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SIDI BEL ABBES

THESE DE DOCTORAT

Présentée par

OUARDANI ABDERRAHMANE

Spécialité : Mathématiques
Option : Equations différentielles ordinaires

Intitulée

Sur le problème des solutions périodiques des équations et inclusions différentielles à retard et de type neutre dans un espace de Banach

Soutenue le 21 Juin 2018
Devant le jury composé de :

Président :	*Benchohra Mouffak*	*Pr*	*Univ. Sidi Bel Abbès*
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Dédicace

À la mémoire de mon père.

À ma mère.

À ma famme.

À mes enfants;
Imad eddine...
Yasser bahaa eddine...
Sid ahmed kamel eddine...
Khaldia belkis...

Contents

Chapter 1

Introduction

The study of differential equations and inclusions with infinite delay has progressed enormously, whether in finite or infinite dimension. This is due to their efficiency in the description of many phenomena in nature compared with equations and inclusions without delay. Since the 70s, the theory of the functional differential equation and inclusions with infinite delay in a finite dimensional Banach space has been developed quickly, a lot of important results have been obtained see [18, 43, 46] and the references therein. The study of functional differential equations and inclusions with infinite delay in a Banach space is more complicated (see [2, 7, 8, 11, 21, 33, 39, 40, 46, 55]), the most general approach, initiated by Hale and Kato [27], consists of taking the initial condition in an abstract seminormed space phase space. This kind of space is defined axiomatically, a concrete choice of this space depends on the particular problem under investigation. Note that such a space has been considered as a phase space in the theory of functional differential equations with infinite delay in [17, 37, 53].

The main objective of this dissertation is to establish some qualitative results in relation with the periodic problem for semilinear functional differential equations and inclusions with infinite delay in a Banach space.

More precisely:

1.1 Existence of periodic mild solutions

Let us consider the periodic boundary value problem for semilinear functional differential nonlinear inclusion with infinite delay, described in the form

$$\begin{cases} x'(t) \in A\,x(t) + F(t, x_t), \sigma < t \leq \sigma + T, \\ x_\sigma = x_{\sigma+T}, \end{cases} \tag{1.1}$$

where, E is a Banach space $A : \mathcal{D}(A) \to E$ is a linear operator not necessarily bounded, $T > 0$ is a fixed time horizon, σ a real number and $F : [\sigma, \sigma + T] \times$

$\mathcal{B} \to Kv(E)$ is an upper-Carathéodory multivalued. Here $\mathcal{B}$ denotes the phase space that will be introduced later.

For $z :]-\infty, +\infty[\to E$, for $t \geq 0$, the function z_t belongs to $\mathcal{B}$ and, is defined by, $z_t(\theta) = z(t+\theta)$, $-\infty < \theta \leq 0$.

Notice that, if $\hat{F}$ is a T-periodic extension of F on the first argument, i.e.,

$$\hat{F}(t,\varphi) = F(t,\varphi), \quad \text{a.e. } t \in [\sigma, \sigma+T], \quad \varphi \in \mathcal{B},$$
$$\hat{F}(t+T,\varphi) = \hat{F}(t,\varphi), \quad \text{a.e. } t \geq \sigma, \quad \varphi \in \mathcal{B},$$

then, the study of the periodic problem

$$x'(t) = Ax(t) + \hat{F}(t, x_t), \quad t \geq \sigma, \tag{1.2}$$

is equivalent to the study of the boundary periodic value problem (1.1).

The periodic problem (1.1) (or equivalently, the periodic boundary value problem (1.2)) was studied by developing the method of the multivalued translation operator (or the translation operator in the single valued case) along the trajectories of solutions. We only mention some works [20, 21, 22, 30, 31, 47, 48]. For more details on the topic see [49] or [62].

In the single valued case the main difficulty occurring in the study of such problems is due to the fact that the translation operator is not compact, even in the case where A generates a compact semigroup. In the multivalued case, another difficulty is added because the multivalued translation operator has non-convex values.

For the study of the problem (1.1) we construct an integral multivalued operator whose fixed points coincide with the (periodic)mild solution and we study its properties. Then, results on the existence and topological structure of the (periodic) mild solutions set are established.

Our constructed operator allows a simpler study of the problem (1.1) and requires less conditions, even in the single valued case, compared to the translation operator. This is due to two reasons:

1. Our constructed operator is condensing under a very simple condition on the phase space without using the data of the studied problem (1.1), as a consequence, it makes possible to give examples, in a very simple way, on the choices of the phase spaces for which the study of the problem (1.1) can be achieved.

2. Our constructed operator is convex valued.

Let us mention that our approach can be considered as a nontrivial direct attempt to extend the approach developed in the recent paper [24], where the periodic problem for functional fully nonlinear differential equations with finite delay was studied.

1.2 The averaging principle

The averaging principle is one of the most efficient methods for the investigation of oscillation phenomena described by differential equations with a small parameter (see [52] and the introduction in [51] for the history of the theory of averaging). It consists in replacing the non autonomous right hand side of an equation with small parameter by its average in time. A key step in the development of this principle started with the works of Bogoliubov-Mitropolskii [13] and Krylov-Bogoliubov [41], where a rigorous justification of this principle was given for finite dimensional nonlinear systems in standard form. Later this principle was justified for several kinds of finite and infinite dimensional differential and functional differential equations with finite delay (see *e.g.*[14, 26, 28, 35, 36, 42, 44, 45, 51, 52, 57, 59] and the references therein).

Among the effective methods for solving the Cauchy problem and proving the averaging principle for infinite dimensional differential equations and inclusions, may be found the method based on the theory of measures of non-compactness (see [4, 19, 36]).

To the best of our knowledge, no averaging result were established for semilinear functional differential equations with infinite delay in a Banach space even in the case of uniqueness of the mild solutions.

A case of non-uniqueness of the mild solutions.

Let us consider a semilinear functional differential equation in a Banach space E with small parameter $\varepsilon > 0$,

$$\begin{cases} z'(t) = A\, z(t) + f(\frac{t}{\varepsilon}, z_t), \quad t \in [0, T], \\ z_0 = \varphi, \end{cases} \tag{P_ε}$$

where the unbounded linear A part generates a noncompact semigroup and the nonlinear part f is single valued map satisfying some boundedness and χ-regularity conditions.

Using the method based on the theory of measures of non-compactness in [21, Theorem 3], a theorem on the existence of mild solutions to (P_ε), $\varepsilon > 0$ was established (with f a multivalued map). One our objectives in this dissertation is to establish averaging results for such i.e., we aim to establish averaging results for the solutions given by [21, Theorem 3] in the single valued case.

Parallel to the problem (P_ε), $\varepsilon > 0$, we consider the averaged problem:

$$\begin{cases} z'(t) = A\, z(t) + f_0(z_t), \quad t \in [0, T], \\ \quad z_0 = \varphi. \end{cases} \tag{P_0}$$

where $f_0 : \mathcal{B} \to E$ is a function for which there exists $\Delta_0 > 0$ such that for all $u \in \mathcal{B}$ and $t_1,\, t_2 \in [0, T]$ with $0 \le t_1 \le t_2 \le t_1 + \Delta_0$,

$$\lim_{\varepsilon \to 0^+} \int_{t_1}^{t_2} S(t_2 - \theta) \left[f(\frac{\theta}{\varepsilon}, u) - f_0(u) \right] d\,\theta = 0. \tag{1.3}$$

Such a condition (1.3) is taken from [54]. We aim to establish a variant of the averaging principle for the problem $(P_\varepsilon)_{\varepsilon>0}$ (Theorem 4.2.1 and Proposition 4.2.1). Note that, because of the lack of the uniqueness of the solutions, the traditional approach, which consists in showing directly that the solutions $(z^\varepsilon)_{\varepsilon>0}$ of the problems $(P_\varepsilon)_{\varepsilon>0}$, converge to the unique solution z^∞ of the problem (P_0) as $\varepsilon \to 0^+$ cannot be applied here.

We replace the condition (1.3) by a more natural hypothesis, that is,

$$f_0(u) = \lim_{t \to +\infty} \frac{1}{t} \int_0^t f(s, u)\, ds,$$

then, we establish a variant of the classical averaging principle (Theorem 4.3.1). Further, we discuss an example that illustrates the applicability of our abstract results.

It should be noted that before establishing our results we explain under which conditions on the space of phase space $\mathcal{B}$ the problem in the normal form

$$\begin{cases} z'(\tau) = \varepsilon \Big[A\, z(\tau) + f(\tau, z_\tau) \Big], & \tau \ge 0, \\ z_0 = \psi \in \mathcal{B}, \end{cases}$$

can be written equivalently as (P_ε).

A case of uniqueness of the mild solutions Let us consider again a semi-linear functional differential equation in E with a small positive parameter ε,

$$\begin{cases} x'(t) = A\, x(t) + f\Big(\frac{t}{\varepsilon}, x_t\Big), & t \in [0, T], \\ x_0 = \varphi. \end{cases} \tag{P'_ε}$$

Parallel to the problem (P'_ε), $\varepsilon > 0$, we consider the averaged problem:

$$\begin{cases} x'(t) = A\, x(t) + f_0(x_t), & t \in [0, T], \\ \quad x_0 = \varphi, \end{cases} \tag{P_0}$$

where, A generates a strongly continuous semigroup $(S(t))_{t\ge 0}$, the function f satisfies a condition with respect to the second argument which is more weaker than the usual Lipschitz condition. The approach which allows to justify the averaging principle for the problem (P'_ε), $\varepsilon > 0$ is exactly the same as that developed for the problem (P_ε), $\varepsilon > 0$ above, but we can not use the same argument in order to establish averaging results. To do that, one can extends the arguments used in [54] to the problem (P'_ε), $\varepsilon > 0$. This was done in this dissertation.

The dissertation is separated into four main chapters. The last three chapters represent the contribution of this dissertation:

Chapter 2. We introduce notations, definitions and we recall some necessary preliminaries from the theory of condensing maps and multivalued analysis which will be used throughout this dissertation.

Chapter 3. We give results on the existence and topological structure of the mild solutions set for the periodic boundary value problem for semilinear functional differential nonlinear inclusion with infinite delay, described in the form

$$\begin{cases} x'(t) \in A\,x(t) + F(t, x_t), \sigma < t \leq \sigma + T, \\ x_\sigma = x_{\sigma+T}, \end{cases}$$

where, A is a linear operator generating a compact semigroup, and F is a upper-Carathéodory multivalued.

We construct a new integral multivalued operator whose fixed points coincide with the (periodic)mild solution and we study its properties.

Chapter 4. We establish averaging results for the problem

$$\begin{cases} z'(t) = A\,z(t) + f(\frac{t}{\varepsilon}, z_t), \quad t \in [0, T], \\ z_0 = \varphi, \end{cases}$$

where the unbounded linear A part generates a noncompact semigroup and the nonlinear part f is single valued map satisfying some boundedness and χ-regularity conditions. Then we establish the averaging principle.

The major difficulty in this chapter comes from the lack of the unicity of the mild solutions, the traditional approach cannot be used.

Chapter 5. We study the same problem described in the previous chapter, but here we suppose that the function f satisfies a condition with respect to the second argument which is more weaker than the usual Lipschitz condition.

Chapter 2

Notations and Preliminaries

2.1 Multivalued maps.

The notion of multimap arises naturally in various branches of modern mathematics, such as mathematical economics, theory of games, convex analysis, etc. Multimaps play a significant role in the description of processes in control theory since the presence of controls provides an intrinsic multivalence in the evolution of the system. For more details see *e.g.*[5, 6, 12, 16, 23, 36, 58].

2.1.1 Definitions and examples.

Definition 2.1.1. *A multivalued map (multimap) F of a set X into a set Y is a correspondence which associates to every $x \in X$ a non-empty subset $F(x) \subseteq Y$, called the value of x. We denote this correspondence as $F : X \longrightarrow \mathcal{P}(Y)$, where $\mathcal{P}(Y)$ is the collection of all non-empty subsets of Y.*
• *If $A \subseteq X$, then the set $F(A) = \bigcup_{x \in A} F(x)$ is called the image of A under F.*
• *For $D \subseteq Y$, the set $F_{+}^{-1}(D) = \{x \in X : F(x) \subset D\}$ is called small preimage of the set D under F.*
• *For $D \subseteq Y$, the set $F_{-}^{-1}(D) = \{x \in X : F(x) \cap D \neq \emptyset\}$ is called complete preimage of the set D under F.*
• *The set $\Gamma_F \subseteq X \times Y$, defined by $\Gamma_F = \{(x, y) : x \in X,\ y \in F(x)\}$ is the graph of F.*
• *A multivalued map F is called closed (resp.convex) if and only if its graph Γ_F is closed (resp. Convex) subset of the space $X \times Y$.*

Example 2.1.1. *The most natural example is that induced by a function (In the classical sense) surjective. Let f be a function defined from X to Y. We define the preimage f^{-1} as a multivalued map, which associates to every*

$y \in Y$, the set of solutions of equation $f(x) = y$;

$$f^{-1}(y) = \{x \in X : f(x) = y\}.$$

Remark 2.1.1. *The multivalued map F is said univalued if for all $x \in X$ the set $F(x)$ is an singleton $\{y\}$. We write in this case $F(x) = y$ instead of $F(x) = \{y\}$.*

We give some properties of the image and of the preimage.

Proposition 2.1.1. *Let $F : X \to \mathcal{P}(Y)$ a multimap, $A \subset X$ and $B \subset Y$ then we have :*

1) $F^{-1}(F(A)) \supset A$;

2) $F(F^{-1}(B)) \subset B$;

3) $X \setminus F^{-1}(B) \supset F^{-1}(Y \setminus B)$;

4) $F_+^{-1}(F(A)) \supset A$;

5) $F(F_+^{-1}(B)) \supset B \cap F(X)$;

6) $X \setminus F_+^{-1}(B) = F^{-1}(Y \setminus B)$.

2.1.2 Continuity of multimaps.

The classical concept of continuity is divided into several concepts in the case of multivalued.

2.1.3 Upper semicontinuity.

Definition 2.1.2. *Let X and Y be topological spaces. A multimap $F : X \to \mathcal{P}(Y)$ is upper semicontinuous at the point $x \in X$ if, for every open set $W \subseteq Y$ such that $F(x) \subset W$, there exists a neighborhood $V(x)$ of x with the property that $F(V(x)) \subset W$.*
A multimap is upper semicontinuous (u.s.c) if it is upper semicontinuous at every point $x \in X$.

Example 2.1.2. *The multimap $F : [0,1] \to \mathcal{P}([0,1])$, defined as*

$$F(x) = \begin{cases} [0, \frac{1}{2}], & \text{if } x \neq \frac{1}{2} \\ [0,1], & \text{if } x = \frac{1}{2} \end{cases}$$

is u.s.c.

Theorem 2.1.1. *The following conditions are equivalent:*

1. *the multimap F is u.s.c;*

2. *the set $F_+^{-1}(W)$ is open for every open set $W \subset Y$;*

3. *the set $F_-^{-1}(Q)$ is closed for every closed set $Q \subset Y$;*

4. $F_-^{-1}(\overline{D}) \supseteq \overline{F_-^{-1}(D)}$ *for every set $D \subset Y$.*

2.1.4 Lower semicontinuity.

Definition 2.1.3. *Let X and Y be topological spaces. A multimap $F : X \to \mathcal{P}(Y)$ is lower semicontinuous at the point $x \in X$ if, for every open set $W \subseteq Y$ such that $F(x) \cap W \neq \emptyset$, there exists a neighborhood $V(x)$ of x with the property that $F(x') \cap W \neq \emptyset$ for all $x' \in V(x)$.*
A multimap is lower semicontinuous (l.s.c) if it is lower semicontinuous at every point $x \in X$.

Example 2.1.3. *The multimap $F : [0, 1] \to \mathcal{P}([0, 1])$, defined as*

$$F(x) = \begin{cases} [0,1], & \text{if } x \neq \frac{1}{2} \\ \left[0, \frac{1}{2}\right], & \text{if } x = \frac{1}{2} \end{cases}$$

is l.s.c.

Theorem 2.1.2. *The following conditions are equivalent:*

1. *the multimap F is l.s.c;*

2. *the set $F_-^{-1}(W)$ is open for every open set $W \subset Y$;*

3. *the set $F_+^{-1}(Q)$ is closed for every closed set $Q \subset Y$;*

4. *if the system of open sets $\{W_j\}_{j \in J}$ forms a base for the topology of Y, then every set $F_-^{-1}(W_j)$ is open;*

5. $F_+^{-1}(\overline{D}) \supseteq \overline{F_+^{-1}(D)}$ *for every set $D \subset Y$;*

6. $F(\overline{A}) \subseteq \overline{F(A)}$ *for every set $A \subset X$;*

7. *for any $x \in X$, if $\{x_\alpha\} \subset X$ is a generalized sequence, $x_\alpha \longrightarrow x$, then for every $y \in F(x)$ there exists a generalized sequence $\{y_\alpha\} \subset Y$, $y_\alpha \in F(x_\alpha)$, $y_\alpha \longrightarrow y$.*

Remark 2.1.2. *When X and Y are metric spaces, in condition (7.) we can take sequences instead of generalized sequences.*

Definition 2.1.4. *A multimap F which is both upper and lower semicontinuous is said to be continuous.*

Example 2.1.4. *Let $u, v : [0,1] \longrightarrow \mathbb{R}$ be continuous functions such that $u(x) \leq v(x)$ for all $x \in [0,1]$. Then the multimap $F : [0,1] \longrightarrow \mathcal{P}(\mathbb{R})$, $F(x) = [u(x), v(x)]$ is continuous.*

Theorem 2.1.3. *The following properties are equivalent:*

1. *the multimap F is closed;*

2. *for any pair $x \in X$, $y \in Y$ such that $y \notin F(x)$ there exist neighborhoods $V(x)$ of x and $W(y)$ of y such that $F(V(x)) \cap W(y) = \emptyset$;*

3. *for any generalized sequences $\{x_\alpha\} \subset X$, $\{y_\alpha\} \subset Y$ if $x_\alpha \longrightarrow x$, and $y_\alpha \in F(x_\alpha)$, $y_\alpha \longrightarrow y$ then $y \in F(x)$.*

Example 2.1.5. *Let X and Y be topological spaces and X be Hausdorff, $f : Y \longrightarrow X$ is a continuous surjective map. Then the inverse multimap $F : X \longrightarrow \mathcal{P}(Y)$, $F(x) = f^{-1}(x)$ is closed.*

Theorem 2.1.4. *Let X be a topological space, Y a regular topological space and $F : X \longrightarrow C(Y)$ an u.s.c. multimap. Then F is closed.*
Where $C(Y) = \{D \in \mathcal{P}(Y) : D \text{ is closed}\}$.

Definition 2.1.5. *A multimap $F : X \longrightarrow \mathcal{P}(Y)$ is*

a. ***compact*** *if its range $F(X)$ is relatively compact in Y, i.e., $\overline{F(X)}$ is compact in Y;*

b. ***locally compact*** *if every point $x \in X$ has a neighborhood $V(x)$ such that the restriction of F to $V(x)$ is compact;*

c. ***quasicompact*** *if its restriction to any compact subset $A \subset X$ is compact.*

Remark 2.1.3. *It is clear that (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c).*

Lemma 2.1.1. *Let X and Y be metric spaces and $F : X \to K(Y)$ be a closed quasicompact multimap. Then F is u.s.c., where $K(Y) = \{D \in \mathcal{P}(Y) : D \text{ is compact}\}$.*

Proof. Set $D \subset Y$ and $x \in \overline{F_-^{-1}(D)}$, then there exists $\{x_\alpha\}_\alpha \subset F_-^{-1}(D)$, such that $x_\alpha \longrightarrow x$, Let $\{y_\alpha\}_\alpha \subset D$, with $y_\alpha \longrightarrow y$ and there exist a subsequence $\{y_{\alpha_i}\}_{\alpha_i}$, such that $y_{\alpha_i} \in F(x_\alpha)$, but F is closed then $y \in F(x) \cap \overline{D}$, hence $x \in F_-^{-1}(\overline{D})$. $\square$

Theorem 2.1.5. *Let $F : X \to C(Y)$ a closed multimap. If $A \subset X$ is a compact set then its image $F(A)$ is a closed subset of Y.*

Definition 2.1.6. *A multimap $F : X \longrightarrow \mathcal{P}(Y)$ is said to be* ***quasi-open*** *at a point $x \in X$ if $intF(x) \neq \emptyset$ and for every $y \in intF(x)$ there exist neighborhoods $W(y) \subset Y$ and $V(x) \subset X$ such that $W(y) \subset F(x')$ for all $x' \in V(x)$.*
A multimap F is said to be ***quasi-open*** *provided it is quasi-open at every point $x \in X$.*

Theorem 2.1.6. *Let X be a topological space and Y a normed space. A multimap $F : X \longrightarrow C_v(Y)$ is quasi-open at a point $x \in X$ if and only if $intF(x) \neq \emptyset$ and F is l.s.c. at x. Where $C_v(Y) = \{D \in \mathcal{P}(Y) : D \text{ is closed and convex}\}$.*

2.1.5 Selections of a multimap.

Definition 2.1.7. *Let X and Y be sets. A single valued map $f : X \longrightarrow Y$ is said to be a selection of a multimap $F : X \longrightarrow \mathcal{P}(Y)$ if*

$$f(x) \in F(x), \text{ for all } x \in X.$$

Theorem 2.1.7. *Let X be a metric space, Y a Banach space. Then every l.s.c. multimap $F : X \to C_v(Y)$ admits a continuous selection.*

2.1.6 Measurable multimaps.

Let $I \subset \mathbb{R}$ be a compact interval, μ a Lebesgue measure on I and $\mathcal{E}$ a Banach space.

Definition 2.1.8. *A multimap $F : I \longrightarrow K(\mathcal{E})$ is said to be measurable if for every open subset $W \subset \mathcal{E}$ the set $F_+^{-1}(W)$ is measurable.*
Recall that $F_+^{-1}(W) = \{x \in I : F(x) \subset W\}$.

Lemma 2.1.2. *Every u.s.c. or l.s.c. multimap is measurable.*

Definition 2.1.9. *A function $f : I \longrightarrow K(\mathcal{E})$ is said to be a* ***measurable selection*** *of a multimap $F : I \longrightarrow K(\mathcal{E})$ if f is measurable and*

$$f(x) \in F(x), \text{ for } \mu\text{-a.e. } x \in I.$$

The set of all measurable selections of F will be denoted as S_F.

The multimap $\widetilde{F} : I \longrightarrow K(\mathcal{E})$ is a **step multimap** if there exists a partition of I into a finite family of disjoint measurable subsets $\{I_j\}$, $\bigcup_j I_j = I$ such that $\widetilde{F}$ is constant on each I_j.

Definition 2.1.10. *A multimap $F : I \longrightarrow K(\mathcal{E})$ is said to be a* ***strongly measurable*** *if there exists a sequence $\{F_n\}_{n=1}^{\infty}$ of step multimap such that*

$$h\left(F_n(t), F(t)\right) \longrightarrow 0$$

as $n \longrightarrow \infty$ for μ-a.e. $t \in I$ where h is the Hausdorff metric on $K(\mathcal{E})$,

$$h\left(F_n(t), F(t)\right) = \max\left\{\sup_{a\in F_n(t)} d(a, F(t)),\ \sup_{b\in F(t)} d(F_n(t), b)\right\},$$

d the metric defined on $\mathcal{E}$.

Remark 2.1.4. *A measurable multimap is not necessarily strongly measurable. But for multimap with compact values acting in a separable Banach space, these two notions coincide.*

2.1.7 Integrable multimaps .

Definition 2.1.11. *Let*

$$S_F^1 = \left\{f \in L^1(I, \mathcal{E}) : f(t) \in F(t)\, for\, \mu\text{-a.e.}\, t \in I\right\}.$$

If $S_F^1 \neq \emptyset$ then the multimap $F : I \longrightarrow \mathcal{P}(\mathcal{E})$, is called integrable and

$$\int_\Omega F(s)\, ds := \left\{\int_\Omega f(s)\, ds : f \in S_F^1\right\}$$

for any measurable set $\Omega \subset I$.

Lemma 2.1.3. *If the multimap $F : I \longrightarrow K(\mathcal{E})$ is*

(i) *F strongly measurable;*

(ii) *there exists $\alpha \in L_+^1(I)$ such that*

$$\| F(t) \| := \max\{\| y \| : y \in F(t)\} \leq \alpha(t),\ for\, \mu\text{-a.e.}\, t \in I.$$

Then F is integrable.

2.1.8 The Carathéodory conditions.

Let E, E_0 be separable Banach spaces.

Definition 2.1.12.

(a) *A multimap $F : I \times E_0 \longrightarrow K(E)$ verifies the upper Carathéodory conditions if*

- (i) *for every $x \in E_0$ the multimap $F(\cdot, x) : I \longrightarrow K(E)$ is measurable;*
- (ii) *for almost every $t \in I$ the multimap $F(t, \cdot) : E_0 \longrightarrow K(E)$ is u.s.c.*

(b) *A multimap $F : I \times E_0 \longrightarrow K(E)$ verifies the Carathéodory conditions if condition (i) holds and condition*

- (ii') *for almost every $t \in I$ the multimap $F(t, \cdot) : E_0 \longrightarrow K(E)$ is continuous is satisfied.*

2.1.9 The superposition multimap.

Every multimap $F : I \times E_0 \longrightarrow \mathcal{P}(E)$ induces a correspondence assigning to every multifunction $Q : I \longrightarrow \mathcal{P}(E_0)$ the multifunction $\Phi : I \longrightarrow \mathcal{P}(E)$ defined by the formula

$$\Phi(t) = F(t, Q(t)).$$

Our object now is to give some properties of this correspondence.

Theorem 2.1.8. *If a multimap $F : I \times E_0 \longrightarrow K(E)$ satisfies the Carathéodory conditions then F is superpositionally measurable in the sense that for every measurable multifunction $Q : I \longrightarrow K(E_0)$ the multifunction Φ is measurable.*

Now we give a sufficient condition for superpositional measurability.

Proposition 2.1.2. *If a multimap $F : I \times E_0 \longrightarrow K(E)$ is u.s.c. or l.s.c. then it is superpositionnally measurable.*

Theorem 2.1.9. *Let E, E_0 be (not necessarily separable) Banach spaces; and let the multimap $F : I \times E_0 \longrightarrow K(E)$ be such that*

- *(i) for every $x \in E_0$ the multifunction $F(\cdot, x) : I \longrightarrow K(E)$ has a strongly measurable selection;*
- *(ii) for μ-a.e. $t \in I$ the multimap $F(t, \cdot) : E_0 \longrightarrow K(E)$ is u.s.c.*

Then for every strongly measurable function $q : I \longrightarrow E_0$ there exists a strongly measurable selection $\varphi : I \longrightarrow E$ of the multifunction $\Phi : I \longrightarrow K(E)$,

$$\Phi(t) = F(t, q(t)).$$

Corollary 2.1.1. *Let the spaces E, E_0 and the multimap F be as in Theorem 2.1.9. Then for every strongly measurable multifunction $Q : I \longrightarrow K(E_0)$ there exists a strongly measurable selection $\varphi : I \longrightarrow E$ of a multifunction Φ, $\Phi(t) = F(t, Q(t))$.*

Definition 2.1.13. *Let $F : I \times E_0 \longrightarrow K(E)$ be a multifunction satisfying the conditions (i) and (ii) of Theorem 2.1.9. The multimap $\mathtt{P}_F$ assigning to every continuous function $q \in C(I, E_0)$ the set of all strongly measurable selections of the multifunction $F(t, q(t))$ is said to be a **superposition multioperator generated** by F.*

$$\mathtt{P}_F : C(I; E_0) \longrightarrow \mathcal{P}\left(L^1(I; E)\right)$$
$$\mathtt{P}_F(q) = S^1_{F(\cdot, q(\cdot))}.$$

2.2 Measure of non-compactness

Definition 2.2.1. *Let $\mathcal{E}$ be a Banach space and $(\mathcal{Y}, \leq)$ a partially ordered set. Denote by $\mathcal{P}(\mathcal{E})$ the collection of all nonempty bounded subsets of $\mathcal{E}$.*

A map $\Psi : \mathcal{P}(\mathcal{E}) \to \mathcal{Y}$ is called a measure of non-compactness (MNC for brevity) in $\mathcal{E}$ if

$$\Psi(\Omega) = \Psi(\bar{co}\, \Omega)$$

for every $\Omega \in \mathcal{P}(\mathcal{E})$, where $\bar{co}\, \Omega$ denotes the closed convex hull of Ω.

