weakly open neighborhood $\mathcal{U}(x)$ of x such that

$$F(\mathcal{U}(x)) \cap X \subseteq \bigcup_{1\leq k\leq \eta} D_{i_k}.$$

The family $\{\mathcal{U}(x); x \in C\}$ is an weakly open covering of C. As C is compact, there exists a finite family $\{x_1, x_2, ..., x_p\}$ in C such that

$$F(C) \subseteq \bigcup_{1\leq j\leq p} F(\mathcal{U}(x_j) \cap X) \subseteq \bigcup_{1\leq j\leq p} \bigcup_{1\leq k\leq \eta(x_j)} D_{i_k}.$$

and this completes the proof.

Lemma 1.2.5 *Let X be a Banach space and K be a nonempty subset of X. Let $F : K \to \mathcal{P}(Y)$ be a strongly weakly u.s.c with nonempty closed and convex values, and let $u_n : [0, b] \to K$ and $f_n \in L^1([0, b], X)$ such that*

$$f_n(t) \in F(u_n(t)) \quad a.e \quad t \in [0, b] \quad \text{for each} \quad n \in \mathbb{N}.$$

If $\lim_{n\to+\infty} u_n(t) = u(t)$ a.e. $t \in [0, b]$ and $\lim_{n\to+\infty} f_n = f$ is weakly in $L^1([0, b], X)$, then

$$f(t) \in F(u(t)) \quad a.e \quad t \in [0, b].$$

The following two theorems are needed in this thesis. The first one is the celebrated Kuratowski-Ryll-Nardzewski selection theorem.

Theorem 1.2.2 *[64] Let $(\Omega, \sum)$ and Y be a separable metric space and let $F : \Omega \to \mathcal{P}_{cl}(Y)$ be a measurable multi-valued. Then F has a measurable selection.*

Theorem 1.2.3 *Let $(\Omega, \sum)$ be a measurable space, let Y be a separable metric space, and let $F : \Omega \longrightarrow \mathcal{P}_{cl}(Y)$ a multi-valued map which is measurable. Then F is graph-measurable.*

The following important result is due to J.R. Aummann (see [61]).

Theorem 1.2.4 *If $G : \Omega \to \mathcal{P}_{cp}(X)$ is a multi-valued map such that the graph $\mathcal{G}r(G)$ of G is measurable, then G has a measurable selector.*

1.3 Fixed point theorems

We recall some classical fixed point theorems which need to prove our existence results of solution. From Schauder's fixed point theorem we obtain the weak form of Kakutani-Fan's fixed point theorem.

Theorem 1.3.1 *[61] Let $(X, \|\cdot\|)$ be a Banach space and let C be weakly compact convex subset of X. Suppose $F : C \to \mathcal{P}(C)$ with nonempty weakly closed, convex values be weakly u.s.c multivalued map. Then there exists $x \in C$ such that $x \in F(x)$.*

The following, so called, nonlinear alternative of Leray-Schauder.

Theorem 1.3.2 *[61] Let X be a Banach space and let C a nonempty bounded, closed and convex subset. Assume $\mathcal{U}$ is an open subset of C with $0 \in \mathcal{U}$ and let $G : \overline{\mathcal{U}} \to C$ be a continuous compact map. Then*
(a) either there is a point $u \in \partial\mathcal{U}$ and $\lambda \in (0, 1)$ with $u = \lambda G(u)$,
(b) or G has a fixed point in $\overline{\mathcal{U}}$.

Chapter 2

Dissipative nonlinear operators in Banach spaces

In this chapter let X will be a real or complex Banach space and X^* will denote its dual. The value of a functional $x^* \in X^*$ at $x \in X$ will be denoted by either (x, x^*) or $x^*(x)$, as is convenient. The norm of X will be denoted by $\|.\|$, and the norm of X^* will be denoted by $\|.\|_*$. If there is no danger of confusion we omit the asterisk from the notation $\|.\|_*$ and denote both the norms of X and X^* by the symbol $\|.\|$.

2.1 Duality mapping

Define on X the mapping $J : X \to \mathcal{P}(X^*)$

$$J(x) = \{x^* \in X^*, \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\} \quad \text{for all} \quad x \in X$$

By the Hahn-Banach theorem we know that for every $x_0 \in X$ there is some x_0^* such that $\langle x_0, x_0^* \rangle = \|x_0\|^2$ and $\|x_0^*\| = \|x_0\|$. Clearly, $x_0^* \in J(x_0)$ and so $J(x) \neq \emptyset$ for every $x \in X$.

Definition 2.1.1 *The mapping $J : X \to \mathcal{P}(X^*)$ is called the duality mapping of the space X. In general, the duality mapping J is multi-valued map.*

From this definition, we easily obtain the following lemma.

Lemma 2.1.1

1. *For $x \in X$ the image $J(x)$ is a closed convex set.*
2. *for all $x \in X$ and α a scalar we have $J(\alpha x) = \alpha J(x)$.*
3. *$Re\langle x - y, f - g \rangle \geq 0$ for all $x, y \in X$ and $f \in J(x)$, $g \in J(y)$.*
4. *If X is reflexive, then J is a mapping from X onto X^*, that is, X^* is the range $J(X)$ of J.*

Theorem 2.1.1 *Let X be a Banach space. If the dual space X^* is strictly convex then the duality mapping $J : X \to \mathcal{P}(X^*)$ is single valued mapping.*

Proof. From the definition, $J(0) = \{0\}$. Next, we let $x \neq 0$, and let $f, g \in J(x)$. Then, from

$$\langle x, f\rangle = \|f\|^2 = \|x\|^2 = \|g\|^2 = \langle x, g\rangle$$

we obtain $\|f + g\|\|x\| \geq \langle x, f + g\rangle = 2\|x\|^2$. That is, $\|f + g\| \geq 2\|x\| = \|f\| = \|g\|$. Therefore, $\|f + g\| = \|f\| + \|g\|$. Since X^* is strictly convex, there is a scalar α such that $g = \alpha f$. From $\langle x, f\rangle = \langle x, g\rangle = \overline{\alpha}\langle x, f\rangle$ and $\langle x, f\rangle \neq 0$, we have $\overline{\alpha} = 1$. Hence, $g = f$.

□

Remark 2.1.1 *Since a uniformly convex Banach space is strictly convex, from the theorem () we have that J is single-valued mapping.*

Lemma 2.1.2 *If X is a Hilbert space, then the duality mapping J coincides with the identity mapping I on X*

Proof. Since $X \simeq X^*$, J is a mapping of X into itself. let $x \in X$ and $f \in J(x)$, then from $\langle x, f\rangle = \|x\|^2 = \|f\|^2$, we have

$$\|x - f\|^2 = \|x\|^2 - 2Re(x, f) + \|f\|^2 = 0$$

Thus we obtain $f = x$, and hence $J = I$.

□

Denote by $[.,.]_+$ the directional derivative of the function $x \to \|x\|$

$$[x, y]_+ := \lim_{h\to 0^+} \frac{1}{h}(\|x + hy\| - \|x\|), \quad x, y \in X. \tag{2.1}$$

Since the function of $h > 0$,

$$\frac{1}{h}(\|x + hy\| - \|x\|)$$

is increasing and bounded from below. Thus

$$\lim_{h\to 0^+} \frac{1}{h}(\|x + hy\| - \|x\|) = \inf_{h>0} \frac{1}{h}(\|x + hy\| - \|x\|)$$

exists and finite valued.

Definition 2.1.2 *We define functions $[.,.]_+ : X\times X \to (-\infty, +\infty)$ $[.,.]_- : X\times X \to (-\infty, +\infty)$ by*

$$[x, y]_+ := \lim_{h\to 0^+} \frac{1}{h}(\|x + hy\| - \|x\|) = \inf_{h>0} \frac{1}{h}(\|x + hy\| - \|x\|)$$

and

$$[x, y]_- := \lim_{h\to 0^-} \frac{1}{h}(\|x + hy\| - \|x\|) = \sup_{h<0} \frac{1}{h}(\|x + hy\| - \|x\|)$$

We have $[x,y]_- \leq [x,y]_+$ for all $x,y \in X$.

Lemma 2.1.3 *We have the following properties:*

1. $[x,y]_+ = -[x,-y]_- = -[-x,y]_-$;
2. $[x,y+z]_+ \leq [x,y]_+ + [x,z]_+$
3. *Let* $x : [0,b] \to X$ *be a given function and let us define* $m : [a,b] \to \mathbb{R}_+$ *by* $m(t) := \|x(t)\|$, *for each* $t \in [0,b]$. *If* x *is differentiable from the right (left) at* $t \in [0,b]$, *then* m *is differentiable from the right (left) at* t *and we have*
$$m'_\pm(t) = [x(t), x'_\pm(t)]_\pm$$

Theorem 2.1.2 *Let* $x,y \in X$.

1. *There exists an element* $f_+ \in J(x)$ *such that*
$$\|x\|[x,y]_+ = \sup\{Re\langle y,f\rangle : f \in J(x)\} = Re\langle y,f_+\rangle.$$
2. *There exists an element* $f_- \in J(x)$ *such that*
$$\|x\|[x,y]_- = \inf\{Re\langle y,f\rangle : f \in J(x)\} = Re\langle y,f_-\rangle.$$

We introduce the following notation. For every $x,y \in X$
$$\langle x,y\rangle_s = \sup\{Re\langle y,f\rangle : f \in J(x)\}$$
and
$$\langle x,y\rangle_i = \inf\{Re\langle y,f\rangle : f \in J(x)\} = -\langle -x,y\rangle_s$$
From the theorem () we have
$\langle x,y\rangle_s = \|x\|[x,y]_+$ and $\langle x,y\rangle_i = \|x\|[x,y]_-$.

Proposition 2.1.1 *We have*

1. $\langle .,.\rangle_s : X \times X \to (-\infty,+\infty)$, *is u.s.c.*
2. $\langle .,.\rangle_i : X \times X \to (-\infty,+\infty)$, *is l.s.c.*

Proof. We have
$$\langle y,x\rangle_s = \inf_{h>0} \|x\|\frac{1}{h}(\|x+hy\| - \|x\|).$$
For each $h > 0$, since $\|x\|\frac{1}{h}(\|x+hy\| - \|x\|)$ is a continuous function $X \times X \to (-\infty,+\infty)$, it follows that $\langle .,.\rangle_s : X \times X \to (-\infty,+\infty)$ is upper semicontinuous, since $\langle y,x\rangle_i = -\langle -y,x\rangle_s$ we obtain the second

$\square$

Proposition 2.1.2 (1) *Let $x, y \in X$. The following properties are equivalent,*

1. $Re(y, f) \leq 0$ *for a certain* $f \in J(x)$.
2. $\|x - hy\| \geq \|x\|$ *for all* $h > 0$.
3. $[y, x]_- \leq 0$.
4. $\langle y, x\rangle_i \leq 0$.

(2) *When $x \neq 0$, The following properties are mutually equivalent,*

1. $Re\langle y, f\rangle \leq 0$ *for all* $f \in J(x)$.
2. $[y, x]_+ \leq 0$.
3. $\langle y, x\rangle_+ \leq 0$.

Proof. Is self-evident. $\square$

Lemma 2.1.4 *If $u(t) : (a, b) \to X$ has weak derivative $u'(t_0)$ at t_0 and also $\|u(t)\|$ is differentiable at t_0, then*

$$\begin{aligned}
[\frac{d}{dt}\|u(t)\|^2]_{t=t_0} &= 2\|u(t_0)\|[\frac{d}{dt}\|u(t)\|]_{t=t_0} && (2.2)\\
&= 2\langle u'(t_0), u(t_0)\rangle_s && (2.3)\\
&= 2\langle u'(t_0), u(t_0)\rangle_i && (2.4)
\end{aligned}$$

Proof. Suppose that $f \in J(u(t_0))$ is given arbitrarily. Since

$$\begin{aligned}
Re\langle u(t) - u(t_0), f\rangle &= Re\langle u(t), f\rangle - \|u(t_0)\|^2 && (2.5)\\
&\leq (\|u(t)\| - \|u(t_0)\|)\|u(t_0)\|, && (2.6)
\end{aligned}$$

if $t > t_0$ we have

$$Re\langle \frac{u(t) - u(t_0)}{t - t_0}, f\rangle \leq \frac{\|u(t)\| - \|u(t_0)\|}{t - t_0}\|u(t_0)\|,$$

and if $t < t_0$ we have

$$Re\langle \frac{u(t) - u(t_0)}{t - t_0}, f\rangle \geq \frac{\|u(t)\| - \|u(t_0)\|}{t - t_0}\|u(t_0)\|.$$

Now we let $t \to_> t_0$ and $t \to_< t_0$, then we have

$$Re\langle u'(t_0), f\rangle \leq \|u(t_0)\|[\frac{d}{dt}\|u(t)\|]_{t=t_0}$$

and

$$Re\langle u'(t_0), f\rangle \geq \|u(t_0)\|[\frac{d}{dt}\|u(t)\|]_{t=t_0}$$

respectively. That is, we obtain

$$\|u(t_0)\|[\frac{d}{dt}\|u(t)\|]_{t=t_0} = Re\langle u'(t_0), f\rangle.$$

$\square$

2.2 Dissipative operators

In this section we consider an operator $A : X \to X$ is not necessarily single valued, we denote by $D(A)$, $R(A)$ its domain and range respectively

Definition 2.2.1 *An operator $A : D(A) \subseteq X \to X$ is called dissipative, if for each $x_i \in D(A)$ and $y_i \in A(x_i)$, $i = 1, 2$, there exists a $f \in J(x_1 - x_2)$ such that $Re\langle y_1 - y_2, f\rangle \leq 0$.*

The operator A is called $m-$dissipative if it is dissipative, and, in addition, $R(I-\lambda A) = X$, for every $\lambda > 0$.

Remark 2.2.1 *Since $J = I$ when X is a Hilbert space in the case the operator A is dissipative if, for each $x, y \in D(A)$, $x' \in Ax$, and $y' \in D(A)$, we have*

$$Re\langle x' - y', x - y\rangle \leq 0.$$

Theorem 2.2.1 *The following (1) and (2) are equivalent,*

1. *A is a dissipative operator.*
2. *For each $x_1, x_2 \in D(A)$, $y_1 \in Ax_1$ and $y_2 \in Ax_2$, any one of the following conditions are holds,*
   - $\|x_1 - x_2 - \lambda(y_1 - y_2)\| \geq \|x_1 - x_2\|$ *for all* $\lambda > 0$
   - $[y_1 - y_2, x_1 - x_2]_- \leq 0$
   - $\langle y_1 - y_2, x_1 - x_2\rangle_i \leq 0$

Proof. Apply proposition (). □

By the theorem () we have the equivalent definition for a dissipative operator.

Definition 2.2.2 *An operator $A : D(A) \subseteq X \to X$ is called dissipative, if for each $x_i \in D(A)$ and $y_i \in A(x_i)$, $i = 1, 2$, we have*

$$[x_1 - x_2, y_1 - y_2]_+ \geq 0.$$

The operator A is called $m-$dissipative if it is dissipative, and, in addition, $R(I - \lambda A) = X$, for every $\lambda > 0$.

Corollary 2.2.1 *Let A be a dissipative operator. Set $J_\lambda = (I - \lambda A)^{-1}$, $\lambda > 0$; then for each $\lambda > 0$, J_λ is a single-valued operator such that $D(J_\lambda) = R(I - \lambda A)$ and $R(J_\lambda) = D(A)$, and J_λ satisfies the following inequality,*

$$\|J_\lambda x - J_\lambda y\| \leq \|x - y\|, \qquad x, y \in D(J_\lambda).$$

We define $|||Ax|||$ by $|||Ax||| = \inf\{\|x'\| : x' \in Ax\}$ for $x \in D(A)$.

Lemma 2.2.1 *Let A be a dissipative operator and let $\lambda > 0$, we have:*

1. *If we set $A_\lambda x = \lambda^{-1}(J_\lambda - I)x$, then $A_\lambda x \in AJ_\lambda x$*
2. $\|A_\lambda x\| \le |||Ax|||$,
3. *If $x \in D(J_\lambda)$ and $\mu > 0$, then*
$$\frac{\mu}{\lambda}x + \frac{\lambda-\mu}{\lambda}J_\lambda x \in D(J_\mu),$$
and
$$J_\lambda x = J_\mu(\frac{\mu}{\lambda}x + \frac{\lambda-\mu}{\lambda}J_\lambda x),$$
4. *If $\lambda \ge \mu > 0$ and $x \in D(J_\lambda)\bigcap D(J_\mu)$, then $\|A_\lambda x\| \le \|A_\mu x\|$.*

Proof. **(1)** Let $x \in D(J_\lambda)$ and set $J_\lambda x = y$. Then $x \in (I - \lambda A)y$, that is, $\lambda^{-1}(y - x) \in Ay$. Thus $A_\lambda x = \lambda^{-1}(y - x) \in Ay = AJ_\lambda x$.
(2) Let $x \in D(J_\lambda)\bigcap D(A)$ and choose $x' \in Ax$ arbitrarily. Then we obtain $J_\lambda(x - \lambda x') = x$ from $x - \lambda x' \in (I - \lambda A)$. Therefore,

$$\begin{aligned} \|A_\lambda x\| &= \lambda^{-1}\|J_\lambda x - x\| = \lambda^{-1}\|J_\lambda x - J_\lambda(x - \lambda x')\| && (2.7)\\ &\le \lambda^{-1}\|x - (x - \lambda x')\| = \|x'\| && (2.8) \end{aligned}$$

Since $x' \in Ax$ is arbitrary, we have $\|A_\lambda x\| \le |||Ax|||$.
(3) Let $x \in D(J_\lambda)$. Then we can write $x = y - \lambda y'$, where $y \in D(A)$ and $y' \in Ay$, and thus $J_\lambda x = y$. Hence,

$$\begin{aligned} \frac{\mu}{\lambda}x + \frac{\lambda-\mu}{\lambda}J_\lambda x &= \frac{\mu}{\lambda}(y - \lambda y') + \frac{\lambda-\mu}{\lambda}y \\ &= y - \mu y' \in R(I - \mu A) = D(J_\mu) \end{aligned}$$

and

$$J_\mu(\frac{\mu}{\lambda}x + \frac{\lambda-\mu}{\lambda}J_\lambda x) = J_\mu(y - \mu y') = y = J_\lambda x.$$

(4) By applying (3) and corollary () we have

$$\begin{aligned} \lambda\|A_\lambda x\| &= \|J_\lambda x - x\| \\ &\le \|J_\lambda x - J_\mu x\| + \|J_\mu x - x\| \\ &= \|J_\mu(\frac{\mu}{\lambda}x + \frac{\lambda-\mu}{\lambda}J_\lambda x) - J_\mu x\| + \|J_\mu x - x\| \\ &\le \|\frac{\mu}{\lambda}x + \frac{\lambda-\mu}{\lambda}J_\lambda x - x\| + \|J_\mu x - x\| \\ &= (\lambda-\mu)\|A_\lambda x\| + \mu\|A_\mu x\|. \end{aligned}$$

Therefore, $\|A_\lambda x\| \le \|A_\mu x\|$.

□

Proposition 2.2.1 *A necessary and sufficient condition for A to be an m-dissipative operator is that A is a dissipative operator and also $R(I-\lambda_0 A) = X$ holds for some $\lambda_0 > 0$.*

Proof. It is self-evident that the condition is necessary. We show that it is sufficient. We have $J_{\lambda_0} = (I-\lambda_0 A)^{-1}$ is a contraction operator defined on all of X. We must note that

$$I - \lambda A = \frac{\lambda}{\lambda_0}[I - (1-\frac{\lambda_0}{\lambda})J_{\lambda_0}](I-\lambda_0 A) \tag{2.9}$$

Now, we choose and fix $x \in X$ arbitrarily and define an operator $T : X \to X$ by

$$Ty = x + (1-\frac{\lambda_0}{\lambda})J_{\lambda_0}y.$$

Since

$$\|Ty - Tz\| \leq |1-\frac{\lambda_0}{\lambda}|\|y-z\|, \qquad x, z \in X$$

we have $|1-\frac{\lambda_0}{\lambda}| < 1$, when $\frac{\lambda_0}{2} < \lambda$ T becomes a strict contraction operator and hence has fixed point $z \in X$. Therefore $Tz = z$, that is, $x = [I-(1-\frac{\lambda_0}{\lambda})J_{\lambda_0}]z$. From this and the hypotheses, we have

$$x \in [I-(1-\frac{\lambda_0}{\lambda})J_{\lambda_0}](I-\lambda_0 A)[D(A)],$$

that is, $X = [I-(1-\frac{\lambda_0}{\lambda})J_{\lambda_0}](1-\lambda_0 A)[D(A)]$. From this (), we obtain

$$R(I-\lambda A) = X \qquad \text{for} \qquad \lambda > \frac{\lambda_0}{2}.$$

Again we apply the above argument to obtain

$$R(I-\lambda A) = X \qquad \text{for} \qquad \lambda > \frac{\lambda_0}{2^2}.$$

We repeat this process and obtain $R(I-\lambda A) = X$.

