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DAOUDI KHELIFA

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Contribution aux équations et inclusions différentielles à retard dépendant de l'état

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Devant le jury composé de :

Président :	*Benchohra Mouffak*	*Professeur*	*à l'UDL*
Examinateurs :	*Abbas Said*	*Professeur*	*à l'université de Saida*
	Baghli Selma-Bendimrad	*Professeur*	*à l'UDL*
	Lazreg Jamal Eddine	*Maitre de conférence*	*à l'UDL*
Directeur de thèse :	*Ouahab Abdelghani*	*Professeur*	*à l'UDL*

Co-Directeur de thèse :……………………………………………………..

Dedication

This thesis is dedicated to

My parents, whose sincerely raised me with their caring and offered me unconditional love, a very special thank for the myriad of ways in which, throughout my life, you have actively supported me in my determination to find and realize my potential.

My brother, sisters who have supported me all the way since the beginning of my study.

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Publications

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Abstract

The objective of this thesis is to establish existence and uniqueness results for various classes of functional problem for partial functional and neutral functional differential equations and inclusions; periodic conditions with delay which may be finite or state-dependent in Banach spaces.
Our results are based upon very recently fixed point theorems.

Key words and phrases:

Mild solutions, state-dependent delays, random variable, impulsive, neutral partial differential equations, fixed point, existence and uniqueness, periodic conditions, functional differential equations system, multivalued map.

AMS (MOS) Subject Classifications: 34A37, 4B15, 34B37, 34K45, 39A12, 34G20, 34K20, 34K30, 47H10, 47H30, 54H25.

Résumé

L'objectif de notre thèse est d'établir des résultats d'existence et d'unicité pour des différentes classes d'équations et inclusions différentielles fonctionnelles, elles peuvent être de :
- type neutre,
- à conditions périodiques,
- à retard fini ou dépendant de état.

Toute cette étude a été faite dans les espaces de Banach.

Nos résultats sont basés sur des théorèmes récents de point fixe.

Mots clés et phrases:

Solution faible, retard dépendant de l'état, variable aléatoire, impulsive, équations partielles fonctionnelles de type neutre, point fixe, conditions périodiques, existence et unicité, système des équations partielles fonctionnelles, operateur aléatoire, multivoque.

AMS (MOS) Subject Classifications: 34A37, 4B15, 34B37, 34K45, 39A12, 34G20, 34K20, 34K30, 47H10, 47H30, 54H25.

ملخص

الهدف من هذه الرسالة مناقشة "وجود و وحدانية" حلول من المسائل ذات معادلات تفاضلية ومعادلات احتواءات من نوع حيادي ; ذات شروط دورية وتأخر محدود أو مرتبط بالحل وقد تمت الدراسة في فضاء بناخ.
النتائج اعتمدت على تقنيات و نظريات نقاط صامدة حديثة.

كلمات و جمل مفْتاحيّة:

حل ضعيف، تأخر مرتبط بالحل، نبضي، متغير عشوائي، معادلات تفاضلية جزئية من نوع حيادي، شروط دورية، وجود و وحدانية، جملة معادلات دالية تفاضلية، متغير عشوائي، متعدد الدوال.

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Introduction

Functional differential equations and inclusions arise in a variety of areas of biological, physical, and engineering applications, and such equations have received much attention in recent years. A good guide to the literature for functional differential equations is the books by Hale [49], Hale and Verduyn Lunel [50], Kolmanovskii and Myshkis [63], and the references therein. During the last decades, existence and uniqueness of mild, strong, classical, almost periodic, almost automorphic solutions of semi-linear functional differential equations and inclusions has been studied extensively by many authors using the semigroup theory, fixed point argument, degree theory, and measures of non-compactness. We mention, for instance, the books by Ahmed [6], Kamenskii et al. [60], Pazy [84], Wu [96], and the references therein.
Neutral functional differential equations arise in many areas of applied mathematics and such equations have received much attention in recent years. A good guide to the literature for neutral functional differential equations is the books by Hale [49], Hale and Verduyn Lunel [50], Kolmanovskii and Myshkis [63], and the references therein. Hernandez in [57] proved the existence of mild, strong, and periodic solutions for neutral equations. Fu in [35, 36] studies the controllability on a bounded interval of a class of neutral functional differential equations. Fu and Ezzinbi [37] considered the existence of mild and classical solutions for a class of neutral partial functional differential equations with nonlocal conditions. Various classes of partial functional and neutral functional differential equations with infinite delay are studied by Adimy et al. [1, 2, 3], Belmekki et al. [17], and Ezzinbi [33]. Henriquez [56] and Hernandez [57, 58] studied the existence and regularity of solutions to functional and neutral functional differential equations with unbounded delay. Balachandran and Dauer have considered various classes of first and second order semi-linear ordinary, functional and neutral functional differential equations on Banach spaces in [15].
Functional differential equation with state-dependent delays appear frequently in applications such as model equations (see, e.g., [7, 12, 24, 69]) and the study of such equations is an active research area (see, e.g., [28, 4, 36, 52, 51, 53, 54, 58, 64, 65, 68, 90, 95].
Abstract neutral differential equations arise in many areas of applied mathematics. For this reason, they have largely been studied during the last few decades. The literature related to ordinary neutral differential equations is very extensive, for which we refer the reader to [50] only, which contains a comprehensive description of such equations. Similarly, for more on partial neutral functionaldifferential equations and related issues

we refer to Adimy and Ezzinbi [4], Hale [49], Wu and Xia [97] and [96] for finite delay equations.

This thesis is structured in seven chapters and each chapter contains more sections. It is arranged as follows:

In **Chapter** 1, we introduce notations, definitions and preliminary facts that will be used through this thesis.

In **Chapter** 2, we introduce a general class of basic theory of retarded functional differential equations. The basic theory of existence, uniqueness, continuation and continuous dependence will be introduced.

In **Chapter** 3, we study the existence and uniqueness of solutions for neutral differential equations with state-dependent delays of the following form,

$$\frac{d}{dt}(x(t)-g(t,x(t-\eta(t)))) = A(x(t)-g(t,x(t-\eta(t))))+f(t,x_t,x(t-\tau(t,x_t))),\ t\in J:=[0,T], \tag{0.0.1}$$

with initial condition

$$x(t)=\varphi(t),\ t\in[-r,0], \tag{0.0.2}$$

where A generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ on a Banach space E, $f : J\times C([-r,0],E)\times E \to E$, $g : J\times E \to E$ are given functions, and $\varphi : [-r,0]\to E$, $\tau : [0,T]\times C([-r,0],E)\to[0,r]$ and $\eta : J\to[0,r]$ are also given continuous functions. This chapter is organized as follows: in Section 3.2, we give one of our main existence results for solutions of (0.0.1)-(0.0.2), with the proof based on Banach's fixed point theorem. In Section 3.3, we give two other existence results for solutions of (0.0.1)-(0.0.2). Their proofs involve the measure of noncompactness paired in one result with a Mönch fixed point theorem and paired in the other result with a Darbo fixed point theorm.

In **Chapter** 4, we consider the periodic boundary value problem,

$$x'(t)=Ax(t)+f(t,x_t,x(t-\tau(t,x_t))),\ t\in J:=[0,b], \tag{0.0.3}$$

$$x_0(\theta)=x_b(\theta),\ \theta\in[-r,0], \tag{0.0.4}$$

where A generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Hilbert space $(H,\|.\|)$,
$f : J\times C([-r,0],H)\times H\to H$, is given function and
$\tau : [0,T]\times C([-r,0],H)\to[0,r]$ is given continuous function.
The proof of the existence of periodic solutions of the problem (0.0.3)-(0.0.4) is based on a generalization of Mawhin coincidence degree theory.

In **Chapter** 5, we shall use a random version of the Perov type theorem for the study of the nonlinear initial value problems of random functional differential equations. Using topological degree methods, we give some existence results for functional differential equations, we study the following system

$$\begin{cases} x'(t,\omega) = f(t, x_t(\omega), y_t(\omega), \omega), & t \in J := [0,T] \\ y'(t,\omega) = g(t, x_t(\omega), y_t(\omega), \omega), & t \in J := [0,T] \\ x_0(\theta) = \varphi(\theta, \omega), & \theta \in [-r, 0] \\ y_0(\theta) = \psi(\theta, \omega), & \theta \in [-r, 0]. \end{cases} \tag{0.0.5}$$

where $f, g : J \times C([-r, 0], \mathbb{R}) \times C([-r, 0], \mathbb{R}) \times \Omega \to \mathbb{R}$, $(\Omega, \mathcal{A})$ is a measurable space and $x_0, y_0 : \Omega \to \mathbb{R}$ are a random variable.

In **Chapter** 6, we prove the existence of solutions for functional differential inclusions system with state-dependent delays of the following form:

$$\begin{cases} x'(t) - A_1 x(t) \in F_1(t, x_t, x(t - \tau_1(t, x_t)), y_t, y(t - \tau_2(t, y_t))), & t \in J := [0, b] \\ y'(t) - A_2 y(t) \in F_2(t, x_t, x(t - \tau_1(t, x_t)), y_t, y(t - \tau_2(t, y_t))), & t \in J \\ x(t) = \varphi(t), & t \in [-r, 0] \\ y(t) = \psi(t), & t \in [-r, 0]. \end{cases} \tag{0.0.6}$$

where the operators A_i, $i = 1, 2$ are infinitesimal generator of a C_0-semigroup $T_i(t)_{t \geq 0}$ on a Banach space E,
$F_1, F_2 : J \times C([-r, T], E) \times E \times C([-r, T], E) \times E \to \mathcal{P}(E)$ are multifunctions,
$\varphi, \psi : [-r, 0] \to E$ and $\tau_1, \tau_2 : [0, T] \times C([-r, 0], E) \to [0, r]$ are given continuous functions.

In the second part of this chapter, we prove the existence of solutions for functional differential inclusions system with state-dependent delays with periodic conditions of the following form:

$$\begin{cases} x'(t) - A_1 x(t) \in F_1(t, x_t, x(t - \tau_1(t, x_t)), y_t, y(t - \tau_2(t, y_t))), & t \in J \\ y'(t) - A_2 y(t) \in F_2(t, x_t, x(t - \tau_1(t, x_t)), y_t, y(t - \tau_2(t, y_t))), & t \in J \\ x_0(\theta) = x_b(\theta), & \theta \in [-r, 0] \\ y_0(\theta) = y_b(\theta), & \theta \in [-r, 0]. \end{cases} \tag{0.0.7}$$

In **Chapter** 7, we prove the existence, uniqueness and multiplicity of solutions for second order impulsive differential equations with parameter. By using the classical fixed point theorem for operators on a cone, the existence of positive solutions is also investigated. We consider the impulsive periodic boundary value problem with a parameter λ of the form

$$y'' - \rho^2 y = -f(t, y, \lambda),\ t \in J := [0, 2\pi],\ t \neq t_k,\ k = 1, ..., m, \tag{0.0.8}$$

$$y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \quad t = 1, ..., m, \tag{0.0.9}$$

$$y'(t_k^+) - y'(t_k^-) = \bar{I}_k(y(t_k^-)), \quad t = 1, ..., m, \tag{0.0.10}$$

$$y(0) = y(2\pi), \quad y'(0) = y'(2\pi). \tag{0.0.11}$$

where $\rho \in \mathbb{R}^*$, λ is a real parameter, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function,
I_k, $\bar{I}_k \in C(\mathbb{R}, \mathbb{R})$, $t_k \in [0, 2\pi]$, $0 = t_0 < t_1 < ... < t_m < t_{m+1} = 2\pi$,
$y(t_k^+) = \lim_{h \to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h \to 0^+} y(t_k - h)$ represent the right and left limits of $y(t)$ at $t = t_k$.