Remark 2.2.1. *If D is dense in Ω then $\bar{co}\, \Omega = \bar{co}\, D$ and hence*

$$\Psi(\Omega) = \Psi(D).$$

Example 2.2.1. *The functions*

$$\psi_1(\Omega) = \begin{cases} 0, & if\, \Omega\, is\ totally\ bounded, \\ 1, & otherwise, \end{cases}$$

and

$$\psi_2(\Omega) = diam \Omega$$

are MNCs.

2.2.1 The Kuratowski measure of non-compactness

In this section we define the Kuratowski and Hausdorff, and We give some their basic properties.
In the following, $\mathcal{E}$ denotes a Banach space (unless otherwise stated); and Ω a subset of $\mathcal{E}$. $B(x,r)$ and $\overline{B}(x,r)$ respectively denote the open ball and the closed ball in $\mathcal{E}$ of center x and radius r and $B = B(0,1)$.

Definition 2.2.2. *The Kuratowski measure of non-compactness $\alpha(\Omega)$ of the set Ω is the infimum of the numbers $d > 0$ such that Ω admits a finite covering by sets of diameter smaller then d.*

Recall that, the diameter of the set A is

$$\mathrm{diam} A = \sup \{\|x - y\| : x, y \in A\};$$

which for A unbounded (empty) is taken to be infinity (resp. zero).

2.2.2 The Hausdorff measure of non-compactness

Definition 2.2.3. *The Hausdorff measure of non-compactness $\chi(\Omega)$ of the set Ω is the infimum of the numbers $\varepsilon > 0$ such that Ω has a finite ε-net in $\mathcal{E}$.*

$$\textbf{\textit{i.e.}} \quad \chi(\Omega) = \inf \{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon\text{-net in } \mathcal{E}\}.$$

The set $S \subset \mathcal{E}$ is an ε-net of Ω if

$$\Omega \subset S + \varepsilon\overline{B} = \left\{s + \varepsilon b : \ s \in S, b \in \overline{B}\right\}.$$

2.2.3 Elementary properties of the measure of non-compactness

Let Ψ is a measure of non-compactness defined in $\mathcal{E}$.
The MNC Ψ enjoy the following properties:

m_1) nonsingularity: $\Psi(\{a\} \cup \Omega) = \Psi(\Omega)$, for every $a \in \mathcal{E}$, $\Omega \in \mathcal{P}(\mathcal{E})$;

m_2) monotonicity: $\Omega_0 \subseteq \Omega_1$ imply $\Psi(\Omega_0) \leq \Psi(\Omega_1)$, for Ω_0, $\Omega_1 \in \mathcal{P}(\mathcal{E})$;

m_3) invariant with respect to reflection through the origin if $\Psi(-\Omega) = \Psi(\Omega)$ for every $\Omega \in \mathcal{P}(\mathcal{E})$;

m_4) semi-additivity: $\Psi(\Omega_1 \cup \Omega_2) = \max \{\Psi(\Omega_1), \Psi(\Omega_2)\}$;

$m_5)$ Lipschitzianity: $|\Psi(\Omega_1) - \Psi(\Omega_2)| \leq L_\psi \rho(\Omega_1, \Omega_2)$, where $L_\chi = 1$, $L_\alpha = 2$ and ρ denotes the Hausdorff metric:

$$\rho(\Omega_1, \Omega_2) = \inf\left\{\varepsilon > 0 : \Omega_1 + \varepsilon\overline{B} \supset \Omega_2,\ \Omega_2 + \varepsilon\overline{B} \supset \Omega_1\right\};$$

$m_6)$ semi-homogeneity: $\psi(t\Omega) = |t|\,\psi(\Omega)$ for any number t;

$m_7)$ invariance under translations: $\psi(\Omega + x_0) = \psi(\Omega)$ for any $x_0 \in \mathcal{E}$;

$m_8)$ continuity: for any $\Omega \subset \mathcal{E}$ and any $\varepsilon > 0$ there is a $\delta > 0$ such that $|\Psi(\Omega) - \Psi(\Omega_1)| < \varepsilon$ for all Ω_1 satisfying $\rho(\Omega, \Omega_1) < \delta$;

$m_9)$ invariant with respect to union with compact sets if $\psi(K \cup \Omega) = \psi(\Omega)$ for every relatively compact set $K \subset \mathcal{E}$, and $\Omega \in \mathcal{P}(\mathcal{E})$.

If $\mathcal{Y}$ is a cone in a Banach space, we say that

$m_{10})$ regularity: $\Psi(\Omega) = 0$ if and only Ω is relatively compactness;

$m_{11})$ algebric semi-additivity: $\Psi(\Omega_1 + \Omega_2) \leq \Psi(\Omega_1) + \Psi(\Omega_2)$.

Remark 2.2.2. *Notice that:*

(i) *if $\mathcal{E}$ is infinite-dimensional then ψ_1 is not continuous;*

(ii) *ψ_2 is not a regular MNC.*

Indeed, if $\Omega = \{x, y\} \subset \mathcal{E},\ \ x \neq y$ then $\operatorname{diam}\Omega \neq 0$, but this set is completely bounded (finite).

Example 2.2.2. *Consider another useful example of MNC in the space of continuous functions $C([a,b];\mathcal{E})$. for $\Omega \subset C([a,b];\mathcal{E})$ set*

$$\phi(\Omega) = \sup_{t\in[a,b]} \chi(\Omega(t))$$

where χ is the Hausdorff MNC in $\mathcal{E}$ and $\Omega(t) = \{y(t) : y \in \Omega,\ t \in [a,b]\}$. It is easy to check that this MNC satisfies all the usual properties of measure of non-compactness except the regularity.

Theorem 2.2.1. *Let B be the unit ball in $\mathcal{E}$. $\alpha(B) = \chi(B) = 0$ if $\mathcal{E}$ is finite-dimensional, and $\alpha(B) = 2$, $\chi(B) = 1$ in the apposite case.*

Theorem 2.2.2. *The Kuratowski and Hausdorff MNCs are related by the inequalities*

$$\chi(\Omega) \leq \alpha(\Omega) \leq 2\chi(\Omega).$$

In the class of all infinite-dimensional spaces these inequalities are sharp.

2.3 Condensing operators.

In infinite dimension, in addition to the continuity one needs the compactness. For example, if one arrives at writing a mathematical problem in the form of an integral equation whose integral operator is completely continuous and whose set of solutions coincides with the fixed points of the latter, we can use the arguments of the theory of fixed points to study this problem. Unfortunately, in infinite dimension, the criteria of compactness are hard to obtain and require very heavy conditions. The theory of condensing operators generalizing that of completely continuous operators is the most appropriate to overcome certain difficulties in infinite dimensions.
In this section we introduce the condensing operators and study some of their properties.

For more details, see *e.g.* [4, 9, 36].

2.3.1 Condensing multimaps.

$\mathcal{E}$ denote a real Banach space. All multimap $F : X \subset \mathcal{E} \longrightarrow \mathcal{P}(\mathcal{E})$ defines a multimap $\Phi : X \longrightarrow \mathcal{P}(\mathcal{E})$; determined by

$$\Phi(x) = x - F(x)$$

which is called the **multivalued vector field** (multifield) corresponding to the multimap F.

Let Λ a space of parameters and $G : \Lambda \times X \longrightarrow \mathcal{P}(\mathcal{E})$ is a family of multimaps, then $\Psi : \Lambda \times X \longrightarrow \mathcal{P}(\mathcal{E})$, given by

$$\Psi(\lambda, x) = x - G(\lambda, x),$$

is said to be a **family of multifields**.

An u.s.c. and compact multimap F is called **completely continuous**.

Definition 2.3.1. *Let β be a measure of non-compactness in $\mathcal{E}$. A multimap $F : X \longrightarrow K(\mathcal{E})$ or a family of multimaps $G : \Lambda \times X \longrightarrow K(\mathcal{E})$ is called β-**condensing** if for every $\Omega \subset X$ that is not relatively compact we have, respectively*

$$\beta(\Omega) \nleq \beta(F(\Omega)) \qquad \textit{or} \qquad \beta(\Omega) \nleq \beta(G(\Lambda \times \Omega)).$$

The following definition is an important class of condensing multimaps.

Definition 2.3.2. *Let β be a real measure of non-compactness and $0 \leq k < 1$. A multimap $F : X \longrightarrow K(\mathcal{E})$ or a family of multimaps $G : \Lambda \times X \longrightarrow K(\mathcal{E})$ is said to be (k, β)-**condensing** if respectively*

$$\beta(F(\Omega)) \leq k\beta(\Omega) \qquad \textit{or} \qquad \beta(G(\Lambda \times \Omega)) \leq k\beta(\Omega) \qquad \textit{for every} \quad \Omega \subset X.$$

Corollary 2.3.1. *Let $X \subset \mathcal{E}$ a bounded closed subset. If the multimap $F_0 : X \longrightarrow K(\mathcal{E})$ is k-Lipschitz, $0 \leq k < 1$, with respect to the Hausdorff metric and the multimap $F_1 : X \longrightarrow K(\mathcal{E})$ is compact then their sum $F_0+F_1 : X \longrightarrow K(\mathcal{E})$ is (k, χ)-condensing.*

2.3.2 Some fixed point theorems of condensing multimap.

Theorem 2.3.1. *If $\mathcal{G}$ is a closed convex subset of a Banach space $\mathcal{E}$, and $\Gamma : \mathcal{G} \to \mathcal{G}$ is a continuous Ψ-condensing map, where Ψ is a nonsingular measure of non-compactness in $\mathcal{E}$, then Γ has at least one fixed point.*

Theorem 2.3.2. *If $\mathcal{M}$ is a bounded convex closed subset of Banach space E, and $\mathcal{F} : \mathcal{M} \to Kv(\mathcal{M})$ is a closed (k, β)-condensing multimap, where β is a real nonsingular and regular measure of noncompactness defined on subsets of $\mathcal{M}$. Then, $Fix\, \mathcal{G} = \{x : x = \mathcal{G}(x)\} \neq \emptyset$.*

Theorem 2.3.3. *If $\mathcal{M}$ is a closed convex subset of Banach space E, and $\mathcal{F} : \mathcal{M} \to Kv(M)$ is a closed β-condensing multimap, where β is a nonsingular measure of noncompactness defined in E, then the fixed points set $Fix\, \mathcal{F} = \{x : x \in \mathcal{F}(x)\}$ is nonempty.*

Proposition 2.3.1. *Let $\mathcal{X} \subset E$ be a closed bounded subset and $\mathcal{F} : \mathcal{X} \to K(E)$ is a closed β-condensing multimap, where β is a monotone measure of noncompactness. Then the fixed points set $Fix\, \mathcal{F}$ is compact.*

2.3.3 Integral multivalued operator and its properties

Definition 2.3.3. *A sequence $\{f_n\}_{n=1}^{\infty} \subset L^1([0,T]; E)$ is* ***semicompact*** *if:*

(i) it is integrably bounded: $\|f_n(t)\| \leq q(t)$ for a.e. $t \in [0,T]$ and for every $n \geq 1$ where $q(.) \in L^1([0,T], \mathbb{R}^+)$;

(ii) the set $\{f_n(t)\}_{n=1}^{\infty}$ is relatively compact for almost every $t \in [0,T]$.

Lemma 2.3.1. *Let $F : X \to K(Y)$ be an upper semicontinuous multivalued map. If $A \subset X$ is a compact set then $F(A) = \cup_{x \in A} F(x)$ is a compact subset of Y.*

Lemma 2.3.2. *Every semicompact sequence is weakly compact in the space $L^1([0,T]; E)$.*

Let the operator

$$\mathcal{S} : L^1([0,d];E) \to C([0,d];E)$$

satisfying the following conditions:

($\mathcal{S}_1$) there exists $M \geq 0$ thuch that, for every $f, g \in L^1([a,b];E), a \leq t \leq b$;

$$\|\mathcal{S}f(t) - \mathcal{S}g(t)\|_E \leq M \int_a^t \|f(s) - g(s)\|_E \, ds,$$

($\mathcal{S}_2$) for any compact $\mathfrak{K} \subset E$ and sequence $\{f_n\}_{n=1}^{\infty} \subset L^1([a,b];E)$ such that $\{f_n(t)\}_{n=1}^{\infty} \subset \mathfrak{K}$ for a.e. $t \in [a,b]$ the weak convergence $f_n \underset{w}{\longrightarrow} f_0$ implies $\mathcal{S}f_n \to \mathcal{S}f_0$.

Note that condition (S_1) implies that the operator $\mathcal{S}$ satisfies the Lipschitz condition

($\mathcal{S}_{1'}$)

$$\| \mathcal{S}f - \mathcal{S}g \|_C \leq M \| f - g \|_{L^1} .$$

Definition 2.3.4. *Let A be the infinitesimal generator of a $\mathcal{C}_0$-semigroup $(S(t))_{t \geq 0}$. The operator $\Gamma : L^1([a,b];\mathcal{E}) \to C([a,b];\mathcal{E})$, defined by,*

$$\Gamma(g)(t) = \int_a^t S(t-s)\, g(s)\, ds.$$

is called the ***Cauchy operator****.*

From [36, Lemma 4.2.1], we have,

Lemma 2.3.3. *The Cauchy operator Γ verify all the conditions ($\mathcal{S}_1$), ($\mathcal{S}_2$) and ($\mathcal{S}_{1'}$).*

From [36, Theorems 4.2.2, 5.1.1] respectively, we have,

Lemma 2.3.4. *Let $\{f_n\}_{n=1}^{\infty} \subset L^1([a,b],\mathcal{E})$ be an integrably bounded sequence satisfying $\chi(\{f_n\}_{n=1}^{\infty}) \leq \zeta(t)$ a.e. $t \in [a,b]$, where $\zeta \in L^1_+([a,b]$, then for every $t \in [a,b]$,*

$$\chi(\Gamma \; \{f_n(t)\}_{n=1}^{\infty}) \leq 2M \int_a^t \zeta(s) ds. \tag{2.1}$$

Corollary 2.3.2. *If the space $\mathcal{E}$ is separable the estimate* (2.1) *has the form*

$$\chi(\Gamma \; \{f_n(t)\}_{n=1}^{\infty}) \leq M \int_a^t \zeta(s) ds.$$

Lemma 2.3.5. *For every semicompact sequence* $\{f_n\}_{n=1}^{\infty} \subset L^1([a,b];\mathcal{E})$, *the sequence* $\{\Gamma(f_n)\}_{n=1}^{\infty}$ *is relatively compact in* $C([a,b];\mathcal{E})$ *and; moreover; if* $f_n \underset{w}{\longrightarrow} f_0$ *implies* $\Gamma f_n \to \Gamma f_0$.

Theorem 2.3.4. *Let* $\mathcal{E}$ *be a separable Banach space and* $\Omega \subset C([0,d];\mathcal{E})$ *a bounded subset. Then the function*

$$\varphi_\Omega : [0,d] \to \mathbb{R}_+, \;\; \varphi_\Omega(t) = \chi(\Omega(t))$$

where $\Omega(t) = \{y(t) : y \in \Omega\}$, *is measurable and hence integrable and*

$$\chi\left(\left\{\int_0^t y(s)\,ds : y \in \Omega\right\}\right) \leq \int_0^t \varphi_\Omega(s)\,ds$$

for all $t \in [0,d]$.

2.4 Semigroups

We present some results from the theory of semigroups of linear operators in Banach spaces. For more details see *e.g.*[3, 50]

Definition 2.4.1. C_0***-semigroup***
A family of bounded linear operators $(S(t))_{t\geq 0}$ *in a Banach space* E *form a* C_0*-semigroup if*

(*i*) $S(0) = I$ *(I the identity operator on E);*

(*ii*) $S(t+s) = S(t)S(s)$ *for every* $t, s \geq 0$;

(*iii*) *the function* $S(\cdot)x : [0,\infty) \to E$, $t \to S(t)x$ *is continuous for every* $x \in E$.

Definition 2.4.2. *A strongly continuous semigroup* $(S(t))_{t\geq 0}$ *is called compact (or immediately compact) if* $S(t)$ *is compact for all* $t > 0$.

Definition 2.4.3. *Let* $(S(t))_{t\geq 0}$ *be a* $\mathcal{C}_0$*-semigroup of linear operators in Banach space* E*. The linear operator* A *defined on the set*

$$D(A) = \left\{x \in E / \lim_{t\to 0} \frac{S(t)x - x}{t} \quad \text{exists}\right\},$$

by

$$Ax = \lim_{t\to 0} \frac{S(t)x - x}{t}, \;\; x \in D(A)$$

is called the infinitesimal generator of the semigroup $(S(t))_{t\geq 0}$, *and we write*

$$S(t) = \exp\{At\} = e^{At}.$$

Lemma 2.4.1. *The infinitesimal generator A of a $\mathcal{C}_0$-semigroup is closed operator, and $D(A)$ is dense in E.*

Remark 2.4.1. *The difference between uniformly continuous and strongly continuous semigroups is just the nature of A.*

Theorem 2.4.1. *Let A be a bounded operator from E to E. Then,*

$$S = \left\{ S(t) = e^{tA} = \sum_{n=0}^{+\infty} \frac{(tA)^n}{n!} \, : \, t \in \mathbb{R}_+ \right\}$$

is a uniformly continuous semigroup.

Proof. Since A is bounded we know $\| A \| < \infty$, so

$$\sum_{n=0}^{+\infty} \frac{(tA)^n}{n!}$$

converges for each $t \geq 0$ to the bounded linear operator $S(t)$. We prove that $S(t)$ as uniformly continuous semigroup:

$$\| S(t) - I \| = \left\| \sum_{n=1}^{+\infty} \frac{(tA)^n}{n!} \right\| \leq \sum_{n=1}^{+\infty} \frac{t^n \| A \|^n}{n!} = e^{t\|A\|} - 1$$

and $e^{t\|A\|} - 1 \longrightarrow 0^+$ as $t \longrightarrow 0$, the proof is complete. $\square$

Definition 2.4.4. *A semigroup* $\exp\{At\}$ *is called* ***decreasing*** *if*

$$\| \exp\{At\} \| = \| e^{At} \| \leq Ce^{at}, \quad t \geq 0$$

where C and a are positive constants. If $C = 1$ then the semigroup $\exp\{At\}$ *is called* ***strongly contractive****.*

We denote by $\mathcal{SG}(C, a)$ the set of $\mathcal{C}_0$-semigroup $\left\{e^{At}\right\}_{t\geq 0}$ for which

$$\| e^{At} \| \leq Ce^{at}, \quad t \geq 0$$

where $a \geq 0$ and $C \geq 1$.

Proposition 2.4.1. *Let A an infinitesimal generator of a decreasing $\mathcal{C}_0$-semigroup $\left\{e^{At}\right\}_{t\geq 0} \in \mathcal{SG}(C, a)$. If $x \in D(A)$, then $e^{At}x \in D(A)$ and we have:*

$$e^{At}Ax = Ae^{At}x, \ \textit{for all} \ \ t \geq 0.$$

Remark 2.4.2. *We say that*

$$e^{At}D(A) \subseteq D(A), \quad \textit{for all} \quad t \geq 0.$$

Proposition 2.4.2. *Let A an infinitesimal generator of a decreasing $\mathcal{C}_0$-semigroup $\left\{e^{At}\right\}_{t\geq 0} \in \mathcal{SG}(C,a)$. Then the map :*

$$[0,\infty) \ni t \longmapsto e^{At}x \in E$$

is derivable on $[0,\infty)$, for all $x \in D(A)$ and we have:

$$\frac{d}{dt}e^{At}x = e^{At}Ax = Ae^{At}x, \quad \textit{for all} \quad t \geq 0.$$

Proposition 2.4.3. *Let $\left\{e^{At}\right\}_{t\geq 0}$ be a $\mathcal{C}_0$-semigroup. Then:*

$$\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} e^{As}x\,ds = e^{At}x$$

for all $x \in E$ and $t \geq 0$

Proposition 2.4.4. *Let A an infinitesimal generator of a decreasing $\mathcal{C}_0$-semigroup $\left\{e^{At}\right\}_{t\geq 0} \in \mathcal{SG}(C,a)$. If $x \in E$, then $\int_0^t e^{As}x\,ds \in D(A)$ and we have:*

$$A\int_0^t e^{As}x\,ds = e^{At}x - x, \quad \textit{for all} \quad t \geq 0.$$

Theorem 2.4.2. *Let A an infinitesimal generator of a decreasing $\mathcal{C}_0$-semigroup $\left\{e^{At}\right\}_{t\geq 0} \in \mathcal{SG}(C,a)$, then $x \in D(A)$, and $Ax = y$ if and only if*

$$e^{At}x - x = \int_0^t e^{As}y\,ds, \quad \textit{for all} \quad t \geq 0.$$

Definition 2.4.5. *Let $\Delta = \{z \in \mathbb{C} : \ \varphi_1 < \arg z < \varphi_2, \ \varphi_1 < 0 < \varphi_2\}$ and for $z \in \Delta$ let $S(z)$ be a bounded linear operator. The family $S(z)$, $z \in \Delta$ is an analytic semigroup in Δ if*

(i) $z \longrightarrow S(z)$ *is analytic in* Δ.

(ii) $S(0) = I$ *and* $\lim\limits_{\substack{z\to 0\\ z\in\Delta}} S(z)x = x$ *for every* $x \in E$.

(iii) $S(z_1 + z_2) = S(z_1)\,S(z_2)$ *for* $z_1,\, z_2 \in E$.

A semigroup $S(z)$ will be called analytic if it is analytic in some sector Δ containing the nonnegative real axis.

Chapter 3

On the periodic boundary value problem for functional semilinear differential inclusions with infinite delay in Banach spaces

Throughout this chapter, E is a Banach space, σ be a real number and $T > 0$ be a fixed time horizon. By $Kv(E)$ we denote the collection of all compact convex subsets of E. By $C([\sigma, \sigma + T]; E)$ we denote the space of continuous functions defined on $[\sigma, \sigma+T]$ with values in a Banach space $(E, \|.\|)$, endowed with the uniform convergence norm.

The propose of this chapter is to give results on the existence and topological structure of the mild solutions set for the periodic boundary value problem for semilinear functional differential nonlinear inclusion with infinite delay, described in the form

$$\begin{cases} x'(t) \in A\,x(t) + F(t, x_t), \sigma < t \leq \sigma + T, \\ x_\sigma = x_{\sigma+T}, \end{cases} \tag{3.1}$$

where, $A : \mathcal{D}(A) \to E$ is a linear operator generating a compact semigroup, and $F : [\sigma, \sigma + T] \times \mathcal{B} \to Kv(E)$ is a upper-Carathéodory multivalued.

For any function $z : (-\infty, \sigma + T] \to E$ and for every $t \in [\sigma, \sigma + T]$, z_t represents the function from $(-\infty, 0]$ into E defined by $z_t(\theta) = z(t+\theta); -\infty < \theta \leq 0$.

Notice that, if $\hat{F}$ is a T-periodic extension of F on the first argument, i.e.,

$$\hat{F}(t, \varphi) = F(t, \varphi), \quad \text{a.e. } t \in [\sigma, \sigma + T], \quad \varphi \in \mathcal{B},$$
$$\hat{F}(t + T, \varphi) = \hat{F}(t, \varphi), \quad \text{a.e. } t \geq \sigma, \quad \varphi \in \mathcal{B},$$

then, the study of the periodic problem

$$x'(t) = Ax(t) + \hat{F}(t, x_t), \quad t \geq \sigma, \tag{3.2}$$

is equivalent to the study of the boundary periodic value problem (3.1).

The phase space Let $\mathcal{B}$ be a linear topological space of functions mapping $(-\infty; 0]$ into E endowed with a seminorm $\|.\|_{\mathcal{B}}$, and satisfying the following axioms:

if $z : (-\infty, \sigma + T] \to E$ is continuous on $[\sigma, \sigma + T]$ and $z_\sigma \in \mathcal{B}$, then, for every $t \in [\sigma, \sigma + T]$, we have

(B1) $z_t \in \mathcal{B}$;

(B2) the function $t \mapsto z_t$ is continuous;

(B3) $\|z_t\|_{\mathcal{B}} \leq K(t - \sigma) \sup_{\sigma \leq s \leq t} \|z(s)\| + N(t - \sigma) \|z_\sigma\|_{\mathcal{B}}$, where $K, N : [0, +\infty) \longrightarrow [0, +\infty)$ are independent of z, K is positive and continuous, and N is locally bounded;

(B4) $\|z(t)\|_E \leq l \|z_t\|_{\mathcal{B}}$, where $l > 0$ is a constant independent of z.

Such a space $\mathcal{B}$ has been considered as a phase space in the theory of retarded functional equations

3.1 Formulation of the problem

Throughout this chapter, the phase space $\mathcal{B}$ is considered as a Banach space satisfying Axioms (B1)-(B4). Let us denote by the symbol $\mathcal{C}((-\infty, \sigma+T]; E)$ the linear topological space consisting of functions $z : (-\infty, \sigma+T] \to E$ such that $z_\sigma \in \mathcal{B}$ and the restriction $z\,|_{[\sigma,\sigma+T]}$ is continuous, endowed with a norm

$$\|z\|_{\mathcal{C}} = \|z_\sigma\|_{\mathcal{B}} + \left\| z\,|_{[\sigma,\sigma+T]} \right\|_{C([\sigma,\sigma+T];E)}. \tag{3.3}$$

Since $\mathcal{B}$ is supposed to be a Banach space, by using Axiom (B4) one can easily show that $\mathcal{C}((-\infty, \sigma + T]; E)$ endowed with $\|\cdot\|_{\mathcal{C}}$ is a Banach space.

In the study of the problem (3.1) will need the following hypotheses:

(A_1) $A : D(A) \subset E \to E$ is a linear operator generating strongly continuous $(\mathcal{T}(t))_{t\geq 0}$ in E which is compact, i.e., $\mathcal{T}(t)$ is compact for all $t > 0$ (see e.g, [50]).

(A_2) there exist constants C and ω with $C \geq 1$, $\omega > 0$ and $C^2 e^{-\omega T} < 1$, such that

$$\|\mathcal{T}(t)\| \leq C e^{-\omega t}.$$

The multimap $F : [\sigma, \sigma + T] \times \mathcal{B} \to Kv(E)$ satisfies:

(F_1) the multimap $F : (\cdot, u) \to Kv(E)$ has a strongly measurable selector for every $u \in \mathcal{B}$;

(F_2) the multimap $F : (t, \cdot) \to Kv(E)$ is u.s.c. for a.e. $t \in [\sigma, \sigma + T]$;

(F_3) for any nonempty bounded set $\Delta \subset \mathcal{B}$ there exists a function $\beta_\Delta(.) \in L^1([\sigma, \sigma + T]; \mathbb{R}^+)$ such that, for all $u \in \Delta$ and almost every where $t \in [\sigma, \sigma + T]$

$$\|F(t, u)\| \leq \beta_\Delta(t).$$

(C2) If $\{\varphi^n(\theta)\} \subset \mathcal{C}_{00}$; where $\mathcal{C}_{00}$ is the space of continuous functions with compact support defined on $(-\infty, 0]$ with values in E, is a uniformly bounded sequence which converges to a function $\varphi(\theta)$ on any compact of $(-\infty, 0]$, so $\varphi \in \mathcal{B}$ and $\|\varphi^n - \varphi\|_{\mathcal{B}} \longrightarrow 0 \quad$ as $\quad n \longrightarrow +\infty$.