□

Example 2.2.1 *Let $\mathbb{R} = (-\infty, +\infty)$ and $g : \mathbb{R} \to \mathbb{R}$ be a decreasing function. Then, g is a single valued dissipative operator defined on $\mathbb{R}$.*
Now, we set

$$g_+(x) = \lim_{h\to 0^+} g(x+h), \qquad g_-(x) = \lim_{h\to 0^-} g(x+h)$$

and define a function $\overline{g}$ *by*

$$\overline{g}(x) = \{z : g_+(x) \leq z \leq g_-(x)\}.$$

Obviously, $\overline{g}$ *is an m-dissipative operator.*

Example 2.2.2 *We consider the following initial boundary value problem for a system of nonlinear partial differential equations of hyperbolic type.*
Given nonnegative function u_0*,* v_0 *defined on* $(\mathbb{R}^+)^2 = \{(x,y); x \geq 0, y \geq 0\}$*, the problem is find nonnegative functions* $u(t,x,y)$ *and* $v(t,x,y)$ *where* $t \geq 0, (x,y) \in (\mathbb{R}^+)^2$ *that satisfy*

$$\begin{cases} u_t + u_x + u^2 - v^2 = 0, \\ v_t + v_y + v^2 - u^2 = 0, \end{cases}$$

$$u(0,x,y) = u_0(x,y), \qquad v_(0,x,y) = v_0(x,y),$$

$$u(t,0,y) = v(t,x,0) = 0$$

We set $D_M(A) = \{[u,v], u, v, u_x, v_y \quad \text{are continuous in} (\mathbb{R}^+)^2, \quad \text{and} u(0,y) = v(x,0) = 0, \quad 0 \leq u(x,y), v(x,y) \leq M\}$ *for* $M > 0$*, and define the operator* A *as follows:*

$$\begin{aligned} D(A) &= (\bigcup_{M>0} D_M(A)) \bigcap \{[u,v] : [u,v] \in L^1((\mathbb{R}^+)^2) \times L^1((\mathbb{R}^+)^2), \\ &\quad [u_x + u^2 - v^2, v_y + v^2 - u^2] \in L^1((\mathbb{R}^+)^2) \times L^1((\mathbb{R}^+)^2)\}, \end{aligned}$$

$$A([u,v]) = [-u_x - u^2 + v^2, -v_y - v^2 + u^2]$$

Then, A *is a operator such that* $D(A) \subset L^1((\mathbb{R}^+)^2) \times L^1((\mathbb{R}^+)^2)$ *and* $R(A) \subset L^1((\mathbb{R}^+)^2) \times L^1((\mathbb{R}^+)^2)$*. We need to show that for* $w_1 = [u_1, v_1]$*,* $w_2 = [u_2, v_2] \in D(A)$ *and* $\lambda > 0$*,*

$$\|(w_1 - \lambda A w_1) - (w_2 - \lambda A w_2)\| \geq \|w_1 - w_2\| \tag{2.10}$$

hold. Here $\|.\|$ *denotes the norm on the space* $L^1((\mathbb{R}^+)^2) \times L^1((\mathbb{R}^+)^2)$*,*

$$\|w\| = \int_0^\infty \int_0^\infty (|u(x,y)| + |v(x,y)|) dx dy.$$

To show (2.10)*, we put* $\widetilde{u} = u_1 - u_2$*,* $\widetilde{v} = v_1 - v_2$*, and consider*

$$\int_0^b \int_0^b |\widetilde{u} + \lambda(\widetilde{u}_x) + u_1^2 - u_2^2 + v_2^2 - v_1^2| dx dy$$

$$\int_0^b \int_0^b |\widetilde{v} + \lambda(\widetilde{v}_y) + v_1^2 - v_2^2 + u_2^2 - u_1^2| dx dy \quad (b > 0).$$

We set $I_1 = \{x : \widetilde{u}(x,y) > 0\} \cap [0,b]$ *and* $I_2 = \{x : \widetilde{u}(x,y) < 0\} \cap [0,b]$, *then*

$$\begin{aligned}\int_0^b |\widetilde{u} + \lambda(\widetilde{u}_x + u_1^2 - u_2^2 + v_2^2 - v_1^2)|dx &\geq \int_{I_1} (\widetilde{u} + \lambda(\widetilde{u}_x + u_1^2 - u_2^2 + v_2^2 - v_1^2))dx \\ &+ \int_{I_2} (-1)(\widetilde{u} + \lambda(\widetilde{u}_x + u_1^2 - u_2^2 + v_2^2 - v_1^2))dx \\ &\geq \int_0^b (|\widetilde{u}| + \lambda(|u_1^2 - u_2^2| - |v_2^2 - v_1^2|))dx.\end{aligned}$$

Here we use the fact that $\int_{I_1} \widetilde{u}_x dx \geq 0$, $\int_{I_2} \widetilde{u}_x dx \leq 0$, *and* $u_1^2 - u_2^2$ *have the same sign as* $\widetilde{u}$. *Integrating with respect to* y *on* $[0,b]$ *we have*

$$\int_0^b \int_0^b |\widetilde{u} + \lambda(\widetilde{u}_x + u_1^2 - u_2^2 + v_2^2 - v_1^2)|dxdy \geq \int_0^b \int_0^b (|\widetilde{u}| + \lambda(|u_1^2 - u_2^2| - |v_2^2 - v_1^2|))dxdy$$

Similarly,

$$\int_0^b \int_0^b |\widetilde{v} + \lambda(\widetilde{v}_y + v_1^2 - v_2^2 + u_2^2 - u_1^2)|dxdy \geq \int_0^b \int_0^b (|\widetilde{v}| + \lambda(|v_1^2 - v_2^2| - |u_2^2 - u_1^2|))dxdy$$

When we add the sides of these two inequalities, respectively, and let $b \to \infty$ *we obtain* (2.10).

Example 2.2.3 *Let* A *be a contraction operator with* $D(A) \subset X$ *and* $R(A) \subset X$. *Then* $A_h = h^{-1}(A - I)$, $h > 0$ *is a dissipative operator, because if* $x, y \in D(A_h)$, *then for every* $f \in J(x-y)$ *we have*

$$\begin{aligned}Re\langle A_h x - A_h y, f\rangle &= h^{-1}[Re\langle Ax - Ay, f\rangle - \|x-y\|^2] \\ &\leq h^{-1}[\|Ax - Ay\|\|f\| - \|x-y\|^2] \\ &\leq h^{-1}[\|x-y\|^2 - \|x-y\|^2] = 0.\end{aligned}$$

Also, let X_0 *be a closed convex subset of* X, *and let* A *be a contraction operator* $X_0 \to X_0$. *Then* A_h *is a dissipative operator, and*

$$R(I - \lambda A_h) \supset X_0 = D(A_h), \quad \textit{for all} \quad \lambda > 0 \tag{2.11}$$

In particular if $X_0 = X$, *then* T_h *is an m-dissipative operator.*
Now, to show () holds,
Let $\lambda > 0$ *and* $x \in X_0$, *and define an operator* A *by*

$$Az = \frac{\lambda}{\lambda + h} Tz + \frac{h}{\lambda + h} x \quad z \in X_0.$$

Then A *is an operator* $X_0 \to X_0$ *such that*

$$\|Az - Ay\| \leq \frac{\lambda}{\lambda + h}\|z - y\|.$$

Since $\frac{\lambda}{\lambda+h} < 1$, A *becomes a strict contraction operator, and from the fixed point theorem, we have* $Az = z$, *that is, there exists* $z \in X_0$ *such that* $(I - \lambda A_h)z = x$. *Then () is proved.*

Definition 2.2.3 *([31]) The $m-$dissipative operator A is called of complete continuous type, if for each sequence $(f_n)_{n\in\mathbb{N}}$ in $L^1([a,b],X)$ and for each sequence $u_n \in C([a,b],X)$, such that u_n is an integral solution of the problem*

$$u_n'(t) \in Au_n(t) + f_n, \ \ a.e. \ \ t \in [a,b], \ \ n \in \mathbb{N},$$

then

$$\lim_{n\to\infty} f_n = f \ \textit{weakly in} \ \ L^1([a,b],X), \textit{and} \lim_{n\to\infty} u_n = u \ \textit{strongly in} \ C([a,b],X),$$

imply that u is an integral solution on $[a,b]$ of the limit problem

$$u'(t) \in Au(t) + f(t), \quad a.e. \ \ t \in [a,b].$$

If A is linear or the topological dual of X is uniformly convex and A generates a compact semigroup, then A is of complete continuous type, see Corollary 2.31, p 49 in Varbie [99]. There are nevertheless examples of $m-$dissipative operators of complete continuous type in Banach spaces, whose duals are not uniformly convex.

Chapter 3

Nonlinear semigroups

In this chapter we are concerned with nonlinear semigroups which have a connection to solutions of nonlinear evolutions with dissipative operators in X.

3.1 Nonlinear semigroups

Definition 3.1.1 *Let X be a real Banach space with norm $\|.\|$. Let X_0 be a closed subset of X. Let $T(t) : X_0 \to X_0$ be a nonlinear operator for every $t \geq 0$.*
The family $\{T(t), t \geq 0\}$ is called a nonlinear semigroup on X_0 of type α if:

1. *$T(0)x = x$ for every $x \in X_0$,*
2. *$T(t+s)x = T(t)T(s)x$ for every $x \in X_0$ and $t, s \geq 0$,*
3. *$\lim_{t\to 0^+} T(t)x = x$ for every $x \in X_0$,*
4. *For every $x, y \in X_0$, $t \geq 0$ and for some $\alpha \in \mathbb{R}$*
$$\|T(t)x - T(t)y\| \leq e^{\alpha t}\|x - y\|.$$

When $\alpha = 0$ $\{S(t), t \geq 0\}$ be a nonlinear semigroup on X_0 of contraction, or simply a nonlinear semigroup on X_0.

Example 3.1.1 *Let $X = (-\infty, +\infty)$ and define an operator $T(t) : X \to X$ for each $t \geq 0$ by*
$$T(t)x = \begin{cases} \max\{0, x-t\}, & x > 0; \\ x, & x \leq 0. \end{cases}$$
Then, $\{T(t), t \geq 0\}$ is a nonlinear semigroup.

Remark 3.1.1 *Let $\{T(t) : t \geq 0\}$ be a semigroup of contraction on X_0 can be extended uniquely to a semigroup of contraction on $\overline{X}_0$. Therefore, when we consider a semigroup of contraction on X_0, we may assume, without loss of generality, that X_0 is closed.*

Lemma 3.1.1 *Let $\{T(t) : t \geq 0\}$ be a semigroup of contraction on X_0 and let $x \in X_0$. If $\liminf_{t\to 0^+} \|T(t)x - x\|/t = L < \infty$, then*

1. $\|T(t+h)x - T(t)x\| \leq hL$*, for each* $t, h \geq 0$;
2. *for each $t \geq 0$, the limit*

$$\lim_{h\to 0^+} \|T(t+h)x - T(t)x\|/h$$

exists and is finite-valued. We denote this limit by $\varphi(t)$, in addition, $\varphi(t) : [0,\infty) \to [0,\infty)$ is a monotone decreasing function.

Proof. (1) For an arbitrary $\varepsilon > 0$, we can select a sequence $\{h_k\}$ of positive numbers that converges to 0 such that

$$\|T(h_k)x - x\| < (L+\varepsilon)h_k, \quad k = 1, 2, ..$$

For every $r_i \geq 0$, $i = 1, 2, .., n$ and $\tau \geq 0$, we have

$$\|T(\tau + \Sigma_{i=1}^n r_i)x - x\| \leq \|T(\tau)x - x\| + \Sigma_{i=1}^n \|T(r_i)x - x\|. \tag{3.1}$$

Also, we note that for $t, h \geq 0$

$$\|T(t+h)x - T(t)x\| \leq \|T(h)x - x\|. \tag{3.2}$$

Let $h > 0$, and $n_k = [h/h_k]$, $\tau = h - n_k h_k$, $r_i = h_k$, $i = 1, 2, ..., n_k$, and apply (). Then,

$$\begin{aligned} \|T(h)x - x\| &\leq \|T(h - n_k h_k)x - x\| + n_k\|T(h_k)x - x\| \\ &\leq \|T(h - n_k h_k)x - x\| + (L+\varepsilon)h. \end{aligned}$$

Here we let $k \to \infty$, then since $0 \leq h - n_k h_k < h_k \to 0$, we have

$$\|T(h)x - x\| \leq (L+\varepsilon)h.$$

Thus $\|T(h)x - x\| \leq Lh$. From this and (), we obtain (1).
(2)From (1) we have

$$\limsup_{h\to 0^+} \|T(h)x - x\|/h \leq L = \liminf_{h\to 0^+} \|T(h)x - x\|/h.$$

Hence, we have obtained that
$\lim_{h\to 0^+} \|T(h)x - x\|/h = L$ exists and finite valued.
Next for every $t > 0$

$$\|T(h)T(t)x - T(t)x\| \leq \|T(h)x - x\|, \quad h > 0,$$

therefore, $\liminf_{h\to 0^+} \|T(h)T(t)x - T(t)x\|/h \le L < \infty$. Hence the statement above in quotes holds even if we replace x by $T(t)x$. That is, for each $t > 0 >$,

$$\lim_{h\to 0^+} \|T(t+h)x - T(t)x\|/h$$

exists and has finite value. Finally, from

$$\|T(t+h)x - T(t)x\| \le \|T(s+h)x - T(s)x\|, \quad s \le t,$$

$\varphi : [0,\infty) \to [0,\infty)$ is monotone decreasing.

□

Definition 3.1.2 *Let $\{T(t), t \ge 0\}$ be a semigroup of contractions on X_0. Set $A_h = [T(h) - I]/h$, $h > 0$.*
Define operators A_0 and A' with domains

$$D(A_0) = \{x \in X_0, \lim_{h\to 0^+} A_h x \quad \textit{exists}\}, \qquad D(A') = \{x \in X_0, w - \lim_{h\to 0^+} A_h x \quad \textit{exists}\}$$

by

$$A_0 x = \lim_{h\to 0^+} A_h x, \qquad A'x = w - \lim_{h\to 0^+} A_h x$$

A_0 and A' are called the infinitesimal generator and the weak infinitesimal generator of $\{T(t), t \ge 0\}$, respectively.

From the definition A' is an extension of A_0.

Lemma 3.1.2 (1) *For each $h > 0$, A_h is a dissipative operator.*
(2) *The weak infinitesimal generator A' is a dissipative operator. Therefore, the infinitesimal generator A_0 is also a dissipative operator.*

Proof. (1) Let $h > 0$ and $x, y \in X_0$. For every $f \in J(x-y)$ we have

$$\begin{aligned} Re\langle A_h x - A_h y, f\rangle &= h^{-1}[Re\langle T(h)x - T(h)y, f\rangle - \|x-y\|^2] && (3.3)\\ &\le h^{-1}(\|x-y\|^2 - \|x-y\|^2) && (3.4)\\ &= 0. && (3.5)\end{aligned}$$

(2) Let $x, y \in D(A')$. From (1), for every $f \in J(x-y)$ we get

$$Re\langle A_h x - A_h y, f\rangle \le 0$$

.
Now let $h \to 0^+$, then we have $Re\langle A'x - A'y, f\rangle \le 0$.

□

Remark 3.1.2 *As we see from the above proof, A_h, A', and A_0 are all strictly dissipative operators.*

Definition 3.1.3 *Let $\{T(t), t \geq 0\}$ be a semigroup of contractions on X_0 and set*

$$\widetilde{D} = \{x \in X_0, \liminf_{h \to 0^+} \|A_h x\| < \infty\}.$$

An operator A is called the (g)-operator of $\{T(t), t \geq 0\}$ if $A_0 \subset A$, $D(A) \subset \widetilde{D}$ and A is a maximal dissipative operator on $\widetilde{D}$.

From Lemma () we have

$$\widetilde{D} = \{x \in X_0, T(t)x : [0, \infty) \to X_0 \qquad \text{is Lipschitz continuous}\}.$$

Also, if $w-\lim_{h \to 0^+} A_h x$ exists, then from the uniform boundedness theorem $\limsup_{h \to 0^+} \|A_h x\| < \infty$. Therefore, we have $D(A_0) \subset D(A') \subset \widetilde{D}$.

Remark 3.1.3 *If $D(A_0) \neq \emptyset$, then the (g)-operator of $\{T(t), t \geq 0\}$ always exists.*

Lemma 3.1.3 *Let $\{T(t), t \geq 0\}$ be a semigroup of contractions on X_0*
(1) *For each $t \geq 0$, $T(t)\widetilde{D} \subset \widetilde{D}$.*
(2) *If X is a reflexive Banach space, then for each $x \in \widetilde{D}$ we have $T(t)x \in D(A_0)$ and*

$$\frac{d}{dt} T(t)x = A_0 T(t)x, \qquad a.e.t, \tag{3.6}$$

and hence $\widetilde{D} \subset \overline{D(A_0)}$.

Proof. (1) Let $t \geq 0$ and $x‘ \in \widetilde{D}$. Since

$$\|A_h T(t)x\| = h^{-1}\|T(t)T(h)x - T(t)x\| \leq \|A_h x\|,$$

we have

$$\liminf_{h \to 0^+} \|A_h T(t)x\| \leq \liminf_{h \to 0^+} \|A_h x\| < \infty.$$

Therefore, $T(t)x \in \widetilde{D}$.
(2) Let $x \in \widetilde{D}$. Since $T(t)x : [0, \infty) \to X_0$ is Lipschitz continuous, $T(t)x$ is absolutely continuous. Thus, since X is reflexive, a theorem of Radon-Nikodym type holds (), that is, $T(t)x$ is strongly differentiable for almost every t $a.e.t.$ And at the point t_0 where $T(t)x$ is strongly differentiable, we have

$$\begin{aligned} [\frac{d}{dt} T(t)x]_{t=t_0} &= \lim_{h \to 0^+} [T(t_0 + h)x - T(t_0)x]/h && (3.7) \\ &= \lim_{h \to 0^+} A_h T(t_0)x = A_0 T(t_0)x. && (3.8) \end{aligned}$$

Thus we obtain (). Next, from $T(t)x \in D(A_0)$ a.e t and from $\lim_{t \to 0^+} T(t)x = x$, we obtain $x \in \overline{D(A_0)}$. Therefore, $\widetilde{D} \subset \overline{D(A_0)}$.

□

Theorem 3.1.1 *[68] Let A be a dissipative operator that satisfies the following condition:*

$$D(A) \subset R(I - \lambda A) \quad \text{for all} \quad \lambda > 0. \tag{3.9}$$

Then, there is a semigroup of contractions $\{T(t), t > 0\}$ on $\overline{D(A)}$ that satisfies:
(1) *For every $x \in R \cap \overline{D(A)}$, where $R = \bigcap_{\lambda > 0} R(I - \lambda A)$,*

$$T(t)x = \lim_{\lambda \to 0^+} (I - \lambda A)^{-[t/\lambda]}, \quad t \geq 0.$$

where convergence is uniform on bounded subintervals of $[0, \infty)$
(2)

$$\|T(t)x - T(s)x\| \leq \||Ax\|||t - s| \quad x \in D(A), \quad t, s \geq 0.$$

3.2 Integral solution of Cauchy problems

Let $A : X \to X$ be an operator, and let b be a positive number. We consider the Cauchy problem for A,

$$\begin{cases} u'(t) \in Au(t), & 0 \leq t \leq b \\ u(0) = x. \end{cases}$$

Definition 3.2.1 *$u(t) : [0, b] \to X$ is called a solution of the Cauchy problem, if*
(1) *$u(0) = x$.*
(2) *$u(t)$ is Lipschitz continuous on $[0, b]$.*
(3) *$u(t)$ is strongly differentiable a.e. $t \in [0, b]$ and satisfies*

$$u'(t) \in Au(t), \quad a.e. \quad t \in [0, b]$$

Theorem 3.2.1 *[68] Let A be a closed dissipative operator satisfying condition () of theorem () and let $\{T(t), t \geq 0\}$ be a nonlinear semigroup of contraction on $\overline{D(A)}$ formulated in theorem () and let $x \in D(A)$.*
(1) *If $T(t)x$ is strongly differentiable a.e. $t \geq 0$, then $T(t)x : [0, \infty) \to X$ is a unique solution of the Cauchy problem.*
(2) *If X is reflexive, then $T(t)x : [0, \infty) \to X$ is a unique solution of the Cauchy problem.*

Let A be a closed dissipative operator satisfying the condition ().
Then, by $T(t)x = \lim_{\lambda \to 0^+} (I - \lambda A)^{-[t/\lambda]} x$, $x \in \overline{D(A)}$, $t > 0$, we can construct a nonlinear semigroup of contraction $\{T(t), t \geq 0\}$ on $\overline{D(A)}$ (Theorem ()). Now, if $T(t)x$, $x \in D(A)$ is strongly differentiable for a.e $t \geq 0$, then $T(t)x : [0, \infty) \to X$ is a unique solution of the Cauchy problem (Theorem). Conversely, the following theorem holds.