Finally, we give a conclusion and a bibliography.

Chapter 1

Preliminaries

1.1 Some Notations and Definitions

In this chapter, we introduce notations, definitions, lemmas and fixed point theorems which are used throughout this thesis. In this section, we introduce notations, definitions, and preliminary facts which are used throughout this section. Let $(E, \|\cdot\|)$ be a Banach space.

$C([-r,T],E)$ is the Banach space of all continuous functions from $[-r,T]$ into E with the norm

$$\|x\|_\infty = \sup_{\theta\in[-r,0]} \sup_{t\in[0,T]} \|x(t+\theta)\|.$$

$L^1([0,T],E)$ denotes the Banach space of measurable functions $x : [0,T] \to E$ which are Bochner integrable and is normed by

$$\|x\|_{L^1} = \int_0^T \|x(t)\|dt.$$

In a normed space $(X, \|\cdot\|_X)$, the open ball around a point x_0 with radius R is denoted by $B_X(x_0,R)$, i.e., $B_X(x_0,R) := \{x \in X : \|x - x_0\|_X < R\}$, and the corresponding closed ball is denoted by $\bar{B}_X(x_0,R)$.

Let $B(E)$ be the Banach space of bounded linear operators from E into E.

Definition 1.1.1. *A linear map $T : E \to Y$ is said to be compact if for any bounded sequence (x_n) in E, $(S(x_n))$ has a convergent subsequence.*

Definition 1.1.2. *Let E be a real normed space. A mapping $T : D(T) \subset E \to E$ is called compact if T maps every bounded subset of $D(T)$ to a relatively compact subset in E. T is said to be completely continuous if T is continuous and compact.*

1.2 Measure of noncompactness

Next, we define in this Section the Kuratowski (1896-1980) and Hausdorf measures of noncompactness (**MNC** for short) and give their basic properties.

Definition 1.2.1. *([62]) Let (X, d) be a complete metric space and $\mathcal{B}$ the family of bounded subsets of X. For every $B \in \mathcal{B}$ we define the Kuratowski measure of noncompactness $\alpha(B)$ of the set B as the infimum of the numbers d such that B admits a finite covering by sets of diameter smaller than d.*

Remark 1.2.1. *The diameter of a set B is the number* $\sup\{d(x, y) : x \in B, y \in B\}$ *denoted by $diam(B)$, with $diam(\emptyset) = 0$.*
It is clear that $0 \leq \alpha(B) \leq diam(B) < +\infty$ for each nonempty bounded subset B of X and that $diam(B) = 0$ if and only if B is an empty set or consists of exactly one point.

Definition 1.2.2. ([*16*]) *Let E be a Banach space and Ω_E the family of bounded subsets of E. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_E \to [0, \infty)$ defined by*

$$\alpha(B) = \inf\{\epsilon > 0 \,:\, B \subseteq \cup_{i=1}^{n} B_i \text{ and } diam(B_i) \leq \epsilon\},$$

where

$$diam(B_i) = sup\{||x - y|| : x, y \in B_i\}.$$

The Kuratowski measure of noncompactness satisfies the following properties:

Lemma 1.2.1. *([16]) Let A and B bounded sets.*

(a) $\alpha(B) = 0 \Leftrightarrow \overline{B}$ *is compact (B is relatively compact), where $\overline{B}$ denotes the closure of B.*

(b) nonsingularity : α is equal to zero on every one element-set.

(c) If B is a finite set, then $\alpha(B) = 0$.

(d) $\alpha(B) = \alpha(\overline{B}) = \alpha(convB)$, *where $convB$ is the convex hull of B.*

(e) monotonicity : $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$.

(f) algebraic semi-additivity : $\alpha(A + B) \leq \alpha(A) + \alpha(B)$, where

$$A + B = \{x + y : x \in A, \quad y \in B\}.$$

(g) semi-homogencity : $\alpha(\lambda B) = |\lambda|\alpha(B)$; $\lambda \in \mathbb{R}$, where $\lambda(B) = \{\lambda x : x \in B\}$.

(h) semi-additivity : $\alpha(A \bigcup B) = max\{\alpha(A), \alpha(B)\}$.

(i) $\alpha(A \bigcap B) = min\{\alpha(A), \alpha(B)\}$.

(j) *invariance under translations :* $\alpha(B + x_0) = \alpha(B)$ *for any* $x_0 \in E$.

Remark 1.2.2. *The* α*-measure of noncompactness was introduced by Kuratowski in order to generalize the Cantor intersection theorem*

Theorem 1.2.2. *([62]) Let* (X, d) *be a complete metric space and* $\{B_n\}$ *be a decreasing sequence of nonempty, closed and bounded subsets of* X *and* $\lim\limits_{n\to\infty} \alpha(B_n) = 0$*. Then the intersection* B_∞ *of all* B_n *is nonempty and compact.*

In [59], Horvath has proved the following generalization of the Kuratowski theorem:

Theorem 1.2.3. *([62]) Let* (X, d) *be a complete metric space and* $\{B_i\}_{i\in I}$ *be a family of nonempty of closed and bounded subsets of* X *having the finite intersection property. If* $\inf\limits_{i\in I} \alpha(B_i) = 0$ *then the intersection* B_∞ *of all* B_i *is nonempty and compact.*

Lemma 1.2.4. *([45]) If* $V \subset C(J, E)$ *is a bounded and equicontinuous set, then*

(i) *the function* $t \to \alpha(V(t))$ *is continuous on* J*, and*

$$\alpha_c(V) = \sup_{0\le t\le T} \alpha(V(t)).$$

(ii) $\alpha\left(\int_0^T x(s)ds : x \in V\right) \le \int_0^T \alpha(V(s))ds,$

where

$$V(s) = \{x(s) : x \in V\},\ s \in J.$$

In the definition of the Kuratowski measure we can consider balls instead of arbitrary sets. In this way we get the definition of the Hausdorff measure:

Definition 1.2.3. *([62]) The Hausdorff measure of noncompactness* $\chi(B)$ *of the set* B *is the infimum of the numbers* r *such that* B *admits a finite covering by balls of radius smaller than* r.

Theorem 1.2.5. *([62]) Let* $B(0, 1)$ *be the unit ball in a Banach space* X*. Then*

$$\alpha(B(0, 1)) = \chi(B(0, 1)) = 0$$

if X *is finite dimensional,*
and $\alpha(B(0, 1)) = 2,\ \chi(B(0, 1)) = 1$ *otherwise.*

Theorem 1.2.6. *([62]) Let* $S(0, 1)$ *be the unit sphere in a Banach space* X*. Then* $\alpha(S(0, 1)) = \chi(S(0, 1)) = 0$ *if* X *is finite dimensional, and* $\alpha(S(0, 1)) = 2$, $\chi(S(0, 1)) = 1$ *otherwise.*

Theorem 1.2.7. *([62]) The Kuratowski and Hausdorff MNCs are related by the inequalities*

$$\chi(B) \leq \alpha(B) \leq 2\chi(B).$$

In the class of all infinite dimensional Banach spaces these inequalities are the best possible.

Example 1.2.1. *Let l^∞ be the space of all real bounded sequences with the supremum norm, and let A be a bounded set in l^∞. Then $\alpha(A) = 2\chi(A)$.*

For further facts concerning measures of noncompactness and their properties we refer to [8, 16, 62, 14] and the references therein.

1.3 Complemented Spaces and Projectors

Definition 1.3.1. *Two subspaces M and N of a vector space X are "algebraic complements" or "are complementary" if*

1. *$M \cap N = \{0\}$, and*
2. *$X = M + N$.*

Under these circumstance each vector $x \in X$ has a unique representation of the form $m + n$ for $m \in M$ and $n \in N$ and we write $X = M \oplus N$.

Definition 1.3.2. *Let X be a vector space and $M \subseteq X$ a closed subspace. If there exists a closed subspace $N \subseteq X$ such that M and N are algebraically complemented subspaces then N is said to be a topological complement (or supplement) of M.*

Properties 1.3.1. *[78] .*

1. *Every finite dimensional subspace is topologically complemented.*
2. *Every algebraic complement of a finite codimension subspace is topologically complemented.*

1.3.1 Projectors

In this part, we consider a class of linear operators of a special kind encountered in the general theory of linear operators and required, in particular for the construction of generalized inverse operators, namely, a class of projectors.

Let X be a Banach space and assume that it is the complemented sum of the two closed subspaces U and V. We note that this implies that $U \cap V = \{0\}$.
We can define two maps

$$P : X \to U \text{ and } \ Q : X \to V$$

where we define $P(x) \in U$ and $Q(x) \in V$ by the equation $x = P(x) + Q(x)$ (which by assumption has a unique solution). Note that P and Q are linear. Indeed if $P(x_1) = u_1, P(x_2) = u_2, Q(x_1) = v_1, Q(x_2) = v_2,$, then for $\lambda, \mu \in \mathbb{K}$ we have

$$\lambda u_1 + \mu u_2 + \lambda v_1 + \mu v_2,$$

and thus by uniqueness

$$P(\lambda x_1 + \mu x_2) = \lambda u_1 + \mu u_2 \text{ and } \ Q(\lambda x_1 + \mu x_2) = \lambda v_1 + \mu v_2.$$

Secondly it follows that $PoP = P$ and $QoQ = Q$. Indeed, for any $x \in X$ we write $P(x) = P(x) + 0 \in U + V$, and since this representation of $P(x)$ is unique it follows that $P(P(x)) = P(x)$. The argument for Q is the same.
Finally it follows that, again using the uniqueness argument, that P is the identity on U and Q is the identity on V. We therefore proved that

a) P is linear,

b) the image of P is U

c) P is idempotent, i.e. $P^2 = P$.

We say in that case that P is a linear projection onto U. Similarly Q is a linear projection onto V , and P and Q are complementary to each other, meaning that $P(X) \cap Q(X) = \{0\}$ and $P + Q = Id$. so we make the following definition.

Definition 1.3.3. *Let X be vector space. A projector (projection operator) is any linear mapping from X to itself which is idempotent, i.e.*

$$P^2 = P.$$

Properties 1.3.2. .

(i) P is a projector if and only if $I - P$ it is.