Remark 3.1.1. *Note that the condition (A_2) is satisfied if the semigroup $(\mathcal{T}(t))_{t \geq 0}$ is strongly contractive i.e., $\|\mathcal{T}(t)\| \leq e^{-\omega t}$ where, $\omega > 0$.*

The superposition operator

Under conditions $(F_1)-(F_3)$, for every continuous function $h : [\sigma, \sigma+T] \to \mathcal{B}$, there exists a summable selection $f : [\sigma, \sigma + T] \to E$ of $t \to F(t, h(t))$. (see e.g. [36, Theorem 1.3.5]). Let us consider the application $\pi : [\sigma, \sigma + T] \times \mathcal{C}((-\infty, \sigma+T]; E) \to \mathcal{B}$ given by $\pi(t, x) = x_t$. By Axiom (B2) the application $\cdot \to \pi(\cdot, x)$ is continuous. Consequently, the superposition operator

$$\begin{aligned}
\mathrm{sel_F} &: \mathcal{C}((-\infty, \sigma + T]; E) \to L^1([\sigma, \sigma + T]; E) \\
\mathrm{sel}_F(x) &= \big\{ f \in L^1([\sigma, \sigma + T]; E) : \; f(t) \in F(t, x_t), \; a.e. t \in [\sigma, \sigma + T] \big\} \\
&= \big\{ f \in L^1([\sigma, \sigma + T]; E) : \; f(t) \in F(t, \pi(t, x)), \; a.e. t \in [\sigma, \sigma + T] \big\}
\end{aligned}$$

is correctly defined.

By Axiom (B3), for all x, $y \in \mathcal{C}((-\infty, \sigma + T]; E)$, we get

$$\|\pi(t, x) - \pi(t, y)\|_{\mathcal{B}} \leq \max(K^*, \, N^*) \, \|x - y\|_{\mathcal{C}((-\infty, \sigma+T]; E)},$$

where

$$K^* = \sup_{t \in [0,T]} K(t); \tag{3.4}$$

$$N^* = \sup_{t \in [0,T]} N(t). \tag{3.5}$$

It follows that $\pi(t,\cdot)$ is Lipschitz continuous uniformly with respect to $t \in [\sigma, \sigma+T]$.

Using the uniform Lipschitz continuity of $\pi(t,\cdot)$ and [36, Lemma 5.1.1] (see also [21]), we deduce that superposition operator $\mathrm{sel_F}$ is weakly closed. More precisely:

Lemma 3.1.1. *If the sequences* $\{x^n\}_{n=1}^{\infty} \subset \mathcal{C}((-\infty, \sigma+T]; E)$, $\{f_n\}_{n=1}^{\infty} \subset L^1([\sigma, \sigma+T]; E)$, $f_n \in \mathrm{sel}_F(x^n)$, $n \geq 1$, *are such that* $x^n \to x^*$, $f_n \xrightarrow{w} f_*$, *then* $f_* \in \mathrm{sel}_F(x^*)$.

A mild solution

Let us clarify the meaning of a mild solution to the problem (3.1). Since A generates a C_0-semigroup, the mild solution operator $S : E \times L^1([\sigma, \sigma+T]; E) \longrightarrow C([\sigma, \sigma+T]; E)$, such that $S(u,g)$ stands for the unique mild solution to the Cauchy problem

$$\begin{cases} x'(t) = Ax(t) + g(t), \ t \in [\sigma, \sigma+T], \\ x(\sigma) = u, \end{cases}$$

is well defined. Moreover, by means of the variation of constants formula, $S(\cdot,\cdot)$ can be expressed explicitly by

$$S(u,g)(t) = \mathcal{T}(t-\sigma)u + \int_{\sigma}^{t} \mathcal{T}(t-s)\, g(s)\, ds. \tag{3.6}$$

Definition 3.1.1. *A function* $x \in \mathcal{C}((-\infty, \sigma+T]; E)$ *is a mild solution to the problem* (3.1) *if* $x_\sigma = x_{\sigma+T}$ *and, there exists* $f \in \mathrm{sel}_F(x)$ *such that,* $x(t) = S(x(\sigma), f)(t)$ *for every* $t \in [\sigma, \sigma+T]$.

3.2 The integral multivalued operator and its properties

Throughout this section we assume that Hypotheses (A_1)-(A_2) and (F_1)-(F_3) are satisfied. We aim to construct an operator whose fixed points coincide with the mild solutions to the problem (3.1) and we study its properties

The construction

Le us construct an integral operator associated with the problem (3.1). To this end we need to introduce some notations. For any $h \in C([\sigma, \sigma+T]; E)$

and $\varphi \in \mathcal{B}$ such that $h(\sigma) = \varphi(0)$, define the function $h[\varphi]$ by

$$h[\varphi](t) = \begin{cases} h(t), & \sigma \le t \le \sigma + T, \\ \varphi(t - \sigma), & -\infty < t \le \sigma. \end{cases}$$

We have

$$h[\varphi]_t(\theta) = \begin{cases} h(t + \theta), & \sigma - t \le \theta \le 0, \\ \varphi(t - \sigma + \theta), & -\infty < \theta \le \sigma - t. \end{cases} \tag{3.7}$$

In particular $h[\varphi]_\sigma = \varphi \in \mathcal{B}$. Hence, by construction, $h[\varphi] \in \mathcal{C}((-\infty, \sigma + T]; E)$ and by Axiom (B1), $h[\varphi]_t \in \mathcal{B}$, for all $t \in [\sigma, \sigma + T]$. Note that each element $z \in \mathcal{C}((-\infty, \sigma + T]; E)$ can be written as $z = h[\varphi]$ with $h = z\mid_{[\sigma,\sigma+T]} \in C([\sigma, \sigma + T]; E)$ and $\varphi = z_\sigma \in \mathcal{B}$. Thus, we have

$$\mathcal{C}((-\infty, \sigma{+}T]; E) = \{h[\varphi] :\ \varphi \in \mathcal{B},\ h \in C([\sigma, \sigma + T]; E) \text{ and, } \varphi(0) = h(\sigma)\}.$$

Recall that $S : E \times L^1([\sigma, \sigma + T];\ E) \to C([\sigma, \sigma + T];\ E)$, is the mild solution operator (see (3.6)). Let us consider the operator $\mathcal{S} : \mathcal{B} \times L^1([\sigma, \sigma + T];\ E) \to \mathcal{C}((-\infty, \sigma + T]; E)$, defined as follow,

$$\mathcal{S}(\varphi, g)(t) = S(\varphi(0), g)[\varphi](t) = \begin{cases} S(\varphi(0), g)(t), & \sigma \le t \le \sigma + T, \\ \varphi(t - \sigma), & -\infty < t \le \sigma. \end{cases} \tag{3.8}$$

Notice that $S(\varphi(0), g)(\sigma) = \varphi(0)$. Moreover, for every $t \in [\sigma, \sigma + T]$

$$\mathcal{S}(\varphi, g)(t) = S(\varphi(0), g)(t) \tag{3.9}$$

and

$$\Big(\mathcal{S}(\varphi, g)\Big)_t(\theta) = \begin{cases} S(\varphi(0), g)(t + \theta), & \sigma - t \le \theta \le 0, \\ \varphi(\theta + t - \sigma), & -\infty < \theta \le \sigma - t. \end{cases}$$

In particular, for $t = \sigma$, we have

$$\Big(\mathcal{S}(\varphi, g)\Big)_\sigma = \varphi. \tag{3.10}$$

Therefore, the operator $\mathcal{S}$ is well defined. Further, from Axiom (B1) and (3.10), we have

$$\Big(\mathcal{S}(\varphi, g)\Big)_{\sigma+T} \in \mathcal{B}. \tag{3.11}$$

Now, consider the operator, $\mathcal{G} : \mathcal{C}((-\infty, \sigma + T]; E) \to \mathcal{C}((-\infty, \sigma + T]; E)$, defined as follow

$$\mathcal{G}(x) = \left\{ \mathcal{S}\Big(\Big(\mathcal{S}(x_\sigma, f)\Big)_{\sigma+T}, f\Big),\ f \in \mathrm{sel}_F(x) \right\}. \tag{3.12}$$

From (3.10) and (3.11), it follows that $\mathcal{G}$ is well defined.

Proposition 3.2.1. *A function $x \in \mathcal{C}((-\infty, \sigma+T]; E)$ is a mild solution of the problem* (3.1) *if and only if x is a fixed point of the operator $\mathcal{G}$.*

Proof. Let x be a fixed point of the operator $\mathcal{G}$, then there exists $f \in \mathrm{sel}_F(x)$ such that

$$x = \mathcal{S}\Big(\Big(\mathcal{S}(x_\sigma, f)\Big)_{\sigma+T}, f\Big),$$

By using (3.10), we have immediately

$$x_\sigma = \Big(\mathcal{S}\Big(\Big(\mathcal{S}(x_\sigma, f)\Big)_{\sigma+T}, f\Big)\Big)_\sigma = \Big(\mathcal{S}(x_\sigma, f)\Big)_{\sigma+T}.$$

Hence,

$$x = \mathcal{S}\Big(\Big(\mathcal{S}(x_\sigma, f)\Big)_{\sigma+T}, f\Big) = \mathcal{S}(x_\sigma, f).$$

It results that

$$x_{\sigma+T} = \Big(\mathcal{S}(x_\sigma, f)\Big)_{\sigma+T} = x_\sigma,$$

and for every $t \in [\sigma, \sigma+T]$ (see (3.9)),

$$x(t) = \mathcal{S}(x_\sigma, f)(t) = S(x(\sigma), f)(t).$$

That is, x is a mild solution of (3.1).

Now, let x be a mild solution of the problem (3.1). Then, there exists $f \in \mathrm{sel}_F(x)$ such that

$$x = \mathcal{S}(x_\sigma, f) \text{ and } x_\sigma = x_{\sigma+T}.$$

We have immediately that $x_{\sigma+T} = \Big(\mathcal{S}(x_\sigma, f)\Big)_{\sigma+T} = x_\sigma$. Therefore, $x = \mathcal{S}\Big(\Big(\mathcal{S}(x_\sigma, f)\Big)_{\sigma+T}, f\Big)$, i.e., x is a fixed point of $\mathcal{G}$. $\square$

Remark 3.2.1. *The idea of such construction of the operator $\mathcal{G}$ was first given by Vrabie [61] in the study of the periodic problem for fully nonlinear differential equations without delay (S is a nonlinear operator) and, was extended in [24] for the study of the same problem with the presence of a finite delay.*

Remark 3.2.2. *Let $x \in \mathcal{C}((-\infty, \sigma+T]; E)$.*

Notice that for any $f \in L^1([\sigma, \sigma+T]; E)$ and any $t \in [\sigma, \sigma+T]$, we have

$$\mathcal{S}\Big(\Big(\mathcal{S}(x_\sigma, f)\Big)_{\sigma+T}, f\Big)(t) = S\left(S(x(\sigma), f)(\sigma+T),\, f\right)(t).$$

Indeed, using (3.9) *twice, we have*

$$\begin{aligned}\mathcal{S}\Big(\Big(\mathcal{S}(x_\sigma, f)\Big)_{\sigma+T}, f\Big)(t) &= S\Big(\Big(\mathcal{S}(x_\sigma, f)\Big)_{\sigma+T}(0), f\Big)(t)\\ &= S\Big(\mathcal{S}(x_\sigma, f)(\sigma+T), f\Big)(t)\\ &= S\left(S(x(\sigma), f)(\sigma+T), f\right)(t).\end{aligned}$$

Notice also that from (3.8) *and* (3.10)*, for any* $f \in L^1([\sigma, \sigma+T]; E)$ *and any* $t \in]-\infty, \sigma]$ *we have*

$$\Big(\mathcal{S}\Big(\Big(\mathcal{S}(x_\sigma, f)\Big)_{\sigma+T}, f\Big)\Big)_\sigma (t-\sigma) = \Big(\mathcal{S}(x_\sigma, f)\Big)_{\sigma+T}(t-\sigma).$$

The some properties

Next we study the properties of the operator $\mathcal{G}$.

Proposition 3.2.2. *The multivalued operator* $\mathcal{G}$ *is with convex values.*

Proof. Let $\lambda \in [0,1]$ and $y_1,\, y_2 \in \mathcal{G}(x)$ with

$$y_i = \mathcal{S}\Big(\Big(\mathcal{S}(x_\sigma, f_i)\Big)_{\sigma+T}, f_i\Big),\ f_i \in \mathrm{sel}_F(x),\ \ i = 1,2.$$

We have to prove that $\lambda y_1 + (1-\lambda)\, y_2 \in \mathcal{G}(x)$.

By using Remark 3.2.2, we have

$$\begin{aligned} y_i(t) &= \mathcal{S}\Big(\Big(\mathcal{S}(x_\sigma, f_i)\Big)_{\sigma+T}, f_i\Big)(t),\ f_i \in \mathrm{sel}_F(x),\ \ i = 1,2.\\ &= \begin{cases} S\Big(S(x(\sigma), f_i)(\sigma+T), f_i\Big)(t),\ t \in [\sigma, \sigma+T],\\ \Big(\mathcal{S}(x_\sigma, f_i)\Big)_{\sigma+T}(t-\sigma),\ t \in]-\infty, \sigma]. \end{cases} \end{aligned} \tag{3.13}$$

From the definition of the operator $S(\cdot,\cdot)$ (see (3.6)), for all $z_1,\, z_2 \in E$ and for every $t \in [\sigma, \sigma+T]$, we have

$$\lambda S(z_1,\, f_1)(t) + (1-\lambda)\, S(z_2,\, f_2)(t) = S\Big(\lambda z_1 + (1-\lambda)\, z_2,\, \lambda f_1 + (1-\lambda)\, f_2\Big)(t). \tag{3.14}$$

By using (3.14) twice, for every $t \in [\sigma, \sigma+T]$, we obtain

$$\begin{aligned} &\lambda\, S\Big(S(x(\sigma), f_1)(\sigma+T),\, f_1\Big)(t) + (1-\lambda) S\Big(S(x(\sigma), f_2)(\sigma+T),\, f_2\Big)(t)\\ &= S\Big(\lambda\, S(x(\sigma), f_1)(\sigma+T) + (1-\lambda) S(x(\sigma), f_2)(\sigma+T), \lambda\, f_1 + (1-\lambda) f_2\Big)(t)\\ &= S\Big(S\Big(x(\sigma), \lambda\, f_1 + (1-\lambda) f_2\Big)(\sigma+T), \lambda\, f_1 + (1-\lambda) f_2\Big)(t). \end{aligned} \tag{3.15}$$

From the definition of the operator $\mathcal{S}$ and (3.15), one can easily show

$$\lambda\,\mathcal{S}(x_\sigma, f_1) + (1-\lambda)\mathcal{S}(x_\sigma, f_2) = \mathcal{S}\left(x_\sigma, \lambda f_1 + (1-\lambda) f_2\right). \tag{3.16}$$

Since the map, $\Pi : \mathcal{C}((-\infty, \sigma+T]; E) \longrightarrow \mathcal{B}$, given by, $\Pi(z) = z_{\sigma+T}$ is linear, (3.16) gives

$$\lambda\Big(\mathcal{S}(x_\sigma, f_1)\Big)_{\sigma+T} + (1-\lambda)\Big(\mathcal{S}(x_\sigma, f_2)\Big)_{\sigma+T} = \Big(\mathcal{S}(x_\sigma, \lambda f_1 + (1-\lambda) f_2)\Big)_{\sigma+T}. \tag{3.17}$$

Consequently, for every $t \in]-\infty, \sigma]$, we have

$$\begin{aligned} &\lambda\Big(\mathcal{S}(x_\sigma, f_1)\Big)_{\sigma+T}(t-\sigma) + (1-\lambda)\Big(\mathcal{S}(x_\sigma, f_2)\Big)_{\sigma+T}(t-\sigma) \\ &= \Big(\mathcal{S}(x_\sigma, \lambda f_1 + (1-\lambda) f_2)\Big)_{\sigma+T}(t-\sigma). \end{aligned} \tag{3.18}$$

From (3.13), (3.15) and (3.18), we obtain

$$\begin{aligned} &\lambda\, y_1(t) + (1-\lambda) y_2(t) = \\ &\begin{cases} S\Big(S\left(x(\sigma), \lambda\, f_1 + (1-\lambda) f_2\right)(\sigma+T), \lambda\, f_1 + (1-\lambda) f_2\Big)(t), \ t \in [\sigma, \sigma+T], \\ \Big(\mathcal{S}(x_\sigma, \lambda f_1 + (1-\lambda) f_2)\Big)_{\sigma+T}(t-\sigma), \ t \in]-\infty, \sigma]. \end{cases} \\ &= \mathcal{S}\Big(\Big(\mathcal{S}(x_\sigma, \lambda f_1 + (1-\lambda) f_2)\Big)_{\sigma+T}, \lambda f_1 + (1-\lambda) f_2\Big). \end{aligned}$$

Since the multivalued map F has convex values, we have

$$\lambda f_1 + (1-\lambda)\, f_2 \in \mathrm{sel}_F(x).$$

Therefore, $\lambda y_1 + (1-\lambda)\, y_2 \in \mathcal{G}(x)$. The proof of proposition 3.2.2 is complete. □

Proposition 3.2.3. *The multivalued operator $\mathcal{G}$ is closed with compact values.*

For the proof we will need the following lemma.

Lemma 3.2.1. *Let $(g_n)_n \subset L^1([\sigma, \sigma+T]; E)$ be a semicompact sequence (see Definition 2.3.3). Assume that A generates a strongly continuous semigroup. If $\varphi^n \to \varphi$ in $\mathcal{B}$ and $g_n \xrightarrow{w} g$ in $L^1([\sigma, \sigma+T]; E)$, then the following assertions are valid:*

(i) $S(\varphi^n(0), g_n) \to S(\varphi(0), g)$ *in* $C([\sigma, \sigma+T], E)$;

(ii) $\mathcal{S}(\varphi^n, g_n) \to \mathcal{S}(\varphi, g)$ *in* $\mathcal{C}((-\infty, \sigma+T]; E)$;

(*iii*) *for every* $t \in [\sigma, \sigma+T]$, $\Big(\mathcal{S}(\varphi^n, g_n)\Big)_t \to \Big(\mathcal{S}(\varphi, g)\Big)_t$ *in* $\mathcal{B}$;

(*iv*) *for every* $t \in [\sigma, \sigma+T]$,

$$\mathcal{S}\Big(\Big(\mathcal{S}(\varphi^n, g_n)\Big)_t, g_n\Big) \to \mathcal{S}\Big(\Big(\mathcal{S}(\varphi, g)\Big)_t, g\Big) \text{ in } \mathcal{C}((-\infty, \sigma+T]; E).$$

Proof. Since A generates a strongly continuous semigroup and $(g_n)_n$ is a semicompact sequence, according to [36, Lemma 4.2.1], we have that

$$\Big(t \to \int_\sigma^t e^{A(t-s)} g_n(s) ds\Big) \longrightarrow \Big(t \to \int_\sigma^t e^{A(t-s)} g(s) ds\Big) \text{ in } C([\sigma, \sigma+T]; E). \tag{3.19}$$

By axiom (B4), we have that $\varphi^n(0) \to \varphi(0)$ in E. Assertion (*i*) follows immediate from (3.19) and the expression (3.6) of the operator S (see (3.6)).

From (3.9) and Assertion (*i*), we obtain

$$\begin{aligned}
&\|\mathcal{S}(\varphi^n, g_n) - \mathcal{S}(\varphi, g)\|_{\mathcal{C}((-\infty,\sigma+T];E)} \\
&\le \|\varphi^n - \varphi\|_{\mathcal{B}} + \sup_{t\in[\sigma,\sigma+T]} \|\mathcal{S}(\varphi^n, g_n)(t) - \mathcal{S}(\varphi, g)(t)\| \\
&= \|\varphi^n - \varphi\|_{\mathcal{B}} + \sup_{t\in[\sigma,\sigma+T]} \|S(\varphi^n(0), g_n)(t) - S(\varphi(0), g)(t)\| \longrightarrow 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus, (ii) is valid.

From Axiom (B3), (3.4), (3.5), (3.10) and Assertion (*ii*), for every $t \in [\sigma, \sigma+T]$, we have

$$\begin{aligned}
&\|(\mathcal{S}(\varphi^n, g_n))_t - (\mathcal{S}(\varphi, g))_t\|_{\mathcal{B}} \\
&\le K^* \sup_{s\in[\sigma,\sigma+T]} \|\mathcal{S}(\varphi^n, g_n)(s) - \mathcal{S}(\varphi, g)(s)\| + N^* \|(\mathcal{S}(\varphi^n, g_n))_\sigma - (\mathcal{S}(\varphi, g))_\sigma\|_{\mathcal{B}} \\
&= K^* \sup_{s\in[\sigma,\sigma+T]} \|\mathcal{S}(\varphi^n, g_n)(s) - \mathcal{S}(\varphi, g)(s)\| + N^* \|\varphi^n - \varphi\|_{\mathcal{B}} \underset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

This proves the validity of the assertion (*iii*). The assertion (*iv*) follows from the Assertions (*iii*) and (*ii*). □

Proof of Proposition 3.2.3. First , let us prove that $\mathcal{G}$ is closed. To this end, consider sequences $\{x^n\}_n$ and $\{z^n\}_n \subset \mathcal{C}((-\infty, \sigma+T]; E)$ such that $x^n \longrightarrow x^*$, $z^n \in \mathcal{G}(x^n)$, $n \ge 1$, and $z^n \longrightarrow z^*$. Let $\{f_n\}_n$ be an any sequence from $L^1([\sigma, \sigma+T], E)$ such that $f_n \in \mathrm{sel}_F(x^n)$ and $z^n = \mathcal{S}\left((\mathcal{S}(x^n_\sigma, f_n))_{\sigma+T}, f_n\right)$, $n \ge 1$. By means of Hypothesis (F_3), the sequence $\{f_n(t)\}_{n=1}^{+\infty})$ is integrably bounded. Setting for every $t \in [\sigma, \sigma+T]$, $\mathcal{K}(t) = \{x^n_t\}_n \cup \{x^*_t\}$, we get

$$\{f_n(t)\}_{n=1}^{+\infty} \subset F(t, \mathcal{K}(t)) \quad \text{a.e. } t \in [\sigma, \sigma+T].$$

Since $x^n \longrightarrow x^*$, Axiom (B2) implies that $x_t^n \to x_t^*$ in $\mathcal{B}$ for every $t \in [\sigma, \sigma+T]$, hence the set $\mathcal{K}(t)$ is a compact set in $\mathcal{B}$ for every $t \in [\sigma, \sigma+T]$. It follows from Hypothesis (F_2) and Lemma 2.3.1 that $F(t, \mathcal{K}(t))$ is compact in E for a.e. $t \in [\sigma, \sigma+T]$. Hence, the sequence $\{f_n(t)\}_{n=1}^{+\infty}$ is semicompact in $L^1([\sigma, \sigma+T]; E)$. Consequently it is weakly compact. Without loss of generality we may assume that $f_n \overset{w}{\to} f_*$. By applying Lemma 3.1.1, we have that $f_* \in \mathrm{sel}_F(x^*)$. Since $x_\sigma^n \to x_\sigma^*$ in $\mathcal{B}$, by applying Lemma 3.2.1-(iv), we obtain

$$z^n = \mathcal{S}\left((\mathcal{S}(x_\sigma^n, f_n))_{\sigma+T}, f_n\right) \to \mathcal{S}\left((\mathcal{S}(x_\sigma^*, f_*))_{\sigma+T}, f_*\right) = z^*.$$

Therefore, $z^* \in \mathcal{G}(x^*)$. This proves the closedness of $\mathcal{G}$.

It remains to show that $\mathcal{G}$ is with compact values. Let $x(.) \in \mathcal{C}((-\infty, \sigma+T]; E)$. Take an any sequence $(y^n)_n \subset \mathcal{G}(x)$ such that $y^n = \mathcal{S}\Big(\Big(\mathcal{S}(x_\sigma, f_n)\Big)_{\sigma+T}, f_n\Big)$, where $f_n \in \mathrm{sel}_F(x)$, $n \geq 1$. By the same reasoning as above, (take $x^n = x$ for all $n \geq 1$), we may assume that $f_n \overset{w}{\to} f_*$. An application of Lemma 3.1.1 yields that $f_* \in \mathrm{sel}_F(x^*)$ and, an application of Lemma 3.2.1-(iv) yields that

$$z^n = \mathcal{S}\left((\mathcal{S}(x_\sigma, f_n))_{\sigma+T}, f_n\right) \to \mathcal{S}\left((\mathcal{S}(x_\sigma, f_*))_{\sigma+T}, f_*\right) = z^* \in \mathcal{G}(x).$$

Therefore, $\mathcal{G}(x)$ is a relatively compact set. Its compactness follows from the fact that it is closed. $\square$

Next we show that the operator $\mathcal{G}$ is condensing.

On bounded subsets of $\mathcal{C}((-\infty, \sigma+T]; E)$, let us define a measure of noncompactness Ψ as follow,

$$\Psi(\Omega) = \max\Big(\alpha_{\mathcal{B}}(\Omega_\sigma),\ \alpha_{C([\sigma,\sigma+T];E)}(\Omega\,|_{[\sigma,\sigma+T]})\Big), \tag{3.20}$$

where $\Omega_\sigma = \{x_\sigma : x \in \Omega\}$.

The range for the function Ψ is the cone $\mathbb{R}_+^2$, max is taken in the sense of the ordering induced by this cone. Ψ is a monotone, nonsingular, and regular measure of noncompactness.

Recall that the function $N(\cdot)$ is from Axiom (B3).

Proposition 3.2.4. *If $N(T) < 1$, then the multivalued operator $\mathcal{G}$ is Ψ-condensing.*

for the proof we will need the following Lemma which gives a fundamental compactness property of the mild solution operator S given by (3.6) (see Vrabie [60, Theorem 2.8.4, p. 194] or Cârjă et al. [15, Lemma 1.5.1, p. 14].)

Lemma 3.2.2. *Assume that $A : D(A) \subset E \to E$ is a linear operator generating a compact semigroup. Then, for each bounded subset Ω in E and each integrably bounded subset G in $L^1([0,T];E)$, the set $S(\Omega \times G) = \{S(y^0, g),\ (y^0, g) \in \Omega \times G\}$ is relatively compact in $C([d,T];E)$ for each $d \in]0,T]$. If in addition, Ω is relatively compact in E, then $S(\Omega \times G)$ is relatively compact in $C([0,T];E)$.*

We will need also the following result of J.S.Shin [56, Theorem 2.1]:

Lemma 3.2.3. *Let Ω be a bounded subset of $\mathcal{C}((-\infty, \sigma+T];E)$. Then for every $t \in [\sigma, \sigma+T]$ the following relation holds:*

$$\alpha_{\mathcal{B}}(\Omega_t) \leq K(t-\sigma)\,\alpha_{C([\sigma,\sigma+t];E)}\Big(\Omega\,|_{[\sigma,\sigma+T]}\Big) + N(t-\sigma)\,\alpha_{\mathcal{B}}(\Omega_\sigma).$$

Proof of Proposition 3.2.4. Let $\Omega \subset \mathcal{C}((-\infty, \sigma+T];E)$ be a bounded subset such that

$$\Psi(\mathcal{G}(\Omega)) \geq \Psi(\Omega). \tag{3.21}$$

We have to show that (3.21) implies that Ω is relatively compact, i.e., $\Psi(\Omega) = 0$.