Theorem 3.2.2 *Let A be a closed dissipative operator satisfying the condition () and $x \in \overline{D(A)}$. If $u(t) : [0, \infty) \to X$ is a solution of the Cauchy problem, then*

$$u(t) = \lim_{\lambda \to 0^+} (I - \lambda A)^{[-t/\lambda]} x, \quad t \geq 0.$$

Proof. Choose an arbitrary $s > 0$, let $0 < 2\lambda < s$, and set $u_\lambda(t) = (I - \lambda A)^{[-t/\lambda]} x, \quad t \geq 0$ and $\lambda^{-1}[u(t) - u(t-\lambda)] - u'(t) = g_\lambda(t)$ a.e. $t \geq \lambda$. Since $\lim_{\lambda \to 0^+} g_\lambda(t) = 0$ a.e. $t > 0$ and $\|g_\lambda(t)\| \leq 2M$, $\lambda \leq t \leq s$, where M is a Lipschitz constant of $u(t)$ on $[0, s]$, $\lim_{\lambda \to 0^+} \int_\lambda^s \|g_\lambda(t)\| dt = 0$. Next, from $u(t-\lambda) + \lambda g_\lambda(t) = u(t) - \lambda u'(t) \in (I - \lambda A) u(t)$, we have $u(t) = (I - \lambda A)^{-1}[u(t-\lambda) + \lambda g_\lambda(t)]$. Hence,

$$\begin{aligned} \|u_\lambda(t) - u(t)\| &\leq \|(I - \lambda A)^{-[(t-\lambda)/\lambda]} x - u(t-\lambda) - \lambda g_\lambda(t)\| \\ &\leq \|u_\lambda(t-\lambda) - u(t-\lambda)\| + \lambda \|g_\lambda(t)\| \quad a.e. t \geq \lambda. \end{aligned}$$

If we integrate this over $[\lambda, s]$, then

$$\lambda^{-1} \int_{s-\lambda}^{s} \|u_\lambda(t) - u(t)\| dt \leq \lambda^{-1} \int_0^\lambda \|u_\lambda(t) - u(t)\| dt + \int_\lambda^s \|g_\lambda(t)\| dt.$$

As $\lambda \to 0^+$, $u_\lambda(t)$ converges uniformly to $T(t)x$ on $[0, s]$. Since $\|T(t)x - u(t)\|$ is continuous on $[0, \infty)$, if we let $\lambda 0^+$ in the above inequality, then we obtain

$$\|T(s)x - u(s)\| \leq 0.$$

Therefore, $u(s) = T(s)x = \lim_{\lambda \to 0^+} (I - \lambda A)^{-[s/\lambda]} x, \quad s > 0.$

□

However, the semigroup of contractions, formulated on $\overline{D(A)}$ through the method of theorem with closed dissipative operator A satisfying the condition (), does not necessarily always give a solution of the Cauchy problem for A we can see the exemple4.7, in [68].
Therefore, it is interesting to generalize the concept of solutions of the Cauchy problem in such a way that a semigroup of contraction on $\overline{D(A)}$, which can be formulated through the method of theorem with a dissipative operator A satisfying the condition (), results in giving a generalized solution in a sense of the Cauchy problem for A.
Let $A : D(A) \subset X \to X$ be an m−accretive operator, let $f : [0, b] \to X$ be a given function and let us consider the inclusion

$$u'(t) + Au(t) \ni f(t) \qquad a \leq t \leq b. \tag{3.10}$$

Definition 3.2.2 *Let $f \in L([0, b], X)$, A function $[a, b] \to X$ is called a strong solution of () on $[0, b]$ if*
(1): $u(t) \in D(A)$ a.e. for $t \in (0, b)$;
(2) : $u(t) \in W^{1,1}_{loc}((0, b], X)$ and there exists $g \in L^1_{loc}((0, b]; X)$, $g(t) \in Au(t)$ a.e. for $t \in (a, b)$such that

$$u'(t) + g(t) = f(t) \quad a.e.\ for \qquad t \in (0, b).$$

Theorem 3.2.3 *Let X be reflexive and let A be an m-dissipative operator. Then for each $u_0 \in D(A)$ and $f \in W^{1,1}([0,b],X)$ there exists a unique strong solution u of () on $[0,b]$ which satisfies: $u(0) = u_0, \quad u \in W^{1,\infty}([0,b];X)$ and*

$$\|u'(t)\| = |Au(t) + f(t)| \leq |Au_0 + f(0)| + \int_0^t \|f'(s)\|ds$$

a.e. for $t \in (0,b)$, where $|Ax + z| = \inf\{\|y + z\|, \quad y \in Ax\}$, for each $x \in D(A)$

See Barbu [16].
If either $u_0 \in \overline{D(A)}$ and $f \in L([0,b];X)$, or X is non-reflexive, there are examples showing that () may have no strong solution u satisfying $u(a) = u_0$. See, Crandall and Liggett [33].
From this remark it becomes clear that once we are interested in developing a powerful theory concerning equations of the form (), first of all we have to define a new concept of solution.
We next indicate there different ways of defining such a concept and we start with the most natural that comes to mind.

Definition 3.2.3 *Let $f \in L^1([0,b],X)$. A function $u : [0,b] \to X$ is called a generalized solution of () on $[0,b]$ if there exists a sequence (u_n, f_n) whose terms belong to $C([0,b],X) \times L^1([0,b],X)$ such that:*
(1) *for each $n \in \mathbb{N}$, u_n is a strong solution of () on $[0,b]$ with f_n instead of f,*
(2) $\lim_{n\to\infty} u_n = u$ *in $C([0,b],X)$ and* $\lim_{n\to\infty} = f$ *in $L^1([0,b],X)$.*

Theorem 3.2.4 *Let X be reflexive and let A be a dissipative operator. Then for each $u_0 \in \overline{D(A)}$ and each $f \in L^1([0,b],X)$ there exists a unique generalized solution u of () on $[0,b]$ which satisfies $u(a) = u_0$.*

We emphasize that, both theorem () and () can not be extended to general Banach spaces. Therefore we are forced to seek another concept of solution. To this aim, let us observe that, whenever u is a strong solution of () on $[0,b]$ we have

$$-u'(\tau) + f(\tau) \in Au(\tau)$$

a.e. for $\tau \in (0,b)$ Since A is accretive, this implies

$$[u(\tau) - x, -u'(\tau) + f(\tau) - y]_+ \geq 0$$

for each $x \in D(A)$, $y \in Ax$ and a.e. for $\tau \in (0,b)$, from (1) and (2) in lemma () it follows

$$[u(\tau) - x, u'(\tau)]_+ \leq [u(\tau) - x, f(\tau) - y]_+$$

for each $x \in D(A)$, $y \in Ax$ and a.e. for $\tau \in (0, b)$. Clearly $\tau \mapsto \|u(\tau) - x\|$ is absolutely continuous on $(0, b)$, and consequently it is almost everywhere differentiable on $(0, b)$. Then (3) in lemma () applies and accordingly

$$[u(\tau) - x, u'(\tau)]_+ = \frac{d^+}{d\tau}(\|u(\tau) - x\|) = \frac{d}{d\tau}(\|u(\tau) - x\|)$$

for each $x \in D()A$, $y \in Ax$ and a.e. for $\tau \in (0, b)$,Integrating both sides of the inequality above over $[s, t] \subset [0, b]$, we finally obtain

$$\|u(t) - x\| \leq \|u(s) - x\| + \int_s^t [u(\tau) - x, f(\tau) - y]_+ d\tau \tag{3.11}$$

for each $x \in D(A)$, $y \in Ax$ and $0 \leq s \leq t \leq b$.
Now, it is quite obvious that the first requirement that a good candidate for solution of () should satisfy ().

Let $f \in \mathcal{L}^1([0, b], X)$ and let us consider the evolution equation

$$u'(t) \in Au(t) + f(t) \qquad a \leq t \leq b. \tag{3.12}$$

Definition 3.2.4 *A function $y : [0, b] \to X$ is called an integral solution of the following evolution problem*

$$\begin{cases} y'(t) & \in & Ay(t) + F(t, y(t)), \quad a.e. \ t \in [0, b], \\ y(0) & = & x, \end{cases} \tag{3.13}$$

if $y \in C([0, b], X)$, $y(t) \in \overline{D(A)}$ for each $t \in [0, b]$ and y satisfies

$$\|y(t) - x\| \leq \|y(s) - x\| + \int_s^t [y(\tau) - x, f(\tau) - z]_+ d\tau$$

for each $x \in D(A)$, $z \in A(x)$, $0 \leq s \leq t \leq b$ and

$$f \in S_{F,y} = \{f \in L([0, b], \overline{D(A)}) : \ f(t) \in F(t, y(t)) \ a.e. \ t \in [0, b]\}.$$

Let $f \in L([0, b], X)$ and we consider the evolution equation

$$y'(t) \in Ay(t) + f(t), \ t \in [0, b]. \tag{3.14}$$

Theorem 3.2.5 *([16], Theorem 2.1, p 124) Let $A : D(A) \subseteq X \to X$ be an m-dissipative operator. Then for each $x \in \overline{D(A)}$ there exists a unique integral solution of () on $[0, b]$ which satisfies $y(0) = x$. If, $f, g \in L^1([0, b], X)$ and y_1, y_2 are two solution of () corresponding to f and g respectively, then:*

$$\|y_1(t) - y_2(t)\| \leq \|y_1(s) - y_2(s)\| + \int_s^t \|f(\tau) - g(\tau)\| d\tau, \ \textit{for each} \ 0 \leq s \leq t \leq b. \tag{3.15}$$

For more details on nonlinear semigroup theory and m-dissipative operators, we refer the reader to the books of Barbu [16, 17], Bénilan [22], Ha [47] and Lakshmikantham and Leela [66].

Chapter 4

Existence results for impulsive nonlinear evolution inclusions

In this chapter we consider the following class of impulsive evolution inclusions

$$\left\{ \begin{array}{rcll} y'(t) & \in & Ay(t) + F(t, y(t)), & \text{a.e. } t \in [0, b], \\ \triangle y|_{t=t_k} & = & I_k(y(t_k)), \quad k = 1, \dots, n, & \\ y(0) & = & x, & \end{array} \right. \tag{4.1}$$

Where $0 = t_0 < t_1 < ... < t_n < t_{n+1} = b$, $F : [0, b] \times E \to \mathcal{P}(E)$ is a multivalued, and $x \in \overline{D(A)}$, The operator A is the infinitesimal generator of a nonlinear semigroup, $I_k \in C(E, E)$ $(k = 1, ..., n)$ and $\triangle y|_{t=t_k} = y(t_k^+) - y(t_k^-)$. The notations $y(t_k^+) = \lim_{h \to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h \to 0^+} y(t_k - h)$ stand for the right and the left limits of the function y at $t = t_k$, respectively.
Let $J_k = (t_k, t_{k+1}], k = 0, ..., n$ and let y_k be the restriction of a function y to J_k. In order to define integral solutions for problem (), we consider the space of piece-wise continuous functions

$$\begin{array}{l} PC = \{y\colon\ [0, b] \to E,\ y_k \in C(J_k, E),\ k = 0, \dots, m, \ \text{ such that} \\ y(t_k^-) \ \text{ and } \ y(t_k^+) \text{ exist and satisfy } \ y(t_k) = y(t_k^-) \ \text{ for } \quad k = 1, \dots, n\}. \end{array}$$

Endowed with the norme

$$\|y\|_{PC} = \max\{\|y_k\|_\infty, \quad k = 0, ..., n\}$$

PC is a Banach space.
where $A : D(A) \subseteq E \to E$ is m-dissipative and such that A generates a compact nonlinear semigroup.

This chapter give to prove the existence results for nonlinear evolution inclusions with impulse effects in the case where the right-hand side multi-valued nonlinearity is either convex or non-convex. . In the first, we assume that the multifunction is almost strongly weakly u.s.c. with convex values, and in the second, we present the non-convex case, where the right-hand said is l.s.c. with decomposable values. In the least subsection we analyze one example referring to the some nonlinear impulsive differential inclusion in bounded domain.

4.1 Sobolev spaces

To fix the idea, let us first recall some notation. If Ω is a nonempty and open subset in $\mathbb{R}^n$ with boundary Γ, we denote by C_0^∞ the space of C^∞-real functions with compact support in Ω. Further, if $1 \leq p < \infty$ and $m \in \mathbb{N}$, $W^{m,p}(\Omega)$ denotes the space of all functions $u : \Omega \to \mathbb{R}$ which, together with their partial derivatives up to the order m, in the sense of distributions over Ω, belong to $L^p(\Omega)$ i.e.

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega); \forall \alpha \in \mathbb{N}^n, \quad |\alpha| \leq m \quad \Rightarrow D^\alpha u \in L^p(\Omega)\}.$$

Endowed with the norm

$$\|u\|_{W^{m,p}(\Omega)} = (\sum_{0 \leq |k| \leq m} \|D^k u\|_L^p),$$

$W^{m,p}(\Omega)$ is a separable real Banach space, densely and continuously imbedded in $L^p(\Omega)$. Here, as usual, if $k = (k_1, k_2, ..., k_n)$ is a multi-index, we denote by

$$D^k u = \frac{\partial^{k_1 + k_2 + ... + k_n} u}{\partial x_1^{k_1} \partial x_2^{k_2} ... \partial x_n^{k_n}},$$

where the partial derivatives are sense of distributions over Ω. We denote by $W_0^{m,p}$ the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$, by $H^1(\Omega) = W^{1,2}(\Omega)$, $H_0^1(\Omega) = W_0^{1,2}(\Omega)$, $H^{-1}(\Omega) = [H_0^1(\Omega)]^*$ and $H^m(\Omega) = W^{m,2}(\Omega)$.
equips $H^m(\Omega)$ with the following equivalent norm:

$$\|u\|_{H^m} = \left(\sum_{|k| \leq m} \|D^k u\|_{L^2}^2 \right)^{\frac{1}{2}}.$$

Then $H^m(\Omega)$ is a Hilbert space with the scalar product

$$\langle u, v \rangle_{H^m} = \sum_{|k| \leq m} \int_\Omega D^k u D^k v dx.$$

If Ω is bounded, there exists a constant C such that

$$\|u\|_{L^2} \leq \|\nabla u\|_{L^2},$$

for all $u \in H_0^1(\Omega)$ (this is Poincare's inequality). It may be more convenient to equip $H_0^1(\Omega)$ with the following scalar product

$$\langle u, v \rangle = \int_\Omega \nabla u . \nabla v dx.$$

Theorem 4.1.1 *(The Lax-Milgram Theorem) Let H be a Hilbert space and let $a : H \times H \to \mathbb{R}$ be a bilinear functional. Assume that there exist two constants $C < \infty$, $\alpha > 0$ such that:*
(i) $|a(u,v)| \leq C\|u\|\|v\|$ for all $(u,v) \in H \times H$ (continuity).
(ii) $a(u,u) \geq \alpha\|u\|^2$ for all $u \in H$ (coerciveness).
Then, for every $f \in H^*$ (the dual space of H), there exists a unique $u \in H$ such that $a(u,v) = \langle f, v\rangle$ for all $v \in H$.

Finally, we make the conventional notation $W^{0,p}(\Omega) = L^p(\Omega)$.

4.2 Examples of m-dissipative operators

Example 4.2.1 ***(The Laplacian in an open subset of $\mathbb{R}^n$)***
We define the linear operator A in $L^2(\Omega)$ by

$$\begin{cases} D(A) = \{u \in H_0^1(\Omega); \Delta u \in L^2(\Omega)\}, \\ Au = \Delta u, \quad \forall u \in D(A). \end{cases}$$

Proposition 4.2.1 *A is m-dissipative.*

We need the following lemma.

Lemma 4.2.1 *We have*

$$\int_\Omega v\Delta u dx = -\int_\Omega \nabla u \nabla v dx \tag{4.2}$$

for all $u \in D(A)$ and $v \in H_0^1(\Omega)$.

Proof. . (4.2) *is satisfied by $v \in \mathcal{D}(\Omega)$. The lemma follows by density, since both terms of* (4.2) *are continuous in v on $H_0^1(\Omega)$.*

□

Proof of proposition. *First, $\mathcal{D}(\Omega) \subset D(A)$, and so $D(A)$ is dense in L^2. let $u \in D(A)$, applying* (4.2) *with $v = u$, we obtain $\langle Au, u\rangle \leq 0$, so that A is dissipative. Let the bilinear mapping*

$$a(u,v) = \int (uv + \nabla u \nabla v) dx.$$

For all $(u,v) \in H_0^1 \times H_0^1$, we have

$$\begin{aligned} |a(u,v)| &\leq \int |uv| dx + \int |\nabla u.\nabla v| dx \\ &\leq \|u\|_{L^2}.\|v\|_{L^2} + \|\nabla u\|_{L^2}.\|\nabla v\|_{L^2} \end{aligned}$$

$$\begin{aligned} &\leq 2(\|u\|_{L^2} + \|\nabla u\|_{L^2})(\|v\|_{L^2} + \|\nabla v\|_{L^2}) \\ &= 2\|u\|_{H_0^1}\|v\|_{H_0^1}. \end{aligned}$$

and

$$\begin{aligned} a(u,u) &= \int_\Omega (u^2(x) + (\nabla u(x))^2)dx \\ &= \|u\|^2_{H_0^1(\Omega)} \\ &\geq \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} \end{aligned}$$

the bilinear mapping is satisfied the condition of the Lax-Milgram theorem then, for all $f \in L^2(\Omega)$, there exists $u \in H_0^1(\Omega)$ such that

$$\int_\Omega (u(x)v(x) + \nabla u(x).\nabla v(x))dx = \int_\Omega f(x)v(x)dx, \quad \forall v \in H_0^1.$$

We obtain

$$u - \Delta u = f,$$

since, in addition $u \in H_0^1(\Omega)$, we obtain $u \in D(A)$ and $u - Bu = f$. Therefore A is m-dissipative.

Example 4.2.2 *:* ***The wave operator (or the Klein-Gordon operator) in*** $H_0^1(\Omega) \times L^2(\Omega)$
Let $X = H_0^1(\Omega) \times L^2(\Omega)$ can be equipped with the scalar product

$$\langle (u,v),(w,z)\rangle = \int_\Omega (\nabla u(x).\nabla v(x) + mu(x)w(x) + v(x)z(x\Omega))dx.$$

We define the linear operator B in X by

$$\begin{cases} D(B) = \{(u,v) \in X, \Delta u \in L^2(\Omega), v \in H_0^1(\Omega)\}, \\ B(u,v) = (v, \Delta u - mu),, \ \forall (u,v) \in D(B). \end{cases}$$

Proposition 4.2.2 *B is m-dissipative operator.*

Proof. *. $\mathcal{D}(\Omega) \times \mathcal{D}(\Omega) \subset D(B)$ and so $D(B)$ is dense in X. On the other hand, for all $((u,v),(w,z)) \in D(B)^2$ and by (4.2)*

$$\begin{aligned} \langle B(u,v),(w,z)\rangle &= \int (\nabla v.\nabla w + mvw + (\Delta u - mu)z)dx \\ &= -\int (\nabla u.\nabla z + muz + (\Delta w - mw)v)dx \\ &= -\langle (u,v), B(w,z)\rangle. \end{aligned}$$

For $(w,z)=(u,v)$ we obtain $\langle B(u,v),(u,v)\rangle = 0$. Hence B is dissipative (Remark ()). Now let $(f,g)\in X$. The equation $(u,v)-B(u,v)=(f,g)$ is equivalent to the following system,

$$\begin{cases} 2u-\Delta u = f+g, \\ v = u-f. \end{cases}$$

By the proposition (), there exists a solution $u\in H_0^1(\Omega)$ of $2u-\Delta u = f+g$, satisfying $\Delta u\in L^2(\Omega)$. Next, we solve $v=u-f$ and we obtain $v\in H_0^1$. Therefore $(u,v)\in D(B)$ and $(u,v)-B(u,v)=(f,g)$, so that B is m-dissipative.

□

Example 4.2.3 *As before, the operator Δ is, in this example, the Laplace operator in the sense of distributions over Ω. If $\phi : D(\phi)\subseteq \mathbb{R}\to \mathcal{P}(\mathbb{R})$, and $u:\Omega\to D(\phi)$, we denote by*

$$S_\phi(u)=\{v\in L^1(\Omega) : v(x)\in\phi(u(x)), \quad a.e. \quad x\in\Omega\}.$$

We say that $\phi : D(\phi)\subseteq\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is maximal monotone if $-\phi$ is $m-$dissipative.

Theorem 4.2.1 *Let Ω be a nonempty, bounded and open subset in $\mathbb{R}^n$ with C^1 boundary Γ and let $\phi : D(\phi)\subseteq\mathbb{R}\to\mathcal{P}(\mathbb{R})$, be maximal monotone with $0\in\phi(0)$. Then the operator $\Delta\phi : D(\Delta\varphi)\subseteq L^1(\Omega)\to\mathcal{P}(L^1(\Omega))$, defined by*

$$D(\Delta\phi)=\{u\in L^1(\Omega) : \exists v\in S_\phi(u)\cap W_0^{1,1}(\Omega), \Delta v\in L^1(\Omega)\}$$

and

$$\Delta\phi(u)=\{\Delta v, v\in S_\phi(u)\cap W_0^{1,1}(\Omega) \quad for \quad u\in D(\Delta\phi)\},$$

is $m-$dissipative on $L^1(\Omega)$.
If, in addition, $\phi:\mathbb{R}\to\mathbb{R}$ is continuous on $\mathbb{R}$ and C^1 on $\mathbb{R}\backslash\{0\}$ and there exist two constant $C>0$ and $a>0$ if $n\leq 2$ and $a>\frac{(n-2)}{n}$ if $n\geq 3$ such that

$$\phi'(r)\geq C|r|^{a-1}$$

for each $r\in\mathbb{R}\backslash\{0\}$, then $\Delta\phi$ generates a compact semigroup.

See Cârjă *et al* [31]

4.3 Existence results

In this section we prove the existence solution for the impulsive evolution inclusion problem (). All the results of this section where considered by Sinacer with Henderson, Ntouyas and Ouahab in [53].

4.3.1 The convex case

Let K be a given subset in $L^1([a,b],X)$, we set $M(K)$ the set of all mild solutions of the problem

$$\begin{cases} u'(t) \in Au(t) + f(t), & a \le t \le b \\ u(a) = x_0. \end{cases}$$

when f ranges over K.