(ii) $ker(P) = Im(I - P)$, $Im(P) = ker(I - P)$.

(iii) If P is a projector, then $P(X)$ is a closed subspace of X and P is the identity on $P(X)$.

Proof. (i) Let P a projector, then for all $x \in X$

$$\begin{aligned}(I-P)^2(x) &= x - 2P(x) + P^2(x)\\ &= x - 2P(x) + P(x)\\ &= x - P(x) = (I-P)(x).\end{aligned}$$

inversely, if $(I-P)$ is a projector, $(I-(I-P)) = P$ it is a projector.

(ii) $\forall x \in Ker(P), (I-P)(x) = x - P(x) = x \Rightarrow x \in Im(I-P)$,
$\forall x \in Im(I-P), x = (I-P)(y), P(x) = P(I-P)(y) = (P-P^2)(y) = 0$
$\Rightarrow x \in Ker(P)$.

(iii) Assume $P(x_k) \to z$. Continuity of P implies $P^2(x_k)$ converges to $P(z)$, thus $z = P(z)$, and thus $z \in P(X)$. We have $x = P(y) \Rightarrow P(x) = P^2(y) = Py = x$ and thus $P(x) = x$.

□

Lemma 1.3.3. *Assume that X is the complemented sum of two closed subspaces U and V. Then the projections P and Q as defined before are bounded.*

Theorem 1.3.4. *Let X be a linear space.*

(a) If $P : X \to X$ is a projection, then $X = Im(P) \oplus Ker(P)$.

(b) If $X = M \oplus N$, where M and N are linear subpaces of X, then there is a projection $P : X \to X$ with $Im(P) = M$ and $Ker(P) = N$.

Proof. To prove (a), we first show that $x \in Im(P)$ if and only if $x = Px$. If $x = Px$, then clearly $x \in Im(P)$. If $x \in Im(P)$, then $x = Py$ for some $y \in X$, and since $P^2 = P$, it follows that $Px = P^2y = Py = x$.
If $x \in Im(P) \cap Ker(P)$ then $x = Px$ and $Px = 0$, so $Im(P) \cap Ker(P) = \{0\}$. If $x \in X$, then we have

$$x = x + (x - Px),$$

where $Px \in Im(P)$ and $(x - Px) \in Ker(P)$, since

$$P(x - Px) = Px - P^2x = Px - Px = 0.$$

Thus $X = Im(P) \oplus Ker(P)$.
To prove (b), we observe that if $X = M \oplus N$, then $x \in X$ has the unique decomposition $x = y + z$ with $y \in M$ and $z \in N$, and $Px = y$ defines the required projection. □

Notice that there is a one to one correspondence between the set of idempotents (or projectors) defined on a vector space X and the set of all pairs of complementary subspaces of X in the following sense:

- Each idempotent P defines a pair of complementary spaces namely $Ker(P)$, and $Im(P)$.
- Every pair of complementary subspaces U and V defines an idempotent namely, the projector onto U along V.

1.3.2 Codimension

Definition 1.3.4. *If M is a subspace of a vector space X, then the codimension of M is the vector space dimension of X/M, i.e.*

$$codim(M) = \dim(X/M).$$

Proposition 1.3.5. *Suppose that X is finite-dimensional. Then X/M is finite-dimensional and*

$$\dim(X/M) = \dim X - \dim M.$$

1.4 Fredholm Mappings

Let X and Z be two vector spaces and let $L : D(L) \subset X \to Z$ be a linear mapping, where $D(L)$ represents the domain of L. Similarly we shall denote the kernel of L by $kerL$, the range of L by ImL and the quotient space Z/ImL, the cokernel of L, by $cokerL$.

Definition 1.4.1. *[81] Let X and Z be real normed vector spaces. A linear mapping $L : D(L) \subset X \to Z$ is called Fredholm if the following conditions hold :*

(i) $Ker(L) = L^{-1}(\{0\})$ has finite dimension;

(ii) $Im(L) = L(D(L))$ is closed and has finite codimension.

Proposition 1.4.1. *[81] Let X be a Banach space, $L : X \to X$ be a linear bounded Fredholm operator and $K : X \to X$ be a linear continuous compact mapping. Then $L + K$ is a Fredholm mapping.*

1.4.1 Index of Fredholm Mappings

The index of L is the integer defined by

$$ind(L) = dim\ Ker(L) - codim\ Im(L).$$

Recall that the codimension of $Im(L)$ is the dimension of $Coker(L) = Y/Im(L)$, where $Y/Im(L)$ is the quotient space of Y under the equivalence relation

$$y \sim y' \Leftrightarrow y - y' \in Im(L).$$

Thus $Coker(L) = \{y + Im(L), y \in Y\}$.

Examples

1. If X and Z have finite dimensions, then every linear operator $L : X \to Z$ is a Fredholm operator and

$$ind(L) = \dim(X) - \dim(Z).$$

2. If X and Z are Banach spaces and $L : X \to Z$ is a linear bijective operator, then L is a Fredholm operator of index 0.

$$\dim(ker(L)) = codim(Im(L)) = 0.$$

3. Identity $I : X \to X$ is a Fredholm operator of index 0.

Theorem 1.4.2. *(Multiplicative property of the index). If we are given two Fredholm operators L_1 and L_2, then L_1L_2 is also a Fredholm operator, and it satisfies*

$$ind(L_1L_2) = ind(L_1) + ind(L_2).$$

Theorem 1.4.3. *(Invariance of Fredholm property and index under small perturbations). Let L be a Fredholm operator. Then there exists a constant $c > 0$ such that for all operators $S : X \to Z$ with norm $< c$, $L + S$ is a Fredholm operator which satisfies*

$$ind(L + S) = ind(L).$$

Theorem 1.4.4. *(Invariance of Fredholm property and index under compact perturbations). Let L be a Fredholm operator. Then for any compact operator $S : X \to Z$, $L + S$ is a Fredholm operator and*

$$ind(L + S) = ind(L).$$

Proposition 1.4.5. *Let L be a Fredholm operator. Then $ind(L) = 0$ if and only if there exist a compact operator K such that $L + K$ is invertible.*

1.4.2 Generalized Inverse of L

Assume that L is a Fredholm mapping. Then there exist two linear continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that

$$Im(P) = Ker(L), \quad Ker(Q) = Im(L).$$

Indeed, since (i) is satisfied then $Ker(L)$ has a topological supplementary denoted G_1 and P is the projection on $Ker(L)$ in a parallel direction with G_1, therefore

$Im(P) = Ker(L)$. Likewise, since (ii) is satisfied then $Im(L)$ has a topological supplementary denoted G_2 and Q is the projection on G_2 in a parallel direction with $Im(L)$, therefore $Ker(Q) = Im(L)$. Also, we have

$$X = Ker(L) \oplus Ker(P), \quad Y = Im(L) \oplus Im(Q)$$

as the topological direct sums.
Since $Im(Q)$ is isomorphic to $Ker(L)$, there exist isomorphism $J : Im(Q) \to Ker(L)$
We define

$$L_P : D(L) \cap Ker(P) \to Im(L)$$

as the restriction $L|_{D(L)\cap Ker(P)}$ of L to $D(L) \cap Ker(P)$, then L_P is an algebraic isomorphism $(dim(D(L) \cap Ker(P)) = dim\ Im(P))$ and its inverse is defined by

$$K_P : Im(L) \to D(L) \cap Ker(P)$$

with

$$K_P = L_P^{-1}.$$

K_P is called the pseudo inverse of L associated with P. K_P is one to one and

$$PK_P = 0,$$

therefore on $Im(L)$

$$LK_P = L(I - P)K_P = L_P K_P = I,$$

and on $D(L)$

$$K_P L = K_P L(I - P) = K_P L_P (I - P) = I - P.$$

Using the projectors P and Q introduced above and letting

$$K_{PQ} = (L|_{D(L)\cap Ker(P)})^{-1}(I - Q) = K_P(I - Q)$$

the right inverse of L associated to P and Q.

Proposition 1.4.6. *[72] Let (P, Q) and (P', Q') be pair of projectors. Then*

$$K_{P'} = (I - P')K_P$$

$$PK_{P'} + P'K_P = 0$$

where K_P and $K_{P'}$ denote the pseudo-inverses of L associated with P and P' respectively.

Proposition 1.4.7. *[72] Let P, P' be projectors of X onto $Ker(L)$ and let $P'' = aP + bP'$ for some real numbers a and b. Then P'' is a projector onto $Ker(L)$ if and only if $a + b = 1$. If this necessary and sufficient condition holds, the pseudo-inverse of L associated with P'' is given by*

$$K_{P''} = aK_P + bK_{P'}.$$

Proposition 1.4.8. *Let X, Y be Banach spaces, $L : D(L) \subset X \to Y$ be a Fredholm mapping of index $m > 0$, $J : Im(Q) \to Ker(L)$ be a monomorphism. Then equation*

$$Lx = y,$$

is equivalent to equation

$$x = Px + JQy + K_{PQ}y.$$

Proof. Let $x \in X; y \in Y$, then we can write

$$x = Px + (I - P)x \text{ and } \; y = Qy + (I - Q)y,$$

so, by substitution in $Lx = y$ we get

$$L(Px + (I - P)x) = Qy + (I - Q)y.$$

Since $Qy = 0$ and $LPx = 0$ ($y \in Im(L)$ and $Px \in Ker(L)$), then

$$L(I - P)x = (I - Q)y,$$

thus

$$x - Px = L_p^{-1}(I - Q)y.$$

since $JQy = 0$, we obtain

$$x = Px + JQy + K_{PQ}y.$$

□

Proposition 1.4.9. *[81] Let X, Y be Banach spaces, $L : D(L) \subset X \to Y$ be a Fredholm mapping of index $m > 0$, $J : Im(Q) \to Ker(L)$ be a monomorphism and $N : D(N) \subset X \to Y$ be a mapping. Then $Lx - Nx = y$ if and only if*

$$x - Px - JQNx - K_{PQ}Nx = K_{PQ}y + JQy.$$

Proof. If $Lx - Nx = y$, then we have

$$L(I - P)x - QNx - (I - Q)Nx = Qy + (I - Q)y,$$

so

$$-QNx = Qy \;\text{ and }\; L(I - P)x - (I - Q)Nx = (I - Q)y,$$

i.e.,

$$-JQNx = JQy \;\text{ and }\; x - Px - K_{PQ}Nx = K_{PQ}y.$$

Thus we have

$$x - Px - JQNx - K_{PQ}Nx = K_{PQ}y + JQy.$$

On the other hand, if

$$x - Px - JQNx - K_{PQ}Nx = K_{PQ}y + JQy,$$

then, since

$$JNx \in Ker(L) = Im(P) \text{ and } K_{PQ}Nx \in D(L) \cap Ker(P),$$

we have

$$x - Px - K_{PQ}Nx = K_{PQ}y \text{ and } -JQNx = JQy,$$

and so,

$$Lx - (I - Q)Nx = (I - Q)y \text{ and } -QNx = Qy.$$

Thus

$$Lx - Nx = y.$$

□

From this Propositions 1.4.8 and 1.4.9 , we can easily check that the equation

$$Lx = Nx$$

is equivalent to equation

$$x = Px + JQNx + K_{PQ}Nx,$$

which is a fixed point problem. In general for each linear continuous mappings of finite rank $A : X \to Y$, the equation

$$Lx + Nx = 0, \tag{1.4.1}$$

is equivalent to the equation

$$(L + A)x + (N - A)x = 0, \tag{1.4.2}$$

and hence, to the fixed point problem

$$x + (L + A)^{-1}(N - A)x = 0, \tag{1.4.3}$$

1.4.3 L-Compact Mappings

Denote by $\mathcal{F}(\mathcal{L})$ the set of linear continuous mappings of finite rank $A : X \to Y$ which are such that $L + A : D(L) \to Y$ is a bijection.