First, let us show that

$$\alpha_{C([\sigma,\sigma+T];E)}(\Omega\,|_{[\sigma,\sigma+T]}) = 0. \tag{3.22}$$

Let $x \in \Omega$. By using Remark 3.2.2, for every $t \in [\sigma, \sigma+T]$, we have,

$$\mathcal{G}(x)(t) = \Big\{S\Big(S(x(\sigma), f)(\sigma+T), f\Big)(t),\ f \in \mathrm{sel}_F(x)\Big\}. \tag{3.23}$$

Set

$$\Theta = \{f :\ f(t) = f(t, x_t)\ \text{a.e.}\ t \in [\sigma, \sigma+T],\ \text{for some}\ x \in \Omega\} \tag{3.24}$$

From (3.23) and (3.24), for every $t \in [\sigma, \sigma+T]$, we have

$$\mathcal{G}(\Omega)(t) \subset \Big\{S\Big(S(\Omega(\sigma), f)(\sigma+T), f\Big)(t),\ f \in \Theta\Big\}.$$

That is,

$$\mathcal{G}(\Omega) \subset \Big\{S\Big(S(\Omega(\sigma), f)(\sigma+T), f\Big),\ f \in \Theta\Big\}. \tag{3.25}$$

Since Ω is bounded the set $\Omega(\sigma) = \{x(\sigma) : x \in \Omega\}$ is bounded too. Now, using Hypothesis (F_3), we have immediately that the set Θ is integrably bounded in $L^1([\sigma, \sigma+T];E)$. According to Lemma 3.2.2, the set

$$\Big\{S\Big(\Omega(\sigma), f\Big)(\sigma+T),\ f \in \Theta\Big\}$$

is relatively compact in E. Using again Lemma 3.2.2, we deduce that the set

$$\Big\{S\Big(S\Big(\Omega(\sigma), f\Big)(\sigma+T),\, f\Big),\ f\in\Theta\Big\}$$

is relatively compact in $C([\sigma,\sigma+T];E)$. Bearing in mind (3.25), we deduce that

$$\alpha_{C([\sigma,\sigma+T];E)}(\mathcal{G}(\Omega)\,|_{[\sigma,\sigma+T]}) = 0. \tag{3.26}$$

Equation (3.22) follows from (3.21) and (3.26).

It remains to show that

$$\alpha_{\mathcal{B}}\Big\{\Big(\mathcal{G}(\Omega)\Big)_{\sigma}\Big\} = 0.$$

From (3.22), the set $\Omega(\sigma) = \{x(\sigma) : x\in\Omega\}$ is relatively compact in E. Hence, by applying Lemma 3.2.2, we deduce that the set $\Big\{S\Big(\Omega(\sigma), f\Big),\ f\in\Theta\Big\}$ is relatively compact in $C([\sigma,\sigma+T];E)$, that is,

$$\alpha_{C([\sigma,\sigma+T];E)}\Big(\,\Big\{S\Big(\Omega(\sigma), f\Big),\ f\in\Theta\Big\}\,\Big) = 0.$$

Bearing in mind the last equality, by applying Lemma 3.2.3, we get

$$\begin{aligned}
&\alpha_{\mathcal{B}}\Big(\Big(\mathcal{G}(\Omega)\Big)_{\sigma}\Big) = \alpha_{\mathcal{B}}\Big(\Big\{\Big(\mathcal{S}(x_\sigma, f)\Big)_{\sigma+T},\ x\in\Omega,\ f\in \mathrm{sel}_F(x)\Big\}\Big)\\
&\le \alpha_{\mathcal{B}}\Big(\,\{(\mathcal{S}(\Omega_\sigma,\Theta))_{\sigma+T}\}\,\Big)\\
&\le M(T)\alpha_{C([\sigma,\sigma+T];E)}\Big(\,\Big\{\mathcal{S}(\Omega_\sigma,\Theta)\,|_{[\sigma,\sigma+T]}\Big\}\,\Big) + N(T)\,\alpha_{\mathcal{B}}\Big(\Big(\mathcal{S}(\Omega_\sigma,\Theta)\Big)_{\sigma}\Big)\\
&\le M(T)\,\alpha_{C([\sigma,\sigma+T];E)}\Big(\,\{S(\Omega(\sigma),\Theta)\}\,\Big) + N(T)\,\alpha_{\mathcal{B}}\Big(\Omega_\sigma\Big)\\
&= N(T)\,\alpha_{\mathcal{B}}\Big(\Omega_\sigma\Big).
\end{aligned}$$

Since $N(T) < 1$ (by hypothesis), from(3.21) we obtain

$$\alpha_{\mathcal{B}}\Big(\Omega_\sigma\Big) = 0. \tag{3.27}$$

Equation (3.22) together with (3.27) imply that Ω is relatively compact in $\mathcal{C}((-\infty,\sigma+T];E)$. The proof of proposition 3.2.4 is complete. □

3.3 Existence results

As previously the phase space $\mathcal{B}$ is considered as a Banach space satisfying Axioms (B1)-(B4).

From the properties of the operator $\mathcal{G}$ described in Propositions 3.2.1 and 3.2.4,Theorems 2.3.2 and 2.3.3 we deduce the following result,

Corollary 3.3.1. *Assume Hypotheses* (A_1)-(A_2) *and* (F_1)-(F_3). *Further, assume that* $N(T) < 1$, *and there exists a convex bounded set* $\Theta \subset \mathcal{C}((-\infty, \sigma + T]; E)$ *such that* $\mathcal{G}(\Theta) \subset \Theta$. *Then the problem* (3.1) *has at least one mild solution. Moreover, if the solution set is bounded then it is compact.*

To ensure the existence of such subset Θ, we strengthen Hypothesis (F_3). More precisely,

Theorem 3.3.1. *Suppose that conditions* (A_1), (A_2), (F_1), (F_2) *are satisfied,* $N(T) < 1$ *and*

(F_3') *there exists a function* $\beta \in L^1([\sigma, \sigma + T], \mathbb{R}^+)$ *such that, for all* $u \in \mathcal{B}$,

$$\|F(t, u)\| \leq \beta(t) \ \text{ a.e. } t \in [\sigma, \sigma + T].$$

Then, the set of all mild solutions of the problem (3.1) *is nonempty and compact.*

Proof. Recall that $C([\sigma, \sigma + T]; E)$ is endowed with sup-norm. In the space $\mathcal{C}((-\infty, \sigma + T]; E)$, let us define an equivalent norm $\|\cdot\|^*$ as follow,

$$\|x\|^* = \max\left(e^{-L}\,\|x_\sigma\|_{\mathcal{B}}\,, \left\|x\,|_{[\sigma,\sigma+T]}\right\|_{C([\sigma,\sigma+T];E)}\right),$$

where $L > 0$ is chosen big a enough so that

$$q = \max\left(C\,M(T)\,e^{-L} + N(T),\ C\,M(T)\,e^{-L}\,\|\beta(\cdot)\|_{L^1}\right) < 1, \tag{3.28}$$

where C is from (A_2) and the functions $M(\cdot)$ and $N(\cdot)$ are from Axiom (B3). Since, by hypothesis $N(T) < 1$, the choice of L is possible. Let us denote by $\mathrm{B}(0,\ r)$ the closed ball in $(\mathcal{C}((-\infty, \sigma + T]; E), \|\cdot\|^*)$, centered at 0 and, with radius r.

$$\mathrm{B}(0,\ r) = \left\{x \in \mathcal{C}((-\infty, \sigma + T]; E) :\ \|x\|^*_{\mathcal{C}((-\infty,\sigma+T];E)} \leq r\right\}$$

Recall that by (A_2), we have $C^2\,e^{-\omega T} < 1$. Let r^* be a positive real number, such that

$$r^* > \max\left(\frac{q}{1-q},\ \frac{C(C+1)\ \|\beta(\cdot)\|_{L^1}}{1 - C^2\,e^{-\omega T}}\right). \tag{3.29}$$

Let us sow that $\mathcal{G}$ maps $\mathrm{B}(0,\ r^*)$ into itself. Let $x \in \mathrm{B}(0,\ r^*)$. We have to show that for all $y \in \mathcal{G}(x)$,

$$\begin{cases} e^{-L}\,\|y_\sigma\|_{\mathcal{B}} \leq\ r^* \\ \qquad\qquad \text{and,} \\ \left\|y\,|_{[\sigma,\sigma+T]}\right\|_{C([\sigma,\sigma+T];E)} \leq\ r^*. \end{cases}$$

Let $y \in \mathcal{G}(x)$. There exists$f \in \mathrm{sel}_F(x)$ such that

$$y = \mathcal{S}\Big(\Big(\mathcal{S}(x_\sigma, f)\Big)_{\sigma+T}, f\Big).$$

By using (3.10), (3.11), Axiom (B3) (3.9) and (3.29), we have

$$\begin{aligned}
&e^{-L}\left\|y_\sigma\right\|_{\mathcal{B}} = e^{-L}\left\|\left(\mathcal{S}\Big(\Big(\mathcal{S}(x_\sigma, f)\Big)_{\sigma+T}, f\Big)\right)_\sigma\right\|_{\mathcal{B}} = e^{-L}\left\|(\mathcal{S}(x_\sigma, f))_{\sigma+T}\right\|_{\mathcal{B}}\\
&\le e^{-L}\left[M(T)\sup_{t\in[\sigma,\sigma+T]}\|\mathcal{S}(x_\sigma, f)(t)\| + N(T)\left\|x_\sigma\right\|_{\mathcal{B}}\right]\\
&= e^{-L}\left[M(T)\sup_{t\in[\sigma,\sigma+T]}\|S(x(\sigma), f)(t)\| + N(T)\ \left\|x_\sigma\right\|_{\mathcal{B}}\right]\\
&\le e^{-L}\left[M(T)\sup_{t\in[\sigma,\sigma+T]}\left[C\,e^{-\omega t}\|x(\sigma)\| + C\int_\sigma^t e^{-\omega(t-r)}\|f(r)\|\,dr\right] + N(T)\left\|x_\sigma\right\|_{\mathcal{B}}\right]\\
&\le C\,M(T)\,e^{-L}\sup_{t\in[\sigma,\sigma+T]}\|x(t)\| + C\,M(T)\,e^{-L}\|\beta\|_{L^1} + N(T)\,e^{-L}\left\|x_\sigma\right\|_{\mathcal{B}}\\
&\le C\,M(T)\,e^{-L}\,r^* + C\,M(T)\,e^{-L}\|\beta(\cdot)\|_{L^1} + N(T)\,r^*\\
&= \Big(C\,M(T)\,e^{-L} + N(T)\Big)\ r^* + C\,M(T)\,e^{-L}\|\beta(\cdot)\|_{L^1} \le q\,r^* + q \le\ r^*.
\end{aligned}$$

Now, using twice Remark 3.2.2 and (3.29), we have for every $t \in [\sigma, \sigma+T]$,

$$\begin{aligned}
&\|y(t)\| = \left\|\mathcal{S}\left((\mathcal{S}(x_\sigma, f))_{\sigma+T}\,,\ f\right)(t)\right\| = \left\|S\Big(\Big(S(x(\sigma), f)\Big)(\sigma+T),\ f\Big)(t)\right\|\\
&\le C\,e^{-\omega t}\left\|S\Big(x(\sigma), f\Big)(\sigma+T)\right\| + C\int_\sigma^t e^{-\omega(t-s)}\|f(s)\|\ ds\\
&\le C\ \left[C\,e^{-\omega T}\|x(\sigma)\| + C\int_\sigma^{\sigma+T} e^{-\omega(T-s)}\|f(s)\|\ ds\right] + C\int_\sigma^t e^{-\omega(t-s)}\|f(s)\|\,ds\\
&\le C^2\,e^{-\omega T}\sup_{t\in[\sigma,\sigma+T]}\|x(t)\| + C(1+C)\ \|\beta(\cdot)\|_{L^1}\\
&\le C^2\,e^{-\omega T}\,r^* + C(C+1)\ \|\beta(\cdot)\|_{L^1} \le\ r^*.
\end{aligned}$$

From Corollary 3.3.1 we conclude that the problem (3.1) has at least one integral solution. Using the global boundedness condition (F_3') and reasoning as above, it is not difficult to show that the set of the mild solutions to the problem (3.1) is bounded.

Since the measure of noncompactness Ψ is monotone, it suffice to apply Proposition 2.3.1 in order to deduce that the mild solutions set solution of the problem (3.1) is compact. This completes the proof of Theorem 3.3.1. $\square$

Remark 3.3.1. *As an example of a phase space $\mathcal{B}$ satisfying Axioms* ($\mathcal{B}_1$)-($\mathcal{B}_4$), *with $N(T) < 1$, one can take the space $\mathcal{L}_\gamma(E)$ defined as follow (see [34, Theorem 1.3.8]):*

for a real constant $\gamma > 0$, the space $\mathcal{L}_\gamma(E)$ is the set of measurable functions $\psi :]-\infty, 0] \to E$ such that $e^{\gamma\theta}\psi(\theta)$ is integrable over $(-\infty, 0]$ *endowed with the norm*

$$\|\psi\|_{\mathcal{L}_\gamma} = \|\psi(0)\| + \int_{-\infty}^{0} e^{\gamma\theta} \|\psi(\theta)\| \, d\theta.$$

For such a space the function $N(\cdot)$ is given by $N(t) = e^{-\gamma t}$.

Periodic solutions

Let us consider the problem

$$x'(t) \in Ax(t) + G(t, x_t), \quad t \geq \sigma, \tag{3.30}$$

where $G : [\sigma, +\infty[\times \mathcal{B} \to Kv(E)$, is a multivalued map which T-periodic on the first argument:

(G_T) for all $u \in \mathcal{B}$ and a.e. $t \in [\sigma, +\infty[, \quad G(t + T, u) = G(t, u)$.

Let $F : [\sigma, \sigma + T] \times \mathcal{B} \to Kv(E)$, be the restriction of G on $[\sigma, \sigma + T] \times \mathcal{B}$., i.e.,

$$F(t, u) = G(t, u), \quad \forall (t, u) \in [\sigma, \sigma + T] \times \mathcal{B}.$$

Corollary 3.3.2. *Assume Hypotheses* (A_1), (A_2), (F_1), (F_2), (F_3') *and* (G_T). *If $N(T) < 1$, then the set of all T-periodic mild solutions of the problem* (3.30) *is nonempty and compact.*

This a direct consequence of Theorem 3.3.1 and the fact that set of all mild solutions of the problem (3.1) extended by T-periodicity on all $\mathbb{R}$ coincides with the set of all T-periodic mild solutions of the problem (3.30).

Corollary 3.3.3. *Assume Hypotheses* (A_1), (F_1), (F_2), (F_3'),(G_T). *Further, suppose that A generates a contraction semigroup. i.e., $\|\mathcal{T}(t)\| \leq 1$ for all $t \geq 0$. If $N(T) < 1$, then for any $\varepsilon > 0$, the set of all T-periodic mild solutions of the problem*

$$x'(t) \in (A - \varepsilon I)\, x(t) + G(t, x_t), \quad t \geq \sigma,$$

is nonempty and compact (here I stands for the identity on E).

Indeed, this is true because the operator $B = A - \varepsilon I$ satisfies Hypothesis (A_1) and Hypothesis (A_2) with $C = 1$ and $\omega = \varepsilon$.

Chapter 4

Averaging results for semilinear functional differential equations with infinite delay in a Banach space: a case of non uniqueness of the solutions

We aim in this chapter to establish averaging results in a case of non uniqueness of the mild solutions for semilinear functional differential equations in a Banach space E with infinite delay on an abstract phase space $\mathcal{B}$ defined axiomatically, of the form,

$$\begin{cases} x'(t) = A\,x(t) + f(\frac{t}{\varepsilon}, x_t), & t \in [0,T], \\ x_0 = \varphi, \end{cases}$$

where, for $x :]-\infty, +\infty[\to E$, for $t \geq 0$, the function x_t defined by, $x_t(\theta) = x(t+\theta)$, $-\infty < \theta \leq 0$, belongs to $\mathcal{B}$, $\varphi \in \mathcal{B}$, A is a linear operator generating a C_0-semigroup in E which is not necessarily compact, ε is a small positive parameter and, $f : [0, +\infty[\times \mathcal{B} \to E$ is a Carathéodory function satisfying some conditions of boundedness and χ-regularity. An example is provided to illustrate our results.

4.1 Formulation of the problem

Let σ be a real number and $T > 0$ be a fixed time. By $C([\sigma, \sigma + T]; E)$ we denote the space of continuous functions defined on $[\sigma, \sigma + T]$ with values in a Banach space $(E, \|.\|)$, endowed with the uniform convergence norm. For any function $z :]-\infty, \sigma+T] \to E$ and for every $t \in [\sigma, \sigma+T]$, z_t represents the function from $]-\infty, 0]$ into E defined by $z_t(\theta) = z(t+\theta); -\infty < \theta \leq 0$.

Phase space

Let $\mathcal{B}$ be a linear topological space of functions mapping $]-\infty, 0]$ into E endowed with a seminorm $\|.\|_{\mathcal{B}}$ and, satisfying the following axioms:

if $z :]-\infty, \sigma+T] \to E$ is continuous on $[\sigma, \sigma+T]$ and $z_\sigma \in \mathcal{B}$, then, for every $t \in [\sigma, \sigma+T]$, we have

(B1) $z_t \in \mathcal{B}$;

(B2) $\|z_t\|_{\mathcal{B}} \leq K(t-\sigma) \sup_{\sigma \leq s \leq t} \|z(s)\| + N(t-\sigma) \|z_\sigma\|_{\mathcal{B}}$, where $K, N : [0, +\infty[\longrightarrow [0, +\infty[$ are independent of z, K is positive and continuous and, N is locally bounded;

(B3) the function $t \mapsto z_t$ is continuous.

Such a space $\mathcal{B}$ was introduced by Hale and Kato [27] and has been considered as a phase space in the theory of functional differential equations with infinite delay.

Existence result

Let us denote by the symbol $\mathcal{C}(]-\infty, T]; E)$ the linear topological space consisting of functions $z :]-\infty, T] \to E$ such that $z_0 \in \mathcal{B}$ and the restriction $z\,|_{[0,T]}$ is continuous, endowed with a seminorm

$$\|z\|_{\mathcal{C}} = \|z_0\|_{\mathcal{B}} + \left\| z\,|_{[0,T]} \right\|_{C([0,T];E)}.$$

Let as consider a semilinear functional differential equation in E:

$$\begin{cases} x'(t) = A\,x(t) + f(t, x_t), & t \in [0, T], \\ x_0 = \varphi, \end{cases} \tag{4.1}$$

where

(A) A is the generator of a $\mathcal{C}_0$-semigroup $(S(t))_{t \geq 0}$ on E.

The function f is acting from $[0, T] \times \mathcal{B}$ to E and satisfies the following hypotheses:

(f_1) For all $u \in \mathcal{B}$, the mapping $t \to f(t, u)$ is measurable.

(f_2) For a.e. $t \in [0,T]$ the map $f(t,\cdot) : \mathcal{B} \to E$ is continuous.

(f_3) There exists a function $\alpha \in L^1([0,T],\mathbb{R}^+)$ such that, for all $u \in \mathcal{B}$,

$$\|f(t,u)\| \leq \alpha(t)(1 + \|u\|_{\mathcal{B}}) \text{ a.e. } t \in [0,T].$$

(f_4) There exists a function $k(\cdot) \in L^1_+([0,T])$ such that for every bounded $\Omega \subset \mathcal{B}$,

$$\chi\left(f(t,\Omega)\right) \leq k(t) \sup_{-\infty<\theta\leq 0} \chi(\Omega(\theta)),$$

a.e. $t \in [0,T]$, where, $\Omega(\theta) = \{u(\theta),\ u \in \Omega\}$.

Definition 4.1.1. *We say that $z \in \mathcal{C}(\,]-\infty,T];E)$ is a* mild solution *to the problem* (4.1) *if $z_0 = \varphi$ and,*

$$z(t) = S(t)\varphi(0) + \int_0^t S(t-s)f(s,z_s)\,ds \quad \textit{for } 0 \leq t \leq T.$$

Taking $F(t,u) = \{f(t,u)\}$ and $\sigma = 0$ in [21, Theorem 3], we deduce immediately,

Theorem 4.1.1. *Assume that the phase space $\mathcal{B}$ satisfies the axioms* (B1)–(B3). *Under Hypotheses* (A), *and $(f_1)-(f_4)$, the set $\Sigma_\varphi \subset \mathcal{C}(]-\infty,T];E)$ of all mild solutions of the problem* (4.1) *on $]-\infty,T]$ is nonempty and compact.*

Note that under hypotheses of Theorem 4.1.1, the mild solutions are not necessarily unique. This may be easily cheeked, for example, in the case where $E = \mathbb{R}$.

System in standard form

Let us consider an infinite delay equation in E with a small positive parameter ε, in normal form,

$$\begin{cases} z'(\tau) = \varepsilon f(\tau, z_\tau), \tau \geq 0 \\ z_0 = \psi, \end{cases} \tag{4.2}$$

where, for $z :]-\infty,+\infty[\to E$, for $\tau \geq 0$, the function z_τ is defined by, $z_\tau(\theta) = z(\tau+\theta)$, $-\infty < \theta \leq 0$ and, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically and $\psi \in \mathcal{B}$. Such a phase space $\mathcal{B}$, was introduced by Hale and Kato [27] and, is defined by suitable axioms as a seminormed space (see Section 3).

Parallel to the problem (4.2), we consider the averaged problem:

$$\begin{cases} z'(\tau) = \varepsilon f_0(z_\tau), \quad \tau \geq 0, \\ \quad z_0 = \psi, \end{cases} \tag{4.3}$$

where $f_0 : \mathcal{B} \to E$ is such that for all $u \in \mathcal{B}$:

$$f_0(u) = \lim_{t\to+\infty} \frac{1}{t} \int_0^t f(\theta, u) d\theta. \tag{4.4}$$

Define the set $\mathfrak{D}(\psi)$ by,

$$\mathfrak{D}(\psi) = \{\hat{z} : [0, +\infty[\to E,\ \hat{z}(0) = \psi(0)\}.$$

Now, for every $\hat{z} \in \mathfrak{D}(\psi)$, let us define the function, $\hat{z}[\psi] :]-\infty, +\infty[\to E$, by

$$\hat{z}[\psi](t) = \begin{cases} \psi(t), & -\infty < t \leq 0, \\ \hat{z}(t), & 0 \leq t < +\infty. \end{cases} \tag{4.5}$$

Then, for every $t \geq 0$ and for every $\theta \leq 0$, we have,

$$\hat{z}[\psi]_t(\theta) = \begin{cases} \psi(t+\theta), & \text{if } -\infty < \theta \leq -t, \\ \hat{z}(t+\theta), & \text{if } -t \leq \theta \leq 0. \end{cases} \tag{4.6}$$

Note that, if z is a solution of the problem (4.2), then z has the form, $z = \hat{z}[\psi]$, for some $\hat{z} \in \mathfrak{D}(\psi)$ solution of the problem,

$$\begin{cases} \hat{z}'(\tau) = \varepsilon f(\tau, \hat{z}[\psi]_\tau), & \tau \geq 0, \\ \hat{z}[\psi]_0 = \psi. \end{cases} \tag{4.7}$$

Thus, the problem (4.2) can be written equivalently as (4.7).

Notice that the expression $\hat{z}[\psi]_0 = \psi$ in (4.7) , means that $\hat{z}(0) = \psi(0)$.

Now, suppose that the function ψ satisfies:

$$\psi(\frac{\cdot}{\varepsilon}) \in \mathcal{B}, \qquad \text{for every} \quad \varepsilon > 0. \tag{4.8}$$

For the problem (4.7), let us consider the change of variable

$$\begin{cases} \theta = \dfrac{\theta'}{\varepsilon}, & \tau = \dfrac{t}{\varepsilon}, \quad \theta \leq 0, \ \tau \geq 0, \\ \\ \psi\Big(\dfrac{\theta'}{\varepsilon}\Big) = \varphi(\theta'), & \hat{z}\Big(\dfrac{t}{\varepsilon}\Big) = y(t). \end{cases} \tag{4.9}$$

Note that

(i) When θ varies in $]-\infty, 0]$, for every $\varepsilon > 0$, the quantity $\theta\varepsilon = \theta'$ varies in $]-\infty, 0]$;

(ii) According to (4.8) and (4.9), we have,

$$\text{the function } \theta' \to \varphi(\theta') \in \mathcal{B}, \text{ and } \hat{z}(0) = \psi(0) = y(0) = \varphi(0).$$

We claim that

$$\hat{z}[\psi]_\tau(\theta) = y[\varphi]_t(\theta').$$

Indeed, we have

$$\begin{aligned}
\hat{z}[\psi]_\tau(\theta) = \hat{z}[\psi]_{t/\varepsilon}(\theta) &= \begin{cases} \psi(\theta + t/\varepsilon), & \text{if } \; -\infty < \theta \leq -t/\varepsilon, \\ \hat{z}(\theta + t/\varepsilon), & \text{if } \; -t/\varepsilon \leq \theta \leq 0. \end{cases} \\
&= \begin{cases} \psi(\frac{1}{\varepsilon}(\theta\varepsilon + t)), & \text{if } \; -\infty < \theta\varepsilon \leq -t, \\ \\ \hat{z}(\frac{1}{\varepsilon}(\theta\varepsilon + t)), & \text{if } \; -t \leq \theta\varepsilon \leq 0. \end{cases} \\
&= \begin{cases} \varphi(\theta\varepsilon + t), & \text{if } \; -\infty < \theta\varepsilon \leq -t, \\ \\ y(\theta\varepsilon + t), & \text{if } \; -t \leq \theta\varepsilon \leq 0. \end{cases} \\
&= \begin{cases} \varphi(\theta' + t), & \text{if } \; -\infty < \theta' \leq -t, \\ \\ y(\theta' + t), & \text{if } \; -t \leq \theta' \leq 0. \end{cases} \\
&= y[\varphi]_t(\theta').
\end{aligned}$$

This proves our claim. Thus, under the change of variable given in (4.9), the problem (4.7) (as a consequence the problem (4.2)) can be written equivalently:

$$\begin{cases} y'(t) = f(\frac{t}{\varepsilon}, y[\varphi]_t), \\ y[\varphi]_0 = \varphi, \end{cases} \tag{4.10}$$

Now, setting in (4.10), $x(t) = y[\varphi](t),\ t \in]-\infty, +\infty[$, we deduce that the problem (4.2) can be written equivalently as:

$$\begin{cases} x'(t) = f(\frac{t}{\varepsilon}, x_t),\ t \geq 0, \\ x_0 = \varphi. \end{cases} \tag{4.11}$$

Similarly, under the condition (4.8), the problem (4.3) can be written equivalently as

$$\begin{cases} x'(t) = f_0(x_t), \quad t \geq 0, \\ \quad x_0 = \varphi. \end{cases}$$

Thus, to establish the averaging principle for the problem (4.2) in the case of finite time interval, we have to show that for any fixed time $T > 0$, the solutions of the problem

$$\begin{cases} x'(t) = f(\frac{t}{\varepsilon}, x_t),\ t \in [0, T], \\ x_0 = \varphi, \end{cases}$$

are approximated (in some sens) by the solutions of the problem,

$$\begin{cases} x'(t) = f_0(x_t), \quad t \in [0, T], \\ \quad x_0 = \varphi, \end{cases}$$

as $\varepsilon \to 0^+$.