Theorem 4.3.1 *[99] If $A : D(A) \subset X \in \mathcal{P}(X)$ is m-dissipative operator, A generates a semigroup, x_0 is a fixed element in $\overline{D(A)}$ and K is a uniformly integrable subset in $L^1([a,b],X)$, then the set $M(K)$ is relatively compact in $C([a,b],X)$.*

Theorem 4.3.2 *[38] Let $\mathcal{A}$ be a bounded set in PC. Assume that*

($\mathcal{R}_1$) *$\mathcal{A}$ is equicontinuous on $[0,b]$; (i.e $\mathcal{A}$ is equicontinuous on $C(J_k,E)$, $k = 1,\dots,n$)*

($\mathcal{R}_2$) *there exists a dense subset D in $[a,b]$ such that, for each $t \in D$, $\mathcal{A}(t) = \{f(t) : f \in \mathcal{A}, t \in [a,b]\}$ is relatively compact in E. Then $\mathcal{A}$ is relatively compact in PC.*

Definition 4.3.1 *A function $y : \overline{[0,b]} \to E$ is called an integral solution of () on $[0,b]$ if $y \in PC([0,b],E)$, $y(t) \in \overline{D(A)}$ for each $t \in [0,b]$ and y satisfies*

$$\|y(t) - x\| \le \|y(s) - x\| + \int_s^t [y(\tau) - x, f(\tau) - z]_+ d\tau,$$

for each $x \in D(A)$, $z \in A(x), 0 \le s \le t \le t_1$, and

$$\|y(t)-y(t_k)-I_k(y(t_k))\| \le \|y(s)-y(t_k)-I_k(y(t_k))\| + \int_s^t [y(\tau)-y(t_k)-I_k(y(t_k)), f(\tau)-z]_+ d\tau$$

$k = 1,\dots,n$, for each $z \in A(y(t_k) + I_k(y(t_k))), t_k \le s \le t_{k+1}$, where

$$f \in S_{F,y} = \{f \in L^1([0,b],E) : \ f(t) \in F(t,y(t)) \ a.e.\ t \in [0,b]\}.$$

Let $f \in L^1([0,b],E)$ and we consider

$$\begin{cases} y'(t) \in Ay(t) + f(t), & \text{a.e. } t \in [0,b], \\ y(t_k^+) - y(t_k) = I_k(y(t_k)), & k = 1,\dots,n. \end{cases} \tag{4.3}$$

We will consider the following two assumptions

($\mathcal{H}_1$) The operator $A\colon D(A) \subseteq E \to E$ is such that
- (a) A is $m-$dissipative, $0 \in A0$ and $\overline{D(A)}$ is convex;
- (b) the semigroup generated by A on $\overline{D(A)}$ is compact;
- (c) A is of complete continuous type.

($\mathcal{H}_2$) $I_k : \overline{D(A)} \to \overline{D(A)}$ are continuous maps, $0 = I_k(0)$, $k = 1,\dots,n$;
- (d) I_k sends bounded sets into bounded sets in $\overline{D(A)}$, $k = 1,\dots,n$;
- (f) $z + I_k(z) \in \overline{D(A)}$ for every $z \in \overline{D(A)}$, $k = 1,\dots,n$.

Lemma 4.3.1 *Assume that the conditions* $(\mathcal{H}_1) - (\mathcal{H}_2)$ *hold. Then for every* $x \in \overline{D(A)}$, *there exists a unique integral solution of () on* $[0, b]$ *which satisfies* $y(0) = x$. *If,* $f, g \in L^1([0, b], E)$ *and* y_1, y_2 *are two solution of () corresponding to* f *and* g *respectively, then:*

$$\|y_1(t) - y_2(t)\| \leq \|y_1(s^+) - y_2(s^+)\| + \int_s^t \|f(\tau) - g(\tau)\| d\tau, \tag{4.4}$$

for each $0 \leq s \leq t \leq b$.

Proof. .
The proof involves several steps.
Step 1: Consider the problem

$$\begin{cases} y'(t) & \in & Ay(t) + f(t), \quad \text{a.e. } t \in [0, t_1], \\ y(0) & = & x \in \overline{D(A)}. \end{cases} \tag{4.5}$$

From Theorem , there exists a unique integral solution denoted by y_0.
Step 2: Consider the problem

$$\begin{cases} y'(t) \in Ay(t) + f(t) \quad t \in (t_1, t_2], \\ y(t_1^+) = y_0(t_1^-) + I_1(y_0(t_1^-)). \end{cases} \tag{4.6}$$

As in Step 1, we can prove that the problem () has unique integral solution denoted by y_1.
Step 3: We continue this process taking into account that $y_n := y\Big|_{(t_n, b]}$ is a solution of the problem

$$\begin{cases} y'(t) & \in & Ay(t) + f(t) \quad \text{a.e. } t \in (t_n, b], \\ y(t_n^+) & = & y_{n-1}(t_n^-) + I_n(y_{n-1}(t_n^-)). \end{cases} \tag{4.7}$$

Now we show that the function y defined by

$$y(t) = \begin{cases} y_0(t), & \text{if } t \in [0, t_1], \\ y_1(t), & \text{if } t \in (t_1, t_2], \\ \dots & \dots \\ y_n(t), & \text{if } t \in (t_n, t_{n+1}], \end{cases}$$

is an integral solution of problem (). Let $t \in [0, b]$. Then $t \in [0, t_1]$ or $t \in (t_k, t_{k+1}], k = 1, \dots, n$.
(1) If $t \in [0, t_1]$, we have

$$\|y(t) - x\| = \|y_0(t) - x\| \leq \|y_0(s) - x\| + \int_s^t [y_0(\tau) - x, f(\tau) - z]_+ d\tau,$$

then, we get

$$\|y(t) - x\| \leq \|y(s) - x\| + \int_s^t [y(\tau) - x, f(\tau) - z]_+ d\tau$$

for each $x \in D(A),\ z \in A(x),\ \ 0 \leq s \leq t \leq t_1$.
(2) If $t \in (t_1, t_2]$, thus $y(t) = y_1(t)$, and for $t_1 \leq s \leq t \leq t_2$, we obtain

$$\begin{aligned} \|y_1(t) - y_0(t_1) - I_1(y_0(t_1))\| &\leq \|y_1(s) - y_0(t_1) - I_1(y_0(t_1))\| \\ &\quad + \int_s^t [y_1(\tau) - y_0(t_1) - I_1(y_0(t_1)), f(\tau) - z]_+ d\tau \end{aligned}$$

Thus

$$\begin{aligned} \|y(t) - y(t_1) - I_1(y(t_1))\| &\leq \|y(s) - y(t_1) - I_1(y(t_1))\| \\ &\quad + \int_s^t [y(\tau) - y(t_1) - I_1(y(t_1)), f(\tau) - z]_+ d\tau, \end{aligned}$$

for each $z \in A(y_0(t_1) + I_1(y_0(t_1))),\ t_1 \leq s \leq t \leq t_2$.
(3) We continue this process to obtain that, for each $t_k \leq s \leq t \leq t_{k+1}$ and $z \in A(y_k(t_{k-1}) + I_k(y_k(t_{k-1})))$, we have

$$\begin{aligned} \|y(t) - y(t_k) - I_k(y(t_k))\| &\leq \|y(s) - y(t_{k-1}) - I_k(y(t_{k-1}))\| \\ &\quad + \int_s^t [y(\tau) - y(t_{k-1}) - I_k(y(t_{k-1})), f(\tau) - z]_+ d\tau. \end{aligned}$$

(4) Let y and $\bar{y}$ be two integral solutions of the problem (). If $t \in [0, t_1]$, then

$$y(t) = \bar{y}(t) \Rightarrow y(t_1) = \bar{y}(t_1) \Rightarrow y(t_1) + I_1(y(t_1)) = \bar{y}(t_1) + I_1(\bar{y}(t_1)).$$

Hence for $t \in (t_k, t_{k+1}],\ k = 1, \dots, n$, $y(t) = \bar{y}(t)$. If $t = t_k^+,\ k = 1, \dots, n$, we have

$$y(t_k^+) - \bar{y}(t_k^+) = I_k(y(t_k)) - I_k(\bar{y}_k(t_k)) = 0.$$

This implies that

$$y(_k^+) = \bar{y}(t_k^+), \quad k = 1, \dots, n.$$

Hence there is a unique integral solution of the problem ().
(5) Let $f, g \in L^1([0, b], E)$ and y, y_* are two solution of () corresponding to f and g, respectively. Then from Theorem

$$\|y(t) - y_*(t)\| \leq \|y(s) - y_*(s)\| + \int_s^t \|f(\tau) - g(\tau)\| d\tau \quad \text{for } 0 \leq s \leq t \leq t_1.$$

For $t \in (t_k, t_{k+1}] \Rightarrow y'(t) - y'_*(t) \in Ay(t) - Ay_*(t) + f(t) - g(t)$ and $y(t_k^+) = y_*(t_k^+)$. From the definition of integral solution and Theorem , we get for $t \in (t_k, t_{k+1}]$ and $s + h \leq t,\ h > 0$

$$\|y(t) - y_*(t)\| \leq \|y(s + h) - y_*(s + h)\| + \int_{s+h}^t \|f(\tau) - g(\tau)\| d\tau.$$

Hence for $h \to 0^+$, we obtain that

$$\|y(t) - y_*(t)\| \leq \|y(s^+) - y_*(s^+)\| + \int_s^t \|f(\tau) - g(\tau)\| d\tau.$$

$\square$

Lemma 4.3.2 *Let $f \in L^1([0,b], E)$, assume that $(\mathcal{H}_1) - (\mathcal{H}_2)$ are satisfied and let the operator $T : \overline{D(A)} \to PC$ defined by*

$$T : x \to T(x) = y^x = \begin{cases} y_0^x, & on\ t \in [0, t_1], \\ y_1^x, & on\ t \in (t_1, t_2], \\ \cdots & \cdots \\ y_n^x, & on\ t \in (t_n, b]. \end{cases}$$

where $y_0^x, y_1^x, \ldots, y_n^x$ are the unique integral solutions of the following evolution problems, respectively,

$$\begin{cases} y'(t) & \in & Ay(t) + f(t), \quad a.e.\ \ t \in [0, t_1], \\ y(0) & = & x \in \overline{D(A)}, \end{cases} \tag{4.8}$$

$$\begin{cases} y'(t) \in Ay(t) + f(t) \quad t \in (t_1, t_2], \\ y(t_1^+) = y_0^x(t_1^-) + I_1(y_0^x(t_1^-)). \end{cases} \tag{4.9}$$

and

$$\begin{cases} y'(t) & \in & Ay(t) + f(t) \quad t \in (t_n, b], \\ y(t_n^+) & = & y_{n-1}^x(t_n^-) + I_n(y_{n-1}^x(t_n^-)). \end{cases} \tag{4.10}$$

If B is a compact set in $\overline{D(A)}$, then $T(B)$ is relatively compact in PC.

- *For fixed $\phi \in L^1([0,b], \mathbb{R}_+)$ and $\mathcal{K}_* = \{f \in L^1([0,b], E) : \|f(t)\| \leq \phi(t)\ a.e. t \in [0,b]\}$, then $\widetilde{T}(B \times \mathcal{K}_*)$ is relatively compact in PC, where*

$$\widetilde{T} : \overline{D(A)} \times \mathcal{K}_* \to PC : (x, f) \to \widetilde{T}(x, f) = y^x(f)$$

and $y^x(f)$ is the unique integral solution of () corresponding to $y(0) = x$ and $F = f$.

Proof. Let B be a compact set in $\overline{D(A)}$.
Part 1. We show that $T(B)$ is relatively compact in PC.
Step 1. From $(\mathcal{H}_1)(b)$, we have for each $\lambda > 0$, $\overline{S(\lambda)}$ is compact. From [15, 99], $\lim\limits_{\lambda \to 0} S(\lambda) y_0^x = y^x = T_0(x)$, where $T_0 : \overline{D(A)} \to C([0, t_1], \overline{D(A)}) : T_0 : x \to T_0(x) = y_0^x$, and y_0^x is the unique integral solution of (). Hence $\overline{T_0(B)}$ is relatively compact in $C([0, t_1], \overline{D(A)})$.
Step 2. Consider

$$T_1 : \overline{D(A)} \to C_1([t_1, t_2], \overline{D(A)}), \quad T_1 : x \to T_1(x) = y_1^x,$$

where
$$C_1([t_1,t_2],\overline{D(A)}) = \{y \in C((t_1,t_2],\overline{D(A)}): \ y(t_1^+) \text{ exists}\}$$
and y_1^x is the unique integral solution of (). Using the fact that $0 \in A(I_1(0))$ and there exists $M_1 > 0$ such that $\|x\| \le M_1$, then from Lemma we have
$$\|y^x(t_1)\| \le M_1 + \int_0^{t_1} \|f(s)\|ds := \bar{M}_1$$
and
$$\sup\{\|I_1(a)\| : \|a\| \le \bar{M}_1\} := \widetilde{M_1} < \infty.$$
Then $T_1(B)$ is bounded independent of x and y_1. As in Step 1, we can prove that T_1 is compact.

Step 3. Continuing this process and taking the operator
$$T_n : \overline{D(A)} \to C_m([t_n,b],\overline{D(A)}), \quad T_n : x \to T_n(x) = y_n^x,$$
where
$$C_1([t_n,b],\overline{D(A)}) = \{y \in C((t_n,b],\overline{D(A)}): \ y(t_n^+) \text{ exists}\}$$
and y_n^x is the unique integral solution of ().

• Consequently, from Steps 1 to 3, we can easily prove that $\overline{T(B)}$ is relatively compact in PC, (see Step 2 in the proof of Theorem 4.3 in [38]).

Part 4. Now, prove that $\widetilde{T}(B \times \mathcal{K}_*)$ is compact in PC. From the definition of integral solution of (), we have $\widetilde{T} : \overline{D(A)} \times \mathcal{K}_* \to PC$ defined by
$$\widetilde{T} : (x,f) \to T(x) = y^x = \begin{cases} y_0^x(f), & \text{on } t \in [0,t_1], \\ y_1^x(f), & \text{on } t \in (t_1,t_2], \\ \dots & \dots \\ y_n^x(f), & \text{on } t \in (t_n,b]. \end{cases}$$
As in , combined with *Part 1* and the Step 2 of Theorem 4.3 in [38], we can easily conclude that $\overline{\widetilde{T}(B \times \mathcal{K}_*)}$ is compact in PC.

Theorem 4.3.3 *Let $F : [0,b] \times \overline{D(A)} \to \mathcal{P}_{wcpcv}(E)$ be a multivalued map almost strongly-weakly upper semicontinuous. Assume that the conditions $(\mathcal{H}_1) - (\mathcal{H}_2)$ and*

$(\mathcal{H}_3)$ *there exist a continuous nondecreasing function $\psi : [0,\infty) \longrightarrow (0,\infty)$ and $p \in L^1(J, \mathbb{R}^+)$ such that*
$$\|F(t,x)\|_{\mathcal{P}} \le p(t)\psi(\|x\|), \ \textit{for a.e. } t \in J \textit{ and each } x \in E,$$
with
$$\int_0^b p(s)ds < \int_{\|x\|}^{\infty} \frac{du}{\psi(u)}.$$
hold, then problem () has at least one integral solution.

Lemma 4.3.3 *If* $(\mathcal{H}_1)-(\mathcal{H}_3)$ *are satisfied, then there exists* $M>0$ *such that*

$$\|y\|_{PC}\leq M \quad \textit{for each} \quad y \textit{ integral solution of } ().$$

Proof. Let $y\in PC$ be an integral solution of (). We consider

$$\begin{cases} u'(t) & \in & Au(t) \quad \text{a.e.} \quad t\in[0,b],\\ u(t_k^+)-u(t_k) & = & I_k(u(t_k^-)).\\ u(0)=0. \end{cases} \tag{4.11}$$

It is clear that the function $u_*=0$ is an integral solution of (). From Lemma , we have

- for $t\in[0,t_1]$

$$\|y(t)\|\leq\|x\|+\int_0^t\|f(\tau)\|d\tau, \quad f\in S_{F,y}.$$

Then

$$\sup\{\|y(t)\|:t\in[0,t_1]\}\leq\Gamma_0^{-1}\left(\int_0^{t_1}p(s)ds\right):=M_0,$$

where

$$\Gamma_0(z)=\int_{\|x\|}^{z}\frac{du}{\psi(u)}.$$

- If $t\in(t_1,t_2]$ we have

$$\begin{aligned} \|y(t)\| &\leq \|y(t_1)-I_1(y(t_1)\|+\|I_k(y(t_1)\|+\int_{t_1}^t\|f(\tau)\|d\tau\\ &\leq M_0+2\|I_1(y(t_1))\|+\int_{t_1}^t p(s)\psi(\|y(s)\|)ds\\ &\leq M_0+2K_1+\int_{t_1}^t p(s)\psi(\|y(s)\|)ds, \end{aligned}$$

where

$$K_1=\sup\{\|I_1(z)\|: \ \|z\|\leq M_0\}.$$

Thus

$$\sup\{\|y(t)\|:t\in[t_1,t_2]\}\leq\Gamma_1^{-1}\left(\int_{t_1}^{t_2}p(s)ds\right):=M_1.$$

where

$$\Gamma_1(z)=\int_{M_0+2K_1}^{z}\frac{du}{\psi(u)}.$$

- Continuing this, we have for $t\in(t_n,b]$

$$\|y(t)\| \leq \|y(t_n)-y(t_1)-\sum_{k=1}^{n-1}I_k(y(t_k))\|+\int_{t_n}^t\|f(\tau)\|d\tau$$

$$\leq \quad M_0 + M_{n-1} + \sum_{i=1}^{n-1} K_i + \int_{t_n}^{t} p(s)\psi(\|y(s)\|)ds.$$

We have that there exists a constant M_n such that

$$\sup\{\|y(t)\| : t \in [t_n, b]\} \leq M_n.$$

Consequently, for each possible integral solution of (), we have

$$\|y\|_{PC} \leq \max(M_i : i = 0, \dots, n) = M.$$

□

Set $\mathcal{K} = \{f \in L^1([0,b], E) : \|f(t)\| \leq p(t)\psi(M) \text{ a.e. } t \in [0,b]\}$. For $\epsilon \in (0,1)$ we consider

$$\begin{cases} y'(t) - Ay(t) & \ni \quad f(t) \quad t \in [0,b], \\ y(t_k^+) - y(t_k) & = \quad (1-\epsilon)I_k(y(t_k^-)). \\ y(0) = (1-\epsilon)x, & \end{cases} \tag{4.12}$$

where p, ψ are defined in $(\mathcal{H}_3)$ and M is the *a priori bound* for all integral solutions of ().

Lemma 4.3.4 *If $(\mathcal{H}_1)-(\mathcal{H}_3)$ hold, then for every fixed $\epsilon \in (0,1)$, the problem () has a unique integral solution $y_\epsilon(f)$. Also the operator $\widetilde{S}_\epsilon : \mathcal{K} \to PC,\ f \to \widetilde{S}_\epsilon(f) = y_\epsilon(f)$ is continuous.*

Proof. From Lemma , we have for every $\epsilon \in (0,1)$ there exists a unique integral solution $y_\epsilon(f)$of (). Now we show that the operator $\widetilde{S}_\epsilon$, is continuous. Indeed, let $f_n \to f$, converge in $L^1([0,b], E)$, as $n \to \infty$, and by Lemma , $(z_n = y_\epsilon(f_n))_{n\in\mathbb{N}}$ is relatively compact in PC.

Then there exists a subsequence of $(z_n)_{n\in\mathbb{N}}$ converging to z in PC. Hence

$$\|z_n(t) - x\| \leq \|z_n(s) - x\| + \int_s^t [z_n(\tau) - x, f_n(\tau) - z_*]_+ d\tau$$

for each $x \in D(A),\ z_* \in A(x), 0 \leq s \leq t \leq t_1$. Since $(z_n, f_n) \to (z, f),\ n \to \infty$ and $[\cdot,\cdot]_+$ is u.s.c., then by Fatou's lemma we have

$$\|z(t) - x\| \leq \|z(s) - x\| + \int_s^t [z(\tau) - x, f(\tau) - z_*]_+ d\tau,$$

for each $x \in D(A),\ z_* \in A(x), 0 \leq s \leq t \leq t_1$.
• $t_1 \leq s \leq t \leq t_2$, we have

$$\begin{aligned} \|z_n(t) - z_n(t_1) - I_1(z_n(t_1))\| \quad &\leq \quad \|z_n(s) - z_n(t_1) - I_1(z_n(t_1))\| \\ &\quad + \int_s^t [z_n(\tau) - z_n(t_1) - I_1(z_n(t_1)), f(\tau) - z_*]_+ d\tau. \end{aligned}$$

Using the fact that I_1 is continuous, $[\cdot,\cdot]_+$ is u.s.c. and by Fatou's lemma, we obtain

$$\begin{aligned}\|z(t)-z(t_1)-I_1(z(t_1))\| &\leq \|z(s)-z(t_1)-I_1(z(t_1))\| \\ &\quad + \int_s^t [z(\tau)-z(t_1)-I_1(z(t_1)), f(\tau)-z_*]_+ d\tau,\end{aligned}$$

for each $z_* \in A(z(t_1)+I_1(z(t_1))), t_1 \leq s \leq t \leq t_2$.