Lemma 1.4.10. *[34] Let E be a metric space and $G : E \to Z$ be a mapping. If there exists some $A \in \mathcal{F}(\mathcal{L})$ such that $(L + A)^{-1}G$ is compact on E, then the same is true for any $B \in \mathcal{F}(\mathcal{L})$.*

Proof. Let $B \in \mathcal{F}(\mathcal{L})$, then

$$\begin{aligned}(L+B)^{-1}N &= (L+B)^{-1}(L+A)(L+A)^{-1}N \\ &= (L+B)^{-1}(L+B+A-B)(L+A)^{-1}N \\ &= (I+(L+B)^{-1}(A-B))(L+A)^{-1}N \\ &= (L+A)^{-1}N+(L+B)^{-1}(A-B)(L+A)^{-1}N\end{aligned}$$

As $A-B$ is continuous and has finite rank, $(L+B)^{-1}(A-B)$ is continuous and has finite rank, and hence $(L+B)^{-1}(A-B)(L+A)^{-1}G$ is compact on E. □

Definition 1.4.2. *[34] We say that $N : E \to Y$ is L-compact on E if there exist $A \in \mathcal{F}(\mathcal{L})$ such that $(L+A)^{-1}N : E \to X$ is compact on E.*

For $E \subset X$, $X = Y$ and $L = I$, this concept reduces to the classical one of compact mapping introduced by Schauder.

Definition 1.4.3. *[81] Let $L : D(L) \subset X \to Z$ be a Fredholm mapping. E be a metric space and $N : E \to Z$. We say that N is L-compact on E if $QN : E \to Z$, $K_{PQ}N : E \to X$ are continuous and $QN(E)$, $K_{PQ}N(E)$ are compact.*

Remark 1.4.1. *It is clear that if $C \subset E$ and if $N : E \to Y$ is L-compact on E, it is also L-compact on C. Also, if $H : E \to Y$ is L-compact on E, the same is true for $N + H$.*

Proposition 1.4.11. *[34] If $A : X \to Z$ is linear, L-completely continuous on X and if $Ker(L + A) = \{0\}$, then $L + A : D(L) \to Z$ is bijective and, for each L-compact mapping $G : E \to Z$, the mapping $(L + A)^{-1}G : E \to X$ is L-compact on E.*

1.5 Coincidence Degree Theory

Let X and Z be real vector normed spaces, $L : D(L) \subset X \to Z$ a linear Fredholm mapping with zero index. Let us denote by C_L the set of couples (F, Ω), where the mapping $F : D(L) \cap \overline{\Omega} \to Z$ has the form $F = L + N$, with $N : \overline{\Omega} \to Z$ L-compact and Ω is an open bounded subset of X satisfying the condition

$$0 \notin F(D(L) \cap \partial\Omega). \tag{1.5.1}$$

A mapping D_L from C_L into Z will be called a degree relatively to L if it is not identically zero and satisfies the following axioms (see [34]).

1. **Addition-excision property.** If $(F, \Omega) \in C_L$ and Ω_1 and Ω_2 are disjoint open subsets in Ω such that

$$0 \notin F[D(L) \cap (\Omega \setminus (\Omega_1 \cup \Omega_2))],$$

then (F, Ω_1) and (F, Ω_2) belong to C_L and

$$D_L(F, \Omega) = D_L(F, \Omega_1) + D_L(F, \Omega_2).$$

2. **Homotopy invariance property.** If Γ is open and bounded in $X \times [0,1]$, $\mathcal{H}$: $(D(L) \cap \overline{\Gamma}) \times [0,1] \to Z$ has the form
$$\mathcal{H}(x,\lambda) = Lx + \mathcal{N}(\S,\lambda)$$
where $\mathcal{N}$: $\overline{\Gamma} \to Z$ is L-compact on $\overline{\Gamma}$ and if
$$\mathcal{H}(x,\lambda) \neq 0$$
for each $x \in (D(L) \cap \partial\Gamma)_\lambda$ and eqch $\lambda \in [0,1]$, where
$$(\partial\Gamma)_\lambda = \{x \in X : (x,\lambda) \in \partial\Gamma\}$$
then the mapping $\lambda \mapsto D_L(\mathcal{H}(.,\lambda),\Gamma_\lambda)$ is constant on $[0,1]$, where Γ_λ denotes the set
$$\{x \in X : (x,\lambda) \in \Gamma\}$$
3. **Normalization property.** If $(F,\Omega) \in C_L$, with F the restriction to $\overline{\Omega}$ of a linear one-to-one mapping from $D(L)$ into Z, then $D_L(F-b,\Omega) = 0$ if $b \notin F(D(L)\cap\Omega)$ and $|D_L(F-b,\Omega)| = 1$ if $b \in F(D(L)\cap\Omega)$).

The definition of D_L is based upon the Leray-Schauder degree and upon a simple lemma. Denote by $\mathcal{C}(\mathcal{L})$ the set of linear completely continuous mappings $A : X \to Z$ such that $Ker(L+A) = \{0\}$. By Proposition 1.4.11, $L+A : D(L) \to Z$ is bijective and $(L+A)^{-1}G$ is compact over $E \to X$ whenever $G : E \to Z$ is L-compact. Furthermore, one has $\mathcal{F}(\mathcal{L}) \subset \mathcal{C}(\mathcal{L})$.

Lemma 1.5.1. *[34] If $A \in C(L)$ and $B \in C(L)$, and if we set $\Delta_{B,A} = (L+B)^{-1}(A-B)$, then $\Delta_{B,A}$ is completely continuous on X and*
$$I + (L+B)^{-1}(N-B) = (I+\Delta_{B,A})[I+(L+A)^{-1}(N-A)].$$

Proof. It is easy to verify that $\Delta_{B,A}$ is completely continuous ($B \in \mathcal{C}(\mathcal{L})$, $(A-B)$ is L-completely continiou on X). We have
$$\begin{aligned} I \;&+\; (L+B)^{-1}(N-B) = I + (L+B)^{-1}(L+A)(L+A)^{-1}(N-A+A-B) \\ &= I + (L+B)^{-1}(L+B+A-B)(L+A)^{-1}(N-A+A-B) \\ &= I + [I+(L+B)^{-1}(A-B)](L+A)^{-1}(N-A) + (L+B)^{-1}(A-B) \\ &= (I+\Delta_{B,A})[I+(L+A)^{-1}(N-A)]. \end{aligned}$$
□

Definition 1.5.1. *[34] If $(F,\Omega) \in C_L$, the degree of F in Ω with respect to L is defined by*
$$D_L(F,\Omega) = D_I(I+(L+A)^{-1}(N-A),\Omega) = deg(I+(L+A)^1(N-A),\Omega,0),$$
for any $A \in \mathcal{C}_+(\mathcal{L})$. where
$\mathcal{C}_+(\mathcal{L})$ is the class containing the application A of the form $\pi_Q^{-1}\Lambda P$, and
$$\Lambda : Ker(L) \to Coker(L)$$
is an orientation preserving isomorphism and π_Q is the restriction to $Im(Q)$ of the canonical projection
$$\pi : Z \to CokerL.$$

1.5.1 The Leray- Schauder Continuation Theorem

Let X be a Banach space and $I = [0,1]$. If $A \subset X \times I$ and $\lambda \in I$, we shall write $A_\lambda = \{x \in X : (x,\lambda) \in A\}$. For $a \in X$ and $r > 0$, $B(a,r)$ will denote the open ball of center a and radius r. Let $\Omega \subset X \times I$ be a bounded open set with closure $\overline{\Omega}$ and boundary $\partial\Omega$ and let $F : \overline{\Omega} \to X$ be a mapping. We denote by Σ the (possibly empty) set defined by

$$\Sigma = \{(x,\lambda) \in \overline{\Omega} : x = F(x,\lambda)\}.$$

The following assumptions were introduced by Leray and Schauder in [67].

(H_0) $F : \overline{\Omega} \to X$ is completely continuous.

(H_1) $\Sigma \cap \partial\Omega = \varnothing$ (A priori estimate).

(H_2) Σ_0 is a finite nonempty set $\{a_1, \ldots, a_\mu\}$ and the corresponding topological degree $deg[I - F(.,0), \Omega_0, 0]$ is different from zero (Degree condition).

Theorem 1.5.2. *[73] If conditions (H_0), (H_1) and (H_2) hold, then Σ contains a continuum C along which λ takes all values in I.*

In other words, under the above assumptions, Σ contains a compact connected subset C connecting Σ_0 to Ω_1. In particular, the equation $x = F(x,1)$ has a solution in Ω_1. Notice that the conclusion of Theorem 1.5.2 still holds if the finiteness of the set Σ_0 is dropped from Assumption (H_2). Hence, from now on, we shall refer to Assumption (H_2) as being the condition

(H_2) $deg[I - F(.,0), \Omega_0, 0] \neq 0$ (Degree condition).

Conditions (H_0) and (H_2) are in general the easiest ones to check. Condition (H_1) requires the a priori knowledge of some properties of the solution set Σ and is in general very difficult to check.
An important special case can be stated as follows. Introduce the condition

(H_1') Σ is bounded (A priori bound).

Corollary 1.5.3. *[73] Assume that conditions (H_0), (H_1') and (H_2) hold. Then the conclusion of Theorem 1.5.2 is valid.*

The next results are an immediate consequence of the continuation theorem of Leray-Schauder.
Let X and Z be real normed vector spaces, $L : D(L) \subset X \to Z$ a linear Fredholm mapping of index zero, Γ open and bounded in $X \times [a,b]$, such that Γ_a is nonempty and bounded, where, for each $\lambda \in [a,b]$,

$$\Gamma_\lambda = \{x \in X : (x,\lambda) \in \Gamma\}.$$

Let $\mathcal{N} : \overline{\Gamma} \to Z$ be L-compact on Γ. Let

$$S = \{(x, \lambda) \in \overline{\Gamma} : x \in D(L) \text{ and } Lx + \mathcal{N}(x, \lambda) \neq 0\}$$

denote the solution set, and for each $\lambda \in [a, b]$, let $S_\lambda = \{x \in X : (x, \lambda) \in S\}$.