Remark 4.1.1. *Using the change of variable given in* (4.9) *and reasoning as above, under the condition* (4.8)*, the problem in the normal form*

$$\begin{cases} z'(\tau) = \varepsilon\Big[A\, z(\tau) + f(\tau, z_\tau)\Big], \quad \tau \geq 0, \\ z_0 = \psi \in \mathcal{B}, \end{cases}$$

can be written equivalently as:

$$\begin{cases} x'(t) = A\, x(t) + f(\frac{t}{\varepsilon}, x_t), \quad t \geq 0, \\ x_0 = \varphi \in \mathcal{B}. \end{cases}$$

Remark 4.1.2. *If we consider the problem* (4.2) *but with finite delay where, the initial condition in* (4.3) *is written as* $z_0 = \psi$ *with,* $\psi \in C([-r, 0], E),\ r > 0$*, then using the same reasoning as above and using the change of variable* (4.9)*, we can define equivalently a problem similar to the problem* (4.11) *but with initial condition* $z(\theta) = \varphi(\theta),\ \theta \in [-\varepsilon r, 0]$*, that is, the effects of the delay is negligible and approaches zero in* ε*, as a consequence the function* $z(\theta + t),\ \theta \in [-\varepsilon r, 0]$ *can be approximated by* $z(t)$ *as* ε *is small enough. This fact can simplify the writing of the averaged equation. In our case (infinite delay), this situation cannot take place since, when* θ *varies in* $]-\infty, 0]$*, for every* $\varepsilon > 0$*, the quantity* $\theta\varepsilon$ *varies in* $]-\infty, 0]$*. It is why we have to choose* φ *defined on* $]-\infty, 0]$*. To guarantee that* φ *belongs to* $\mathcal{B}$ *we have added condition given in* (4.8)*.*

Remark 4.1.3. *Condition* (4.8) *is not satisfied for any choice of a phase space* $\mathcal{B}$ *introduced by Hale and Kato [27], as shows this simple example:*

For $\gamma > 0$, take $\mathcal{B} = C_\gamma$, where C_γ is the space of continuous functions $\psi :]-\infty, 0] \to \mathbb{R}$ such that $e^{\gamma\theta}\psi(\theta)$ has a limit in $\mathbb{R}$ as $\theta \to -\infty$, endowed with the norm,

$$\|\psi\|_{\mathcal{B}} = \sup\left\{e^{\gamma\theta}\,|\psi(\theta)|\ :\ \theta \in]-\infty, 0]\right\}.$$

It is clear that the function $\psi(\theta) = e^{-\gamma\theta}$ belongs to $\mathcal{B}$, but for every $0 < \varepsilon < 1$,

$$\left\|\psi(\frac{\cdot}{\varepsilon})\right\|_{\mathcal{B}} = \sup_{\theta\in]-\infty,0]} e^{\gamma\theta}e^{-\gamma(\theta/\varepsilon)} = \sup_{\theta\in]-\infty,0]} e^{\gamma\theta(1-1/\varepsilon)} = +\infty.$$

4.2 Averaging results

Throughout this section, the phase space $\mathcal{B}$ is considered as a Banach space satisfying Axioms (B1)-(B3). Examples of such phase spaces can be found in [32] (see also [33, page 20]). We give only one example: As an example of such phase space $\mathcal{B}$, one can take the Banach space $C_\infty(E)$ (See [32]), constituted of continuous functions $\psi :]-\infty, 0] \to E$, such that $\lim_{t\to-\infty}\psi(\theta)$ exists, endowed with the sup-norm,

$$\|\psi\|_{C_\infty} = \sup\{\|\psi(\theta)\| : \theta \in]-\infty, 0]\}.$$

Let us consider a semilinear differential equation in E with a small positive parameter ε, $\varepsilon \in]0, 1]$, of the form

$$\begin{cases} z'(t) = A\,z(t) + f(\frac{t}{\varepsilon}, z_t), & t \in [0, T], \\ z_0 = \varphi, \end{cases} \qquad (P_\varepsilon)$$

Now parallel to the problem (P_ε), $\varepsilon > 0$, we consider the averaged problem

$$\begin{cases} z'(t) = A\,z(t) + f_0(z_t), \\ \quad z_0 = \varphi. \end{cases} \qquad (P_0)$$

Suppose that the unbounded linear operator A satisfies the condition (A) and the function f is acting from $\mathbb{R}^+ \times \mathcal{B}$ to E. Let us consider the following hypotheses,

($\mathtt{f}_1$) For all $u \in \mathcal{B}$, the mapping $t \to f(t, u)$ is measurable.

($\mathtt{f}_2$) The function f is continuous in the second variable uniformly with respect to first one, i.e., for every $u \in \mathcal{B}$ and for every $\delta > 0$ there exists $\beta > 0$ such that,

$$\forall\, u' \in \mathcal{B},\ \|u - u'\| < \beta \Rightarrow \|f(t, u) - f(t, u')\| < \delta \quad \text{a.e.}\, t \in [0, +\infty[.$$

($\mathtt{f}_3$) There exists a constant $C > 0$ such that, for all $u \in \mathcal{B}$,

$$\|f(t,u)\| \leq C(1 + \|u\|_{\mathcal{B}}) \ \text{ a.e. } t \in \mathbb{R}^+. \tag{4.12}$$

($\mathtt{f}_4$) There exists a positive constant p such that for every bounded $D \subset \mathcal{B}$ and for any negligible subset $\mathcal{N}$ of $[0, +\infty[$ such that, $f([0, +\infty[\setminus \mathcal{N},\, D)$ is bounded,

$$\chi\left(f([0, +\infty[\setminus \mathcal{N},\, D)\right) \leq p \sup_{-\infty < \theta \leq 0} \chi(D(\theta)). \tag{4.13}$$

(see Remark 4.2.2).

($\mathtt{Af}$) There exist $\Delta_0 > 0$ and a continuous function $f_0 : \mathcal{B} \to E$ such that, for all $u \in \mathcal{B}$ and $t_1,\, t_2 \in [0, T]$ with $0 \leq t_1 \leq t_2 \leq t_1 + \Delta_0 \leq T$,

$$\lim_{\varepsilon \to 0^+} \int_{t_1}^{t_2} S(t_2 - \theta) \left[f(\frac{\theta}{\varepsilon}, u) - f_0(u) \right] d\theta = 0. \tag{4.14}$$

Theorem 4.2.1. *Assume the hypotheses* (A), $(\mathtt{f}_1) - (\mathtt{f}_4)$ *and* $(\mathtt{Af})$. *Let* $(\varepsilon_k)_{k \geq 1}$ *be a sequence of positive numbers which converges to* 0. *Then, any sequence* $(z^k)_{k \geq 1}$, *where* z^k *is a mild solution to the problem* (P_{ε_k}) $(\varepsilon = \varepsilon_k)$ *has a subsequence which converges in* $\mathcal{C}(]-\infty, T]; E)$ *to a mild solution of the problem* (P_0).

Remark 4.2.1. *Such a condition* $(\mathtt{Af})$ *is directly inspired from [54, Theorem 3]. Note that here, we do not suppose that* f *or* f_0 *is a Lipschitz function and the uniqueness of the mild solutions to the problems* $(P_\varepsilon)_{\varepsilon \geq 0}$ *is not guaranteed.*

Remark 4.2.2. *The existence of such a negligible subset* $\mathcal{N}$ *in* $(\mathtt{f}_4)$ *is guaranteed by* $(\mathtt{f}_3)$. *Since the right hand side in* (4.13) *does not depend on* $t \in [0, +\infty[$ *(on the first argument of the function* f*), Inequality* (4.13) *is independent on the choice of such a subset* $\mathcal{N}$.

Note that, if $f(\cdot, \cdot)$ *is continuous then* (4.12) *is satisfied for all* $t \in \mathbb{R}^+$ *and,* (4.13) *can be written as*

$$\chi\left(f([0, +\infty[,\, D)\right) \leq p \sup_{-\infty < \theta \leq 0} \chi(D(\theta)).$$

Remark 4.2.3. *Since* $(S(t))_{t \geq 0}$ *is a strongly continuous semigroup, there exists a constant* $M > 0$ *such that*

$$\|S(t)\| \leq M, \quad 0 \leq t \leq T \tag{4.15}$$

(See e.g. *[50, Theorem 2.2]).*

Remark 4.2.4. *Note that:*

(i) Hypothesis ($\mathtt{f}_2$) *implies that, for any compact subset $\mathcal{W}$ of $\mathcal{B}$, the function f is uniformly continuous for $u \in \mathcal{W}$, uniformly with respect to a.e. $t \in]-\infty, +\infty[$, i.e.,*

$$\forall \delta > 0 \; \exists \beta > 0 \;\; \forall u, \, u' \in \mathcal{W}: \\ \|u - u'\| < \beta \Rightarrow \|f(t,u) - f(t,u')\| < \delta \quad a.e.\, t \in [0, +\infty[.$$

This is due to the fact that $\mathcal{W}$ is a compact set.

(ii) Under Hypothesis ($\mathtt{Af}$)*, the estimation* (4.14) *is satisfied uniformly with respect to u in compact sets.*

One can easily show (ii) by bearing in mind (i), the fact that the restriction of f_0 to any compact subset of $\mathcal{B}$ is uniformly continuous, the estimation (4.15) *and the fact that the time T in* ($\mathtt{Af}$) *is bounded, i.e., $T < +\infty$.*

Remark 4.2.5. *If in addition, we suppose that the phase space satisfies the axiom:*

($\mathcal{B}_{av}$) *If $\psi \in \mathcal{B}$ then, for any $\varepsilon > 0$, $\psi(\frac{\cdot}{\varepsilon}) \in \mathcal{B}$,*

then Remark 4.1.1 is valid. It is clear that the axiom ($\mathcal{B}_{av}$) is obviously satisfied by the space $C_\infty(E)$.

Before giving the proof of Theorem 4.2.1, we need some auxiliaries results. We will use the same notations as in [21].

In the space $C([0,T],E)$, let us define the set

$$\mathfrak{D}(\varphi, T) = \{x \in C([0,T];E), \; x(0) = \varphi(0)\}. \tag{4.16}$$

It is clear that $\mathfrak{D}(\varphi, T)$ is a closed convex subset in $C([0,T];E)$.

For every $x \in \mathfrak{D}(\varphi, T)$, let us define the function $x[\varphi] \in \mathcal{C}(\,]-\infty, T]; E)$ by

$$x[\varphi](t) = \begin{cases} \varphi(t), & -\infty < t \leq 0, \\ x(t), & 0 < t \leq T. \end{cases} \tag{4.17}$$

Then

$$x[\varphi]_t(\theta) = \begin{cases} \varphi(t+\theta), & \text{if } \; -\infty < \theta \leq -t, \\ x(t+\theta), & \text{if } \; -t < \theta \leq 0. \end{cases} \tag{4.18}$$

Let us fix a sequence $(\varepsilon_k)_{k \geq 1}$ of positive number which converges to 0. Denote by Φ_k, $1 \leq k \leq \infty$, the mappings which, with every element $x \in \mathfrak{D}(\varphi, T)$ associate the element $\Phi_k(x) \in \mathfrak{D}(\varphi, T)$, given by, for $1 \leq k < \infty$,

$$\Phi_k(x)(t) = S(t)\varphi(0) + \int_0^t S(t-s) f(\frac{s}{\varepsilon_k}, x[\varphi]_s)\, ds, \quad 0 \leq t \leq T, \tag{4.19}$$

and, for $k = \infty$,

$$\Phi_\infty(x)(t) = S(t)\varphi(0) + \int_0^t S(t-s) f_0(x[\varphi]_s)\, ds, \quad 0 \le t \le T. \tag{4.20}$$

Remark 4.2.6. *According to Theorem 4.1.1, under hypotheses* ($\mathtt{f}_1$)-($\mathtt{f}_4$), *for each $k \ge 1$, the problem (P_{ε_k}) has at least one mild solution. Note that, for each $k \ge 1$, a mild solution of (P_{ε_k}) has the form $z^k = x^k[\varphi]$, for some fixed point $x^k \in \mathfrak{D}(\varphi, T)$of the operator Φ_k (i.e., $x^k = \Phi_k(x^k)$). Note that, if z^∞ is a mild solution of the problem (P_0) then, $z^\infty = x^\infty[\varphi]$, for some fixed point $x^\infty \in \mathfrak{D}(\varphi, T)$ of the operator Φ_∞.*

Lemma 4.2.1. *Let $(y^k)_{k\ge1}$ be an any sequence of $\mathfrak{D}(\varphi, T)$. Assume that $y^k \to y^0$, for some $y^0 \in C([0,T]; E)$. Then, we have,*

$$\lim_{k\to\infty} \sup_{t\in[0,T]} \left\| \int_0^t S(t-\theta) \left[f(\frac{\theta}{\varepsilon_k}, y^k[\varphi]_\theta) - f_0(y^0[\varphi]_\theta) \right] d\theta \right\| = 0.$$

With others words,

$$\Phi_k(y^k) \underset{k\to\infty}{\longrightarrow} \Phi_\infty(y^0) \ \text{ in } \ C([0,T]; E).$$

Proof. Because the set $\mathfrak{D}(\varphi, T)$ is closed, we have necessarily that $y^0 \in \mathfrak{D}(\varphi, T)$. Moreover, by Axioms (B3) and (B2), we have, $\lim_{k\to\infty} \left\| y^k[\varphi]_{(.)} - y^0[\varphi]_{(.)} \right\|_{C([0,T];\mathcal{B})} = 0$. Denote by $\mathcal{K}$ the compact subset of $C([0,T]; \mathcal{B})$, given by, $\mathcal{K} = \left\{ y^k[\varphi]_{(.)}, \quad k \ge 0 \right\}$ and, by $\mathcal{Q}$ the compact subset of $\mathcal{B}$, given by,

$$\mathcal{Q} = \left\{ y^k[\varphi]_s, \quad s \in [0,T], \quad k \ge 0 \right\}.$$

The function f_0 is uniformly continuous on $\mathcal{Q}$ and from Remark 4.2.4-(i), the continuity of f is uniform for $u \in \mathcal{Q}$, uniformly with respect to a.e. $t \in]-\infty, +\infty[$. Let $\delta > 0$. There exists $\gamma \in]0, \delta]$ such that for all u, $v \in \mathcal{Q}$ with $\|u - v\|_{\mathcal{B}} \le \gamma$, one has,

$$\|f_0(u) - f_0(v)\| \le \delta \quad \text{and} \quad \|f(t,u) - f(t,v)\| \le \delta \ \text{ a.e. } t \in]-\infty, +\infty[. \tag{4.21}$$

Now, using Arzela-Ascoli Theorem, we can find a finite set $\mathcal{V}$ of step functions with values in $\mathcal{Q}$, corresponding to a partition $0 = t_0 < t_1 < ... < t_q = T$ of $[0,T]$, which forms a finite γ-net of $\mathcal{K}$. It is clear that one can assume that

$$\max\{t_i - t_{i-1},\, i = 1, ...q\} \le \min\{\delta,\, \Delta_0\}, \tag{4.22}$$

where, Δ_0 is defined in Hypothesis ($\mathtt{Af}$). For each $y^k[\varphi]_{(.)} \in \mathcal{K}$, $k \ge 0$, there exists $\bar{Y} \in \mathcal{V}$ with $\bar{Y}(t_{i-1}) \in \mathcal{Q}, i = 1, ..., q$ and,

$$\left\| y^k[\varphi]_t - \bar{Y}(t_{i-1}) \right\| \le \gamma \ \text{ for all } t \in [t_{i-1}, t_i[,\ i = 1, ..., q. \tag{4.23}$$

Set $\tau(t) = \max\{i,\, t_i \leq t\}$ and, $\vartheta(t) = t_{\tau(t)}$. We have, for every $t \in [0, T]$,

$$\begin{aligned}
&\int_0^t S(t-\theta)\left[f(\frac{\theta}{\varepsilon_k}, y^k[\varphi]_\theta) - f_0(y^0[\varphi]_\theta)\right] d\theta \\
&= \int_{\vartheta(t)}^t S(t-\theta)\left[f(\frac{\theta}{\varepsilon_k}, y^k[\varphi]_\theta) - f_0(y^0[\varphi]_\theta)\right] d\theta \\
&+ \int_0^{\vartheta(t)} S(t-\theta)\left[f(\frac{\theta}{\varepsilon_k}, y^k[\varphi]_\theta) - f_0(y^0[\varphi]_\theta)\right] d\theta \\
&\qquad = I_1 + I_2.
\end{aligned}$$

Using ($\mathtt{f}_3$), (4.22) and the fact that f_0 is continuous on the compact set $\mathcal{Q}$, we get,

$$\begin{aligned}
\|I_1\| &\leq M \int_{\vartheta(t)}^t \left\| f(\frac{\theta}{\varepsilon_k}, y^k[\varphi]_\theta) - f_0(y^0[\varphi]_\theta)\right\| d\theta \\
&\leq M \int_{\vartheta(t)}^t \left[\left\| f(\frac{\theta}{\varepsilon_k}, y^k[\varphi]_\theta)\right\| + \|f_0(y^0[\varphi]_\theta)\|\right] d\theta \\
&\leq M\,(t-\vartheta(t))\left[C\left(1+\max_{x\in\mathcal{Q}}\|x\|\right) + \max_{x\in\mathcal{Q}}\|f_0(x)\|\right] \\
&\leq M\,\min\{\delta,\,\Delta_0\}\left[C\left(1+\max_{x\in\mathcal{Q}}\|x\|\right) + \max_{x\in\mathcal{Q}}\|f_0(x)\|\right] \\
&\leq M\,\delta\left[C\left(1+\max_{x\in\mathcal{Q}}\|x\|\right) + \max_{x\in\mathcal{Q}}\|f_0(x)\|\right].
\end{aligned}$$

It remains to estimate I_2. For every $t \in [0, T]$, we have

$$\begin{aligned}
I_2 &= \int_0^{\vartheta(t)} S(t-\theta)\left[f(\frac{\theta}{\varepsilon_k}, y^k[\varphi]_\theta) - f(\frac{\theta}{\varepsilon_k}, \bar{Y}(\theta))\right] d\theta \\
&+ \int_0^{\vartheta(t)} S(t-\theta)\big[f_0(\bar{Y}(\theta)) - f_0(y^0[\varphi]_\theta)\big]\, d\theta \\
&+ \int_0^{\vartheta(t)} S(t-\theta)\left[f(\frac{\theta}{\varepsilon_k}, \bar{Y}(\theta)) - f_0(\bar{Y}(\theta))\right] d\theta \\
&= J_1 + J_2 + J_3.
\end{aligned}$$

Using (4.23) and (4.21), we get

$$\begin{aligned}
\|J_1\| &\le M \int_0^{\vartheta(t)} \left\| f(\frac{\theta}{\varepsilon_k}, y^k[\varphi]_\theta) - f(\frac{\theta}{\varepsilon_k}, \bar{Y}(\theta)) \right\| d\theta \\
&\le M \sum_{i=1}^{q} \int_{t_{i-1}}^{t_i} \left\| f(\frac{\theta}{\varepsilon_k}, y^k[\varphi]_\theta) - f(\frac{\theta}{\varepsilon_k}, \bar{Y}(\theta)) \right\| d\theta \\
&\le M \sum_{i=1}^{q} \int_{t_{i-1}}^{t_i} \left\| f(\frac{\theta}{\varepsilon_k}, y^k[\varphi]_\theta) - f(\frac{\theta}{\varepsilon_k}, \bar{Y}(t_{i-1})) \right\| d\theta \\
&\le M\,T\,\delta.
\end{aligned}$$

Again, using (4.23) and (4.21), we get

$$\begin{aligned}
\|J_2\| &\le M \sum_{i=1}^{q} \int_{t_{i-1}}^{t_i} \left\| f_0(y^0[\varphi]_\theta) - f_0(\bar{Y}(\theta)) \right\| d\theta \\
&\le M \sum_{i=1}^{q} \int_{t_{i-1}}^{t_i} \left\| f_0(y^0[\varphi]_\theta) - f_0(\bar{Y}(t_{i-1})) \right\| d\theta \\
&\le M\,T\,\delta.
\end{aligned}$$

It remains to estimate the term J_3. Since q is finite and $\mathcal{Q}$ is a compact set, using the hypothesis ($\mathtt{Af}$) and Remark 4.2.4-(ii), we get,

$$\begin{aligned}
\|J_3\| &= \left\| \sum_{i=1}^{\tau(t)} \int_{t_{i-1}}^{t_i} S(t-\theta) \left[f(\frac{\theta}{\varepsilon}, \bar{Y}(\theta)) - f_0(\bar{Y}(\theta)) \right] d\theta \right\| \\
&\le M \sum_{i=1}^{q} \left\| \int_{t_{i-1}}^{t_i} S(t_i-\theta) \left[f(\frac{\theta}{\varepsilon}, \bar{Y}(t_{i-1})) - f_0(\bar{Y}(t_{i-1})) \right] d\theta \right\| \\
&\le M\,\delta \text{ as } \varepsilon \to 0,
\end{aligned}$$

uniformly with respect to $\bar{Y} \in \mathcal{V}$. □

Proof of Theorem 4.2.1. Let $(z^k)_{k\ge1}$ be an any sequence with, $z^k = x^k[\varphi]$ where, x^k is a fixed point of the operator Φ_k (see (4.19)). First, let us prove that the sequence $(x^k)_{k\ge1}$ is bounded in $C([0,T];E)$. Set

$$\lambda_K = \sup_{0\le s\le T} K(s); \tag{4.24}$$

$$\lambda_N = \sup_{0\le s\le T} N(s). \tag{4.25}$$

Let $k \geq 1$. Using ($\mathtt{f}_3$) and Axiom (B2), we obtain, for any $t \in [0,T]$,

$$\begin{aligned}
\left\|x^k(t)\right\| = \left\|\Phi_k(x^k)(t)\right\| &\leq M\left\|\varphi(0)\right\| + M\int_0^t \left\|f(\frac{\tau}{\varepsilon_k}, x^k[\varphi]_\tau)\right\| d\tau \\
\leq & M\left\|\varphi(0)\right\| + M\int_0^t C\left(1+\lambda_K \sup_{0\leq\theta\leq\tau}\left\|x^k(\theta)\right\| + \lambda_N \left\|\varphi\right\|_{\mathcal{B}}\right) d\tau \\
\leq & M\left[\left\|\varphi(0)\right\| + T\,C(1+\lambda_N\left\|\varphi\right\|_{\mathcal{B}})\right] + MC\lambda_K\int_0^t \sup_{0\leq\theta\leq\tau}\left\|x^k(\theta)\right\| d\tau.
\end{aligned}$$

where M is from (4.15). Since the last expression does not decrease, setting

$$\omega = M\left[\left\|\varphi(0)\right\| + TC(1+\lambda_N\left\|\varphi\right\|_{\mathcal{B}})\right] + MC\lambda_K,$$

we get,

$$e \sup_{0\leq\tau\leq t}\left\|x^k(\tau)\right\| \leq \omega + \omega\int_0^t \sup_{0\leq\theta\leq\tau}\left\|x^k(\theta)\right\| d\tau.$$

Hence, by Gronwall Lemma, we obtain, for any $t \in [0,T]$,

$$\sup_{0\leq\tau\leq t}\left\|x^k(\tau)\right\| \leq \omega\, e^{\omega t}.$$

Now, let us show that the sequence $(x^k)_{k\geq 1}$ is relatively compact.
For every $k \geq 1$, set,

$$f^k(t) = f(\frac{t}{\varepsilon_k}, x^k[\varphi]_t), \text{ a.e. } t \in [0,T]. \tag{4.26}$$

It is clear that for each $k \geq 1$ and, for every $t \in [0,T]$,

$$x^k(t) = \Phi_k(x^k)(t) = S(t)\varphi(0) + \Gamma(f^k)(t),$$

where, $\Gamma : L^1([0,T];E) \to C([0,T];E)$, is the Cauchy operator, defined by

$$\Gamma(g)(t) = \int_0^t S(t-s)\,g(s)\,ds.$$

Now, on bounded subsets of $\mathfrak{D}(\varphi,T)$, consider the following measure of non-compactness,

$$\Psi(\Omega) = \max_{\mathcal{D}\in\Delta(\Omega)}\Big(\Theta(\mathcal{D}),\, \mathrm{mod}_c(\mathcal{D})\Big),$$

where, $\Delta(\Omega)$ is the collection of all denumerable subsets of Ω,

$$\mathrm{mod}_c(\mathcal{D}) = \lim_{\delta\to 0}\sup_{x\in\mathcal{D}}\max_{|t_1-t_2|\leq\delta}\left\|x(t_1)-x(t_2)\right\|;$$

$$\Theta(\mathcal{D}) = \sup_{t\in[0,T]} e^{-L\,t}\chi(\mathcal{D}(t)),$$

where $L > 0$, is chosen so that

$$\max_{t\in[0,T]} 2\,M\,p \int_0^t e^{-L(t-s)}\,ds \leq J < 1, \tag{4.27}$$

where p is from $(\mathtt{f}_4)$.

The range for the measure of noncompactness Ψ is the cone $\mathbb{R}^2_+$, max is taken in the sense of the ordering induced by this cone. The measure of noncompactness Ψ is well defined, monotone, nonsingular, and regular (see [36, Example 2.1.4]).

Since the sequence $(x^k)_{k\geq 1}$ is bounded, by (B2), the set

$$V = \left\{x^k[\varphi]_t,\, t \in [0,T],\, k \geq 1\right\}, \tag{4.28}$$

is bounded too in $\mathcal{B}$.

Let $\mathcal{N}_1$ be an any negligible subset of $[0,+\infty[$, such that, $f([0,+\infty[\ \mathcal{N}_1, V)$ is bounded in E ($\mathcal{N}_1$ exists by $(\mathtt{f}_3)$). Let $\mathcal{N}_2$ be a subset of $[0,+\infty[$, defined by

$$\mathcal{N}_2 = \left\{\frac{s}{\varepsilon_k} \in [0,+\infty[,\ s \in [0,T] \text{ and } k \geq 1, \text{ such that, } f^k(s) \neq f\left(\frac{s}{\varepsilon_k},\, x^k[\varphi]_s\right)\right\},$$

where, $f^k,\ k \geq 1$ is defined in (4.26). Set

$$\mathcal{N} = \left\{s \in [0,T] :\ \frac{s}{\varepsilon_k} \in \mathcal{N}_1 \cup \mathcal{N}_2 \text{ for some } k \geq 1\right\}.$$

As a countable union of negligible subsets, the sets $\mathcal{N}_2$ and $\mathcal{N}$ are negligible.