• We continue this process, and we obtain that, for each $t_k \leq s \leq t \leq t_{k+1}$ and $z_* \in A(z_k(t_{k-1})+I_k(z_k(t_{k-1}))$, we have

$$\begin{aligned}\|z(t)-z(t_k)-I_k(z(t_k))\| &\leq \|z(s)-z(t_{k-1})-I_k(z(t_{k-1}))\| \\ &\quad + \int_s^t [y(\tau)-z(t_{k-1})-I_k(z(t_{k-1})), f(\tau)-z_*]_+ d\tau.\end{aligned}$$

□

Proof. Proof of Theorem Since F is almost strongly-weakly u.s.c., then for $\epsilon \in (0,1)$, there exists an open set $J_\epsilon \subset J$ whose Lebesgue measure $\mu(J_\epsilon) \leq \epsilon$ and such that $F|_{J\backslash J_\epsilon \times \overline{D(A)}}$ is strongly-weakly u.s.c. Let $F_\epsilon : [0,b] \times \overline{D(A)} \to \mathcal{P}(E)$ be defined by

$$F_\epsilon(t,x) = \begin{cases} F(t,x) & \text{if } (t,x) \in D_\epsilon(F), \\ 0, & \text{if } (t,x) \notin D(F)\backslash D_\epsilon(F), \end{cases}$$

where

$$D(F) = J \times \overline{D(A)}, \quad D_\epsilon(F) = J\backslash J_\epsilon \times \overline{D(A)}.$$

As in Lemma , there exists $M > 0$ such that for every integral solution of () we have $\|y\| \leq M$. Set

$$\mathcal{K}_* = \{f \in L^1(J, \overline{D(A)} : \|f(t)\| \leq p(t)\psi(M)\}.$$

Consider the following impulsive evolution problem

$$\left\{\begin{array}{lll} y'(t) & \in & Ay(t)+F_\epsilon(t,y(t)) \quad t \in [0,b], \\ y(t_k^+)-y(t_k) & = & (1-\epsilon)I_k(y(t_k^-)). \\ y(0)=(1-\epsilon)x. & & \end{array}\right. \tag{4.13}$$

It clear that $\mathcal{K}_*$ is uniformly integral bounded in $L^1(J,E)$. From Lemma

$$G_\epsilon = \{y_\epsilon(f) : f \in \mathcal{K}_*\}$$

is relatively compact in PC. Then

$$\widetilde{G}_\epsilon = \{y_\epsilon(f)(t) : f \in \mathcal{K}_*, \text{ a.e. } t \in [0,b]\}$$

is relatively compact in E. Since the restriction of F_ϵ to $D_\epsilon(F)$ is strongly-weakly u.s.c., from Lemma (), the set

$$H_\epsilon = \overline{co}(F_\epsilon(J\backslash J_\epsilon \times \overline{\widetilde{G}_\epsilon} \cup \{0\}))$$

is weakly compact in E. Let

$$\mathcal{K}_*^\epsilon = \{f \in \mathcal{K}_* : f(t) \in H_\epsilon, \text{ a.e. } t \in J\}.$$

It is clear that $\mathcal{K}_*^\epsilon \neq \emptyset$ and is weakly compact in L^1. Now we define the following operator $S_{F_\epsilon}^\epsilon : \mathcal{K}_*^\epsilon \to L^1([0,b],E)$ by

$$S_{F_\epsilon}^\epsilon(f) = \{h \in L^1([0,b],E) : h(t) \in F_\epsilon(t, y_\epsilon(f)(t)), \text{ a.e. } t \in [0,b]\},$$

where $y_\epsilon(f)$ is the unique integral solution of (). We can easily show that $S_{F_\epsilon}^\epsilon(.) \in \mathcal{P}_{cv,wcp}(L^1([0,b],E))$. Moreover, it the operator $S_{F_\epsilon}^\epsilon$ has fixed points, then the Problem () have at least one integral solutions. We shall show that $S_{F_\epsilon}^\epsilon$ satisfies the assumptions of Kakutani-Ky Fan fixed point theorem.

• $S_{F_\epsilon}^\epsilon(\mathcal{K}_*^\epsilon) \subseteq \mathcal{K}_*^\epsilon$. Indeed, if $f \in \mathcal{K}_*^\epsilon$ and $h \in S_{F_\epsilon}^\epsilon(f)$ then

$$h(t) \in F_\epsilon(t, y_\epsilon(f)(t)) \subseteq H_\epsilon, \text{ a.e. } t \in J.$$

Hence

$$\|h(t)\| \leq \|F(t, y_\epsilon(f))\|_{\mathcal{P}} \Longrightarrow \|h(t)\| \leq p(t)\psi(\|y_\epsilon(f))\|).$$

From Lemma we conclude that

$$\|h(t)\| \leq p(t)\psi(M) \Longrightarrow h \in \mathcal{K}_*,$$

and so, $S_{F_\epsilon}^\epsilon(\mathcal{K}_*^\epsilon). \subseteq \mathcal{K}_*^\epsilon$. Also $S_{F_\epsilon}^\epsilon(\mathcal{K}_*^\epsilon)$ is weakly compact in $L^1(J,E)$.

• $S_{F_\epsilon}^\epsilon$ has a closed graph. Let $h_n \rightharpoonup h_*$, $h_n \in S_{F_\epsilon}^\epsilon(f_n)$ and $f_n \rightharpoonup f_*$. We shall prove that $h_* \in S_{F_\epsilon}^\epsilon(f_*)$. For every $h_n \in S_{F_\epsilon}^\epsilon(f_n)$, then there exists $y_\epsilon(f_n) \in PC$ a solution of problem () such that

$$h_n(t) \in F_\epsilon(t, y_\epsilon(f_n)(t)) \subseteq H_\epsilon, \text{ a.e. } t \in J.$$

From Lemma , we have $\{y_\epsilon(f_n) : n \in \mathbb{N}\}$ is relatively compact in PC, and using the fact that A is completely continuous type and I_k are continuous, we get

$$y_\epsilon(f_n) \to y_\epsilon(f_*), \text{ as } n \to \infty \text{ in } PC.$$

Since

$$f_n(t) \in F_\epsilon(t, y_\epsilon(f_n)), \text{ a.e.}, t \in J.$$

Hence, by Lemma ()

$$f_*(t) \in F_\epsilon(t, y_\epsilon(f_*)), \text{ a.e.}, t \in J \backslash J_\epsilon.$$

On the other hand, we have $f_n(t) = f_*(t) = 0$, for each $t \in J_\epsilon$. Thus

$$f_*(t) \in F_\epsilon(t, y_\epsilon(f_*)), \text{ a.e.}, t \in J.$$

This implies that the graph of $S^{\epsilon}_{F_\epsilon}$ is weakly sequentially closed. By the Kakutani-Ky Fan fixed point theorem (), $S^{\epsilon}_{F_\epsilon}$ has a fixed point $f \in \mathcal{K}_*$ such that $y_\epsilon(f) \in PC$ which is solution of ().

Next, let $\epsilon_n \in (0,1)$, $\epsilon_n \to 0$. Then there exists a sequence $\{y_{\epsilon_n}(f_n)\}_{n\in\mathbb{N}}$ of integral solutions of (). From $(\mathcal{H}_3)$, we deduce that $\{f_n : n \in \mathbb{N}\}$ is uniformly integrable in $L^1([0,b],E)$ and $B = \{(1-\epsilon_n)x : n \in \mathbb{N}\}$ is relatively compact in $\overline{D(A)}$. By Lemma the set $\{y_{\epsilon_n} : n \in \mathbb{N}\}$ is relatively compact in PC. Hence

$$\overline{\{y_{\epsilon_n}(f_n)(t) : n \in \mathbb{N},\ t \in J\}}$$

is compact in E. Let $\alpha \in (0,1)$. Then there exists a Lebesgue measurable set J_α in J, $\mu(J_\alpha) \leq \gamma$ such that $F|_{J\backslash J_\alpha \times \overline{D(A)}}$ is strongly weakly convergent. Let

$$L_\alpha = \overline{\{(t, y_{\epsilon_n}(f_n)(t)) : n \in \mathbb{N},\ t \in J\backslash J_\alpha\}}.$$

It is clear that L_α is compact. Then

$$\widetilde{L}_\alpha = F(L_\alpha) \cup \{0\}$$

is weakly compact and $F|_{L_\gamma}$ is strongly weakly convergent. Let $\alpha \in (0,1)$. Then there exists J_α such that $\mu(J\backslash J_\alpha) \leq \alpha$ and

$$f_n(J_\alpha) \subset \bigcup_{t\in J\backslash J_\alpha} F(t, y_{\epsilon_n}(f_n)(t)) \subseteq \widetilde{L}_\alpha.$$

Hence by Theorem () $\{f_n : n \in \mathbb{N}\}$ is weakly relatively compact in $L^1(J,E)$. So, there exists a subsequence such that

$$\begin{cases} f_n \rightharpoonup f & \text{in } L^1, \\ y_{\epsilon_n}(f_n) \to y & \text{in } PC. \end{cases}$$

Then from Lemma (), we get

$$f(t) \in F_{\epsilon_n}(t, y(t)), \text{ for each } n \in \mathbb{N} \text{ and } t \in J\backslash J_{\epsilon_n}.$$

Since $\lim_n \mu(J_{\epsilon_n}) = 0$, then

$$f(t) \in F(t, y(t)), \text{ a.e. } t \in J.$$

Using the fact that A is completely continuous type and I_k are continuous functions, it follows that y is the integral solution of the problem () corresponding to the selection f of $F(\cdot, y(\cdot))$.

□

4.3.2 Non-convex case

In all this part, we assume that E is a real separable Banach space.

Definition 4.3.2 *A is called $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $I \times D$ where I is Lebesgue measurable in J and D is Borel measurable in E.*

Definition 4.3.3 *A subset $A \subset L^1(J, E)$ is decomposable if for all $u, v \in A$ and for every Lebesgue measurable set $I \subset J$, we have*

$$u\chi_I + v\chi_{J \setminus I} \in A,$$

where χ_A stands for the characteristic function of the set A. We denote by Dco the family of decomposable sets.

More details on the previous sets can be found in the monograph by Fryszkowski [40].

Let $F: \ J \times E \to \mathcal{P}(E)$ be a multi-valued map with nonempty closed values. Assign to F the multi-valued operator $\mathcal{F}: \ PC(J, E) \to \mathcal{P}(L^1(J, E))$ defined by

$$\mathcal{F}(y) = \{v \in L^1(J, E) : v(t) \in F(t, y(t)), \text{ a.e. } t \in J\}.$$

The operator $\mathcal{F}$ is called the Nemyts'kiĭ operator associated to F.

Definition 4.3.4 *Let $F : J \times E \to \mathcal{P}(E)$ be a multi-valued map with nonempty compact values. We say that F is of lower semi-continuous type (l.s.c. type) if its associated Nemyts'kiĭ operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.*

Next, we state a classical selection theorem due to Bressan and Colombo.

Lemma 4.3.5 *(see [28, 35, 55]) Let X be a separable metric space and let E be a Banach space. Then every l.s.c. multi-valued operator $N : X \to \mathcal{P}_{cl}(L^1(J, E))$ with closed decomposable values has a continuous selection, i.e. there exists a continuous single-valued function $f : X \to L^1(J, E)$ such that $f(x) \in N(x)$ for every $x \in X$.*

Let us introduce the following hypothesis.

($\mathcal{H}_4$) $F : [0, b] \times E \longrightarrow \mathcal{P}_{cp}(E)$ is a multi-valued map such that
(a) the mapping $(t, y) \mapsto F(t, y)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable;
(b) the mapping $y \mapsto F(t, y)$ is lower semi-continuous for a.e. $t \in [0, b]$.

The following lemma is crucial in the proof of our existence theorem.

Lemma 4.3.6 *(see e.g. [35, 39]) Let $F : J \times E \to \mathcal{P}_{cp}(E)$ be an integrably bounded multi-valued map satisfying ($\mathcal{H}_4$). Then F is of lower semi-continuous type.*

Let $f : J \times E \to E$ be a Carathéodory function. Consider the problem

$$\left\{\begin{array}{rcll} y'(t) - Ay(t) & \ni & f(t, y(t)), \text{ a.e.} & t \in [0, b], \\ \Delta y|_{t=t_k} & = & I_k(y(t_k^-)), & k = 1, \dots, n, \\ y(0) & = & x \in D(A), & \end{array}\right. \tag{4.14}$$

Lemma 4.3.7 *In addition to conditions $(\mathcal{H}_1) - (\mathcal{H}_4)$, assume that f satisfies the following condition:*

$(\mathcal{H}_5)$ *there exist a continuous nondecreasing function $\psi : [0, \infty) \longrightarrow (0, \infty)$ and $p \in L^1(J, \mathbb{R}^+)$ such that*

$$\|f(t, x)\| \le p(t)\psi(\|x\|), \text{ for a.e. } t \in J \text{ and each } x \in \overline{D(A)},$$

with

$$\int_0^b p(s)ds < \int_{\|x\|}^{\infty} \frac{du}{\psi(u)}.$$

Then the problem () has at least one integral solution.

Proof.
For $y \in PC$, we consider the following problem

$$\left\{\begin{array}{rcll} u'(t) - Au(t) & \ni & f(t, y(t)), \text{ a.e.} & t \in [0, b], \\ \Delta u|_{t=t_k} & = & I_k(y(t_k^-)), & k = 1, \dots, n, \\ u(0) & = & x \in D(A), & \end{array}\right. \tag{4.15}$$

From Lemma , there exists a unique integral solution u_y of (). This fact allows us to introduce the operator $\mathcal{G} : PC \to PC$ defined as follows, $\mathcal{G} : y \to \mathcal{G}(y) = u_y$. We shall show that $\mathcal{G}$ satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof will be given in several steps.

Step 1: $\mathcal{G}$ *is continuous.*

Let $\{y_p\}$ be a sequence such that $y_p \to y$ in PC. Then, from from Lemma , we obtain

$$\|\mathcal{G}(y_p)(t) - \mathcal{G}(y)(t)\| \le \int_0^{t_1} \|f(s, y_p(s)) - f(s, y(s))ds\|, \ t \in [0, t_1],$$

and for $k = 1, \dots, n$,

$$\begin{aligned} \|\mathcal{G}(y_p)(t) - \mathcal{G}(y)(t)\| \ \le \ & \|y_p(t_k) - y(t_k)\| + \|I_k(y_p(t_k)) - I_k(y(t_k))\| \\ & + \int_{t_k}^{t} \|f(s, y_p(s)) - f(s, y(s))ds\|, \ t \in (t_k, t_{k+1}], \end{aligned}$$

and hence,

$$\begin{aligned}\|\mathcal{G}(y_p)-\mathcal{G}(y)\|_{PC} &\leq n\|y_p-y\|_{PC}+\sum_{k=1}^{n}\|I_k(y_p(t_k))-I_k(y(t_k))\| \\ &\quad +\int_0^b \|f(s,y_p(s))-f(s,y(s))ds\| \to 0, \text{ as } p\to\infty.\end{aligned}$$

Hence $\mathcal{G}$ is continuous. By $(\mathcal{H}_2)$ and $(\mathcal{H}_4)$, we can easily prove that $\mathcal{G}$ maps bounded sets into bounded sets.

Step 2: $\mathcal{G}$ maps bounded set into equicontinuous sets of PC. Let C be a bounded set in PC. Then there exists $r>0$ such that

$$\|\mathcal{G}(y)\|_{PC}\leq r \Rightarrow \|f(t,y(t))\|\leq p(t)\psi(r), \quad \text{for every } x\in C.$$

Hence from the second part of Lemma , we deduce that $\mathcal{G}(\mathcal{C})$ is relatively compact in PC.

Step 3: By $(\mathcal{H}_1)$ and $(\mathcal{H}_5)$, it is easy to prove that there exists $M>0$ such that for every $y\in PC$ integral solution of (), we have $\|y\|_{PC}\leq M$. Let

$$U:=\{y\in PC:\|y\|_{PC}<\widetilde{M}+1\}$$

and consider the operator $\mathcal{G}:\overline{U}\to PC$. From the choice of U, there is no $y\in\partial U$ such that $y=\lambda\mathcal{G}(y)$ for some $\lambda\in(0,1)$. As a consequence of of the nonlinear alternative of Leray-Schauder type, $\mathcal{G}$ has a fixed point y in U which is a solution of problem ().

□

Theorem 4.3.4 *Suppose that the hypotheses $(\mathcal{H}_1),(\mathcal{H}_2),(\mathcal{H}_3)$ and $(\mathcal{H}_4)$ are satisfied. Then Problem () has at least one integral solution.*

Proof. $(\mathcal{H}_3)$ and $(\mathcal{H}_4)$ imply, by Lemma , that F is of lower semi-continuous type. From Lemma , there is a continuous selection $f:PC\to L^1([0,b],E)$ such that $f(y)\in\mathcal{F}(y)$ for all $y\in PC$. Consider the problem

$$\left\{\begin{array}{rcll} y'(t)-Ay(t) & \ni & f(y)(t), \text{ a.e.} & t\in[0,b], \\ \Delta y|_{t=t_k} & = & I_k(y(t_k^-)), & k=1,\ldots,n, \\ y(0) & = & x\in D(A). & \end{array}\right. \tag{4.16}$$

As in Lemma , we can prove that the problem () has at least one integral solution which is an integral solution ().

□

4.4 Example

Let Ω be a nonempty, bounded and open subset in $\mathbb{R}^p$, $p \geq 1$, with C^1 boundary $\partial\Omega$, let $\omega > 0$ and $\phi : D(\phi) \subset \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be maximal monotone with $0 \in \phi(0)$. Let us consider the following impulsive problem

$$\left\{ \begin{array}{rcll} \frac{\partial u}{\partial t}(t,x) & \in & \Delta\phi(u(t,x)) + \omega u(t,x) + F(t,x,u(t,x)), & (t,x) \in [0,b] \times \Omega, \\ u((t_k^+,x) - u(t_k^-,x) & = & b_k u(t_k^-,x)), & k = 1,\ldots,n, \\ u(0,x) & = & 0, & x \in \partial\Omega, \end{array} \right. \tag{4.17}$$

where $b_k \in \mathbb{R}, k = 1,\ldots,n,$ $I_k(x) = b_k x$, Δ be the Laplace operator in the sense of distribution over Ω, $\phi : D(\phi) \subset \mathbb{R} \to \mathcal{P}(\mathbb{R})$ defined by

$$S_\phi(u) = \{v \in L^1(\Omega) : v(x) \in \phi(u(x)), \text{ a.e. for } x \in \Omega\}.$$

Take $E = L^1(\Omega)$ and define $A : D(\Delta\phi) \subset L^1(\Omega) \to \mathcal{P}(L^1(\Omega))$ by

$$\left\{ \begin{array}{rcl} D(\Delta\phi) & = & \{u \in L^1(\Omega) : \exists v \in S_\phi(u) \cap W_0^{1,1}(\Omega),\ \Delta v \in L^1(\Omega)\}, \\ \Delta(\phi) & = & \{\Delta v : v \in S_\phi(u) \cap W_0^{1,1}(\Omega), \text{ for } u \in D(\Delta\phi)\} \end{array} \right.$$

It is well known that A is $m-$dissipative (see Brezis and Straus [29]). If we assume that $\phi : \mathbb{R} \to \mathbb{R}$ is continuous function on $\mathbb{R}$ and C^1 on $\mathbb{R}\backslash\{0\}$ and there exist two constants $C > 0$ and $a > 0$ if $p \leq 2$, and $a > \frac{p-2}{p}$ if $p \geq 3$ such that

$$\phi'(r) \geq C|r|^{a-1} \text{ for each } r \in \mathbb{R}\backslash\{0\},$$

then $\Delta\phi$ is of completely continuous type (see Badii *et al* [14]) and generates a compact semigroup (see the theorem). Hence the problem () can be written as,

$$\left\{ \begin{array}{rcll} u'(t) & \in & \Delta\phi(u(t)) + \omega u(t) + F(t,u(t)), & t \in [0,b], \\ u((t_k^+) - u(t_k^-) & = & b_k u(t_k^-), & k = 1,\ldots,n, \\ u(t) & = & 0. & \end{array} \right. \tag{4.18}$$

Assume that F satisfies the conditions of Theorem or Theorem . Then, the problem () has at least one solution which is a solution of ().

Chapter 5

Random fixed point theorem in generalized Banach space and applications

5.1 Introduction

Probabilistic functional analysis is an important mathematical area of research due to its applications to probabilistic models in applied problems. Random operator theory is needed for the study of various classes of random equations. Indeed,in many cases the mathematical models or equations used to describe phenomena in the biological, physical, engineering, and systems sciences contain certain parameters or coefficients which have specific interpretations, but whose values are unknown. Therefore, it is more realistic to consider such equations as random operator equations. These equations are much more difficult to handle mathematically than deterministic equations. Important contributions to the study of the mathematical aspects of such random equations have been undertaken in [24, 90, 72] among others. The problem of fixed points for random mappings was initialed by the Prague school of probabilistes. The first results were studied in 1955-1956 by Špaček and Hanš in the context of Fredholm integral equations with random Kernel. In a separable metric space, random fixed point theorems for contraction mappings were proved by Hanš [49, 50], Špaček [89], Hanš and, Špaček [51] and Mukherjee [70, 71]. Then random fixed point theorems of Schauder or Krasnosel'skii type were given by Mukherjea (cf. Bharucha-Reid [24], p. 110), Prakasa Rao [81] and Bharucha-Reid [25]. Now it has become the full fledged research area and vast amount of mathematical activities have been carried out in this direction see for examples [7, 8, 18, 87, 88]. In 1958, Krasnosel'skii [63] established his famous fixed point theorem.