Corollary 1.5.4. *[34] If $(L + \mathcal{N}(.a), \Gamma_a) \in C_L$ and*

1. *$D_L(L + \mathcal{N}(.a), \Gamma_a) \neq 0$*
2. *$L + \mathcal{N}(x, \lambda) \neq 0$ for each $x \in (\partial\Gamma)_\lambda$ and each $\lambda \in]a, b]$.*

then there exists a closed connected subset Σ of S which connects $\Gamma_a \times \{a\}$ to $\Gamma_b \times \{b\}$.

Theorem 1.5.5. *[34] Let $(H, \Omega) \in C_L$ and $F = L + N$ with $N : \overline{\Omega} \to Z$ L-compact and Ω open and bounded in X. Assume that the following conditions are satisfied.*

(i) $\lambda Fx + (1 - \lambda)Hx \neq 0$ for each $(x, \lambda) \in (D(L) \cap \partial\Omega) \times]0, 1[$.

(ii) $D_L(H, \Omega) \neq 0$.

Then the equation $Lx + Nx = 0$ has at least one solution in $(D(L) \cap \overline{\Omega})$.

Proof. For each $x \in (D(L) \cap \overline{\Omega})$ and each $\lambda \in [0, 1]$, If $H = L + K$ with K L-compact in $\overline{\Omega}$,

$$\lambda Fx + (1 - \lambda)Hx = Lx + \lambda Nx + (1 - \lambda)Kx = Lx + \mathcal{N}(x, \lambda),$$

with $\mathcal{N} : \overline{\Omega} \to Z$ L-compact, as easily verified. If we take $\Gamma = \Omega \times [0, 1]$. We have Γ bounded and hence S is bounded too. Thus, either the equation $Lx + Nx = 0$ has a solution in $D(L) \cap \partial\Omega$ or the set Γ satisfies all the conditions of Corollary 1.5.4. $\square$

1.5.2 Coincidence Degree (Mawhin 1972)

Let X, Y be real normed spaces, $L : D(L) \subset X \to Y$ be a Fredholm mapping of index zero and Ω be an open bounded subset of X. Suppose that $F = L + N : D(L) \cap \overline{\Omega} \to Y$ is a mapping and $N : \overline{\Omega} \to Y$ is L-compact on $\overline{\Omega}$. Suppose also that $0 \notin F(D(L) \cap \partial\Omega)$. Let $J : Im(Q) \to Ker(L)$ be an isomorphism. Put

$$H^J_{PQ} = JQ + K_{PQ}.$$

It is easy to check that

$$H^J_{PQ}F = K_{PQ}L + H^J_{PQ}N = I - P + (JQ + K_{PQ})N.$$

Consequently, $0 \notin H^J_{PQ}F(D(L) \cap \partial\Omega)$ (if $0 \in H^J_{PQ}F(D(L) \cap \partial\Omega)$ then $K_{PQ}(Lx+Nx)+JQNx = 0$ for some $x \in D(L)\partial\Omega$, so $QNx = 0$ and $(I-Q)(Lx+Nx) = 0$. Thus $Lx + Nx = 0$, which is a contradiction). By the L-compactness of N, the

Leray Schauder degree $deg(I-P+(JQ+K_{PQ})N,\Omega,0)$ is well defined.
Now, we define a degree by

$$D_L(L+N,\Omega,0)=deg(I-P+(JQ+K_{PQ})N,\Omega,0),$$

which is called the coincidence degree of L and $-N$ on $\Omega\cap D(L)$. One can easily prove that this definition does not depend on the choice of P,Q.

Remark 1.5.1. *[81].*

(1) If $\dim(X)=\dim(Y)<+\infty$ and we take $L=0$, then any continuous mapping N on $\overline{\Omega}$ is L-compact. If we take $P=I$ and $Q=I$, then it follows that $K_{PQ}=0$, so $H^J_{PQ}F=JN$ and thus we have

$$D_L(T,\Omega,0)=deg(JN,\Omega,0)=sign(\det J)deg(N,\Omega,0).$$

Therefore, if we only take those J such that $detJ>0$, then we have $D_L(T,\Omega,0)=deg(N,\Omega,0)$, which is the Brouwer degree.

(2) If $X=Y$ and we take $L=I$, then any continuous compact mapping N on $\overline{\Omega}$ is L-compact. If we take $P=Q=0$, then $K_{PQ}=I, J=0:\{0\}\to\{0\}$ and $H^J_{PQ}F=I+N$. Thus

$$D_L(I+N,\Omega,0)=deg(I+N,\Omega,0),$$

which is the Leray Schauder degree.

Theorem 1.5.6. *[81] The coincidence degree of L and $-N$ on has the following properties:*

(1) If Ω_1 and Ω_2 are disjoint open subsets of Ω such that $0\notin F(D(L)\cap\Omega\setminus\Omega_1\cup\Omega_2)$, then

$$D_L(L+N,\Omega,0)=D_L(L+N,\Omega_1,0)+D_L(L+N,\Omega_2,0);$$

(2) If $H(t,x):[0,1]\times\overline{\Omega}\to Y$ is L-compact on $[0,1]\times\overline{\Omega}$ and $0\neq Lx+H(t,x)$ for all $(t,x)\in[0,1]\times\partial\Omega$, then $D_L(L+H(t,.),\Omega,0)$ does not depend on $t\in[0,1]$.

(3) If $D_L(L+T,\Omega,0)\neq 0$, then $0\in(L+T)(D(L)\cap\Omega)$.

Corollary 1.5.7. *[81] If T_1,T_2 are L-compact mappings on $\overline{\omega}$ and $T_1x=T_2x$ for all $x\in D(L)\cap\partial\Omega$, then*

$$D_L(L+T_1,\Omega,0)=D_L(L+T_2,\Omega,0).$$

Proposition 1.5.8. *[81] Let X,Y be real normed spaces, $L:D(L)\subset X\to Y$ be a Fredholm mapping of index zero, Y_0 be a finite dimensional subspace of Y satisfying $Y=Im(L)\oplus Y_0$ algebraically and Ω be an open bounded subset of X. If T is L-compact on $\overline{\Omega}\cap D(L)$, and $T(\overline{\Omega})\subset Y_0$, then*

$$D_L(L+T,\Omega,0)=signdet(J)deg(T,\Omega\cap Ker(L),0),$$

where $deg(.,.,.)$ is the Brouwer degree.

1.5.3 Continuation Theorems for $Lx = Nx$

In this part we will present an extension in the frame of coincidence degree theory, the well-known Leray-Shauder continuation theorem.
Let $L : D(L) \subset X \to Z$ be a Fredholm mapping of index zero and

$$N^* : \overline{\Omega} \times [0,1] \to Z$$

$$(x, \lambda) \mapsto N^*(x, \lambda),$$

be a L-compact mapping in $\overline{\Omega} \times [0,1]$, and let us write $N = N^*(.,1)$.
Let $y \in Im(L)$ and consider the family of equations

$$Lx = \lambda N^*(x, \lambda) + y \quad \lambda \in [0,1]. \tag{1.5.2}$$

Lemma 1.5.9. *[38] For each $\lambda \in]0,1]$, the set of solutions of equation (1.5.2) is equal to the set of solutions of equation*

$$Lx = QN^*(x, \lambda) + \lambda(I - Q)N^*(x, \lambda) + y, \tag{1.5.3}$$

and if $\lambda = 0$, every solution of (1.5.3) is a solution of (1.5.2).

Proof. If $\lambda \in]0,1]$, (1.5.2) is equivalent to

$$0 = QN^*(x, \lambda), \qquad Lx = \lambda(I - Q)N^*(x, \lambda) + y,$$

and hence to (1.5.3). If $\lambda = 0$, (1.5.3) is equivalent to

$$0 = QN^*(x, 0), \qquad Lx = y,$$

and the result is clear. □

Let

$$\pi : Z \to coker(L)$$

$$z \mapsto z + Im(L)$$

the canonical surjection.

Theorem 1.5.10. *([38],Generalized continuation theorem). Let L and N^* be like above and such that the following conditions are verified:*

(1) $Lx \neq \lambda N^*(x, \lambda) + y$ for every $x \in D(L) \cap \partial\Omega$ and every $\lambda \in]0,1[$;

(2) $\pi N^*(x, 0) \neq 0$ for every $x \in L^{-1}\{y\} \cap \partial\Omega$;

(3) $deg(\pi N^*(.,0) \mid_{L^{-1}\{y\}}, \Omega \cap L^{-1}\{y\}, 0) \neq 0$;

which this last number is the Brouwer degree at 0. *Then, for each* $\lambda \in [0, 1[$, *Equation (1.5.2) has at least one solution in* Ω *and equation*

$$Lx = Nx + y$$

has at least one solution in $\overline{\Omega}$.

Conditions (2) and (3) implying quotient spaces can seem difficult to verify in practice. The next result is an equivalent form which avoids introduction of quotient spaces.

Corollary 1.5.11. *[38] Suppose that condition (1) of Theorem 1.5.10 holds and that for some continuous projector* $Q : Z \to Z$ *such that* $Ker(Q) = Im(L)$ *and some isomorphism* $J : Im(Q) \to Ker(L)$ *one has*

(2') $QN^*(x, 0) \neq 0$ *for every* $x \in L^{-1} \cap \partial\Omega$;

(3') $deg(JQN^*(. + K_P y)\mid_{Ker(L)}, (-K_P y + \Omega) \cap Ker(L), 0) \neq 0$,

then the conclusion of Theorem 1.5.10 holds.

Remark 1.5.2. *[38] (1) if* $y = 0$, *(2') and (3') become respectively:*

(2") $QN^*(x, 0) \neq 0$ *for every* $x \in \Omega \cap Ker(L)$;

(3") $deg(JQN^*(., 0)\mid_{Ker(L)}, \Omega \cap Ker(L), 0) \neq 0$.

Corollary 1.5.12. *[81] Let* $N : \overline{\Omega} \to Z$ *be L-compact. Suppose that the following conditions hold:*

(1) $Lx - \lambda N$ *for all* $(\lambda, x) \in (0, 1) \times (D(L) \setminus Ker(L)) \cap \partial\Omega$,

(2) $Nx \notin Im(L) = 0$ *for all* $x \in Ker(L) \cap \partial\Omega$,

(3) $deg(QN_{Ker(L)}, \Omega \cap Ker(L), 0) \neq 0$, *where* $Q : Z \to Q$ *is the projection such that* $Ker(Q) = Im(L)$.

Then $Lx = Nx$ *has a solution in* $D(L) \cap \overline{\Omega}$.

The above continuation theorems are generalized in [70] and extended in [39] to the case when L is nonlinear operator in view of its application to problems involving $p-$Laplacian like operators.

Let X and Z be two real Banach spaces with norm $\|.\|_X$ and $\|.\|_Z$ respectively. A continuous operator $M : X \cap domM \to Z$ is said to be quasi-linear if

(i) $dimKerM = dimM^{-1}(0) = n < \infty$;

(ii) $ImM = M(X \cap domM)$ is a closed subset of Z.