By $(\mathtt{f}_4)$, we have for every $s \in [0,T]\,/\mathcal{N}$ and $k \geq 1$,

$$\begin{aligned}
\chi\left(\{f^k(s)\}_{k=1}^{+\infty}\right) &= \chi\left(\left\{f(\frac{s}{\varepsilon_k},\, x^k[\varphi]_s)\right\}_{k=1}^{+\infty}\right)\\
&\leq \chi\left(f(\,[0,+\infty[\,/\mathcal{N}_1 \cup \mathcal{N}_2,\ \{x^k[\varphi]_s\}_{k=1}^{+\infty})\right)\\
&\leq p \sup_{-\infty\leq\theta\leq 0} \chi(\{x^k[\varphi]_s(\theta)\}_{k=1}^{+\infty})\\
&= p \sup_{-\infty\leq\theta\leq 0} \chi(\{x^k[\varphi](\theta+s)\}_{k=1}^{+\infty})\\
&\leq p\left[\sup_{-\infty\leq\theta\leq -s} \chi(\{x^k[\varphi](\theta+s)\}_{k=1}^{+\infty}) + \sup_{-s\leq\theta\leq 0} \chi(\{x^k[\varphi](\theta+s)\}_{k=1}^{+\infty})\right]\\
&= p\left[\sup_{-\infty\leq\theta\leq 0} \chi(\{\varphi(\theta)\}_{k=1}^{+\infty}) + \sup_{-s\leq\theta\leq 0} \chi(\{x^k[\varphi](\theta+s)\}_{k=1}^{+\infty})\right]\\
&= p \sup_{-s\leq\theta\leq 0} \chi(\{x^k[\varphi](\theta+s)\}_{k=1}^{+\infty}) = p \sup_{0\leq\theta\leq s} \chi(\{x^k(\theta)\}_{k=1}^{+\infty})\\
&\leq p\, e^{L\,s}\, e^{-L\,s} \sup_{0\leq\theta\leq s} \chi(\{x^k(\theta)\}_{k=1}^{+\infty})\\
&\leq p\, e^{L\,s} \sup_{0\leq\theta\leq s} e^{-L\,\theta}\chi(\{x^k(\theta)\}_{k=1}^{+\infty})\\
&\leq p\, e^{L\,s}\, \Theta\left(\{x^k\}_{k=1}^{+\infty}\right).
\end{aligned}$$

Thus,

$$\chi\left(\{f^k(s)\}_{k=1}^{+\infty}\right) \leq p\, e^{L\,s}\, \Theta\left(\{x^k\}_{k=1}^{+\infty}\right) \text{ a.e. } s \in [0,T]. \tag{4.29}$$

Using the last inequality and Lemma 2.3.4, we get

$$\chi\left(\left\{\Gamma(f^k)(t)\,\right\}_{k=1}^{+\infty}\right) \leq \Theta\left(\{x^k\}_{k=1}^{+\infty}\right)\, 2\,M\,p \int_0^t e^{L(s)} ds.$$

Hence,

$$\begin{aligned}
\Theta\left(\{x^k\}_{k=1}^{+\infty}\right) =&\Theta\left(\{\Phi_k(x^k)\}_{k=1}^{+\infty}\right)\\
&= \sup_{t\in[0,T]}\left[e^{-Lt}\chi\left(\left\{S(t)\varphi(0) + \Gamma(f^k)(t)\right\}_{k=1}^{+\infty}\right)\right]\\
&= \sup_{t\in[0,T]}\left[e^{-Lt}\chi\left(\left\{\Gamma(f^k)(t)\right\}_{k=1}^{+\infty}\right)\right]\\
&\leq\ \Theta\left(\{x^k\}_{k=1}^{+\infty}\right)\, 2\,M\,p \int_0^t e^{-L(t-s)} ds \leq J\,\Theta\left(\{x^k\}_{k=1}^{+\infty}\right).
\end{aligned}$$

where J is from (4.27). Because $J < 1$, we have $\Theta\left(\{x^k\}_{k=1}^{+\infty}\right) = 0$. From (4.29), we have immediately that, $\chi\left(\{f^k(t)\}_{k=1}^{+\infty}\right) = 0$ a.e $t \in [0,T]$. Now, since the set V is bounded (see (4.28)), by $(\mathtt{f}_3)$, the sequence $(f^k(\cdot))_{k\geq 1}$ is integrably bounded in $L^1([0,T];E)$. Thus, the sequence $\{f^k(\cdot)\}_{k=1}^{+\infty}$ is semicompact. According to Lemma 2.3.5, the sequence $\big(\Gamma(f^k)(\cdot)\big)_{k=1}^{\infty}$ is relatively compact in $C([0,T];E)$. As a consequence, the sequence $(x^k)_{k\geq 1} = \big(S(\cdot)\varphi(0) + \Gamma(f^k)\big)_{k\geq 1}$ is relatively compact in $C([0,T];E)$ (recall that $(S(t))_{t\geq 0}$ is a strongly continuous semigroup). Therefore, the sequence $(x^k)_{k\geq 1}$ admits a subsequence, still denoted by $(x^k)_{k\geq 1}$, which converges to some $x^\infty \in \mathfrak{D}(\varphi,T)$ (recall that $\mathfrak{D}(\varphi,T)$ is closed). Now, Lemma 4.2.1, ensures that $\Phi_k(x^k)$ converges to $\Phi_\infty(x^\infty)$ as $k\to\infty$. But, for every $k\geq 1$, we have, $\Phi_k(x^k) = x^k$, thus $x^\infty = \Phi_\infty(x^\infty)$, i.e., x^∞ is a fixed point of the operator Φ_∞. Thus $z^\infty = x^\infty[\varphi]$ is a mild solution to the problem (P_0). By (B2), we have,

$$z^k = x^k[\varphi] \underset{k\to\infty}{\longrightarrow} x^\infty[\varphi] = z^\infty.$$

The proof is complete. □

Let us denote by $\Sigma_\varphi^\varepsilon$, $\varepsilon\in[0,1]$, the mild solutions set of the problem (P_ε) and by $\mathfrak{C}(\mathcal{C}(]-\infty,T];E))$ the collection af all compact subsets of $\mathcal{C}(]-\infty,T];E)$.

Proposition 4.2.1. *Assume the hypotheses of Theorem 4.2.1. Moreover, assume that f_0 satisfies the following additional hypotheses,*

$(\mathtt{f}_5)$ *There exists a constant $C_0>0$ such that, for all $u\in\mathcal{B}$,*

$$\|f_0(u)\| \leq C_0(1+\|u\|_{\mathcal{B}}) \quad a.e.\ t\in\mathbb{R}^+;$$

$(\mathtt{f}_6)$ *There exists a positive constant p_0 such that for every bounded $D\subset\mathcal{B}$,*

$$\chi\left(f_0(D)\right) \leq p_0 \sup_{-\infty\leq\theta\leq 0} \chi(D(\theta)).$$

Then, the map,

$$\begin{aligned} \Pi : [0,1] &\longrightarrow \mathfrak{C}(\mathcal{C}(]-\infty,T];E)) \\ \varepsilon &\longrightarrow \Sigma_\varphi^\varepsilon, \end{aligned} \tag{4.30}$$

is well defined and upper semicontinuous at the point $\varepsilon = 0$.

Proof. Invoking Theorem 4.1.1, we deduce that Π is well defined. Note that, we have added Hypotheses $(\mathtt{f}_5)$-$(\mathtt{f}_6)$ in order to guarantee, by Theorem 4.1.1, that Σ_φ^0 is nonempty compact set. Under hypotheses of Theorem 4.2.1, the set Σ_φ^0 is nonempty but there is no reason that it is compact. Let as show that Π is upper semicontinuous at the point $\varepsilon=0$. Let $(\varepsilon_k)_k$ be a sequence such that $\varepsilon_k \to 0$. Take $z^k \in \Pi(\varepsilon_k)$, $k \geq 1$. Then, for each $k \geq 1$, $z^k = x^k[\varphi]$, for some fixed point x^k of the operator Φ_k. From the proof of Theorem 4.2.1, we deduce that the sequence $(x^k)_{k\geq 1}$ is relatively compact. One can assume without lose of generality that $x^k \to x^\infty$, which is a fixed point of the operator Φ_∞. By (B2), we have,

$$z^k = x^k[\varphi] \underset{k\to\infty}{\longrightarrow} x^\infty[\varphi] = z^\infty \in \Pi(0).$$

The proof is complete. $\square$

4.3 A variant of the classical averaging principle

As the previous section, the phase space $\mathcal{B}$ is considered as a Banach space satisfying Axioms (B1)-(B3). We aim to establish a variant of the classical averaging principle for the problem (P_ε).

Let us replace Hypothesis ($\mathtt{Af}$)(see Theorem 4.2.1) by more natural one,

($\mathtt{Avf}$) For every $u \in \mathcal{B}$, the limit,

$$f_0(u) = \lim_{t\to+\infty} \frac{1}{t}\int_0^t f(s,u)\,ds, \tag{4.31}$$

exists uniformly for u in compact sets.

Theorem 4.3.1. *Assume Hypotheses (A), $(\mathtt{f}_1) - (\mathtt{f}_4)$ (see Theorem 4.2.1) and $(\mathtt{Avf})$. Moreover, assume that E is reflexive and, the semigroup $(S(t))_{t\geq 0}$ is analytic. Then the map Π defined by (4.30) is well defined and upper semicontinuous at the point $\varepsilon = 0$.*

For the proof of Theorem 4.3.1 we need some lemmas.

Lemma 4.3.1. *Assume Hypotheses $(\mathtt{f}_1) - (\mathtt{f}_4)$ and $(\mathtt{Avf})$. Then the function f_0 is continuous and satisfies Hypotheses $(\mathtt{f}_5)$ and $(\mathtt{f}_6)$.*

Proof. Hypothesis $(\mathtt{f}_5)$ follows easily from $(\mathtt{f}_3)$ and $(\mathtt{Avf})$. Note that, if a sequence $(u_n)_{n\geq 1}$ in $\mathcal{B}$ is such that, $u_n \to u_0$, then, $\mathcal{W} = \{u_n,\ n \geq 0\}$ is a compact set in $\mathcal{B}$. The continuity of f_0 follows easily from $(\mathtt{Avf})$ and Remark 4.2.4-(i).

Let us proof that f_0 satisfies $(\mathtt{f}_6)$. For a bounded subset Ω of $\mathcal{B}$, the expression,

$$\int_0^t f(s,\Omega)\,ds, \quad \text{means,} \quad \left\{\int_0^t f(s,u)\,ds,\ u \in \Omega\right\}.$$

For $t > 0$, let $(x_n)_n \subset E$ be an any denumerable subset of $\int_0^t f(s,\Omega)\,ds$. Then,

$$x_n = \frac{1}{t}\int_0^t f(s,u_n)\,ds, \quad \text{for some } u_n \in \Omega.$$

Set,

$$f_n(s) = f(s,u_n), \quad \text{a.e.}, s \in [0,t].$$

Let $\mathcal{N}$ be an any negligible subset of $[0,+\infty[$, such that, $f([0,+\infty[/\mathcal{N},\Omega)$ is bounded in E ($\mathcal{N}_1$ exists by $(\mathtt{f}_3)$). By $(\mathtt{f}_4)$, we have, for every $s \in [0,t]/\mathcal{N}$,

$$\chi(\{f_n(s)\}_{n=1}^{\infty})) \leq \chi(f([0,+\infty[/\mathcal{N},\Omega)) \leq p \sup_{-\infty<\theta\leq 0} \chi(\Omega(\theta)).$$

That is,

$$\frac{1}{t}\,\chi(\{f_n(s)\}_{n=1}^{\infty})) \leq \frac{1}{t}\,p \sup_{-\infty<\theta\leq 0} \chi\left(D(\theta)\right) \text{ a.e. } s \in [0,t].$$

Using lemma 2.3.4, (with, $A \equiv 0$, $(M=1)$), we get, for every $t > 0$,

$$\chi\left(\{x_n\}_{n=1}^{\infty}\right) = \chi\left(\left\{\frac{1}{t}\int_0^t f_n(s)\,ds\right\}_{n=1}^{\infty}\right) \leq 2\,p \sup_{-\infty<\theta\leq 0} \chi\left(D(\theta)\right).$$

Hence, for every $t > 0$,

$$\chi\left(\frac{1}{t}\int_0^t f(s,\Omega)\,ds\right) \leq 2\,p \sup_{-\infty<\theta\leq 0} \chi\left(D(\theta)\right).$$

Thus,

$$\begin{aligned}\chi\left(f_0(\Omega)\right) = \chi\left(\lim_{t\to+\infty} \frac{1}{t}\int_0^t f\left(s,\Omega\right)\,ds\right) &= \overline{\lim_{t\to+\infty}}\,\chi\left(\frac{1}{t}\int_0^t f\left(s,\Omega\right)\,ds\right)\\ &\leq 2\,p \sup_{-\infty<\theta\leq 0} \chi\left(D(\theta)\right).\end{aligned}$$

□

Now, invoking [54, Lemma 3], we deduce the following result

Lemma 4.3.2. *Assume Hypotheses* $(\mathtt{f}_1)$, $(\mathtt{f}_3)$. *Moreover, assume that E is reflexive and, the semigroup $(S(t))_{t\geq 0}$ is analytic. Then, under the hypothesis* $(\mathtt{Avf})$, *for every $u \in \mathcal{B}$ and t_1, t_2 such that $0 \leq t_1 \leq t_2 \leq T$,*

$$\lim_{\varepsilon\to 0^+} \int_{t_1}^{t_2} S(t_2-\theta)\left[f(\frac{\theta}{\varepsilon},u) - f_0(u)\right]d\,\theta = 0.$$

Note that, since E is reflexive the proof of Lemma 4.3.2 is exactly the same as in [54, Lemma 3]. From Lemmas 4.3.2 and 4.3.1, we have

Corollary 4.3.1. *Under the hypotheses of Theorem 4.3.1 we have,* (`Avf`) $\Rightarrow$ (`Af`).

Proof of Theorem 4.3.1. The proof is a consequence of Lemma 4.3.1, Corollary 4.3.1 and Proposition 4.2.1. $\square$

4.4 Example

In this section we give an example that illustrates the applicability of our abstract results. Let us consider the following partial differential equation of the form:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}(t,x) + F\left(\dfrac{t}{\varepsilon},\, Q(u(t,\cdot))(x),\, \displaystyle\int_{-\infty}^{0} a(\theta)\, u(\theta+t,x)\, d\,\theta\right) \\ \qquad u(t,0) = u(t,1) = 0, \quad t \in [0,1], \\ \qquad u(\theta,x) = \varphi(\theta)(x), \quad \theta \le 0, \quad x \in [0,1] \text{ a.e}, \end{cases} \tag{$\hat{P}_\varepsilon$}$$

where ε is a small positive parameter, $F : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, the initial condition $\varphi(\cdot)(\cdot) \in C_\infty(L^2([0,1]))$ (see section 4) and Q is a linear compact operator from $L^2([0,1])$ to $L^2([0,1])$.We assume that the following conditions hold:

(F_1) $\cdot \to F(\cdot,x,y)$ is measurable for all $x, y \in \mathbb{R}$;

(F_2) $(\cdot,\cdot) \longrightarrow F(t,\cdot,\cdot)$ is continuous uniformly with respect to $t \in [0,+\infty[$ a.e.;

(F_3) there exist a constant $c > 0$ such that for all $x, y \in \mathbb{R}$,

$$|F(t,x,y)| \le c\,(1 + |x| + |y|), \text{ a.e.}\, t \in \mathbb{R}_+;$$

(F_4) there exists a constant $k > 0$ such that for all y_1, y_2, $x \in \mathbb{R}$,

$$|F(t,x,y_1) - F(t,x,y_2,)| \le k\,|y_1 - y_2|\,, \text{ a.e.}\, t \in \mathbb{R}_+;$$

(F_T) the map F is 1-periodic in the first argument, i.e., for a.e. $t \in \mathbb{R}_+$

$$F(t+1,x,y) = F(t,x,y),$$

for all $x, y \in \mathbb{R}$;

(H_a) $a(\cdot) : \mathbb{R}_+ \to \mathbb{R}$, is a non-negative measurable function such that

$$\int_{-\infty}^{0} a(\theta)\, d\theta = q < +\infty.$$

Let $E = L^2([0,1])$ and $\mathcal{B} = C_\infty(E)$. Set $X(t) = u(t, \cdot)$, $t \in]-\infty, 1]$. Consider the operator $A = \frac{d^2}{dx^2}$, with domain

$$D(A) = \{h \in E : \; h',\, h'' \in E \text{ and } \; h(0) = h(1) = 0\}. \tag{4.32}$$

It is well know that A generates a strongly continuous-analytic semigroup. With the above data, the problem $(\hat{P}_\varepsilon)$ can be reformulated as

$$\begin{cases} dX(t) = AX(t) + f(\frac{t}{\varepsilon}, X_t)\, dt, \; t \in [0,\, 1], \\ X_0 = \varphi, \end{cases}$$

where

$$f(t, \psi)(x) = F\left(t,\, Q(\psi(0)(\cdot))(x),\, \int_{-\infty}^{0} a(\theta)\, \psi(\theta)(x)\, d\,\theta\right) \tag{4.33}$$

for a.e. $x \in [0,1]$ and every $\psi \in C_\infty(E)$. Note that

$$\|\psi\|_{C_\infty(L^2([0,1]))} = \sup_{\theta \le 0} \left(\int_0^1 |\psi(\theta)(x)|^2 \; dx\right)^{\frac{1}{2}}.$$

It is clear that for any bounded subset in $C_\infty(L^2([0,1]))$, we have

$$\chi_{C_\infty(L^2([0,1]))}(\Omega) = \sup_{\theta \le 0} \chi_{L^2([0,1])}(\Omega(\theta, \cdot)).$$

Parallel to the problem $(\hat{P}_\varepsilon)$ we consider the averaged problem

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}(t,x) + F_0\left(Q(u(t,\cdot))(x)\,,\, \displaystyle\int_{-\infty}^{0} a(\theta)\, u(\theta + t, x)\, d\,\theta\right) \\ \qquad u(t,0) = u(t,1) = 0, \quad t \in [0,1], \\ \qquad u(\theta, x) = \varphi(\theta)(x), \quad \theta \le 0, \quad x \in [0,1] \text{ a.e}, \end{cases} \tag{$\hat{P}_0$}$$

where $F_0 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$F_0(x,y) = \int_0^1 F(t,x,y)\, dt.$$

The problem $(\hat{P}_0)$ can be reformulated as

$$\begin{cases} dX(t) = AX(t) + f_0(X_t)\,dt, \ t \in [0,\ 1], \\ X_0 = \varphi, \end{cases}$$

where $f_0 : C_\infty(L^2([0,1])) \to L^2([0,1])$, is defined by

$$f_0(\psi)(x) = F_0\left(Q(\psi(0)(\cdot))(x),\ \int_{-\infty}^{0} a(\theta)\,\psi(\theta)(x)\,d\,\theta\right). \tag{4.34}$$

The mild solutions to the problems $(\hat{P}_\varepsilon)$, $\varepsilon \in [0,1]$, are observed in the space $\mathcal{C}(]-\infty,1];L^2([0,1])$ (see Section 3). For every $\varepsilon \in [0,1]$, by the symbol $\Sigma_\varphi^\varepsilon$ we denote the mild solutions set of the problem $(\hat{P}_\varepsilon)$. By the symbol $\mathfrak{C}(\mathcal{C}(]-\infty,1];L^2([0,1]))$ we denote the collection af all compact subsets of $\mathcal{C}(]-\infty,1];L^2([0,1])$.

Theorem 4.4.1. *Under the hypotheses (F_1)-(F_4), (F_T) and (H_a), the map*

$$\begin{aligned} \Pi : [0,1] &\longrightarrow \mathfrak{C}(\mathcal{C}(]-\infty,1];L^2([0,1])) \\ \varepsilon &\longrightarrow \Sigma_\varphi^\varepsilon, \end{aligned}$$

is well defined and upper semicontinuous at the point $\varepsilon = 0$.

Proof. We have only to prove that the map f given by (4.33) satisfies hypotheses $(\mathtt{f}_1) - (\mathtt{f}_4)$ of Theorem 4.3.1. Without loss of generality assume that

$$\|Q\|_{\mathcal{L}(L^2([0,1]))} \leq 1, \tag{4.35}$$

where $\mathcal{L}(L^2([0,1]))$ denotes the space of bounded linear operators from $L^2([0,1])$ to $L^2([0,1])$.

Note that the functions, $x \to Q(\psi(0)(\cdot))(x)$ and $x \to \int_{-\infty}^{0} a(\theta)\,\psi(\theta)(x)d\theta$, are measurable.

Note that the linear maps:

$$\begin{cases} \Gamma_1 : C_\infty(L^2(\Omega)) \longrightarrow L^2(\Omega) \\ \qquad\qquad \psi \longrightarrow Q\Big(\psi(0)(\cdot)\Big)(\cdot) \end{cases} \quad \text{and} \quad \begin{cases} \Gamma_2 : C_\infty(L^2(\Omega)) \longrightarrow L^2(\Omega) \\ \psi \longrightarrow \int_{-\infty}^{0} a(\theta)\,\psi(\theta)(\cdot)\,d\theta, \end{cases}$$

are continuous. Indeed, the map Γ_1 is continuous because

$$\begin{aligned} \|\Gamma_1(\psi)\|_{L^2([0,1])} &= \left\|Q\Big(\psi(0)(\cdot)\Big)(\cdot)\right\|_{L^2([0,1])} \\ &\leq \|\psi(0)(\cdot)\|_{L^2([0,1])} \leq \|\psi(\cdot)(\cdot)\|_{C_\infty(L^2(\Omega))}\,. \end{aligned} \tag{4.36}$$

By using Hypothesis (H_a), (4.35) and, by applying Jensen's inequality and Fubini's theorem, for every $\psi \in C_\infty(L^2(\Omega))$, we get

$$\begin{aligned}
\int_0^1 \left| \int_{-\infty}^0 a(\theta)\, \psi(\theta)(x) d\theta \right|^2 dx &= \int_0^1 \left| q \int_{-\infty}^0 \psi(\theta)(x)\, \frac{a(\theta)}{q} d\theta \right|^2 dx \\
&\leq q^2 \int_0^1 \int_{-\infty}^0 |\psi(\theta)(x)|^2\, \frac{a(\theta)}{q}\, d\theta\, dx \\
&\leq q \int_{-\infty}^0 \int_0^1 |\psi(\theta)(x)|^2\, a(\theta) d\theta\, dx \\
&\leq q \int_{-\infty}^0 \int_0^1 |\psi(\theta)(x)|^2\, a(\theta) d\theta\, dx \\
&\leq q \sup_{\theta \leq 0} \int_0^1 |\psi(\theta)(x)|^2\, dx \int_{-\infty}^0 a(\theta)\, d\theta \\
&= q^2\, \|\psi\|^2_{C_\infty(L^2(\Omega))} .
\end{aligned}$$

Hence,

$$\|\Gamma_2(\psi)\|_{L^2([0,1])} = \left\| \int_{-\infty}^0 a(\theta)\, Q(\psi)(\theta)(\cdot) \right\|_{L^2([0,1])} \leq q\, \|\psi\|_{C_\infty(L^2(\Omega))} . \qquad (4.37)$$

Therefore, the map Γ_2 is continuous too. Arguing as in the proof of (iii) in [60, Lemma A.6.1, p. 313], one can show that f is well defined and satisfies Hypotheses $(\mathtt{f}_1)$ and $(\mathtt{f}_2)$.

Using Hypothesis (F_3), (4.36) and (4.37), we obtain

$$\begin{aligned}
|f(t,\psi)(\cdot))|_{L^2([0,1])} &\leq c \left[1 + \|Q(\psi(0)(\cdot))(\cdot)\|_{L^2([0,1])} + \left\| \int_{-\infty}^0 a(\theta)\, \psi(\theta)(\cdot) \right\|_{L^2([0,1])} \right] \\
&= c \left[1 + \|\Gamma_1(\psi)\|_{L^2([0,1])} + \|\Gamma_2(\psi)\|_{L^2([0,1])} \right] \\
&\leq c \left[1 + (1+q)\, \|\psi\|_{C_\infty(L^2(\Omega))} \right] .
\end{aligned}$$

Thus, f satisfies Hypothesis $(\mathtt{f}_3)$.

Note that the map

$$\psi \to F\left(t,\, Q(\psi(0)(\cdot))(\cdot),\, \int_{-\infty}^0 a(\theta)\, \psi(\theta)(\cdot)\, d\theta\right),$$

can be presented as

$$F\left(t,\, Q(\psi(0)(\cdot))(\cdot),\, \int_{-\infty}^0 a(\theta)\, \psi(\theta)(\cdot)\, d\theta\right) = G(t,\psi,\psi),$$

where $G : \mathbb{R}_+ \times \mathcal{B} \times \mathcal{B} \to L^2([0,1])$ is defined by

$$G(t, \psi_1, \psi_2) = F\left(t,\, Q(\psi_1(0)(\cdot))(\cdot),\, \int_{-\infty}^{0} a(\theta)\, \psi_2(\theta)(\cdot)\, d\,\theta\right).$$

It is clear that G is 1-periodic with respect to the first argument and compact with respect to the second argument. Moreover G is $k\,q$ Lipschitz with respect to the third argument. Indeed, from Hypothesis (F_4), for all ψ, ψ_1, ψ_2 in $C_\infty(L^2([0,1]))$ we have (see (4.37))

$$\begin{aligned}\|G(t,\psi,\psi_1) - G(t,\psi,\psi_2)\|_{L^2([0,1]} &\le k\,\|\Gamma_2(\psi_1) - \Gamma_2(\psi_2)\|_{L^2([0,1]} \\ &\le k\,q\,\|\psi_1 - \psi_2\|_{C_\infty(L^2(\Omega))}\,, \quad \text{a.e. } t \in [0,1].\end{aligned}$$

Let Ω be a bounded subset in $C_\infty(L^2([0,1])$. Let $\mathcal{N}$ be an any negligible subset in $[0,1]$ such that the set $f([0,1]/\mathcal{N}, \Omega)$ is bounded in $L^2([0,1])$. Invoking [36, Proposition 2.2.2] we get

$$\begin{aligned}&\chi_{L^2([0,1])}\left\{f\Big([0,1]/\mathcal{N}, \Omega\Big)\right\} \\ &= \chi_{L^2([0,1])}\left\{F\left([0,1]/\mathcal{N},\, Q(\Omega(0)(\cdot))(\cdot),\, \int_{-\infty}^{0} a(\theta)\,\Omega(\theta)(\cdot)\,d\,\theta\right)\right\} \\ &= \chi_{L^2([0,1])}\left\{G\Big([0,1]/\mathcal{N}, \Omega, \Omega\Big)\right\} \\ &\le k\,q\,\chi_{C_\infty(L^2([0,1]))}\Big(\Omega\Big) = k\,q\,\sup_{\theta\le 0}\chi_{L^2([0,1])}\Big(\Omega(\theta)\Big).\end{aligned}$$

Since f is 1-periodic we conclude that f satisfies Hypothesis $(\mathtt{f}_4)$.

The proof of Theorem 4.4.1 is complete. □

Chapter 5

On the averaging principle for semilinear functional differential equations with infinite delay in a Banach space

In the present chapter, averaging results are established for semilinear functional differential equations with infinite delay on an abstract phase space of Banach space valued functions defined axiomatically, where the unbounded linear part generates a non-compact semi-group and the nonlinear part satisfies a condition with respect to the second argument which is more weaker than the usual Lipschitz condition. As preliminary result, using the technique of the theory of condensing maps, a theorem on the existence and uniqueness of mild solutions is established for such equations.

More precisely, we consider a semi-linear functional differential equation in E with a small positive parameter ε, of the form

$$\begin{cases} x'(t) = A\,x(t) + f\left(\frac{t}{\varepsilon}, x_t\right), \quad t \in [0,T], \\ x_0 = \varphi. \end{cases} \tag{P_ε}$$

Parallel to the problem (P_ε), $\varepsilon > 0$, we consider the averaged problem:

$$\begin{cases} x'(t) = A\,x(t) + f_0(x_t), \quad t \in [0,T], \\ \quad x_0 = \varphi, \end{cases} \tag{P_0}$$

where, A generates a strongly continuous semigroup $(S(t))_{t\geq 0}$, the function f satisfies a condition with respect to the second argument which is more weaker than the usual Lipschitz condition and the function $f_0 : \mathcal{B} \to E$, is such that, for all $u \in \mathcal{B}$ and t_1, $t_2 \in [0,T]$ with $0 \leq t_1 \leq t_2 \leq t_1 + \Delta_0$, for some $\Delta_0 > 0$,

$$\lim_{\varepsilon \to 0^+} \int_{t_1}^{t_2} S(t_2 - \theta) \left[f\Big(\frac{\theta}{\varepsilon}, u\Big) - f_0(u) \right] d\,\theta = 0. \tag{5.1}$$

By strengthening the conditions on f and, adding axioms on the phase space $\mathcal{B}$, we establish averaging result for the problem $(P_\varepsilon)_{\varepsilon>0}$ (Theorem 5.2.1). Then, we replace the condition (5.1) by a more natural hypothesis, then we establish the averaging principle in the traditional form (Theorem 5.3.1). Note that such a condition (5.1) is inspired from [54].