The result combined the Banach contraction principle and Schauder's fixed point theorem. The existence of fixed points for the sum of two deterministic operators has attracted tremendous interest, and their applications are frequent in nonlinear analysis. Many improvements of Krasnosel'skii's theorem have been established in the literature in the course of time by modifying the above assumptions; see, for example, [42, 44, 43].

Rao [81] obtained a probabilistic version of Krasnoselskii's theorem which is a sum of a contraction random operator and a compact random operator on a closed convex subset of a separable Banach space. Moreover, Itoh [57] extended Rao's result to a sum of a nonexpansive random operator and a completely continuous random operator on a weakly compact convex subset of a separable uniformly convex Banach space. Very recently in [9] Arunchai and Plubtieng obtained a random fixed point theorem for the sum of a weakly-strongly continuous random operator and a nonexpansive random operator which contains as a special Krasnoselskii type.

Several well-known fixed point theorems of single-valued mappings such as Banach's and Schauder's have been extended in generalized Banach spaces; see [6, 45].

Very recently, in generalized Banach space a Krasnoselskii type fixed point theorem of single and multi-valued operator was studied by Petre [79], Petre and Petruşel [80] and Ouahab [85].

Our goal in this chapter is to give a random version of Perov-type, Schauder-type and Krasnoselskii-type fixed point theorems in generalized Banach space.

The chapter is organized as follows. In Sections , we introduce all the background material needed in this chapter such as generalized metric space, some fixed point theorems and random variable.

The aim of section is devoted to establishing a random version of the Pervo, Schauder and Krasnoselskii fixed point theorems in generalized metric and Banach space. In Section , we apply our results to random Cauchy problem and second order boundary value problems. All the results of this chapter where considered by Sinacer *et al* [92].

5.2 Generalized metric space

In this section we recall from the literature some notations, definitions, and auxiliary results which will be used throughout this chapter. If, $x, y \in \mathbb{R}^n$, with $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, then by $x \leq y$ we mean $x_i \leq y_i$ for all $i = 1, \dots, n$. Also we set $|x| = (|x_1|, \dots, |x_n|)$, $\max(x, y) = (\max(x_1, y_1), \dots, \max(x_n, y_n))$ and $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : \ x_i > 0\}$. If $c \in \mathbb{R}$, then $x \leq c$ means $x_i \leq c$ for each $i = 1, \dots, n$.

Definition 5.2.1 *Let X be a nonempty set. By a vector-valued metric on X we mean a map $d : X \times X \to \mathbb{R}^n$ with the following properties:*

(i) $d(u, v) \geq 0$ *for all* $u, v \in X$; *if* $d(u, v) = 0$ *then* $u = v$;

(ii) $d(u, v) = d(v, u)$ *for all* $u, v \in X$;

(iii) $d(u, v) \leq d(u, w) + d(w, v)$ *for all* $u, v, w \in X$.

We call the pair (X, d) a generalized metric space with $d(x,y) := \begin{pmatrix} d_1(x,y) \\ \cdots \\ d_n(x,y) \end{pmatrix}$.
Notice that d is a generalized metric space on X if and only if $d_i,\ i = 1,\ldots,n$ are metrics on X.

For $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_+$, we will denote by

$$B(x_0, r) = \{x \in X : d(x_0, x) < r\} = \{x \in X : d_i(x_0, x) < r_i, i = 1, ..., n\}$$

the open ball centered in x_0 with radius r and

$$\overline{B(x_0, r)} = \{x \in X : d(x_0, x) \leq r\} = \{x \in X : d_i(x_0, x) \leq r_i, i = 1, ..., n\}$$

the closed ball centered in x_0 with radius r. We mention that for generalized metric space, the notions of open subset, closed set, convergence, Cauchy sequence and completeness are similar to those in usual metric spaces.

Let (X, d) be a generalized metric space we define the following metric spaces: Let $X_i = X,\ i = 1, \ldots, n$. Consider $\prod_{i=1}^n X_i$ with $\bar{d}$:

$$\bar{d}((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sum_{i=1}^n d_i(x_i, y_i).$$

The diagonal space of $\prod_{i=1}^n X_i$ defined by

$$\widetilde{X} = \{(x, \ldots, x) \in \prod_{i=1}^n X_i : x \in X,\ i = 1, \ldots, n\}.$$

Thus it is a metric space with the following distance

$$d_*((x, \ldots, x), (y, \ldots, y)) = \sum_{i=1}^n d_i(x, y), \text{ for each } x, y \in X.$$

It is clear that $\widetilde{X}$ is closed set in $\prod_{i=1}^n X_i$.

Lemma 5.2.1 *Let (X, d) be a generalized metric space. Then there exists a homeomorphism map $h : X \to \widetilde{X}$.*

Proof. Consider $h : X \to \widetilde{X}$ defined by

$$h(x) = (x, \ldots, x) \quad \text{for all } x \in X.$$

Obviously h is bijective.
• To prove that h is a continuous map.
Let $x, y \in X$. Thus

$$d_*(h(x), h(y)) \ \leq \ \sum_{i=1}^n d_i(x, y).$$

For $\epsilon > 0$ we take $\delta = (\frac{\epsilon}{n}, \ldots, \frac{\epsilon}{n})$, let $x_0 \in X$ be fixed and $B(x_0, \delta) = \{x \in X : d(x_0, x) < \delta\}$,. Then for every $x \in B(x_0, \delta)$ we have

$$d_*(h(x_0), h(x)) \leq \epsilon.$$

• Now we show that $h^{-1} : \widetilde{X} \to X$ defined by

$$h^{-1}(x, \ldots, x) = x, \qquad (x, \ldots, x) \in \widetilde{X}.$$

is a continuous map.
Let $(x, \ldots, x), (y, \ldots, y) \in \widetilde{X}$, then

$$d(h^{-1}(x, \ldots, x), h^{-1}(y, \ldots, y)) = d(x, y).$$

Let $\epsilon = (\epsilon_1, \ldots, \epsilon_n) > 0$ we take $\delta = \frac{\min\limits_{1 \leq i \leq n} \epsilon_i}{n}$ and we fix $(x_0, \ldots, x_0) \in \widetilde{X}$. Set

$$B((x_0, \ldots, x_0), \delta) = \{(x, \ldots, x) \in \widetilde{X} : d_*((x_0, \ldots, x_0), (x, \ldots, x)) < \delta\}.$$

For $(x, \ldots, x) \in B((x_0, \ldots, x_0), \delta)$ we have

$$d_*((x_0, \ldots, x_0), (x, \ldots, x)) < \delta \Rightarrow \sum_{i=1}^{n} d_i(x_0, x) < \frac{\min\limits_{1 \leq i \leq n} \epsilon_i}{n}.$$

Then

$$d_i(x_0, x) < \frac{\min\limits_{1 \leq i \leq n} \epsilon_i}{n}, \; i = 1, \ldots, n \Rightarrow d(x_0, x) < \epsilon.$$

Hence h^{-1} is continuous.

□

Definition 5.2.2 *A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc i.e. $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $det(M - \lambda I) = 0$, where I denote the unit matrix of $\mathcal{M}_{n \times n}(\mathbb{R})$.*

Theorem 5.2.1 ([86], p.12,p.88) . *Let $M \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$. The following assertions are equivalent:*

(i) M is convergent towards zero;

(ii) $M^k \to 0$ as $k \to \infty$;

(iii) The matrix $(I - M)$ is nonsingular and

$$(I - M)^{-1} = I + M + M^2 + \ldots + M^k + \ldots,$$

(iv) The matrix $(I - M)$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements.

Definition 5.2.3 *Let (X, d) be a generalized metric space. An operator $N : X \to X$ is said to be contractive if there exists a convergent to zero matrix M such that*

$$d(N(x), N(y)) \leq Md(x, y) \text{ for all } x, y \in X.$$

For $n = 1$ we recover the classical Banach's contraction fixed point result.

Definition 5.2.4 *We say that a nonsingular matrix $A = (a_{ij})_{1\leq i,j\leq n} \in \mathcal{M}_{n\times n}\mathbb{R}$ has the absolute value property if*

$$A^{-1}|A| \leq I,$$

where

$$|A| = (|a_{ij}|)_{1\leq i,j\leq n} \in \mathcal{M}_{n\times n}(\mathbb{R}_+).$$

Some examples of matrices $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ convergent to zero which also satisfies the property $(I - A)^{-1}|I - A| \leq I$ are as follows

1) $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ and $\max(a, b) < 1$

2) $A = \begin{pmatrix} a & -c \\ 0 & b \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ and $a + b < 1,\ c < 1$

3) $A = \begin{pmatrix} a & -a \\ b & -b \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ and $|a - b| < 1,\ a > 1, b > 0$.

Definition 5.2.5 *A matrix $Q \in \mathcal{M}_{2\times 2}(\mathbb{R})$ is said to be order preserving (or positive) if $p_1 \geq p_0$, $q_1 \geq q_0$ imply*

$$Q\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \leq Q\begin{pmatrix} p_1 \\ q_1 \end{pmatrix}.$$

in the sense of components.

Lemma 5.2.2 *[100] Let*

$$Q = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

where $a, b, c, d \geq 0$ and $detQ > 0$. Then Q^{-1} is order preserving.

Theorem 5.2.2 *[77] Let (X, d) be a complete generalized metric space and $N : X \to X$ a contractive operator with Lipschitz matrix M. Then N has a unique fixed point x_* and for each $x_0 \in X$ we have*

$$d(N^k(x_0), x_*) \leq M^k(I - M)^{-1}d(x_0, N(x_0)) \text{ for all } k \in \mathbb{N}.$$

As a consequence of Perov fixed point theorem we have the following result

Theorem 5.2.3 *[80] Let (X, d) be a generalized Banach space and $N : X \to X$ an M-contraction. Then $I_X - N$ is a homeomorphism, i.e.,$I_X - N$ bijective and its inverse is continuous, $(I_X - N)^{-1}$ is continuous too.*

Theorem 5.2.4 *[34] Let E be a generalized Banach space, let $C \subset E$ be a nonempty closed convex subset of E and let $N : C \to C$ be a continuous operator with relatively compact range. Then N has at least fixed point in C.*

5.3 Random variable and some selection theorems

In this section, we introduce notations, definitions, and preliminary of random variable which are used throughout this chapter. Let (X, d) be a metric space or generalized metric space and Y be a subset of X. We denote: Let $(\Omega, \sum)$ be a measurable space and $F : \Omega \to \mathcal{P}(X)$ a multi-valued mapping. The mapping F is called measurable if $F_+^{-1}(Q) = \{\omega \in \Omega : F(\omega) \subset Q\}$ for every $Q \in \mathcal{P}_{cl}(X)$ or equivalently if for every open subset U of X the set $F_-^{-1}(U) = \{\omega \in \Omega : F(\omega) \cap U \neq \emptyset\}$ is measurable.

If X is a metric space, we shall use $B(X)$ denote the Borel $\sigma-$algebra on X. Then $\sum \otimes B(X)$ denotes the smallest σ-algebra on $\Omega \times X$ which contains all the sets $A \times S$, where $Q \in \sum$ and $S \in B(X)$. Let $F : X \to \mathcal{P}(Y)$ be a multi-valued map.

Definition 5.3.1 *Recall that a mapping $f : \Omega \times X \to X$ is said to be a random operator if, for any $x \in X$, $f(.,x)$ is measurable.*

Definition 5.3.2 *A random fixed point of f is a measurable function $y : \Omega \to X$ such that*

$$y(\omega) = f(\omega, y(\omega)) \text{ for all } \omega \in \Omega.$$

Equivalently, it is a measurable selection for the multi-valued map $FixF_\omega : \Omega \to \mathcal{P}(X)$ defined by

$$FixF_\omega(x) = \{x \in X : x = f(\omega, x)\}.$$

As a consequence of Kuratowski-Ryll-Nardzewski and J.R. Aummann selection theorems we can conclude the following results

Theorem 5.3.1 *Let $(\Omega, \sum)$ and X be a separable generalized metric space and let $F : \Omega \to \mathcal{P}_{cl}(X)$ be a measurable multi-valued map. Then F has a measurable selection.*

Proof. Consider $F_* : \Omega \to \mathcal{P}_{cl}(\widetilde{X})$ by

$$F_*(\omega) = (h \circ F)(\omega), \quad \text{for all } \omega \in \Omega$$

where h defined in lemma . Let $C \subset \widetilde{X}$ be a open set, then

$$\begin{aligned} F_*^{-1}(C) &= \{\omega \in \Omega : \ (h \circ F)(\omega) \cap C \neq \emptyset\} \\ &= \{\omega \in \Omega : \ F(\omega) \cap h^{-1}(C) \neq \emptyset\}. \end{aligned}$$

Since F is a measurable multi-valued function, we have $F_*^{-1}(C) \in \sum$. By Theorem there exists a measurable single-valued function $x : \Omega \to \widetilde{X}$ such that

$$x(\omega) \in (h \circ F)(\omega), \quad \text{for all } \omega \in \Omega \Rightarrow (h^{-1} \circ x)(\omega) \in F(\omega), \quad \text{for all } \omega \in \Omega.$$

Using the fact that h^{-1} is a continuous function, then $h^{-1} \circ x : \Omega \to X$ is a measurable selection of F.

□

Theorem 5.3.2 *Let* $(\Omega, \sum)$ *and* X *be a separable generalized metric space. If* $G : \Omega \to \mathcal{P}_{cp}(X)$ *is a multi-valued map such that the graph* $\mathcal{G}r(G)$ *of* G *is measurable, then* G *has a measurable selector.*

Proof. Let $G_* : \Omega \to \mathcal{P}_{cp}(\widetilde{X})$ defined by

$$G_*(\omega) = (h \circ G)(\omega), \quad \text{for all } \omega \in \Omega.$$

Then

$$\begin{aligned}
\mathcal{G}r(G_*) &= \{(\omega, y) \in \Omega \times \widetilde{X} : \ y \in G_*(\omega)\} \\
&= \{(\omega, y) \in \Omega \times \widetilde{X} : \ y \in (h \circ G)(\omega)\} \\
&= \{(\omega, y) \in \Omega \times \widetilde{X} : \ h^{-1}(y) \in G(\omega)\} \\
&= \{(\omega, y) \in \Omega \times \widetilde{X} : \ h^{-1}(y) \in G(\omega)\} \\
&= \{(\omega, z) \in \Omega \times h^{-1}(\widetilde{X}) : \ z \in G(\omega)\}.
\end{aligned}$$

Hence $\mathcal{G}r(G_*)$ is measurable. By theorem there exists a measurable single-valued function $x : \Omega \to \widetilde{X}$ such that

$$x(\omega) \in (h \circ G_*)(\omega), \quad \text{for all } \omega \in \Omega.$$

So $h^{-1} \circ x$ is a measurable selection of the multi-valued map G.

□

Definition 5.3.3 *A random operator* $f : \Omega \times X \to X$ *is said to be continuous at* $x_0 \in X$ *if* $\lim_{n\to\infty} \|x_n - x_0\| = 0$ *implies* $\lim_{n\to\infty} \|f(\omega, x_n) - f(\omega, x)\| = 0$ *a.s.*

5.4 Random fixed points theorems

In this section, we give the random versions of Perov, Schauder and Krasnoselskki fixed point theorems in a generalized metric space.

Theorem 5.4.1 *Let* $(\Omega, \mathcal{F}, \mu)$ *be a probability space, let* X *be a real separable generalized Banach space and let* $F : \Omega \times X \to X$ *be a continuous random operator. Let* $M(\omega) \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ *be a random variable matrix such that* $M(\omega)$ *converges to* 0 *a.s. and*

$$d(F(\omega, x_1), F(\omega, x_2)) \leq M(\omega) d(x_1, x_2) \text{ for each } x_1, x_2 \in X, \ \omega \in \Omega.$$

then there exists a random variable $x : \Omega \to X$ *which is the unique random fixed point of* F.

Proof. Let $E = \{\omega \in \Omega : d(F(\omega, x_1), F(\omega, x_2)) \leq M(\omega)d(x_1, x_2)$ for each $x_1, x_2 \in X\}$, then $\mu(E) = 1$. From theorem , we have for every fixed $\omega \in E$ that there exists a unique $x(\omega) \in X$ such that $F(\omega, x(\omega)) = x(\omega)$. Let $y : \Omega \to X$ be any arbitrary measurable function, we define $(x_n(\omega))_{n\in\mathbb{N}}$, with $x_0(\omega) = y(\omega)$, by

$$x_n(\omega) = F(\omega, F^{n-1}(\omega, y(\omega)), \quad n \in \mathbb{N}.$$

It is clear that x_n is a random variable and for each $n, m \in \mathbb{N}$ we have

$$d(x_n(\omega), x_{n+k}(\omega)) \leq (M^k(\omega) + \dots M^{n+k}(\omega))d(x_0(\omega), x_1(\omega)).$$

By lemma , we get

$$d(x_n(\omega), x_{n+k}(\omega)) \leq M^k(\omega)(I - M(\omega))^{-1}d(x_0(\omega), x_1(\omega)).$$

Hence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Then there exists a random variable $y_* : \Omega \to X$ such that

$$d(x_n(\omega), y_*(\omega)) \to 0 \text{ as } n \to \infty.$$

We obtain

$$d(y_*(\omega), F(\omega, y_*(\omega)) \leq d(y_*(\omega), x_n(\omega)) + M(\omega)d(x_n(\omega), y_*(\omega)) \to 0 \text{ as } n \to \infty.$$

Thus

$$y_*(\omega) = F(\omega, y_*(\omega)) \text{ for each } \omega \in E,$$

so

$$y_*(\omega) = x(\omega), \qquad \omega \in E.$$

□

By a simple modification we conclude the following result.

Theorem 5.4.2 *Let $(\Omega, \mathcal{F})$ be a measurable space, let X be a real separable generalized Banach space and let $F : \Omega \times X \to X$ be a continuous random operator. Let $M(\omega) \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ be a random variable matrix such that for every $\omega \in \Omega$ the matrix $M(\omega)$ converges to 0 and*

$$d(F(\omega, x_1), F(\omega, x_2)) \leq M(\omega)d(x_1, x_2) \textit{ for each } x_1, x_2 \in X, \ \omega \in \Omega.$$

Then there exists a random variable $x : \Omega \to X$ which is the unique random fixed point of F.

Theorem 5.4.3 *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, let X be a real separable generalized Banach space and let $T : \Omega \times X \to X$ be a continuous random operator. Let $M(\omega) \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ be a nonnegative real matrix random variable such that $\rho(M(\omega)) < 1$ a.s. and*

$$\|T(\omega, x_1) - T(\omega, x_2)\| \leq M(\omega)\|x_1 - x_2\| \textit{ for each } x_1, x_2 \in X, \ \omega \in \Omega.$$

Then there exists a random variable $y : \Omega \to X$ which is the unique fixed point of T.

Proof. Let $E = \{\omega \in \Omega : \rho(M(\omega)) < 1\}$, $F = \{\omega \in \Omega : T(\omega, .)$ is a continuous$\}$ and

$$G_{x,y} = \{\omega \in \Omega : \|T(\omega, x) - T(\omega, y)\| \leq M(\omega)\|x - y\|\} .$$

Let D be a countable dense subset of X. We first show that

$$\left(\bigcap_{x,y \in X} G_{x,y} \cap E \cap F \right) = \left(\bigcap_{a,b \in D} G_{a,b} \cap E \cap F \right).$$

Let $\omega \in \left(\bigcap_{a,b \in D} G_{a,b} \cap E \cap F \right)$. Then

$$\|T(\omega, a) - T(\omega, b)\| \leq M(\omega)\|a - b\|, \quad \text{for all } a, b \in D. \tag{5.1}$$

Let $x, y \in X$, thus there exist two sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}} \in D$ such that

$$a_n \to x, \quad b_n \to y \text{ as } n \to \infty.$$

From (), we get

$$\|T(\omega, a_n) - T(\omega, b_n)\| \leq M(\omega)\|a_n - b_n\|, \quad n \in \mathbb{N}.$$

By the continuity of $T(\omega, .)$, we obtain

$$\|T(\omega, x) - T(\omega, y)\| \leq M(\omega)\|x - y\|,$$

as $n \to \infty$.
This implies that

$$\left(\bigcap_{a,b \in D} G_{a,b} \cap E \cap F \right) \subseteq \left(\bigcap_{x,y \in X} G_{x,y} \cap E \cap F \right). \tag{5.2}$$

Also it is obvious that

$$\left(\bigcap_{x,y \in X} G_{x,y} \cap E \cap F \right) \subseteq \left(\bigcap_{a,b \in D} G_{a,b} \cap E \cap F \right). \tag{5.3}$$

From () and (), we have

$$\left(\bigcap_{x,y \in X} G_{x,y} \cap E \cap F \right) = \left(\bigcap_{a,b \in D} G_{a,b} \cap E \cap F \right).$$

Hence

$$\left(\bigcap_{x,y \in X} G_{x,y} \cap E \cap F \right) \in \mathcal{F}.$$

Since $\mu(E) = 1$ and $\mu(F) = 1$ we have

$$\mu(\Omega \backslash E) = 0, \quad \mu(\Omega \backslash F) = 0.$$

It is clear that

$$\bigcap_{x,y \in X} G_{x,y} = \{\omega \in \Omega : \|T(\omega, x) - T(\omega, y)\| \leq M(\omega)\|x - y\|, \quad \text{for all } x, y \in X\}.$$

Hence

$$\mu(\bigcap_{x,y \in X} G_{x,y}) = 1 \Rightarrow \mu(\Omega \backslash \bigcap_{x,y \in X} G_{x,y}) = 0.$$

Therefore,

$$\mu(E_*) = 1, \quad E_* = \bigcap_{x,y \in X} G_{x,y} \cap E \cap F.$$

Thus for every $\omega \in E_*$, $T(\omega, .)$ is a deterministic operator and hence has a unique fixed point in deterministic case. We denote by $\xi(\omega)$. Let $x : \Omega \to X$ be a function defined by

$$x(\omega) = \begin{cases} \xi(\omega), & \text{if } \omega \in E_*, \\ \\ 0, & \text{if } \omega \in \Omega \backslash E_*. \end{cases}$$

By the same method used in Theorem we can prove that x is the unique random fixed point of T.