Let $X = X_1 \oplus X_2$ and $Z = Z_1 \oplus Z_2$, where $X_1 = KerM$, $Z_2 = ImM$ and X_2, Z_1 are respectively the complementary spaces of X_1 in X, Z_2 in Z. Assume that $dimX_1 = dimZ_1$ then we can define $P : X \to X_1$ and $Q : Z \to Z_1$ as the corresponding orthogonal projectors such that

$$ImP = KerM, \quad KerQ = ImM.$$

Denoted by $J : Z_1 \to X_1$ a homeomorphism with $J(0) = 0$.

Let Ω be a bounded open subset of X, with $0 \in \Omega$ such that $domM \cap \Omega \neq \emptyset$, and consider a parameter family of perturbation (generally) nonlinear $N_\lambda : [0,1] \times \bar{\Omega} \to Z$ with $N_1 = N$. The continuous operator N_λ is said to be $M-$compact in $\bar{\Omega}$ with respect to M if there is an operator $K : ImM \to X_2$ with $K(0) = 0$ such that for $\lambda \in [0,1]$,

$$(I-Q)N_\lambda(\bar{\Omega}) \subset ImM,$$

$$(I-Q)N_0 = 0, \qquad QN_\lambda x = 0 \Leftrightarrow QNx = 0, \quad \lambda \in (0,1),$$

$$KM = I-P, \qquad K(I-Q)N_\lambda : \bar{\Omega} \to X_2 \quad \textit{is compact},$$

$$M[P + K(I-Q)N_\lambda] = (I-Q)N_\lambda.$$

We introduce the intermediate map

$$F(\lambda,.) = P + K(I-Q)N + JQN, \tag{1.5.4}$$

which is clearly compact under the above assumptions.
consider the abstract equation

$$Mx = N_\lambda x, \qquad \lambda \in (0,1]. \tag{1.5.5}$$

Lemma 1.5.13. *([39]) Let X and Z be Banach spaces, $\Omega \subset X$ an open and bounded nonempty set, M a quasi-linear operator and N_λ a $M-$compact operator in $\bar{\Omega}$. Then (1.5.5) has a solution $x \in \bar{\Omega}$ [resp $x \in \partial\Omega$] if and only if $x \in \bar{\Omega}$ [resp $x \in \partial\Omega$] is a fixed point of $F(\lambda,.)$ defined in (1.5.4).*

1.6 Some Fixed Point Theorems

Theorem 1.6.1 (Banach's fixed point theorem (1922) [32]). *Let C be a non-empty closed subset of a Banach space X, then any contraction mapping T of C into itself has a unique fixed point.*

Theorem 1.6.2 (**Schaefer's** fixed point theorem [32]). *Let X be a Banach space, and $N : X \longrightarrow X$ completely continuous operator.*
If the set $\mathcal{E} = \{y \in X : \ y = \lambda Ny, \ forsome \ \lambda \in (0,1)\}$ is bounded, then N has fixed points.

Theorem 1.6.3 (Darbo's Fixed Point Theorem [40, 32]). *Let X be a Banach space and C be a bounded, closed, convex and nonempty subset of X. Suppose a continuous mapping $T : C \to C$ is such that for all closed subsets D of C,*

$$\alpha(T(D)) \le k\alpha(D), \tag{1.6.1}$$

where $0 \le k < 1$, and α is the Kuratowski measure of noncompactness. Then T has a fixed point in C.

Remark 1.6.1. *Mappings satisfying the Darbo-condition (1.6.1) have subsequently been called k-set contractions.*

Theorem 1.6.4 (Mönch's Fixed Point Theorem [5, 77]). *Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv}N(V) \quad or \quad V = N(V) \cup \{0\} \Rightarrow \alpha(V) = 0$$

holds for every subset V of D, then N has a fixed point.
Here α is the Kuratowski measure of noncompactness.

The next lemma and theorem concern the existence of positive solutions.

Definition 1.6.1. *Let $(E, \|.\|)$ be a real Banach space. A nonempty, closed, convex set $C \subset E$ is said to be a cone provided the following are satisfied:*

(a) if $y \in C$ and $\lambda \ge 0$, then $\lambda y \in C$;

(b) if $y \in C$ and $-y \in C$, then $y = 0$.

We say that C is a solid cone if C° is not empty, where C° is the interior of C.

Every cone $C \subset E$ induces a partial ordering "$\le$" on E defined by

$$x \le y \quad iff \quad y - x \in C.$$

Definition 1.6.2. *Let C be a solid cone of a real Banach space E and $N : C^\circ \to C^\circ$ be an operator.*
We say that N is called an α-concave operator(-α-convex operator), if

$$N(tx) \ge t^\alpha Nx \quad (N(tx) \le t^{-\alpha} Nx) \quad for\ any\ x \in C^\circ\ and\ 0 < t < 1,$$

where $0 \le \alpha < 1$.

Lemma 1.6.5. *[44] Suppose N is a normal solid cone of a real Banach space, $N : C^\circ \to C^\circ$ is α-concave increasing(or α-convex decreasing) operator. Then N has only one fixed point in C°.*

Theorem 1.6.6 (Krasnosel'skii twin fixed point theorem [55]). *Let E be a Banach space, $C \subset E$ a cone of E, and $R > 0$ a constant. Let $C_R = \{y \in C : \|y\| < R\}$ and let $N : C_R \to C$ be a completely continuous operator, where $0 < r < R$. If*

(A_1) $\|N(y)\| \leq \|y\|$ for all $y \in \partial C_r$;

(A_2) $\|N(y)\| \geq \|y\|$ for all $y \in \partial C_R$.

Then N has at least two fixed points y_1, y_2, in C_R. Furthermore, $\|y_1\| < r$, $r < \|y_2\| \leq R$.

1.7 Perov's Fixed Point Theorem

Definition 1.7.1. *Let X be a nonempty set. By a vector-valued metric on X we mean a map $d : X \times X \to \mathbb{R}^n_+$ with the following properties:*

(i) $d(u, v) \geq 0$ for all $u, v \in X$; if $d(u, v)$ then $u = v$;

(ii) $d(u, v) = d(v, u)$ for all $u, v \in X$;

(iii) $d(u, v) \leq d(u, w) + d(w, v)$ for all $u, v, w \in X$.

We call the pair (X, d) a generalized metric space with $d(x, y) := \begin{pmatrix} d_1(x, y) \\ \cdots \\ d_n(x, y) \end{pmatrix}$.

Notice that d is a generalized metric space on X if and only if $d_i,\ i = 1, \ldots, n$ are metrics on X.

For $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_+$, we will denote by

$$B(x_0, r) = \{x \in X : d(x_0, x) < r\}$$

the open ball centered in x_0 with radius r and

$$\overline{B(x_0, r)} = \{x \in X : d(x_0, x) \leq r\}$$

the closed ball centered in x_0 with radius r. We mention that for generalized metric space, the notation of open subset, closed set, convergence, Cauchy sequence and completeness are similar to those in usual metric spaces.

Definition 1.7.2. *A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc i.e. $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $det(M - \lambda I) = 0$, where I denote the unit matrix of $\mathcal{M}_{n\times n}(\mathbb{R})$.*

Theorem 1.7.1 ([93], pages 12,88). *Let $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$. The following assertions are equivalent:*

(i) *M is convergent towards zero;*

(ii) $M^k \to 0$ *as* $k \to \infty$;

(iii) *The matrix* $(I - M)$ *is nonsingular and*

$$(I - M)^{-1} = I + M + M^2 + \ldots + M^k + \ldots,$$

(iv) *The matrix* $(I - M)$ *is nonsingular and* $(I - M)^{-1}$ *has nonnegative elements.*

Definition 1.7.3. *Let* (X, d) *be a generalized metric space. An operator* $N : X \to X$ *is said to be contractive if there exists a convergent to zero matrix* M *such that*

$$d(N(x), N(y)) \leq Md(x, y) \text{ for all } x, y \in X.$$

For $n = 1$ *we recover the classical Banach's contraction fixed point result.*

Definition 1.7.4. *We say that a non-singular matrix* $A = (a_{ij})_{1\leq i,j\leq n} \in \mathcal{M}_{n\times n}(\mathbb{R})$ *has the absolute value property if*

$$A^{-1}|A| \leq I,$$

where

$$|A| = (|a_{ij}|)_{1\leq i,j\leq n} \in \mathcal{M}_{n\times n}(\mathbb{R}_+).$$

Some examples of matrices convergent to zero $A \in \mathcal{M}_{n\times n}(\mathbb{R})$, which also satisfies the property $(I - A)^{-1}|I - A| \leq I$ are:

1) $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ and $\max(a, b) < 1$

2) $A = \begin{pmatrix} a & -c \\ 0 & b \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ and $a + b < 1,\ c < 1$

3) $A = \begin{pmatrix} a & -a \\ b & -b \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ and $|a - b| < 1,\ a > 1, b > 0.$

Theorem 1.7.2. *[85] Let* (X, d) *be a complete generalized metric space and* $N : X \to X$ *a contractive operator with Lipschitz matrix* M*. Then* N *has a unique fixed point* x_* *and for each* $x_0 \in X$ *we have*

$$d(N^k(x_0), x_*) \leq M^k(I - M)^{-1}d(x_0, N(x_0)) \text{ for all } k \in \mathbb{N}.$$

1.8 Multi-Valued Analysis

Let (X, d) be a metric space and Y be a subset of X. Denote by

- $\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\}$.
- $\mathcal{P}_p(X) = \{Y \in \mathcal{P}(X) : Y$ has the property "p"$\}$ where p could be: *cl*=closed, *b*=bounded, *cp*=compact, *cv*=convex, etc. Thus,
- $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$.
- $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$.
- $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$.
- $\mathcal{P}_{cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ convex}\}$ where X is a Banach space.
- $\mathcal{P}_{cv,cp}(X) = \mathcal{P}_{cv}(X) \cap \mathcal{P}_{cp}(X)$.

Let (X, d_*) be a metric space, we will denote by H_{d_*} the Hausdorff pseudo-metric distance on $\mathcal{P}(X)$, defined as

$$H_{d_*}\colon\ \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\},\ H_{d_*}(A, B) = \max\left\{\sup_{a\in A} d_*(a, B),\ \sup_{b\in B} d_*(A, b)\right\}.$$

where $d_*(A, b) = \inf_{a\in A} d_*(a, b)$ and $d_*(a, B) = \inf_{b\in B} d_*(a, b)$. Then $(\mathcal{P}_{b,cl}(X), H_{d_*})$ is a metric space and $(\mathcal{P}_{cl}(X), H_{d_*})$ is a generalized metric space. In particular, H_{d_*} satisfies the triangle inequality.