In the proofs of our averaging results we will not need to have recourse to the concept of measure of non-compactness, we will make direct estimations by generalizing the approach given in [54] to our case. the concept of measure of non-compactness is used only to establish an existence result.

5.1 Existence results

Let σ be a real number and $T > 0$ be a fixed time. By $C([\sigma, \sigma+T];\, E)$ we denote the space of continuous functions defined on $[\sigma, \sigma+T]$ with values in a Banach space $(E, \|.\|)$, endowed with the uniform convergence norm. For any function $z :]-\infty, \sigma+T] \to E$ and for every $t \in [\sigma, \sigma+T]$, z_t represents the function from $]-\infty, 0]$ into E defined by $z_t(\theta) = z(t+\theta)$, $-\infty < \theta \leq 0$. Let $\mathcal{B}$ be a linear topological space of functions mapping $]-\infty, 0]$ into E endowed with a seminorm $\|.\|_{\mathcal{B}}$ and, satisfying the following axioms:

if $z :]-\infty, \sigma+T] \to E$ is continuous on $[\sigma, \sigma+T]$ and $z_\sigma \in \mathcal{B}$, then, for every $t \in [\sigma, \sigma+T]$, we have

(B1) $z_t \in \mathcal{B}$;

(B2) $\|z_t\|_{\mathcal{B}} \leq K(t-\sigma) \sup_{\sigma \leq s \leq t} \|z(s)\| + N(t-\sigma)\, \|z_\sigma\|_{\mathcal{B}}$, where $K, N : [0, +\infty[\longrightarrow [0, +\infty[$, are independent of z, K is positive and continuous and, N is locally bounded;

(B3) the function $t \mapsto z_t$ is continuous.

Such a space $\mathcal{B}$ was introduced by Hale and Kato [27] and has been considered as a phase space in the theory of functional differential equations with infinite delay.

Let us denote by the symbol $\mathcal{C}(]-\infty, \sigma+T]; E)$ the linear topological space consisting of functions $z :]-\infty, \sigma+T] \to E$ such that $z_\sigma \in \mathcal{B}$ and the restriction $z\,|_{[\sigma,\sigma+T]}$ is continuous, endowed with a seminorm

$$\|z\|_{\mathcal{C}} = \|z_\sigma\|_{\mathcal{B}} + \left\| z\,|_{[\sigma,\sigma+T]} \right\|_{C([\sigma,\sigma+T];E)}.$$

Let us consider a semilinear functional differential equation in E, of the form

$$\begin{cases} x'(t) = A\,x(t) + f(t, x_t), & t \in [\sigma, \sigma+T], \\ x_\sigma = \varphi, \end{cases} \tag{5.2}$$

where

(A) A is the generator of a $\mathcal{C}_0$ semigroup $(S(t))_{t\geq 0}$ on E.

The function f is acting from $[\sigma, \sigma+T] \times \mathcal{B}$ to E and satisfies,

(f_1) For all $u \in \mathcal{B}$, the mapping $t \to f(t,\, u)$ is measurable.

(f_2) For all $(u,v) \in \mathcal{B} \times \mathcal{B}$,

$$\|f(t,u) - f(t,v))\| \leq L\Big(t, \|u-v\|_{\mathcal{B}}\Big) \quad \text{a.e. } t \in [\sigma, \sigma+T],$$

where $L : [\sigma, \sigma+T] \times [0, +\infty[\to [0, +\infty[$ is a given mapping such that

(i) $L(\cdot, \theta)$ is locally integrable for each $\theta \in [0, +\infty[$ and for a.e. $t \in [\sigma, \sigma+T]$, $L(t, \cdot)$ is continuous, monotone nondecreasing and, $L(t, 0) = 0$;

(ii) for every nonnegative continuous function $h : [\sigma, \sigma+T] \to [0, +\infty[$ and for every constant ζ, the following implication holds true:

$$\left[\forall t \in [\sigma, \sigma+T],\ h(t) \leq \zeta \int_\sigma^t L\Big(s, h(s)\Big)\, ds\right] \Rightarrow h \equiv 0 \tag{5.3}$$

(f_3) There exists a function $\alpha \in L^1([\sigma, \sigma+T], \mathbb{R}^+)$ such that, for all $u \in \mathcal{B}$,

$$\|f(t,u)\| \leq \alpha(t)\Big(1 + \|u\|_{\mathcal{B}}\Big) \quad \text{a.e. } t \in [\sigma, \sigma+T].$$

Remark 5.1.1. *Concrete examples of such a function L can be found* e.g. *[1, 10, 29].*

Remark 5.1.2. *Let $L : [a,b] \times [0, +\infty[\to [0, +\infty[$ be a mapping satisfying Hypotheses (i) and (ii) of (f_2). According to [10, Lemma 2.2], if a nonnegative monotone nondecreasing function $h : [a,b] \to [0, +\infty[$ such that $h(a) = 0$ satisfies* (5.3) *for some constant $\zeta > 0$, then $h(t) = 0$ for all $t \in [a,b]$.*

Definition 5.1.1. *We say that* $z \in \mathcal{C}(\,]-\infty, \sigma+T]; E)$ *is a* mild solution *to the problem* (5.2) *if* $z_\sigma = \varphi$ *and,*

$$z(t) = S(t-\sigma)\varphi(0) + \int_\sigma^t S(t-s) f(s, z_s)\, ds \quad \text{for } \sigma \le t \le \sigma + T.$$

We can now state our preliminary result.

Theorem 5.1.1. *Assume that the phase space* $\mathcal{B}$ *satisfies the axioms* (B1)-(B3). *Under Hypotheses* (A) *and* $(f_1)-(f_3)$*, the problem* (5.2) *has a unique mild solution in* $\mathcal{C}(\,]-\infty, \sigma+T]; E)$.

Remark 5.1.3. *Theorem 5.1.1 is a deterministic version of [25, Theorem 3.2], its proof may be deduced from the latter's proof, except some modifications. As this is not the same context (the conditions on the phase space* $\mathcal{B}$ *are not exactly the same) and for easier reading, we prefer to give the proof. Instead of observing the solution as a limit of a sequence of approximating solutions constructed via Tonelli's scheme (the approach that was used in [25]), here, for the proof of Theorem 5.1.1, the mild solutions are observed as a fixed points of an integral operator which is condensing with respect to a measure of non-compactness with good properties. Then, to end the proof, a fixed point theorem for such class of operators (Theorem 2.3.1) is invoked.*

Another reason for which we prefer to give the proof of Theorem 5.1.1, is due to the fact that in the literature, we did not find existence results for the problem (5.2) *with a general non Lipschitz condition given by* (f_2).

Before giving the proof of Theorem 5.1.1, we prove some auxiliary results. We will use the same notation as in [21].

Let $t \in [\sigma, \sigma+T]$. In the space $C([\sigma, t], E)$, let us define the set

$$\mathfrak{D}(\varphi, t) = \{x \in C([\sigma, t]; E),\ x(\sigma) = \varphi(0)\}\,. \tag{5.4}$$

It is clear that $\mathfrak{D}(\varphi, t)$ is a closed convex subset in $C([\sigma, t]; E)$.

For every $x \in \mathfrak{D}(\varphi, \sigma+T)$, let us define the function $x[\varphi] \in \mathcal{C}(\,]-\infty, \sigma + T]; E)$ by

$$x[\varphi](t) = \begin{cases} \varphi(t-\sigma), & -\infty < t < \sigma, \\ x(t), & \sigma \le t \le \sigma + T. \end{cases} \tag{5.5}$$

Then

$$x[\varphi]_t(\theta) = \begin{cases} \varphi(t-\sigma+\theta), & \text{if } \ -\infty < \theta < \sigma - t, \\ x(t+\theta), & \text{if } \ \sigma - t \le \theta \le 0. \end{cases} \tag{5.6}$$

The function $x[\varphi]\,|_{[\sigma,\sigma+T]} = x(\cdot)$ is continuous and $x[\varphi]_\sigma = \varphi$, hence by Axiom (B1), $x[\varphi]_t \in \mathcal{B}$ for all $t \in [\sigma, \sigma+T]$.

Let us denote by Φ the mapping which, with every element $x \in \mathfrak{D}(\varphi, \sigma + T)$ associates the element $\Phi(x) \in \mathfrak{D}(\varphi, \sigma+T)$, given by

$$S(t-\sigma)\varphi(0) + \int_\sigma^t S(t-s) f\Big(s, x[\varphi]_s\Big)\, ds, \quad \sigma \le t \le \sigma + T.$$

Remark 5.1.4. *It is clear that each mild solution to* (5.2) *has the form* $x[\varphi](.)$*, for some a fixed point* $x \in \mathfrak{D}(\varphi, \sigma+T)$ *of* Φ*. Thus, for the proof of Theorem 5.1.1, it is enough to prove that the mapping* Φ *has a unique fixed point in* $\mathfrak{D}(\varphi, \sigma+T)$*.*

Remark 5.1.5. *Since* $(S(t))_{t\ge 0}$ *is a* $\mathcal{C}_0$*-semigroup, there exists, for each* $t \in [0,T]$*, a constant* $M_t > 0$ *such that*

$$\sup_{0\le s\le t} \|S(s)\| \le M_t. \tag{5.7}$$

(See e.g. *[50, Theorem 2.2]).*

Lemma 5.1.1. *Let* $L : [\sigma, \sigma+T] \times [0, +\infty[\to [0, +\infty[$ *be a mapping satisfying Hypotheses (i)-(ii) of* (f_2)*. Let* $(v_n)_n$ *be a sequence of positive number such that* $v_n \to 0$ *as* $n \to \infty$*. Then for any* $\mu \ge 0$*, we have*

$$\int_\sigma^T L\Big(s, \mu\, v_n\Big)\, ds \to 0, \quad \text{as} \quad n \to \infty.$$

Proof. The result follows immediately from the Lebesgue dominated convergence theorem. □

Lemma 5.1.2. *The map* $\Phi : \mathfrak{D}(\varphi, \sigma+T) \to \mathfrak{D}(\varphi, \sigma+T)$ *is continuous.*

Proof. Let

$$\lambda_K = \sup_{0\le s\le T} K(s), \tag{5.8}$$

where the function K is introduced in Axiom (B2). It is clear that $\lambda_k > 0$.

Let $(x_n)_n$ be a sequence of $\mathfrak{D}(\varphi, \sigma+T)$, such that $x_n \to x$ as $n \to \infty$. Using Axiom (B2) and the fact that the function $L(s, \cdot)$ is monotone nondecreasing a.e $s \in [\sigma, \sigma+T]$, we get

$$\begin{aligned}
\|\Phi(x)(t) - \Phi(x_n)(t)\| \le & M_t \int_\sigma^t \Big\| f\Big(s, x[\varphi]_s\Big) - f\Big(s, x_n[\varphi]_s\Big)\Big\|\, ds \\
\le & M_t \int_\sigma^t L\Big(s, K(s-\sigma) \sup_{\sigma\le\tau\le s} \|x(\tau) - x_n(\tau)\|\Big)\, ds \\
\le & M_T \int_\sigma^T L\Big(s, \lambda_K\, \|x - x_n\|_{C([\sigma,\sigma+T],E)}\Big)\, ds.
\end{aligned}$$

where M_T is from (5.7). It remains to apply Lemma 5.1.1. $\square$

Let Λ be a subset of $C([\sigma, \sigma + T]; E)$. For any $t \in [\sigma, \sigma + T]$, by $\Lambda_{|_t}$ we denote the restrictions to $[\sigma, t]$ of elements of Λ. It is clear that $\Lambda_{|_t}$ is a subset of $C([\sigma, t]; E)$. Let $\mathcal{T}[\sigma, \sigma + T]$ denote the partially ordered linear space of all real monotone nondecreasing functions defined on $[\sigma, \sigma + T]$. Let us consider a measure of non-compactness $\Psi_{C([\sigma,\sigma+T];E)}$ defined on bounded subsets of $C([\sigma, \sigma + T]; E)$ and with values in $\mathcal{T}[\sigma, \sigma + T]$, given by

$$\Psi_{C([\sigma,\sigma+T];E)}(\Lambda)(t) = \inf \Big\{ \epsilon > 0;\ \Lambda_{|_t} \text{ has a finite } \epsilon\text{–net in } C([\sigma, t]; E) \Big\} .$$

Now on bounded subsets of $\mathfrak{D}(\varphi, \sigma + T)$, let us define a new measure of non-compactness $\Psi_{\mathfrak{D}}$ with values in $\mathcal{T}[\sigma, \sigma + T]$, given by:

$$\Psi_{\mathfrak{D}}(\Lambda)(t) = \inf \Big\{ \epsilon > 0;\ \Lambda_{|_t} \text{ has a finite } \epsilon\text{–net in } \mathfrak{D}(\varphi, t) \Big\} .$$

Remark 5.1.6. *Note that for each $t \in [\sigma, \sigma + T]$, the set $\mathfrak{D}(\varphi, t)$ endowed with the metric $d_{\mathfrak{D}(\varphi,t)}$, given by,*

$$d_{\mathfrak{D}(\varphi,t)}(x, y) = \|x - y\|_{C([\sigma,t];E)} ,$$

is a (complete) metric space.

Remark 5.1.7. *It is obvious that a subset Λ of $C([\sigma, \sigma + T]; E)$ is relatively compact if and only if*

$$\Psi_{C([\sigma,\sigma+T];E)}(\Lambda)(T) = 0.$$

Remark 5.1.8. *One can easily show that the restriction of $\Psi_{C([\sigma,\sigma+T];E)}$ on bounded subsets of $\mathfrak{D}(\varphi, \sigma + T)$ satisfies for each $t \in [\sigma, \sigma + T]$ and for any bounded set $\Lambda \subset \mathfrak{D}(\varphi, \sigma + T)$, the relation*

$$\Psi_{C([\sigma,\sigma+T];E)}\,(\Lambda)\,(t) \leq \Psi_{\mathfrak{D}}(\Lambda)(t) \leq 2\,\Psi_{C([\sigma,\sigma+T];E)}\,(\Lambda)\,(t). \tag{5.9}$$

Lemma 5.1.3. *Let Λ be a bounded subset of $\mathfrak{D}(\varphi, \sigma+T)$. Assume Hypotheses* (A), $(f_1) - (f_3)$ *and* (B1)–(B3). *Then for every $t \in [\sigma, \sigma + T]$, we have*

$$\Psi_{\mathfrak{D}}(\Phi \circ \Lambda)(t) \leq M_T \int_{\sigma}^{t} L\Big(s, K(s)\,\Psi_{\mathfrak{D}}(\Lambda)(s)\Big)\, ds,$$

where M_T is from (5.7) *and $K(\cdot)$ from Axiom* (B2).

Proof. Lemma 5.1.3 is a deterministic version of [25, Lemma 3.4] (see Remark 5.1.3).

Let Λ be a bounded subset of $\mathfrak{D}(\varphi, \sigma+T)$. Let $\epsilon > 0$. As the function $t \mapsto \Psi_{\mathfrak{D}}(\Lambda)(t)$ is nondecreasing and bounded, it admits at most a finite number of points $\sigma \leq t_1 \leq \cdots \leq t_n \leq \sigma+T$ for which

$$|\Psi_{\mathfrak{D}}(\Lambda)(t_i+0) - \Psi_{\mathfrak{D}}(\Lambda)(t_i-0)| \geq \epsilon, \;\; i = 1, ..., n.$$

Remove these points with their disjoint δ_1-neighborhoods from the segment $[\sigma, \sigma+T]$. If $t_1 \neq \sigma$, remove also the segment $[\sigma, \sigma+\delta_1[$ from the segment $[\sigma, \sigma+T]$. Using points β_j, $j = 1, ...m$, divide the remaining part into intervals on which the increment of the function $\Psi_{\mathfrak{D}}(\Lambda)(\cdot)$ is smaller than ϵ, i.e.,

$$\sup_{s,t \in [\beta_{j-1}, \beta_j]} |\Psi_{\mathfrak{D}}(\Lambda)(s) - \Psi_{\mathfrak{D}}(\Lambda)(t)| < \epsilon, \;\; j = 2, ..., m. \tag{5.10}$$

Now, in order to be able to construct a net of continuous functions, surround the points β_j, $j = 1, \ldots, m$, by δ_2-neighborhoods and consider the family $\mathcal{R}$ in $\mathfrak{D}(\varphi, \, \sigma+T)$ obtained by taking all continuous functions which coincide on each $[\beta_{j-1}+\delta_2, \beta_j - \delta_2]$ $(2 \leq j \leq m)$ with some element of a finite $(\Psi_{\mathfrak{D}}(\Lambda)(\beta_j)+\epsilon)$–net of Λ in $\mathfrak{D}(\varphi, \, \beta_j)$ and which have affine trajectories on the complementary segments. By construction, the elements of $\mathcal{R}$ are affine on $[\sigma, \sigma+\delta_1]$, to stay in $\mathfrak{D}(\varphi, \, \sigma+T)$, we choose them equal to $\varphi(0)$ at $t = \sigma$.

The set $\mathcal{R}$ in $\mathfrak{D}(\varphi, \, \sigma+T)$ is finite. Denote by $z_k, \; k = 1, ..., p$ its elements. For each $j = 1, \ldots, m$, let $(z_l^j)_{1 \leq l \leq q_j}$ be a $(\Psi_{\mathfrak{D}}(\Lambda)(\beta_j)+\epsilon)$–net of $\Lambda\left|_{\beta_j}\right.$ in $\mathfrak{D}(\varphi, \beta_j)$. Consider a fixed $x \in \Lambda$. Then, one can find an element z_l^j such that,

$$d_{\mathfrak{D}(\varphi, \beta_j)}\left(x, z_l^j\right) := \left\|x - z_l^j\right\|_{C([\sigma, \beta_j];\, E)} \leq \Psi_{\mathfrak{D}}(\Lambda)(\beta_j) + \epsilon. \tag{5.11}$$

By construction, for each $j \in \{1, \ldots, m\}$ and fixed $l \in \{1, \cdots, q_j\}$, one can find an element z_k of $\mathcal{R}$, such that $z_l^j\left|_{[\beta_{j-1}+\delta_2, \beta_j-\delta_2]}\right. = z_k\left|_{[\beta_{j-1}+\delta_2, \beta_j-\delta_2]}\right.$. From (5.11) and (5.10), it results that for every $t \in [\beta_{j-1}+\delta_2, \beta_j-\delta_2]$

$$\begin{aligned} \|x(t) - z_k(t)\| = \left\|x(t) - z_l^j(t)\right\| &\leq d_{\mathfrak{D}(\varphi, \beta_j)}\left(x, z_l^j\right) \\ &\leq \Psi_{\mathfrak{D}}(\Lambda)(\beta_j) + \epsilon \\ &\leq \Psi_{\mathfrak{D}}(\Lambda)(t) + 2\epsilon. \end{aligned} \tag{5.12}$$

Because the function $\Psi_{\mathfrak{D}}(\Lambda)(\cdot)$ is nondecreasing, for every $t \in [\beta_{j-1}+\delta_2, \beta_j-\delta_2]$, we get

$$\sup_{\beta_{j-1}+\delta_2 \leq s \leq t} \|x(s) - z_k(s)\| \leq \Psi_{\mathfrak{D}}(\Lambda)(t) + 2\epsilon. \tag{5.13}$$

By (B2), we have for every $t \in [\sigma, \sigma+T]$,

$$\begin{aligned}
&\sup_{\sigma\le s\le t} \|\Phi(x)(s) - \Phi(z_k)(s)\| \\
&\quad\le \sup_{\sigma\le s\le t} \Big\| \int_\sigma^s S(s-\tau)\Big(f(\tau, x[\varphi]_\tau) - f(\tau, z_k[\varphi]_\tau)\Big)\, d\tau \Big\| \\
&\quad\le M_t \int_\sigma^t \|f(\tau, x[\varphi]_\tau) - f(\tau, z_k[\varphi]_\tau)\|\, d\tau \\
&\quad\le M_t \int_\sigma^t L\Big(\tau, K(\tau-\sigma) \sup_{\sigma\le\theta\le\tau} \|x(\theta) - z_k(\theta)\|\ \Big) d\tau
\end{aligned}$$

where M_t is from (5.7). Let us denote by

$$J(t) = [\sigma, t] \cap \Big([\sigma, \sigma+\delta_1[\cup (\cup_{1\le i\le n}]t_i - \delta_1, t_i + \delta_1[) \cup (\cup_{1\le j\le m}]\beta_j - \delta_2, \beta_j + \delta_2[) \Big),$$

$$I(t) = [\sigma, t] \setminus J(t).$$

Because the sets Λ and $\mathcal{R}$ are bounded in $\mathfrak{D}(\varphi,\ \sigma+T)$, using the last estimation and taking δ_1 and δ_2 small enough, one can ensure that

$$d_{\mathfrak{D}(\varphi,\, t)}\Big(\Phi(x), \Phi(z)\Big) \le M_T\Big(\int_{I(t)} L(s, K(s) \sup_{\sigma\le\tau\le s} \|x(\tau) - z(\tau)\|)\, ds + \epsilon\Big).$$

Now, using (5.13), we deduce

$$d_{\mathfrak{D}(\varphi,\, t)}\Big(\Phi(x), \Phi(z)\Big) \le M_T\Big(\int_\sigma^t L\left(s, K(s)\left(\Psi(\Lambda)(s) + 2\epsilon\right)\right)\, ds + \epsilon\Big). \quad (5.14)$$

The result follows from the arbitrariness in the choice of ϵ and x.

□

Corollary 5.1.1. *The map Φ is $\Psi_{C([\sigma,\sigma+T];E)}$-condensing on bounded subsets of $\mathfrak{D}(\varphi, \sigma+T)$.*

Indeed, Let Λ be a bounded subset of $\mathfrak{D}(\varphi, \sigma+T)$ such that

$$\Psi_{C([\sigma,\sigma+T];E)}\left(\Phi \circ \Lambda\right) \ge \Psi_{C([\sigma,\sigma+T];E)}\left(\Lambda\right). \quad (5.15)$$

By Lemma 5.1.3, (5.9), (5.15), and the fact that $L(t,\cdot)$ monotone nondecreasing, we have for every $t \in [\sigma, \sigma+T]$,

$$\begin{aligned}
\frac{1}{2}\Psi_{\mathfrak{D}}\left(\Lambda\right)(t) &\le \Psi_{C([\sigma,\sigma+T];E)}\left(\Lambda\right)(t) \\
&\le \Psi_{C([\sigma,\sigma+T];E)}\left(\Phi\circ\Lambda\right)(t) \\
&\le \Psi_{\mathfrak{D}}\left(\Phi\circ\Lambda\right)(t) \\
&\le M_T \int_\sigma^t L\Big(s, K(s)\, \Psi_{\mathfrak{D}}(\Lambda)(s)\Big)\, ds.
\end{aligned}$$

Hence
$$\lambda_K \Psi_{\mathfrak{D}}(\Lambda)(t) \leq 2\lambda_K M_T \int_\sigma^t L\Big(s, \lambda_K\, \Psi_{\mathfrak{D}}(\Lambda)(s)\Big)\, ds,$$
where λ_K is from (5.8). Because $\lambda_K > 0$, from Remark 5.1.2, it results that for every $t \in [\sigma, \sigma + T]$, $\Psi_{\mathfrak{D}}(\Lambda)(t) = 0$. As a consequence, for every $t \in [\sigma, \sigma + T]$, $\Psi_{C([\sigma,\sigma+T];E)}(\Lambda)(t) = 0$. Thus, Λ is relatively compact.

Proof of Theorem 5.1.1. Let us begin with the existence problem. According to Remark 5.1.4, is it enough to prove that Φ has a fixed point. In the space $C([\sigma, \sigma+T]; E)$, let us consider an equivalent norm $\|\cdot\|_*$ given by the formula
$$\|y\|_* = \max_{t\in[\sigma,\sigma+T]} e^{-\mathcal{L}(t-\sigma)}\|y(t)\|,$$
where $\mathcal{L} > 0$ is chosen so that
$$\max_{t\in[\sigma,\sigma+T]} M_T\, \lambda_K \int_\sigma^t e^{-\mathcal{L}(t-s)}\alpha(s)\, ds \leq q < 1, \tag{5.16}$$
where $\alpha(\cdot)$ is from (f_3). With this norm, let us consider the closed ball $\bar{B}_{\mathfrak{D}}(\bar{\varphi}, r) \subset \mathfrak{D}(\varphi, \sigma + T)$ of the radius $r > 0$ centered at the function $\bar{\varphi} \in \mathfrak{D}(\varphi, \sigma + T)$, where $\bar{\varphi}(t) = \varphi(0)$ for all $t \in [\sigma, \sigma + T]$, i.e.,
$$\bar{B}_{\mathfrak{D}}(\bar{\varphi}, r) = \left\{ x \in \mathfrak{D}(\varphi, \sigma + T) : \max_{t\in[\sigma,\sigma+T]} e^{-\mathcal{L}(t-\sigma)}\, \|x(t) - \varphi(0)\| \leq r \right\},$$
Choose $r > 0$ such that
$$r \geq \big[(M_T + 2)\, \|\varphi(0)\|_E + M_T\, \|\alpha\|_{L^1} (1 + \lambda_N\, \|\alpha\|_{\mathcal{B}})\big](1 - q)^{-1}$$
where q is from (5.16) and,
$$\lambda_N = \sup_{0\leq s\leq T} N(s). \tag{5.17}$$
The last inequality implies
$$(M_T + 2)\, \|\varphi(0)\|_E + M_T\, \|\alpha\|_{L^1} (1 + \lambda_N\, \|\alpha\|_{\mathcal{B}}) + qr \leq r, \tag{5.18}$$

Let us chow that Φ maps $\bar{B}_{\mathfrak{D}}(\bar{\varphi}, r)$ into itself. Let $x \in \bar{B}_{\mathfrak{D}}(\bar{\varphi}, r)$. For all $t \in [\sigma, \sigma + T]$, we have

$$
\begin{aligned}
e^{-\mathcal{L}(t-\sigma)} \|\Phi(x)(t) - \bar{\varphi}(t)\| &= e^{-\mathcal{L}(t-\sigma)} \|\Phi(x)(t) - \varphi(0)\| \\
&\leq \|S(t-\sigma)\varphi(0) - \varphi(0)\| \\
&\quad + e^{-\mathcal{L}(t-\sigma)} M_T \int_\sigma^t \alpha(s)(1 + \|x[\varphi]_s\|)\, ds \\
&\leq (M_T + 1) \|\varphi(0)\|_E \\
&\quad + e^{-\mathcal{L}(t-\sigma)} M_T \int_\sigma^t \alpha(s)(1 + \lambda_K \sup_{\theta\in[\sigma,s]} \|x(\theta)\| + \lambda_N \|\varphi\|_{\mathcal{B}})\, ds \\
&\leq (M_T + 1) \|\varphi(0)\|_E + M_T \|\alpha\|_{L^1} (1 + \lambda_N \|\varphi\|_{\mathcal{B}}) \\
&\quad + M_T \lambda_K \int_\sigma^t e^{-\mathcal{L}(t-\sigma)} e^{\mathcal{L}(s-\sigma)} \alpha(s) e^{-\mathcal{L}(s-\sigma)} \sup_{\theta\in[\sigma,s]} \|x(\theta)\|\, ds \\
&\leq (M_T + 1) \|\varphi(0)\| + M_T \|\alpha\|_{L^1} (1 + \lambda_N \|\varphi\|_{\mathcal{B}}) \\
&\quad + M_T \lambda_K \int_\sigma^t e^{-\mathcal{L}(t-s)} \alpha(s) \sup_{\theta\in[\sigma,s]} e^{-\mathcal{L}(\theta-\sigma)} [\|x(\theta) - \varphi(0)\| + \|\varphi(0)\|_E]\, ds \\
&\leq (M_T + 1) \|\varphi(0)\|_E + M_T \|\alpha\|_{L^1} (1 + \lambda_N \|\varphi\|_{\mathcal{B}}) + qr + q \|\varphi(0)\|_E \\
&\leq (M_T + 2) \|\varphi(0)\|_E + M_T \|\alpha\|_{L^1} (1 + \lambda_N \|\varphi\|_{\mathcal{B}}) + qr \leq r.
\end{aligned}
$$

It results that the operator Φ maps $\bar{B}_{\mathfrak{D}}(\bar{\varphi}, r)$ into itself, moreover, according to Corollary 5.1.1, $\Phi : \bar{B}_{\mathfrak{D}}(\bar{\varphi}, r) \to \bar{B}_{\mathfrak{D}}(\bar{\varphi}, r)$ is $\Psi_{C([\sigma,\sigma+T];E)}$-condensing. From Theorem 2.3.1, it follows that Φ admits a fixed point.