□

Theorem 5.4.4 *Let $T : \Omega \times X \to X$ be an almost surly continuous random operator. Assume that there exists a real matrix random variable $M(\omega) \in \mathcal{M}_{2\times 2}(\mathbb{R}_+)$ such that*

$$\mu(\{\omega : \|T(\omega, x_1) - T(\omega, x_2)\| \leq M(\omega)\|x - y\|\}) = 1.$$

Then for every real number $\lambda \neq 0$ such that $\rho(M(\omega)) < |\lambda|$ and

$$\mu\{\omega \in \Omega : \rho(M(\omega)) < |\lambda|\} = 1,$$

there exists a random operator S that is the inverse of the random operator $(T(\omega, .) - \lambda I)$, I_X denote the identity operator.

Proof. Let $\lambda \neq 0$ and $y \in X$. We consider $T_y(\omega, .) : X \to X$ defined by

$$T_y(\omega) = \frac{1}{|\lambda|} T(\omega, x) - y, \ x \in X.$$

It is clear that T_y is a random contraction operator. Therefore by Theorem there exists a unique random operator $x_y : \Omega \to X$ such that

$$x_y(\omega) = \frac{1}{|\lambda|} T(\omega, x_y(\omega)) - y, \text{ a.s.}$$

Then there exists a random operator $S_1(\omega) : X \to X$ such that

$$S_1(\omega)\left(\frac{1}{|\lambda|}T(\omega) - I\right) = I.$$

So $S(\omega) = \frac{1}{|\lambda|}S_1(\omega)$ is the inverse of $T(\omega, .) - \lambda I$.

□

Remark 5.4.1 *We can replace the constant λ of above theorem by a real-valued random operator $\lambda : \Omega \to \mathbb{R}$ such that*

$$\rho(M(\omega)) < \lambda(\omega) \ a.s.$$

In 1966 Mukherjea gave a random version of the Schauder fixed point theorem on an atomic measure space. Then Prakasa Rao extended this result and obtained a theorem of Krasnoselskki type on a same measure spaces. Bharucha-Reid generalized results of Mukherjea and Prakasa Rao.

Theorem 5.4.5 *Let X be a generalized Banach space, let C be a separable closed convex subset of X and let $F : \Omega \times C \to C$ be a continuous random operator. Suppose that for every $\omega \in \Omega$, $F(\omega, C)$ is compact. Then there exists a random fixed point $x : \Omega \to C$ of F.*

Proof. Let $\omega \in \Omega$. We consider $F(\omega, .) : C \to C$ defined by

$$F_\omega(x) = F(\omega, x), \ x \in X.$$

Therefore, by Theorem there exists $x(\omega) \in C$ such that

$$x(\omega) = F(\omega, x(\omega)). \tag{5.4}$$

Now, we define $\widetilde{F_*} : \Omega \to \mathcal{P}(\widetilde{X})$ by

$$\widetilde{F_*}(\omega) = \{(x, x, \ldots, x) \in X : (x, \ldots, x) = h \circ F(\omega, h^{-1}(x))\}$$

where $h : X \to \widetilde{X}$ is defined by

$$h(x) = (x, \ldots, x), \quad x \in X.$$

and

$$\widetilde{X} = \{(x, \ldots, x) \in \prod_{i=1}^{n} X_i : x \in X, \ i = 1, \ldots, n\}.$$

Thus is a Banach space with the following norm

$$\|x\| = \sum_{i=1}^{n} \|x\|_i, \text{ for each } x \in X.$$

From () and h is homeomorphism mapping, we get

$$\widetilde{F}_*(\omega) \in \mathcal{P}_{cl}(\widetilde{X}) \text{ for all } \omega \in \Omega.$$

Since for each $\omega \in \Omega$, $F(\omega, C)$ is a compact, it follows that

$$\widetilde{F}_*(\omega) \in \mathcal{P}_{cp}(\widetilde{C}) \text{ for all } \omega \in \Omega,$$

where

$$\widetilde{C} = \{(x, \ldots, x) : x \in C\}.$$

Let K be a nonempty closed subset of $\widetilde{C}$, then

$$\begin{aligned}
\widetilde{F}_*^{-1}(K) &= \{\omega \in \Omega : \widetilde{F}_*(\omega) \cap K \neq \emptyset\} \\
&= \cup_{x \in K}\{\omega \in \Omega : (x, \ldots, x) = h \circ F(\omega, h^{-1}(x))\} \\
&= \cup_{x \in C}\{\omega \in \Omega : h^{-1}(x, \ldots, x) = F(\omega, h^{-1}(x))\}
\end{aligned}$$

and so

$$\widetilde{F}_*^{-1}(K) = \cap_{m=1}^{\infty} \cup_{x_i \in h^{-1}(K_m)} \left\{\omega \in \Omega : \|h^{-1}(x_i, \ldots, x_i) - F(\omega, h^{-1}(x_i, \ldots, x_i))\| < \epsilon_m\right\}$$

where

$$\epsilon_m := \left(\begin{array}{c} \frac{1}{m} \\ \cdots \\ \frac{1}{m} \end{array}\right) \text{ and } K_n = \left\{(x, \ldots, x) \in \widetilde{C} : d((x, \ldots, x), K) < \frac{1}{m}\right\}.$$

Then

$$\widetilde{F}_*^{-1}(K) = \bigcap_{m=1}^{\infty} \bigcup_{x_i \in h^{-1}(K_m)} F^{-1}(B(x_i, \epsilon_m), x_i) \in \mathcal{F}.$$

Then from theorem there exists a measurable function $x : \Omega \to C$ such that

$$x(\omega) = F(\omega, x(\omega)), \quad \omega \in \Omega.$$

□

By the above result we can present the following random nonlinear alternative.

Theorem 5.4.6 *Let X be a separable generalized Banach space and let $F : \Omega \times X \to X$ be a completely continuous random operator. Then either of the following holds:*

(i) the random equation $F(\omega, x) = x$ has a random solution, i.e., there is a measurable function $x : \Omega \to X$ such that $F(\omega, x(\omega)) = x(\omega)$ for all $\omega \in \Omega$, or

(ii) the set $\mathcal{M} = \{x : \Omega \to X \text{ is measurable} | \lambda(\omega)F(\omega, x) = x\}$ is unbounded for some measurable $\lambda : \Omega \to X$ with $0 < \lambda(\omega) < 1$ on Ω.

Finally, we prove a random Krasnoselskki-type fixed point theorem.

Theorem 5.4.7 *Let $C \subset X$ be a nonempty compact convex subset of a separable generalized Banach space X. Suppose that T, $B : \Omega \times C \to X$ are random operators such that*

($\mathcal{A}_1$) *T is a continuous random operator.*

($\mathcal{A}_2$) *B is a continuous random and $M(\omega)-$contraction operator.*

($\mathcal{A}_3$) *the matrix $I - M(\omega)$ has the absolute value property*

($\mathcal{A}_4$) *$B(\omega, C) + T(\omega, C) \subset C, \quad \omega \in \Omega.$*

Then $B + T$ has at least one random fixed point.

Proof. Let $\omega \in \Omega, y \in C$ and let $F_y : C \to C$ be given by

$$F_{\omega,y}(x) = B(\omega, x) + T(\omega, y).$$

From theorem there exists a unique fixed point of $F_{\omega,y}(.)$ and by theorem $(I - B(\omega, .))^{-1}$ exists. We define the operator $N_\omega : C \to C$ by $N_\omega(y) = (I-B(\omega, .))^{-1}T(\omega, y)$. It is easy to see that by the Schauder fixed point theorem N_ω has at least one fixed point. Define a mapping $S : \Omega \to \mathcal{P}(C)$ by

$$S(\omega) = \{y \in C : y = B(\omega, y) + T(\omega, y)\}.$$

Let K be closed subset of C then

$$\begin{aligned} S_-^{-1}(K) &= \{\omega \in \Omega : S(\omega) \cap K \neq \emptyset\} \\ &= \{\omega \in \Omega : y = B(\omega, y) + T(\omega, y),\ y \in K\}. \end{aligned}$$

Since X is a separable generalized Banach space, there exists $\{y_i : i \in \mathbb{N}\} \subset K$ such that

$$\overline{\{y_i : i \in \mathbb{N}\}} = K.$$

Hence

$$S_-^{-1}(K) = \bigcup_{i=1}^{\infty} \bigcap_{n=1}^{\infty} \{\omega \in \Omega : \|y_i - B(\omega, y_i) - T(\omega, y_i)\| < \epsilon_n\}.$$

where

$$\epsilon_n = \begin{pmatrix} \frac{1}{n} \\ \vdots \\ \frac{1}{n} \end{pmatrix}, \quad (x, y) \in X \times X.$$

Therefore $S_-^{-1}(K)$ is measurable. Since $B(\omega, .) + T(\omega, .)$ is a continuous mapping, it follows that $S(\omega) \in \mathcal{P}_{cl}(C)$ for every $\omega \in \Omega$. From theorem , there exists a measurable selection $y : \Omega \to C$ of S which is a random fixed point of $B + T$.

□

An immediate lemma to the theorem above in applicable form is as follows.

Lemma 5.4.1 *Let X be a separable generalized Banach space X. Suppose that T, $B : \Omega \times X \to X$ are random operators such that:*

($\bar{\mathcal{A}}_1$) *T is a continuous random operator.*

($\bar{\mathcal{A}}_2$) *B is a continuous random and $M(\omega)-$contraction operator.*

($\bar{\mathcal{A}}_3$) *the matrix $I - M$ has the absolute value property.*

If

$$\mathcal{M} = \{x : \Omega \to X \text{ is measurable}; \quad \lambda(\omega)T(\omega, x) + \lambda(\omega)B\left(\frac{x}{\lambda(\omega)}, \omega\right) = x\}$$

is bounded for all measurable mappings $\lambda : \Omega \to \mathbb{R}$ with $0 < \lambda(\omega)) < 1$ on Ω, then the random equation

$$x = T(\omega, x) + B(\omega, x), \quad x \in X,$$

has at least one solution.

5.5 Random Cauchy problem

In this section we shall use a random version of the Perov type and study the nonlinear initial value problems of random differential equations of first order for different aspects of the solutions under suitable conditions.

In this section we study the following systems

$$\begin{cases} x'(t, \omega) = f(t, x(t, \omega), y(t, \omega), \omega) \\ y'(t, \omega) = g(t, x(t, \omega), y(t, \omega), \omega) \\ x(0, \omega) = x_0(\omega), \ \omega \in \Omega \\ y(0, \omega) = x_0(\omega). \end{cases} \tag{5.5}$$

where $f, g : [0, b] \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$, $(\Omega, \mathcal{A})$ is a measurable space and $x_0, y_0 : \Omega \to \mathbb{R}$ are a random variable.

Definition 5.5.1 *A function $f : [0, b] \times \mathbb{R} \times \Omega \to \mathbb{R}$ is called random Carathéodory if the following conditions are satisfied:*

(*i*) *the map $(t, \omega) \to f(t, x, \omega)$ is jointly measurable for all $x \in \mathbb{R}$,*

(*ii*) *the map $x \to f(t, x, \omega)$ is continuous for all $t \in [0, b]$ and $\omega \in \Omega$.*

Definition 5.5.2 *A Carathéodory function $f : [0, b] \times \mathbb{R} \times \Omega \to \mathbb{R}$ is called random L^1-Carathéodory if for each real number $r > 0$ there is a measurable and bounded function $h_r \in L^1([0, b], \mathbb{R})$ such that*

$$|f(t, x, \omega)| \leq h_r(t, \omega), \quad a.e. \;\; t \in [0, b]$$

for all $\omega \in \Omega$ and $x \in \mathbb{R}$ with $|x| \leq r$.

Theorem 5.5.1 *(Carathéodory) Let X be a separable metric space and let $G : \Omega \times X \to X$ be a mapping such that $G(., x)$ is measurable for all $x \in X$ and $G(\omega, .)$ is continuous for all $\omega \in \Omega$. Then the map $(\omega, x) \to G(\omega, x)$ is jointly measurable.*

As consequence of the above theorem we can easily prove the following result.

Lemma 5.5.1 *Let X be a separable generalized metric space and $G : \Omega \times X \to X$ be a mapping such that $G(., x)$ is measurable for all $x \in X$ and $G(\omega, .)$ is continuous for all $\omega \in \Omega$. Then the map $(\omega, x) \to G(\omega, x)$ is jointly measurable.*

Proof. Let $C \subset X$ be a closed set of X, then

$$\begin{aligned} G^{-1}(C) &= \{(\omega, x) \in \Omega \times X : \; G(\omega, x) \in C\} \\ &= \{(\omega, x) \in \Omega \times X : \; d(G(\omega, x), C) = 0\} \end{aligned}$$

where

$$d(x, y) = \begin{pmatrix} d_1(x, y) \\ \vdots \\ d_n(x, y) \end{pmatrix}, \quad (x, y) \in X \times X.$$

Since X is a separable generalized metric space, there exists a countable dense set D of X. Let

$$C_m = \{x \in X : d(x, C) < \epsilon_m\},$$

where

$$\epsilon_m := \begin{pmatrix} \frac{1}{m} \\ \vdots \\ \frac{1}{m} \end{pmatrix}.$$

Hence $G(\omega, x) \in C$ if and only if for every $m \geq 1$ there exists an $a \in D$ such that

$$d(x, a) < \epsilon_m \text{ and } G(\omega, x) \in C_m,$$

therefore

$$G^{-1}(C) = \bigcap_{m \geq 1} \bigcup_{a \in D} \{\omega \in \Omega : G(\omega, a) \in C_m\} \times \{x \in X : d(x, a) < \epsilon_m\} \in \mathcal{A} \bigotimes B(X).$$

This implies that $G(., .)$ is joint measurable.

□

Theorem 5.5.2 *Let $f, g : [0, b] \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$ be two Carathédory functions. Assume that the following condition*

$(\mathcal{L}_1)$ *There exist random variables $p_1, p_2, p_3, p_4 : \Omega \to \mathbb{R}$ such that*

$$|f(t, x, y, \omega) - f(t, \widetilde{x}, \widetilde{y}, \omega)| \leq p_1(\omega)|x - \widetilde{x}| + p_2(\omega)|y - \widetilde{y}|$$

and

$$|g(t, x, y, \omega) - g(t, \widetilde{x}, \widetilde{y}, \omega)| \leq p_3(\omega)|x - \widetilde{x}| + p_4(\omega)|y - \widetilde{y}|$$

for all $x, y, \widetilde{x}, \widetilde{y} \in \mathbb{R}$
where

$$M(\omega) = \begin{pmatrix} bp_1(\omega) & bp_2(\omega) \\ bp_3(\omega) & bp_4(\omega) \end{pmatrix}.$$

If $M(\omega)$ converge to 0, then problem () has a unique random solution.

Proof. Consider the operator $N : C([0, b], \mathbb{R}) \times C([0, b], \mathbb{R}) \times \Omega \to C([0, b], \mathbb{R}) \times C([0, b], \mathbb{R})$ given by

$$(x, y) \mapsto (A_1(t, x, y, \omega), A_2(t, x, y, \omega))$$

where

$$A_1(x, y, \omega) \quad = \quad \int_0^t f(s, x(s, \omega), y(s, \omega), \omega)ds + x_0(\omega)$$

and

$$A_2(x, y, \omega) \quad = \quad \int_0^t g(s, x(s, \omega), y(s, \omega), \omega)ds + y_0(\omega).$$

First we show that N is a random operator on $C([0, b], \mathbb{R}) \times C([0, b], \mathbb{R})$. Since f and g are Carathédory functions, it follows that $\omega \to f(t, x, y, \omega)$ and $\omega \to g(t, x, y, \omega)$ are measurable maps in view of lemma . Further, the integral is a limit of a finite sum of measurable functions,therefore the following two maps are measurable:

$$\omega \to A_1(x(t, \omega), y(t, \omega), \omega), \ \omega \to A_2(x(t, \omega), y(t, \omega), \omega).$$

As a result, N is a random operator on $C([0, b], \mathbb{R}) \times C([0, b], \mathbb{R}) \times \Omega$ into $C([0, b], \mathbb{R}) \times C([0, b], \mathbb{R})$.

We show that the random operator N satisfies the conditions of theorem on $C([0, b], \mathbb{R}) \times C([0, b], \mathbb{R})$. Let $(x, y), (\widetilde{x}, \widetilde{y}) \in C([0, b], \mathbb{R}) \times C([0, b], \mathbb{R})$. Then

$$|A_1(t, x(t), y(t), \omega) - A_1(t, \widetilde{x}(t), \widetilde{y}(t), \omega)| \quad = \quad \left| \int_0^t f(s, x(s, \omega), y(s, \omega), \omega)ds \right.$$

$$
\begin{aligned}
& - \int_0^t f(s, \widetilde{x}(s,\omega), \widetilde{y}(s,\omega), \omega) ds \Bigg| \\
\leq & \int_0^t |f(s, x(s,x), y(s,\omega), \omega) \\
& - f(s, \widetilde{x}(s,\omega), \widetilde{y}(s,\omega), \omega)| ds \\
\leq & \int_0^t p_1(\omega) |x(s,\omega) - \widetilde{x}(s,\omega)| ds \\
& + \int_0^t p_2(\omega) |y(s,\omega) - \widetilde{y}(s,\omega)| ds
\end{aligned}
$$

and so

$$\|A_1(.,x,y,\omega) - A_1(t,\widetilde{x},\widetilde{y},\omega))\|_\infty \leq \|x-\widetilde{x}\|_\infty p_1(\omega)b + \|y-\widetilde{y}\|_\infty p_2(\omega)b.$$

Similarly, we obtain

$$\|A_2(x,y,\omega) - A_2(\widetilde{x},\widetilde{y},\omega)\|_\infty \leq \|x-\widetilde{x}\|_\infty p_3(\omega)b + \|y-\widetilde{y}\|_\infty p_4(\omega)b.$$

Hence

$$d(N(x,y,\omega), N(\widetilde{x},\widetilde{y},\omega)) \leq \widetilde{M}(\omega) d((x,y),(\widetilde{x},\widetilde{y}))$$

where

$$d(x,y) = \begin{pmatrix} \|x-y\|_\infty \\ \|x-y\|_\infty \end{pmatrix}$$

and

$$\widetilde{M}(\omega) = \begin{pmatrix} bp_1(\omega) & bp_2(\omega) \\ bp_3(\omega) & bp_4(\omega) \end{pmatrix} = bM(\omega).$$

From theorem there exists unique random solution of problem ().

□

The following result is known as Gronwal-Bihari Theorem.

Lemma 5.5.2 *[26] Let u, $\bar{g}\colon [a,b] \to \mathbb{R}$ be positive real continuous functions. Assume there exist $c > 0$ and a continuous nondecreasing function $\phi\colon \mathbb{R} \to (0,+\infty)$ such that*

$$u(t) \leq c + \int_a^t \bar{g}(s)\phi(u(s))\, ds, \quad \forall\, t \in J.$$

Then

$$u(t) \leq H^{-1}\left(\int_a^t \bar{g}(s)\, ds\right), \quad \forall\ t \in J$$

provided

$$\int_c^{+\infty} \frac{dy}{\phi(y)} > \int_a^b \bar{g}(s)\, ds.$$

Here H^{-1} refers to the inverse of the function $H(u) = \int_c^u \frac{dy}{\phi(y)}$ for $u \geq c$.

We consider the following set of hypotheses in what follows:

(H_1) The functions f and g are random Carathéodory on $[0,b]\times\mathbb{R}\times\Omega$

(H_2) There exist a measurable and bounded functions $\gamma_1,\gamma_2:\Omega\to L^1([0,b],\mathbb{R})$ and a continuous and nondecreasing function $\psi_1,\psi_2:\mathbb{R}_+\to(0,\infty)$ such that

$$|f(t,x,y)|\leq\gamma_1(t,\omega)\psi(|x|+|y|),\ |g(t,x,y)|\leq\gamma_2(t,\omega)\psi_2(|x|+|y|)\quad\text{a.e. } t\in[0,b]$$

for all $\omega\in\Omega$ and $x,y\in\mathbb{R}$.

Now,we give prove of the existence result of problem () by using Leary-Schauder random fixed point theorem type in generalized Banach space.

Theorem 5.5.3 *Assume that the hypotheses* (H_1) *and* (H_2) *hold. If*

$$\int_0^b|\gamma_1(\omega)|+|\gamma_2(\omega)|<\int_{|x_0(\omega)|+|y_0(\omega)|}^{\infty}\frac{du}{\psi_1(u)+\psi_2(u)},\quad \textit{for all } \omega\in\Omega.$$

Then the problem () has a random solution defined on $[0,b]$.