Consider the generalized Hausdorff pseudo-metric distance

$$H_d\colon\ \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}^n_+ \cup \{\infty\}$$

defined by

$$H_d(A, B) := \begin{pmatrix} H_{d_1}(A, B) \\ \cdots \\ H_{d_n}(A, B) \end{pmatrix}.$$

Definition 1.8.1. *Let (X, d) be a generalized metric space. A multivalued operator $N\colon X \to \mathcal{P}_{cl}(X)$ is said to be contractive if there exists a matrix $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ such that*

$$M^k \to 0 \quad as \quad k \to \infty$$

and

$$H_d(N(u), N(v)) \leq M d(u, v), \quad \text{for all } u, v \in X.$$

Let (X, d) and (Y, ρ) be two metric spaces and $F : X \to \mathcal{P}(Y)$ be a multi-valued mapping. Then F is said to be *lower semi-continuous* (*l.s.c.*) if the inverse image of V by F

$$F^{-1}(V) = \{x \in X \,:\, F(x) \cap V \neq \emptyset\}$$

is open for any open set V in Y. Equivalently, F is *l.s.c.* if the core of V by F

$$F^{+1}(V) = \{x \in X \,:\, F(x) \subset V\}$$

is closed for any closed set V in Y.

Likewise, the map F is called *upper semi-continuous* (*u.s.c.*) on X if for each $x_0 \in X$ the set $F(x_0)$ is a nonempty, closed subset of X, and if for each open set N of Y containing $F(x_0)$, there exists an open neighborhood M of x_0 such that $F(M) \subseteq Y$. That is, if the set $F^{-1}(V)$ is closed for any closed set V in Y. Equivalently, F is *u.s.c.* if the set $F^{+1}(V)$ is open for any open set V in Y.

The mapping F is said to be *completely continuous* if it is *u.s.c.* and, for every bounded subset $A \subseteq X$, $F(A)$ is relatively compact, i.e., there exists a relatively compact set $K = K(A) \subset X$ such that

$$F(A) = \bigcup\{F(x) : x \in A\} \subset K.$$

Also, F is *compact* if $F(X)$ is relatively compact, and it is called *locally compact* if for each $x \in X$, there exists an open set U containing x such that $F(U)$ is relatively compact.

We denote the graph of F to be the set $Graph(F) = \{(x, y) \in X \times Y,\ y \in F(x)\}$, and we recall the following facts.

Definition 1.8.2. *A multivalued map $F : [a, b] \to \mathcal{P}(Y)$ is said measurable if for every open $U \subset Y$, the set*

$$F_+^{-1}(U) = \{x \in Y : F(x) \subset U\}$$

is Lebesgue measurable.

Definition 1.8.3. *A multi-map F is called a Carathéodory function if*
(a) the multi-map $t \mapsto F(t, x)$ is measurable for each $x \in X$;
(b) for a.e. $t \in J$, the map $x \mapsto F(t, x)$ is upper semi-continuous.
Furthermore, F is L^1-Carathéodory if it is further locally integrably bounded, i.e., for each positive r, there exists $h_r \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t, x)\|_{\mathcal{P}} \leq h_r(t), \quad \textit{for a.e. } t \in J \textit{ and all } |x| \leq r.$$

Lemma 1.8.1. ([25, 41]) *The multivalued map $F : [a, b] \to \mathcal{P}_{cl}(Y)$ is measurable if and only if for each $x \in Y$, the function $\zeta : [a, b] \to [0, +\infty)$ defined by*

$$\zeta(t) = \mathrm{dist}(x, F(t)) = \inf\{\|x - y\| \,:\, y \in F(t)\}, \quad t \in [a, b],$$

is Lebesgue measurable.

The following two lemmas are needed. The first one is the celebrated Kuratowski-Ryll-Nardzewski selection theorem.

Lemma 1.8.2. *([41], Theorem 19.7) Let Y be a separable metric space and $F : [a,b] \to \mathcal{P}(Y)$ a measurable multi-valued map with nonempty closed values. Then F has a measurable selection.*

Lemma 1.8.3. *[66] Let I be a compact interval and E be a Banach space. Let F be an L^1-Carathéodory multi-valued map with $S_{F,y} \neq \emptyset$, and let Γ be a linear continuous mapping from $L^1(I,E)$ to $C(I,E)$. Then, the operator*

$$\Gamma \circ S_F : C(I,E) \longrightarrow \mathcal{P}_{cp,c}(E), \quad y \longmapsto (\Gamma \circ S_F)(y) = \Gamma(S_{F,y}),$$

is a closed graph operator in $C(I,E) \times C(I,E)$, where $S_{F,y}$ is known as the selectors set from F and given by

$f \in S_{F,y} = \{f \in L^1(I,E) : f(t) \in F(t,y(t)) \ \ for\ a.e.t \in I\}$.

Lemma 1.8.4. *[66] Let I be a compact interval and E be a Banach space. Let F be an L^1-Carathéodory multi-valued map with $S_{F,y} \neq \emptyset$, and let Γ be a linear continuous mapping from $L^1(I,E)$ to $C(I,E)$. Then, the operator*

$$\Gamma \circ S_F : C(I,E) \longrightarrow \mathcal{P}_{cp,c}(E), \quad y \longmapsto (\Gamma \circ S_F)(y) = \Gamma(S_{F,y}),$$

is a closed graph operator in $C(I,E) \times C(I,E)$, where $S_{F,y}$ is known as the selectors set from F and given by

$f \in S_{F,y} = \{f \in L^1(I,E) : f(t) \in F(t,y(t)) \ \ for\ a.e.t \in I\}$.

Lemma 1.8.5. *(See e.g. [13], Theorem 1.4.13). If $G : X \to \mathcal{P}_{cp}$ is u.s.c, then for any $x_0 \in X$,*

$$\lim_{x \to x_0} \sup G(x) = G(x_0).$$

Lemma 1.8.6. *(See e.g. [13], Lemma 1.1.9). Let $(k_n)_{n\in\mathbb{N}} \subset k \subset X$ be a sequence of subsets where K is compact in the separable Banach space X. Then*

$$\overline{co}(\lim_{n\to\infty} \sup k_n) = \bigcap_{N>0} \overline{co}(\bigcup_{n\geq N} k_n),$$

where $\overline{co}A$ refers to the closure of the convex hull of A.

Lemma 1.8.7. *[13] Every semi-compact sequence $L^1([0,b],E)$ is weakly compact in $L^1([0,b],E)$.*

Lemma 1.8.8. *(Mazur's Lemma, [79], Theorem 21.4). Let E be a normed space and $x_{k k\in\mathbb{N}} \subset E$ be a sequence weakly converging to a limit $x \in E$. Then there exists a sequence of convex combinations $y_m = \sum_{k=1}^{k=m} \alpha_{mk} x_k$ with $\alpha_{mk} > 0$ for $k = 1, 2, ..., m$ and $\sum_{k=1}^{k=m} \alpha_{mk} = 1$, which converges strongly to x.*

Theorem 1.8.9. *[82] Let (X,d) be a complete generalized metric space and $F : X \to \mathcal{P}_{cl,b}(X)$ a contractive multivalued operator with Lipschitz matrix M. Then N has at least one fixed point.*

Theorem 1.8.10. *[82] Let (X,d) be a complete generalized metric space and $F : X \to \mathcal{P}_{cl}(X)$ be a multivalued map. Assume that there exist $A, B, C \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ such that*

$$H_d(F(x), F(y)) \leq Ad(x,y) + Bd(y, F(x)) + Cd(x, F(x)))$$

where $A + C$ converge to zero. Then there exist $x \in X$ such that $x \in F(x)$.

Theorem 1.8.11. *[82] Let $(X, \|\cdot\|)$ be a generalized Banach space and $F : X \to \mathcal{P}_{cp,cv}(X)$ be a completely continuous multivalued mapping and u.s.c. Moreover assume that the set*

$$\mathcal{A} = \{x \in X : x \in \lambda F(x) \quad \text{for some } \lambda \in (0,1)\}$$

is bounded. Then F has a fixed point.

1.9 Semigroup of Linear Operator

Definition 1.9.1. *A one-parameter family $S(t)$ for of bounded linear operators on a Banach space X is a C_0-semigroup (or strongly continuous) on X if*

(i) $S(t) \circ S(s) = S(t+s)$, for $t, s \geq 0$, (semigroup property),

(ii) $S(0) = I$, (the identity on X);

(iii) the map $t \to S(t)x$ is strongly continuous, for each $x \in X$, i.e;

$$\lim_{t\to 0} S(t)(x) = x, \forall x \in X.$$

Remark 1.9.1. *A semigroup of bounded linear operators $(S(t))_{t\geq 0}$ is uniformly continuous if*

$$\lim_{t\to 0} \|S(t) - I\| = 0.$$

Here I denotes the identity operator in E. A strongly continuous semigroup of bounded linear operators on X will be called a semigroup of class C_0 or simply a C_0 semigroup. If only (i) and (ii) are satisfied we say that the family $(S(t))_{t\geq 0}$ of bounded linear operators is a semigroup.

Definition 1.9.2. *Let $S(t)$ be a semigroup of class (C_0) defined on X. The infinitesimal generator A of $S(t)$ is the linear operator defined by*

$$A(x) = \lim_{h\to 0} = \frac{S(h)(x) - x}{h}, \text{ for } x \in D(A),$$

where $D(A) = \{x \in X \mid \lim_{h\to 0} = \dfrac{S(h)(x) - x}{h}$ *exists in* $X\}$.

Let us recall the following property:

Theorem 1.9.1. *[84] If $S(t)$ is a C_0-semigroup, then there exist $\omega \geq 0$ and $M \geq 1$ such that*

$$\|S(t)\|_{B(E)} \leq M\exp(\omega t), \text{ for } 0 \leq t < \infty \tag{1.9.1}$$

Proof. We show first there is $\eta \in (0, 1]$ such that

$$\sup_{t\in[0,\eta]} \|S(t)\| < +\infty.$$

Assume the contrary, i.e $\forall \eta = \frac{1}{n} \in (0, 1]$ with $n \in \mathbb{N} : \sup_{t\in[0,\eta]} \|S(t)\| = +\infty$. It follows that

$$\forall n \in \mathbb{N},\ \exists t_n \in [0, \tfrac{1}{n}] \text{ such that } \sup_{n\geq 1} \|S(t_n)\| = +\infty.$$