It remain to show the uniqueness. Invoking Remark 5.1.4, we deduce that it is enough to prove that Φ has a unique fixed point. Suppose that Φ has two fixed points in $\mathfrak{D}(\varphi, \sigma + T)$, say x and y, then following the same line of calculations as above, we get for any $t \in [\sigma, \sigma + T]$,

$$
\begin{aligned}
\sup_{\sigma\leq s\leq t} \|x(s) - y(s)\| &= \sup_{\sigma\leq s\leq t} \|\Phi(x)(s) - \Phi(y)(s)\| \\
&\leq M_T \int_\sigma^t \|f(\tau, x[\varphi]_\tau) - f(\tau, y[\varphi]_\tau)\|\, d\tau \\
&\leq M_T \int_\sigma^t L\Big(\tau, K(\tau - \sigma) \sup_{\sigma\leq\theta\leq\tau} \|x(\theta) - y(\theta)\|\Big)\, d\tau.
\end{aligned}
$$

Consequently

$$
\lambda_K \sup_{\sigma\leq s\leq t} \|x(s) - y(s)\| \leq \lambda_K M_T \int_\sigma^t L\Big(\tau, \lambda_K \Big(\sup_{\sigma\leq\theta\leq\tau} \|x(\theta) - y(\theta)\|\Big)\, d\tau.
$$

where, M_T is from (5.7) and, λ_K is from (5.8). Since $\lambda_K > 0$, by (f_2), it results that,

$$\sup_{\sigma \leq s \leq t} \|x(s) - y(s)\| = 0, \text{ for any } t \in [\sigma, \sigma + T].$$

Thus, the problem (5.2) has a unique mild solution. □

5.2 Averaging result

Throughout this section, the phase space $\mathcal{B}$ is considered as a normed space satisfying Axioms (B1)-(B3).

As an example of such phase space $\mathcal{B}$, one can take the Banach space C_∞ (See [32]), constituted of continuous functions $\psi :]-\infty, 0] \to E$, such that $\lim_{\theta \to -\infty} \psi(\theta)$ exists, endowed with the sup-norm,

$$\|\psi\|_{C_\infty} = \sup \{\|\psi(\theta)\| : \theta \in]-\infty, 0]\}.$$

Let us consider a semilinear functional differential equation in E with a small positive parameter ε, of the form

$$\begin{cases} z'(t) = A\,z(t) + f\left(\frac{t}{\varepsilon}, z_t\right), & t \in [0, T], \\ z_0 = \varphi, \end{cases} \tag{P_ε}$$

Now parallel to the problem (P_ε), $\varepsilon > 0$, we consider the averaged problem

$$\begin{cases} z'(t) = A\,z(t) + f_0(z_t), \\ z_0 = \varphi. \end{cases} \tag{P_0}$$

Remark 5.2.1. *It is clear that if in addition the phase space $\mathcal{B}$ satisfies Axiom $(\mathcal{B}_{av})$ then Remark 4.2.5 remains valid, that is the problems in the normal form*

$$\begin{cases} w'(\tau) = \varepsilon \left[A\,w(\tau) + f(\tau, w_\tau)\right], & \tau \in [0, \dfrac{T}{\varepsilon}], \\ w_0 = \psi, \end{cases} \tag{5.19}$$

can be written equivalently as

$$\begin{cases} w'(t) = \varepsilon \left[A\,w(t) + f_0(w_t)\right], \\ w_0 = \psi, \end{cases} \tag{5.20}$$

Hypotheses

Suppose that the unbounded linear operator A satisfies the condition (A) and the function f is acting from $\mathbb{R}_+ \times \mathcal{B}$ to E.

Let us consider the following hypotheses,

($\mathtt{f}_1$) For all $u \in \mathcal{B}$, the mapping $t \to f(t,\, u)$ is measurable.

($\mathtt{f}_2$) For all $(u,v) \in \mathcal{B} \times \mathcal{B}$,

$$\|f(t,u) - f(t,v))\| \leq \mathbb{L}\Big(\, \|u-v\|_{\mathcal{B}}\Big) \;\text{ a.e. } t \in \mathbb{R}^+,$$

where $\mathbb{L} : \; [0,+\infty[\to [0,+\infty[$ is a given mapping such that:

(i) $\mathbb{L}(\cdot)$ is continuous, monotone nondecreasing and, $\mathbb{L}(0) = 0$;

(ii) for every nonnegative continuous mapping $h : \; [0,T] \to [0,+\infty[$ and for every constant ζ, the following implication holds true:

$$\left[\forall t \in [\sigma, \sigma+T],\; h(t) \leq \zeta \int_\sigma^t L(s,h(s))\, ds\right] \Rightarrow h \equiv 0 \tag{5.21}$$

($\mathtt{f}_3$) There exists a constant $C > 0$, such that for all $u \in \mathcal{B}$

$$\|f(t,u)\| \leq C\Big(1 + \|u\|_{\mathcal{B}}\Big) \;\text{ a.e. } t \in \mathbb{R}^+.$$

($\mathtt{Af}$) There exist $\Delta_0 > 0$ and a function $f_0 : \mathcal{B} \to E$ satisfying conditions similar to ($\mathtt{f}_2$) and ($\mathtt{f}_3$), and for all $u \in \mathcal{B}$ and $t_1,\, t_2 \in [0,T]$ with $0 \leq t_1 \leq t_2 \leq t_1 + \Delta_0 \leq T$,

$$\lim_{\varepsilon \to 0^+} \int_{t_1}^{t_2} S(t_2-\theta)\Big[f\Big(\frac{\theta}{\varepsilon}, u\Big) - f_0(u)\Big] d\,\theta = 0. \tag{5.22}$$

Theorem 5.2.1. *Assume the hypotheses* (*A*), ($\mathtt{f}_1$)$-$($\mathtt{f}_3$) *and* ($\mathtt{Af}$). *Then, the sequence* $(z^\varepsilon)_{\varepsilon>0}$*, where* z^ε *is the mild solution to the problem* (P_ε)*, converges to the mild solution* z^∞ *of the problem* (P_0) *as* $\varepsilon \to 0^+$.

Remark 5.2.2. *In Hypothesis* ($\mathtt{Af}$) *by,* f_0 *satisfies a condition similar to* (f_2) *and* (f_3)*, we mean that,* f_0 *satisfies*

(1) *For all* $(u,v) \in \mathcal{B} \times \mathcal{B}$,

$$\|f_0(u) - f_0(v))\| \leq \mathbb{L}\Big(\, \|u-v\|_{\mathcal{B}}\Big);$$

(2) *There exists a constant $C' > 0$, such that for all $u \in \mathcal{B}$*

$$\|f(u)\| \leq C'\Big(1 + \|u\|_{\mathcal{B}}\Big).$$

It is clear that f_0 is a continuous function.

Remark 5.2.3. *Hypothesis* (`Af`) *is inspired from [54].*

Proof of Theorem 5.2.1. According to Theorem 5.1.1, for each $\varepsilon > 0$ there exists a unique mild solution z^ε to the problem (P_ε), and there exists a unique mild solution z^∞ to the problem (P_0). Moreover, for each $\varepsilon > 0$, $z^\varepsilon = x^\varepsilon[\varphi]$, where x^ε is the unique fixed point of the operator, $\Phi_\varepsilon : \mathfrak{D}(\varphi, T) \to \mathfrak{D}(\varphi, T)$, defined by:

$$\Phi_\varepsilon(x)(t) = S(t)\varphi(0) + \int_0^t S(t-s) f(\frac{s}{\varepsilon}, x[\varphi]_s)\, ds, \quad 0 \leq t \leq T,$$

and $z^\infty = x^\infty[\varphi]$, where x^∞ is the unique fixed point of the operator, $\Phi_\infty : \mathfrak{D}(\varphi, T) \to \mathfrak{D}(\varphi, T)$, defined by

$$\Phi_\infty(x)(t) = S(t)\varphi(0) + \int_0^t S(t-s) f_0(x[\varphi]_s)\, ds, \quad 0 \leq t \leq T. \tag{5.23}$$

Note that, if $x^\varepsilon \to x^\infty$ in $C([0,T]; E)$, then $x^\varepsilon[\varphi] \to x^\infty[\varphi]$ in $\mathcal{C}(]-\infty, T]; E)$. Therefore, for the proof of Theorem 5.2.1, we have only to prove that

$$\lim_{\varepsilon \to 0^+} \sup_{t \in [0,T]} \|x^\varepsilon(t) - x^\infty(t)\| = 0.$$

First, let us prove that the sequence $(x^\varepsilon)_{\varepsilon > 0}$ is bounded. Let $\varepsilon > 0$. Using (`f`$_3$) and Axiom (B2), for any $t \in [0,T]$, we have

$$\begin{aligned}
\|x^\varepsilon(t)\| = \|\Phi_\varepsilon(x^\varepsilon)(t)\| &\leq M_T \|\varphi(0)\| + M_T \int_0^t \left\| f\Big(\frac{\tau}{\varepsilon}, x^\varepsilon[\varphi]_\tau\Big)\right\| d\tau \\
&\leq M_T \|\varphi(0)\| + M_T \int_0^t C \left(1 + \lambda_K \sup_{0 \leq \theta \leq \tau} \|x^\varepsilon(\theta)\| + \lambda_N \|\varphi\|_{\mathcal{B}}\right) d\tau \\
&\leq M_T \Big[\|\varphi(0)\| + T\, C\Big(1 + \lambda_N \|\varphi\|_{\mathcal{B}}\Big)\Big] + M_T\, C\, \lambda_K \int_0^t \sup_{0 \leq \theta \leq \tau} \|x^\varepsilon(\theta)\|\, d\tau.
\end{aligned}$$

where M_T is from (5.7), λ_K is from (5.8), λ_N is from (5.17), and C from Hypothesis (`f`$_3$). Since the last expression does not decrease, setting

$$\omega = M_T\, [\|\varphi(0)\| + T\, C\, (1 + \lambda_N \|\varphi\|_{\mathcal{B}})] + M_T\, C\, \lambda_K,$$

we get

$$\sup_{0\le\tau\le t}\|x^\varepsilon(\tau)\| \le \omega + \omega\int_0^t \sup_{0\le\theta\le\tau}\|x^\varepsilon(\theta)\|\, d\tau.$$

Hence, by Gronwall Lemma, we obtain, for any $t\in[0,T]$,

$$\sup_{0\le\tau\le t}\|x^\varepsilon(\tau)\| \le \omega\, e^{\omega t},$$

which proves the boundedness of the sequence $(x^\varepsilon)_{\varepsilon>0}$.

By Axiom (B3), the function, $x^\infty[\varphi]_{(.)} : t \to x^\infty[\varphi]_t$ is continuous. Let $\delta>0$. There exists a partition $0=t_0<t_1<\cdots<t_q=T$ of $[0,T]$, such that

$$\max_{1\le i\le q}(t_i-t_{i-1}) \le \min\{\delta,\Delta_0\}; \tag{5.24}$$

$$\|x^\infty[\varphi]_t - x^\infty[\varphi]_{t_i}\| \le \delta, \quad \forall t\in[t_{i-1},t_i], \quad i=1,\cdots,q, \tag{5.25}$$

where Δ_0 is from ($\mathtt{Af}$). Define the function $\overline{x}^\infty[\varphi]_{(.)}$ by

$$\text{for } t\in[t_{i-1},t_i[, \quad \overline{x}^\infty[\varphi]_t = x^\infty[\varphi]_{t_{i-1}} \quad i=1,\cdots,q. \tag{5.26}$$

Setting $\tau(t)=\max\{i,\, t_i\le t\}$ and, $\vartheta(t)=t_{\tau(t)}$, for every $t\in[0,T]$ we have

$$\begin{aligned}
x^\varepsilon[\varphi](t) - x^\infty[\varphi](t) &= \Phi_\varepsilon(x^\varepsilon)(t) - \Phi_\infty(x^\infty)(t)\\
&= \int_{\vartheta(t)}^t S(t-\theta)\Big[f\Big(\frac{\theta}{\varepsilon}, x^\varepsilon[\varphi]_\theta\Big) - f_0\Big(x^\infty[\varphi]_\theta\Big)\Big]\, d\theta\\
&+ \int_0^{\vartheta(t)} S(t-\theta)\Big[f\Big(\frac{\theta}{\varepsilon}, x^\varepsilon[\varphi]_\theta\Big) - f\Big(\frac{\theta}{\varepsilon}, x^\infty[\varphi]_\theta\Big)\Big]\, d\theta\\
&+ \int_0^{\vartheta(t)} S(t-\theta)\Big[f\Big(\frac{\theta}{\varepsilon}, x^\infty[\varphi]_\theta\Big) - f\Big(\frac{\theta}{\varepsilon}, \overline{x}^\infty[\varphi]_\theta\Big)\Big]\, d\theta\\
&+ \int_0^{\vartheta(t)} S(t-\theta)\Big[f\Big(\frac{\theta}{\varepsilon}, \overline{x}^\infty[\varphi]_\theta\Big) - f_0\Big(\overline{x}^\infty[\varphi]_\theta\Big)\Big]\, d\theta\\
&+ \int_0^{\vartheta(t)} S(t-\theta)\Big[f_0\Big(\overline{x}^\infty[\varphi]_\theta\Big) - f_0\Big(x^\infty[\varphi]_\theta\Big)\Big]\, d\theta\\
&= I_1+\cdots+I_5.
\end{aligned}$$

Since the sequence $(x^\varepsilon)_{\varepsilon>0}$ is bounded, by Axiom (B2), the sequence $(x^\varepsilon[\varphi]_{(.)})_{\varepsilon>0}$ is bounded too in $C([0,T];\mathcal{B})$. Then, setting

$$\varpi = M_T\left[\, C\Big(1+\sup_{\varepsilon>0}\big\|x^\varepsilon[\varphi]_{(.)}\big\|_{C([0,T];\mathcal{B})}\Big) + C'\Big(1+ \mathop{max}_{t\in[0,T]}\|x^\infty[\varphi]_t\|\,\Big)\right]$$

where C is from $(\mathtt{f}_3)$ and C' is from Remark 5.2.2, by (5.24) we get

$$\|I_1\| \leq (t-\vartheta(t))\,\varpi \leq \min\{\delta,\, \Delta_0\}\,\varpi \leq \,\varpi\,\delta.$$

Using (5.7), Axiom (B2) and Hypothesis $(\mathtt{f}_2)$, we get

$$\begin{aligned}\|I_2\| &\leq \int_0^t S(t-\theta)\Big[f\Big(\frac{\theta}{\varepsilon}, x^{\varepsilon}[\varphi]_\theta\Big) - f\Big(\frac{\theta}{\varepsilon}, x^{\infty}[\varphi]_\theta\Big)\Big]\,d\theta \\ &\leq M_T \int_0^t \mathbb{L}\Big(K(\theta) \sup_{0\leq s\leq\theta} \|x^{\varepsilon}[\varphi](s) - x^{\infty}[\varphi](s)\|\,\Big)d\theta.\end{aligned}$$

Now, using (5.7), $(\mathtt{f}_2)$, (5.25) and (5.26), we have

$$\begin{aligned}\|I_3\| &\leq M_T \sum_{i=1}^{q} \int_{t_{i-1}}^{t_i} \Big\| f\Big(\frac{\theta}{\varepsilon}, x^{\infty}[\varphi]_\theta\Big) - f\Big(\frac{\theta}{\varepsilon}, \overline{x}^{\infty}[\varphi]_\theta\Big)\Big\|\,d\theta \\ &\leq M_T \sum_{i=1}^{q} \int_{t_{i-1}}^{t_i} \Big\| f\Big(\frac{\theta}{\varepsilon}, x^{\infty}[\varphi]_\theta\Big) - f\Big(\frac{\theta}{\varepsilon}, x^{\infty}[\varphi]_{t_{i-1}}\Big)\Big\|\,d\theta \\ &\leq M_T\, T\, \mathbb{L}(\delta).\end{aligned}$$

Since the function f_0 satisfies a condition similar to $(\mathtt{f}_2)$, the term I_5 can be estimated as I_3,

$$\|I_5\| \leq M_T\, T\, \mathbb{L}(\delta).$$

It remains to estimate the term I_4. We have

$$\begin{aligned}\|I_4\| &= \Big\| \int_0^{\vartheta(t)} S(t-\theta)\Big[f\Big(\frac{\theta}{\varepsilon}, \overline{x}^{\infty}[\varphi]_\theta\Big) - f_0\Big(\overline{x}^{\infty}[\varphi]_\theta\Big)\Big]\,d\theta \Big\| \\ &\leq M_T \sum_{i=1}^{\tau(t)} \Big\| \int_{t_{i-1}}^{t_i} S(t_i-\theta)\Big[f\Big(\frac{\theta}{\varepsilon}, x^{\infty}[\varphi]_{t_{i-1}}\Big) - f_0\Big(x^{\infty}[\varphi]_{t_{i-1}}\Big)\Big]\,d\theta \Big\|\end{aligned}$$

Bearing in mind (5.24), from Hypothesis $(\mathtt{Af1})$ we have

$$\max_{1\leq i\leq q} \lim_{\varepsilon\to 0^+} \int_{t_{i-1}}^{t_i} S(t_i-\theta)\Big[f\Big(\frac{\theta}{\varepsilon}, x^{\infty}[\varphi]_{t_{i-1}}\Big) - f_0\Big(x^{\infty}[\varphi]_{t_{i-1}}\Big)\Big]\,d\theta = 0.$$

Thus

$$\max_{1\leq i\leq q} \Big\| \int_{t_{i-1}}^{t_i} S(t_i-\theta)\Big[f\Big(\frac{\theta}{\varepsilon}, x^{\infty}[\varphi]_{t_{i-1}}\Big) - f_0\Big(x^{\infty}[\varphi]_{t_{i-1}}\Big)\Big]\,d\theta \Big\| \leq \frac{\delta}{q} \quad \text{as} \quad \varepsilon \to 0^+.$$

Thus

$$\|I_4\| \leq M_T\, \frac{\tau(t)}{q}\,\delta \leq M_T\,\delta.$$

From the estimations of the terms $I_1, \cdots, I_5$, for every $t \in [0, T]$, we have

$$\begin{aligned}
&\|x^{\varepsilon}[\varphi](t) - x^{\infty}[\varphi](t)\| \\
&\leq (\varpi + M_T)\,\delta + 2M_T\, T\, \mathbb{L}(\delta) + M_T \int_0^t \mathbb{L}\Big(K(\theta) \sup_{0\leq s\leq \theta} \|x^{\varepsilon}[\varphi](s) - x^{\infty}[\varphi](s)\|\Big) d\theta,
\end{aligned}$$

as $\varepsilon \to 0^+$. Since the last expression does not decrease, setting

$$\rho(\delta) = (\varpi + M_T)\delta + 2\, M_T\, T\, \mathbb{L}(\delta),$$

we get for every $t \in [0, T]$

$$\begin{aligned}
&K(t) \sup_{0\leq s\leq t} \|x^{\varepsilon}[\varphi](s) - x^{\infty}[\varphi](s)\| \\
&\qquad \leq \lambda_K \rho(\delta) + \lambda_K M_T \int_0^t \mathbb{L}\Big(K(\theta) \sup_{0\leq s\leq \theta} \|x^{\varepsilon}[\varphi](s) - x^{\infty}[\varphi](s)\|\Big) d\theta,
\end{aligned}$$

as $\varepsilon \to 0^+$, where λ_K is from (5.8) (recall that $K(\cdot)$ is with positive values). From ($\mathtt{f}_2$), the function $\delta \to \lambda_K\, \rho(\delta)$, is continuous on $[0, +\infty[$ and $\lambda_K\, \rho(0) = 0$. Then, from the arbitrariness of δ, we have necessarily that,

$$\begin{aligned}
&K(t) \sup_{0\leq s\leq t} \|x^{\varepsilon}[\varphi](s) - x^{\infty}[\varphi](s)\| \\
&\qquad \leq \lambda_K M_T \int_0^t \mathbb{L}\Big(K(\theta) \sup_{0\leq s\leq \theta} \|x^{\varepsilon}[\varphi](s) - x^{\infty}[\varphi](s)\|\Big) d\theta \text{ as } \varepsilon \to 0^+.
\end{aligned}$$

Using ($\mathtt{f}_2$) again, we have, for every $t \in [0, T]$,

$$K(t) \sup_{0\leq s\leq t} \|x^{\varepsilon}[\varphi](s) - x^{\infty}[\varphi](s)\| = 0 \text{ as } \varepsilon \to 0^+.$$

Since $K(\cdot)$ is a positive function, we deduce that for every $t \in [0, T]$,

$$\sup_{0\leq s\leq t} \|x^{\varepsilon}[\varphi](s) - x^{\infty}[\varphi](s)\| = 0 \text{ as } \varepsilon \to 0^+.$$

The proof is complete. □

5.3 Averaging principle in the traditional form

Now we aim to establish the averaging principle in traditional form for the problem (P_ε). Recall that the phase space $\mathcal{B}$ is considered as a normed space satisfying Axioms (B1) – (B3).

Assume the hypotheses ($\mathtt{f}_1$)-($\mathtt{f}_3$). Let us replace the hypothesis ($\mathtt{Af}$) by a more natural one:

($\mathtt{Avf}$) For every $u \in \mathcal{B}$,

$$f_0(u) = \lim_{t\to+\infty} \frac{1}{t} \int_0^t f(s,u)\, ds. \tag{5.27}$$

We have immediately

Lemma 5.3.1. *The function f_0 satisfies hypothesis similar to* ($\mathtt{f}_2$) *and* ($\mathtt{f}_3$)*.*

The proof of Lemma 5.3.1 is easy.

From Lemma 5.3.1 and Lemma 4.3.2, we deduce

Corollary 5.3.1. *Assume the hypotheses* ($\mathtt{f}_1$)–($\mathtt{f}_3$)*. Moreover, assume that E is reflexive, and the semigroup $(S(t))_t$ is analytic. Then,* ($\mathtt{Avf}$) $\Rightarrow$ ($\mathtt{Af}$)*.*

From Lemma 5.3.1, Corollary 5.3.1 and Theorem 5.2.1, we have immediately,

Theorem 5.3.1. *Assume Hypotheses* (A)*,* ($\mathtt{f}_1$)-($\mathtt{f}_3$) *and* ($\mathtt{Avf}$)*. Moreover, assume that E is reflexive and, the semigroup $(S(t))_t$ is analytic. Then*

$$\lim_{\varepsilon\to 0} \|z^\varepsilon - z^\infty\|_{C(]-\infty,T];E)} = 0,$$

where z^ε is the unique mild solution to the problem (P_ε), $\varepsilon > 0$, and z^∞ is the unique mild solution to the problem (P_0).

Conclusion and perspectives

In this thesis, qualitative studies, related to the problem of periodic solutions, were made for the semilinear functional differential equations in a Banach space and with an infinite delay on an axiomatically defined abstract phase space. More precisely, three questions were answered:

1. What are the most optimal and possible conditions on the phase space (in the sense that the choice on the phase space is made in a very simple way) that allow to complete the study of the periodic problem in the multivalued case (inclusion) where the linear part generates a compact semi-group. This was done thanks to the introduction and the study of a new multivalued integral operator whose fixed points coincide with mild-periodic solutions.
2. How to justify the averaging principle for the single valued case:
   2.1. in the case of uniqueness of solutions;
   2.2. in the case of non-uniqueness of solutions.

As perspective,

We note that all the approaches and constructions used in this thesis can be adapted in the study of the periodic problem for fractional equations and inclusions with infinite delay when the derivative is understood in the sense of Riemann-Liouville.

To the best of our knowledge, results on the averaging principle does not exist in the literature in the case of an infinite delay differential equations and inclusions in Banach spaces. It is for this reason that we have limited ourselves to the non-neutral case. Our approach can be extended to justify this principle in the case of neutral equations.

Further, it should be noted that the study of the averaging problem in the case of inclusions in infinite dimensional space is more complicated than that of equations. This is due to the fact that we do not know practical conditions on the multivalued part, except the periodicity, which allow the existence of averaged selections. The last point remains an open problem.

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الخلاصة

هذه الأطروحة تحتوي على بعض الدراسات النوعية للمعادلات و الإحتواءات التفاضلية النصف خطية بوجود تأخر زمني غير محدود في فضاء باناخ. في حالة الإحتواءات ندرس الوجود والبنية الطوبولوجية للحلول الدورية الضعيفة من خلال بناءودراسة مؤثر تكاملي متعددالقيم جديد. في حالة المعادلات، يتم اثبات بعض النتائج المتعلقة بمبدأ المتوسط وهذا في حالة وحدانية و عدم وحدانية الحلول الضعيفة.

الكلمات المفتاحية

قياس عدم التراص، مؤثر التكثيف، الإحتواءات المجردة، نصف زمرةديناميكية، الحلول الضعيفة، التأخر غير المحدود، فضاء المرحلة، مبدأ المتوسط.

Abstract

This thesis contains some qualitative results for semi-linear functional differential equations and inclusions with infinite delay in a Banach space. In the case of an inclusion we study the existence and the topological structure of the mild-periodic solutions by means of the construction and the study of a new multivalued integral operator. In the case of an equation, some averaging results are established in the case of uniqueness and the case of non-uniqueness of the mild solutions.

Keywords

Measure of non-compactness, Condensing operator, Semigroups, Abstract inclusion, Mild solution, Infinite delay, Phase space, Averaging principle.

Résumé

Cette thèses contient quelques résultats qualitatives pour des équations et inclusions différentielles fonctionnelles semi-linéaires à retard infini dans un espace de Banach. Dans le cas d’une inclusion on étudie l’existence et la structure topologique des mild-périodiques solutions à travers la construction et l’étude d’un nouveau opérateur intégral multivoque. Dans le cas d’une équation, on établi certains résultats de moyennisation dans le cas de l’unicité et le cas de non-unicité des mild solutions.

Mots clés

Mesure de noncompacité, Opérateur condensant, Semi- groupes, Inclusion abstraite, Mild solution, Retard infini, Espace de phase, Principe de moyennisation.