Proof. Let $N:C([0,b],\mathbb{R})\times\times\Omega\to C([0,b],\mathbb{R}])\times C([0,b],\mathbb{R})$ be a random operator defined in Theorem .

Clearly, the random fixe points of N are solutions to (). In order to apply theorem , we first show that N is completely continuous. The proof will be given in several steps.

- **Step 1.** $N(.,.,\omega)=(A_1(.,.,\omega),A_2(.,.,\omega))$ is continuous.
  Let (x_n,y_n) be a sequence such that $(x_n,y_n)\to(x,y)\in C([0,b],\mathbb{R})\times C([0,b],\mathbb{R})$ as $n\to\infty$. Then

  $$\begin{aligned}|A_1(x_n(t,\omega),y_n(t,\omega),\omega)-A_1(x(t,\omega),y(t,\omega),\omega)|\leq\ & \textstyle\int_0^t|f(s,x_n(s,\omega),y_n(s,\omega),\omega)\\ & -f(s,x(s,\omega),y(s,\omega),\omega)|ds\end{aligned}$$

  and so

  $$\begin{aligned}\|A_1(x_n(.,\omega),y_n(.,\omega),\omega)-A_1(x(.,\omega),y(.,\omega),\omega)\|_\infty\leq\ & \textstyle\int_0^b|f(s,x_n(s,\omega),y_n(s,\omega),\omega)\\ & -f(s,x(s,\omega),y(s,\omega),\omega)|ds\end{aligned}$$

  Since f is a Carathéodory function. By Lebesgue dominated convergence theorem we get

  $$\|A_1(x_n(.,\omega),y_n(.,\omega),\omega)-A_1(x(.,\omega),y(.,\omega),\omega)\|_\infty\to 0 \text{ as } n\to\infty.$$

  Similarly

  $$\|A_2(x_n(.,\omega),y_n(.,\omega),\omega)-A_2(x(.,\omega),y(.,\omega),\omega)\|_\infty\to 0 \text{ as } n\to\infty.$$

  Thus N is continuous.

- **Step 2.** N maps bounded sets into bounded sets in $C([0,b],\mathbb{R}) \times C([0,b],\mathbb{R})$. Indeed, it is enough to show that for any $q > 0$ there exists a positive constant l such that
$$\|N(x,y,\omega)\|_\infty \le l = (l_1, l_2)$$
for each $(x,y) \in B_q = \{(x,y) \in C([0,b],\mathbb{R}) \times C([0,b],\mathbb{R}) : \|x\|_\infty \le q,\ \|y\|_\infty \le q\}$.
Then for each $t \in [0,b]$, we get
$$\begin{aligned} |A_1(x(t,\omega), y(t,\omega),\omega)| &= \left| x_0(\omega) + \int_0^t f(s, x(s,\omega), y(s,\omega),\omega)ds \right| \\ &\le |x_0(\omega)| + \int_0^b |f_1(s, x(s,\omega), y(s,\omega))|ds. \end{aligned}$$
From (H_2), we get
$$\|A_1(x(.,\omega), y(.,\omega),\omega)\|_\infty \le |x_0(\omega)| + \psi_1(2q)\int_0^b \gamma_1(s,\omega)ds := l_1.$$
Similarly, we have
$$\|A_2(x(.,\omega), y(.,\omega),\omega)\|_\infty \le |y_0(\omega)| + \psi_2(2q)\int_0^b \gamma_2(s,\omega)ds := l_2.$$

- **Step 3.** N maps bounded sets into equicontinuous sets of $C([0,b],\mathbb{R}) \times C([0,b],\mathbb{R})$.
Let B_q be a bounded set in $C([0,b],\mathbb{R}) \times C([0,b],\mathbb{R})$ as in Step 2. Let $r_1, r_2 \in J, r_1 < r_2$ and $u \in B_q$. Thus we have
$$|A_1(x(r_2,\omega), y(r_2,\omega),\omega) - A_1(x(r_1,\omega), y(r_1,\omega),\omega)| \le \int_{r_1}^{r_2} |f(s, x(s,\omega), y(s,\omega),\omega)|ds$$
Hence
$$|A_1(x(r_2,\omega), y(r_2,\omega),\omega) - A_1(x(r_1,\omega), y(r_1,\omega),\omega)| \le \psi_1(2q)\int_{r_1}^{r_2} \gamma_1(s,\omega)ds$$
and
$$|A_2(x(r_2,\omega), y(r_2,\omega),\omega) - A_2(x(r_1,\omega), y(r_1,\omega),\omega)| \le \psi_2(2q)\int_{r_1}^{r_2} \gamma_2(s,\omega)ds.$$
The right-hand term tends to zero as $|r_2 - r_1| \to 0$. As a consequence of Steps 1 to 3 together with the Arzela-Ascoli, we conclude that N maps B_q into a precompact set in $C([0,b],\mathbb{R}) \times C([0,b],\mathbb{R})$.

- **Step 4.** It remains to show that

$$\mathcal{A} = \{(x,y) \in C([0,b],\mathbb{R}) \times C([0,b],\mathbb{R}) : \ (x,y) = \lambda(\omega)N(x,y), \lambda(\omega) \in (0,1)\}$$

is bounded.
Let $(x,y) \in \mathcal{A}$. Then $x = \lambda(\omega)A_1(x,y)$ and $y = \lambda(\omega)A_2(x,y)$ for some $0 < \lambda < 1$. Thus, for $t \in [0,b]$, we have

$$\begin{aligned} |x(t,\omega)| &\leq |x_0(\omega)| + \int_0^t |f_1(s,x(s,\omega),y(s,\omega),\omega)|ds \\ &\leq |x_0(\omega)| + \int_0^t \gamma_1(s,\omega)\psi_1(|x(s,\omega)| + |y(s,\omega)|)ds \end{aligned}$$

Hence

$$|x(t,\omega)| \leq |x_0(\omega)| + \int_0^t \gamma_1(s,\omega)\psi_1(|x(s,\omega)| + |y(s,\omega)|)ds$$

and

$$|y(t,\omega)| \leq |y_0(\omega)| + \int_0^t \gamma_2(s,\omega)\psi_2(|x(s,\omega)| + |y(s,\omega)|)ds.$$

Therefore

$$|x(t,\omega)| + |y(t,\omega)| \leq |x_0(\omega)| + |y_0(\omega)| + \int_0^t p(s)\phi(|x(s,\omega)| + |y(s,\omega)|)ds,$$

where

$$c = |x_0(\omega)| + |y_0(\omega)|, \phi = \psi_1 + \psi_2, \text{ and } p = \gamma_1 + \gamma_2.$$

By lemma , we have

$$|x(t,\omega)| + |y(t,\omega)| \leq \Gamma^{-1}\left(\int_c^b p(s)ds\right) := K_*, \text{ for each } t \in [0,b],$$

where

$$\Gamma(z) = \int_c^z \frac{du}{\phi(u)}.$$

Consequently

$$\|x\|_\infty \leq K_* \text{ and } \|y\|_\infty \leq K_*.$$

This shows that $\mathcal{A}$ is bounded. As a consequence of Theorem we deduce that N has a random fixed point (x,y) which is a solution to the problem ().

□

5.6 Boundary value problem

Nonlocal boundary problems arise in many applied sciences. For example, the vibrations of a guy wire of a uniform cross-section and composed of N parts of different densities, and some problems in the theory of elastic stability can be modelled by multi-point boundary value problems (see [69, 74]). The existence of solutions for systems of local and nonlocal boundary value problems (BVPs) has received an increased attention by researchers, see for example the papers of Agarwal, O'Regan and Wong [2, 3, 4], Henderson, Ntouyas and Purnaras [52], Precup [82, 83] and references therein. Very recently the coupled systems of BVPs with local and nonlocal conditions was studied, e.g, by Bolojan-Nica *et al* [27], Precup [84] and references therein.

In this section we interesting by a random solution of the following coupled systems:

$$\begin{cases} x''(t,\omega) = f(t,x(t,\omega),y(t,\omega),\omega), & t\in(0,1) \\ y''(t,\omega) = g(t,x(t,\omega),y(t,\omega),\omega), & t\in(0,1) \\ x(0,\omega)=0, \quad x(1,\omega) = L_1(\int_0^1 x(t,\omega)d\alpha(t)), & \omega\in\Omega \\ y(0,\omega)=0, \quad y(1,\omega) = L_2(\int_0^1 y(t,\omega)d\beta(t)), & \omega\in\Omega. \end{cases}$$

where $L_1, L_2 \in C(\mathbb{R},\mathbb{R})$ and $\int_0^1 x(t,\omega)d\alpha(t)$, $\int_0^1 x(t,\omega)d\alpha(t)$ denote the Riemann-Stieltjes integrals.

Lemma 5.6.1 *Let $f,g:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are two continuous functions*

$$\begin{cases} x''(t) = f(t,x(t),y(t)), & t\in(0,1) \\ y''(t) = g(t,x(t),y(t)), & t\in(0,1) \\ x(0)=0, \quad x(1) = L_1\left(\int_0^1 x(t)d\alpha(t)\right) \\ y(0)=0, \quad y(1) = L_2\left(\int_0^1 y(t)d\beta(t)\right). \end{cases}$$

where $L_1, L_2 \in C(\mathbb{R},\mathbb{R})$ and $\int_0^1 x(t)d\alpha(t)$, $\int_0^1 x(t)d\alpha(t)$ denote the Riemann-Stieltjes integrals. The couple $(x,y)\in C([0,1],\mathbb{R})\times C([0,1],\mathbb{R})$ is a solution of problem () if and only if

$$\begin{cases} x(t) = \int_0^1 \bar{K}(t,s)f(s,x(s),y(s))ds + L_1\left(\int_0^1 x(s)d\alpha(s)\right)t, & t\in(0,1) \\ y(t) = \int_0^1 \bar{K}(t,s)g(s,x(s),y(s))ds + L_2\left(\int_0^1 y(s)d\beta(s)\right)t, & t\in(0,1) \end{cases}$$

where

$$\bar{K}(t,s) = \begin{cases} t(1-s), & 0\le t\le s\le 1 \\ s(1-t), & 0\le s\le t\le 1. \end{cases}$$

Theorem 5.6.1 *Assume that $(\mathcal{L}_1)$ and the following condition*

(H_3) *there exist* $0 < K_1 < 1, 0 < K_2 < 0$ *two positives real constants such that*

$$|L_1(x) - L_1(y)| \leq K_1|x-y|, \quad |L_2(x) - L_2(y)| \leq K_1|x-y|,$$

for each $x, y \in \mathbb{R}$ *are satisfies. If*

$$M_*(\omega) = \begin{pmatrix} p_1(\omega) + K_1 & p_2(\omega) + K_2 \\ p_3(\omega) + K_1 & p_4(\omega) + K_2 \end{pmatrix}$$

converge to 0. *Then problem () has unique random solution.*

Proof. Consider the operator $N_* : C([0,b],\mathbb{R}) \times C([0,b],\mathbb{R}) \times \Omega \to C([0,b],\mathbb{R}) \times C([0,b],\mathbb{R})$,

$$(x, y, \omega) \mapsto (\bar{A}_1(t,x,y,\omega), \bar{A}_2(t,x,y,\omega))$$

where

$$A_1(t,x,y,\omega) = \int_0^1 \bar{K}(t,s) f(s, x(s,\omega), y(s,\omega), \omega) ds + L_1\left(\int_0^1 x(s,\omega) d\alpha(s)\right) t$$

and

$$\bar{A}_2(t,x,y,\omega) = \int_0^1 \bar{K}(t,s) g(s, x(s,\omega), y(s,\omega), \omega) ds + L_1\left(\int_0^1 y(s,\omega) d\alpha(s)\right) t.$$

Clear from lemma the random fixed point of N_* are solution of problem ().

As in theorem , we can prove that N_* has unique random fixed point which is solution of problem ()

□

Theorem 5.6.2 *Assume that* (H_1), (H_3) *with* $C_1 + K_1, \bar{C}_2 + K_2 \in [0,1)$ *and the following condition*

(H_4) *there exist* $C_1, C_2, C_3, \bar{C}_1, \bar{C}_2, \bar{C}_3 > 0$ *are positives real constants such that*

$$|f(t,x,y,\omega)| \leq C_1|x| + C_2|y| + C_3, \; |g(t,x,y,\omega)| \leq \bar{C}_1|x| + \bar{C}_2|y| + \bar{C}_3 \quad a.e. \;\; t \in [0,b]$$

for all $\omega \in \Omega$ *and* $x, y \in \mathbb{R}$, *hold. If*

$$\bar{M} = \begin{pmatrix} 1 - C_1 - K_1 & -C_2 \\ -\bar{C}_1 & 1 - \bar{C}_2 - K_2 \end{pmatrix}$$

if det$\bar{M} > 0$ *Then problem () has at least one random solution.*

Proof. Let $N : C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R}) \times \Omega \to C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$ be operator defined by

$$N(x,y,\omega) = F(x,y,\omega) + B(x,y,\omega), \quad (x,y,\omega) \in C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R}) \times \Omega,$$

where

$$F(x,y,\omega) = (F_1(x,y,\omega), F_2(x,y,\omega)),\ B(x,y,\omega) = (B_1(x,y,\omega), B_2(x,y,\omega))$$

and

$$F_1(t,x,y,\omega) = \int_0^1 \bar{K}(t,s) f(s,x(s,\omega),y(s,\omega),\omega)ds, \quad B_1(x,y,\omega) = L_1\left(\int_0^1 x(s,\omega)d\alpha(s)\right)t$$

$$F_2(t,x,y,\omega) = \int_0^1 \bar{K}(t,s) g(s,x(s,\omega),y(s,\omega),\omega)ds, \quad B_2(x,y,\omega) = L_2\left(\int_0^1 y(s,\omega)d\beta(s)\right)t.$$

Since $K_1, K_2 \in [0,1)$ then

$$\bar{M} = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}$$

converges to zero. This implies that B is contraction operator. By the same method used in theorem we can prove that F is completely continuous operator. Now, we show that the following set

$$\mathcal{M} = \{(x,y) : \Omega \to C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R}) \text{ is measurable} | \\ \lambda(\omega)F(x,y,\omega) + B(\frac{x}{\lambda(\omega)}, \frac{y}{\lambda(\omega)}, \omega) = (x,y)\}$$

is bounded for some measurable $\lambda : \Omega \to \mathbb{R}$ with $0 < \lambda(\omega)) < 1$ on Ω. Let $(x,y) \in \mathcal{M}$, then

$$x(t,\omega) = \lambda(\omega)\int_0^1 \bar{K}(t,s) f(s,x(s,\omega),y(s,\omega),\omega)ds + \lambda(\omega)L_1\left(\int_0^1 \frac{x(s,\omega)}{\lambda(\omega)}d\alpha(s)\right)t$$

and

$$y(t,\omega) = \lambda(\omega)\int_0^1 \bar{K}(t,s) g(s,x(s,\omega),y(s,\omega),\omega)ds + \lambda(\omega)L_2\left(\int_0^1 \frac{y(s,\omega)}{\lambda(\omega)}d\beta(s)\right)t.$$

Thus

$$|x(t,\omega)| \leq C_1|x(t,\omega)| + C_2|y(t,\omega)| + C_3 + K_1|x(t,\omega)| + |L_1(0)|$$

and

$$|y(t,\omega)| \leq \bar{C}_1|x(t,\omega)| + \bar{C}_2|y(t,\omega)| + \bar{C}_3 + K_2|y(t,\omega)| + |L_2(0)|.$$

This implies that

$$\begin{pmatrix} 1 - C_1 - K_1 & -C_2 \\ -\bar{C}_1 & 1 - \bar{C}_2 - K_2 \end{pmatrix} \begin{pmatrix} x(t,\omega) \\ y(t,\omega) \end{pmatrix} \leq \begin{pmatrix} C_3 + |L_1(0)| \\ \bar{C}_3 + |L_2(0)| \end{pmatrix}.$$

Therefore

$$\bar{M} \begin{pmatrix} x(t,\omega) \\ y(t,\omega) \end{pmatrix} \leq \begin{pmatrix} C_3 + |L_1(0)| \\ \bar{C}_3 + |L_2(0)| \end{pmatrix}. \tag{5.6}$$

Since $\bar{M}$ satisfies the hypotheses of lemma thus $(\bar{M})^{-1}$ is order preserving. We apply $(\bar{M})^{-1}$ to both sides of the inequality we obtain

$$\begin{pmatrix} x(t,\omega) \\ y(t,\omega) \end{pmatrix} \leq (\bar{M})^{-1} \begin{pmatrix} C_3 + |L_1(0)| \\ \bar{C}_3 + |L_2(0)| \end{pmatrix}.$$

Hence, from lemma , the operator N has at least one random fixed point which is solution of problem ()

□

5.7 Example

Let $\Omega = \mathbb{R}$ be equipped with the usual $\sigma-$ algebra consisting of Lebesgue measurable subsets of $(-\infty, 0)$ and $J := [0,1]$.
Consider the following random differential equation system.

$$\begin{cases} x'(t,\omega) = \frac{t\omega^2 x^2(t,\omega)}{2(1+\omega^2)(1+x^2(t,\omega)+y^2(t,\omega))} \\ y'(t,\omega) = \frac{t\omega^2 y^2(t,\omega)}{2(1+\omega^2)(1+x^2(t,\omega)+y^2(t,\omega))} \\ x(0,\omega) = \sin\omega, \ \ \omega \in \Omega \\ y(0,\omega) = \cos\omega. \end{cases} \tag{5.7}$$

Here

$$f(t,x,y,\omega) = \frac{t\omega^2 x^2}{2(1+\omega^2)(1+x^2+y^2)}$$

$$g(t,x,y,\omega) = \frac{t\omega^2 y^2}{2(1+\omega^2)(1+x^2+y^2)}$$

Clearly, the map $(t,\omega) \mapsto f(t,x,y,\omega)$ is jointly continuous for all $x, y \in [1,\infty)$. The same for the map g. Also the maps $x \mapsto f(t,x,y,\omega)$ and $y \mapsto f(t,x,y,\omega)$ are continuous for all $t \in J$ and $\omega \in \Omega$. Similarly for the maps corresponding to function g. Thus the functions f and g are Carathéodory on $J \times [1,\infty) \times [1,\infty) \times \Omega$. Firstly, we show that f and g are Lipschitz functions. Indeed, let $x, y \in \mathbb{R}$, then

$$\begin{aligned} |f(t,x,y,\omega) - f(t,\widetilde{x},\widetilde{y},\omega)| &= \left| \frac{t\omega^2 x^2}{2(1+\omega^2)(1+x^2+y^2)} - \frac{t\omega^2 x^2}{2(1+\omega^2)(1+\widetilde{x}^2+\widetilde{y}^2)} \right| \\ &= \left| \frac{t\omega^2[(1+\widetilde{x}^2+\widetilde{y}^2)x^2 - (1+x^2+y^2)\widetilde{x}^2}{2(1+\omega^2)(1+x^2+y^2)(1+\widetilde{x}^2+\widetilde{y}^2)} \right| \end{aligned}$$

$$
\begin{aligned}
&= \frac{t\omega^2}{2(1+\omega^2)(1+x^2+y^2)(1+\widetilde{x}^2+\widetilde{y}^2)} \left|x^2+\widetilde{y}^2x^2-\widetilde{x}^2-y^2\widetilde{x}^2\right| \\
&\leq \frac{\omega^2}{2(1+\omega^2)}|x-\widetilde{x}| + \frac{\omega^2}{2(1+\omega^2)}|y-\widetilde{y}|
\end{aligned}
$$

Then

$$|f(t,x,y,\omega)-f(t,\widetilde{x},\widetilde{y},\omega)| \leq \frac{\omega^2}{2(1+\omega^2)}|x-\widetilde{x}| + \frac{\omega^2}{2(1+\omega^2)}|y-\widetilde{y}|.$$

Analogously for the function g, we get

$$|g(t,x,y,\omega)-g(t,\widetilde{x},\widetilde{y},\omega)| \leq \frac{\omega^2}{2(1+\omega^2)}|x-\widetilde{x}| + \frac{\omega^2}{2(1+\omega^2)}|y-\widetilde{y}|.$$

We take,

$$p_1(\omega)=p_2(\omega)=p_3(\omega)=p_4(\omega)=\frac{\omega^2}{2(1+\omega^2)}$$

and

$$M(\omega)=\begin{pmatrix} \frac{\omega^2}{2(1+\omega^2)} & \frac{\omega^2}{2(1+\omega^2)} \\ \frac{\omega^2}{2(1+\omega^2)} & \frac{\omega^2}{2(1+\omega^2)} \end{pmatrix}.$$

We remark that

$$|\rho(M(\omega))| = \frac{\omega^2}{2(1+\omega^2)} < 1,$$

then

$$M(\omega), \text{ converge to } 0.$$

Therefore, all the conditions of theorem are satisfied. Hence the problem () has a unique random solution.

Conclusion and perspective

This thesis is devised in two parts, in the first we considered the problem of the existence of integral solution for some class of nonlinear evolution inclusions with impulses effect, in the case where nonlinearity may be either convex or non convex. Our results are obtained by means of technique of fixe point theorems. In the second part, we proved some random fixed point theorems in generalized Banach spaces (random version of Pevor-type, Schauder-type and Krasnoselskii-type fixed point theorems). The results are used to prove the existence of solution for random differential equations with initial and boundary conditions.
We will study propriety of the integrals solutions set of the problem considered in the thesis. And we will study the problem of the existence of integral solution for some class of random nonlinear evolution inclusions with impulses effect.

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