By uniform boundedness principle $\exists x \in X : \sup_{n\geq 1} \|S(t_n)x\| = +\infty$ that $\|S(t_n)x\|$ is unbounded.
On the other hand $\forall x \in X, \mathbb{R} \ni t \to S(t)x \in X$ is continuous at 0; that is $\forall \epsilon > 0$, $\exists \delta > 0$: $|t| < \delta \Rightarrow \|S(t)x - x\| < \epsilon$.
In particular, let $\epsilon = 1$.
Then,

$$\|S(t)x - x\| < 1.$$

Hence we obtain the estimates:

$$\|S(t)x\| - \|x\| \leq |\|S(t)x\| - \|x\|| \leq \|S(t)x - x\| < 1.$$

This implies that

$$\|S(t)x\| \leq 1 + \|x\|.$$

But one has $0 \leq t_n \leq \frac{1}{n}$ and then $t_n \to 0$ as $n \to +\infty$ i.e take $\epsilon = \delta$,

$$\exists n_0 \in \mathbb{N} : |t_n| < \delta;\ \forall n > n_0,$$

then

$$\|S(t_n)x\| \leq 1 + \|x\|,\ n > n_0;$$

it follows that

$$\sup_{n\geq n_0} \|S(t_n)x\| \leq 1 + \|x\|, n > n_0. \tag{1.9.2}$$

Now let $n = 1, 2, ..., n_0 - 1$ there is only a finite number of $S(t_n)x$.
Let $M^* = \max \|S(t_n)x\|, n = 1, 2, ..., n_0 - 1$. Then for these,

$$\sup \|S(t_n)x\| \leq M^* \text{ for } n = 1, 2, ..., n_0 - 1. \tag{1.9.3}$$

So from (1.9.2) and (1.9.3) we have $\sup_{n\geq 1}\|S(t_n)x\| \leq 1+\|x\|+M^*$.
Hence we get the contradiction,
Thus,

$$\exists \eta \in (0,1]: \sup_{t\in[0,\eta]}\|S(t)\| < +\infty.$$

Let $M := \sup_{t\in[0,\eta]}\|S(t)\|$, since $\|S(0)\| = 1$ then $M \geq 1$.
Let $\omega = \eta^{-1}\log M$. Given $t \geq 0$ with $t > \eta$ we have $t = n(t)\eta + \delta$, where $0 \leq \delta < \eta$ and $n(t) \in \mathbb{N}$.
By semigroup property

$$\begin{aligned}
\|S(t)\| &= \|S(\eta)^{n(t)}S(\delta)\| \\
&\leq \|S(\eta)^{n(t)}\|\|S(\delta)\| \\
&\leq MM^{n(t)} = MM^{\frac{t-\delta}{\eta}} \\
&\leq MM^{\frac{t}{\eta}} = M\exp(\omega\eta\frac{t}{\eta}) = M\exp(\omega t).
\end{aligned}$$

This completes the proof. □

Remark 1.9.2. *If, $M = 1$ and $\omega = 0$, i.e; $\|S(t)\|_{B(E)} \leq 1$, for $t \geq 0$, then the semigroup $S(t)$ is called a contraction semigroup (C_0)*

Theorem 1.9.2. *If $(S(t))_{t\geq 0}$ is a C_0 semigroup then $t \to S(t)x$ is continuous, for each $x \in X$ is continuous from $\mathbb{R}^+$ (the positive real line) into X.*

Proof. Let $t_0 > 0$, $x \in X$.
We want to show that $\lim_{t\to t_0} S(t)(x) = S(t_0)x$.
Case 1: $t > t_0$

$$\begin{aligned}
S(t)(x) - S(t_0)x &= S(t_0)[S(t-t_0)x - x] \\
\|S(t)(x) - S(t_0)x\| &\leq \|S(t_0)\|\|S(t-t_0)x - x\| \to 0 \text{ as } t \to t_0.
\end{aligned}$$

Therefore, $\lim_{t\to t_0^+} S(t)(x) = S(t_0)x$.
Case 2: $t < t_0$

$$\begin{aligned}
\|S(t)(x) - S(t_0)x\| &= S(t)[S(t_0-t)x - x] \\
\|S(t)(x) - S(t_0)x\| &\leq \|S(t)\|\|S(t_0-t)x - x\| \to 0 \text{ as } t \to t_0.
\end{aligned}$$

Therefore, $\lim_{t\to t_0^-} S(t)(x) = S(t_0)x$.

□

Theorem 1.9.3. *Let $S(t)_{t\geq 0}$ be a C_0 semigroup and A be its infinitesimal generator. Then*

(a) *For* $x \in X$,

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} S(s)xds = S(t)x. \quad (1.9.4)$$

(b) *For* $x \in X$, $\int_0^t S(s)xds \in D(A)$ *and*

$$A(\int_0^t S(s)xds) = S(t)x - x. \quad (1.9.5)$$

(c) *For* $x \in D(A)$, $S(t) \in D(A)$ *and*

$$\frac{d}{dt} S(t)(x) = A(S(t)(x)) = S(t)(A(x)). \quad (1.9.6)$$

(d) *For* $x \in D(A)$

$$S(t)x - S(s)x = \int_s^t S(\tau)Axd\tau = \int_s^t AS(\tau)xd\tau. \quad (1.9.7)$$

$$\lim_{t \to 0} S(t)(x) = x, \, \forall x \in X.$$

Proof. (a) Let $x \in X$ and $h > 0$; let's write the estimates

$$\begin{aligned} \|\frac{1}{h} \int_t^{t+h} S(s)xds - S(t)x\| &= \|\frac{1}{h} \int_t^{t+h} S(s)xds - \frac{1}{h} \int_t^{t+h} S(t)xds\| \\ &\leq \frac{1}{h} \int_t^{t+h} \|S(s)x - S(t)x\|ds. \quad (1.9.8) \end{aligned}$$

Changing the variable, set $u + t = s$, $du = ds$; if $s = u$ then $u = 0$ and $s = t + h$ then $u = h$.

$$\frac{1}{h} \int_t^{t+h} S(s)xds\|S(s)x - S(t)x\|ds = \frac{1}{h} \int_0^h \|S(t+u)x - S(t)x\|du.$$

Since u is a dummy variable one can write

$$\frac{1}{h} \int_t^{t+h} S(s)xds\|S(s)x - S(t)x\|ds = \frac{1}{h} \int_0^h \|S(s+t)x - S(t)x\|ds.$$

Since $t \mapsto S(t)x$ is a continuous function from $\mathbb{R}^+$ to X i.e, given $\epsilon > 0$, $\exists \delta > 0$ such that $|t - t_0| < \delta$ then $\|S(t)x - S(t_0x\| < \epsilon$. Take $h = t_0 - t$, we can write the continuity of $t \mapsto S(t)x$ equivalentely as follows given $\epsilon > 0$, $\exists \delta > 0$ such that $|h| < \delta$ then $\|S(t)x - S(t+h)x\| < \epsilon$.

$$\frac{1}{h} \int_0^h \|S(s+t)x - S(t)x\|ds < \frac{1}{h} \int_0^h \epsilon ds = \epsilon.$$

It is then natural to write

$$\frac{1}{h}\int_t^{t+h}\|S(s)x-S(t)x\|ds=\frac{1}{h}\int_0^h\|S(s+t)x-S(t)x\|ds<\frac{1}{h}\int_0^h\epsilon ds=\epsilon.$$

Therefore

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h}S(s)xds=S(t)x.$$

(b) Let $x\in X$ and $h>0$;

$$\begin{aligned}\frac{S(h)-I}{h}\int_0^t S(s)xds &= \frac{1}{h}\int_0^t S(s+h)xds-\frac{1}{h}\int_0^t S(s)xds\\ &= \frac{1}{h}\int_h^{t+h}S(s)xds-\frac{1}{h}\int_0^t S(s)xds.\end{aligned}$$

In the right hand side we have set for the first integral $u=s+h$; $du=ds$; if $s=0$ then $u=h$ and if $s=t$ then $u=t+h$.

$$\begin{aligned}\frac{1}{h}\int_h^{t+h}S(s)xds-\frac{1}{h}\int_0^t S(s)xds &= \frac{1}{h}\int_h^t S(s)xds+\frac{1}{h}\int_t^{t+h}S(s)xds-\frac{1}{h}\int_0^t S(s)xds\\ &= \frac{1}{h}\int_h^0 S(s)xds+ac1h\int_0^t S(s)xds\\ &\quad +\frac{1}{h}\int_t^{t+h}S(s)xds-\frac{1}{h}\int_0^t S(s)xds\\ &= \frac{1}{h}\int_t^{t+h}S(s)xds-\frac{1}{h}\int_0^t S(s)xds.\end{aligned}$$

and letting $h\to 0$ the right-hand side tends to $S(t)x-x\in X$, which proves (b).

(c) Let $x\in D(A)$ and $h>0$; then

$$\begin{aligned}\frac{S(h)-I}{h}S(t)x &= \frac{S(t+h)-S(t)}{h}x\\ &= S(t)\frac{S(h)-I}{h}x\to S(t)A(x)\text{ as }\ h\to 0.\end{aligned}\tag{1.9.9}$$

Thus, $S(t)x\in D(A)$ and $AS(t)x=S(t)x$. (1.9.9) implies also that

$$\frac{d+}{dt}S(t)x=AS(t)x=S(t)Ax,$$

i.e the right derivative of $S(t)x$ is $S(t)Ax$, to prove (1.9.6) we have to show that for $t>0$ the left derivative of $S(t)x$ exists and equals $S(t)Ax$.
This follows from,

$$\lim_{h\to 0}[\frac{S(t)x-S(t-h)x}{h}-S(t)x]=\lim_{h\to 0}S(t-h)[\frac{S(h)x-x}{h}-S(t)x]+\lim_{h\to 0}(S(t-h)Ax-S(t)Ax)$$

and the fact that both terms on the right-hand side are zero, the first since $x \in D(A)$ and $\|S(t-h)\|$ is bounded on $0 \leq h \leq t$ and the second by continuity of $S(t)$. This conclude the proof of (c).

(d) Integrating (1.9.6) from s to t we obtain (d). □

Corollary 1.9.4. *If A is the infinitesimal generator of a C_0 semigroup $(S(t)_{t\geq 0})$ then $D(A)$ the domain of A, is dense in X and A is closed linear operator.*

Proof. Let $x \in X$, set $x_t = \frac{1}{t}\displaystyle\int_0^t S(t)xds$. By part (c) of Theorem 1.9.3, $x_t \in D(A)$ for $t > 0$ and by part (a) of the same theorem $x_t \to x$ as $t \to 0$. Thus $\overline{D(A)} = X$.
Let $(x, y) \in \overline{A}$ then there exist $(x_n)_{n\geq 1} \subset D(A)$ such that $(x_n, Ax_n) \to (x, y)$ i.e $x_n \to x$ and $Ax_n \to y$.
By part (b) of Theorem 1.9.3, we have

$$S(t)x_n - x_x = \int_0^t S(s)Ax_n ds \tag{1.9.10}$$

Claim: $\displaystyle\int_0^t S(s)Ax_n ds \to \int_0^t S(s)yds$ uniformly on bounded interval. Let $t \in [0, a]$ with $a > 0$;

$$\begin{aligned}\|\int_0^t S(s)Ax_n ds - \int_0^t S(s)yds\| &\leq \int_0^t \|S(s)(Ax_n - y\|ds \\ &\leq \int_0^t \|S(s)\|\|Ax_n - y\|ds \\ &\leq Me^{\omega t}\|Ax_n - y\|.\end{aligned}$$

Since $Ax_n \to y$, it follows that

$$\lim_{n\to+\infty} \sup_{t\in[0,a]} \|\int_0^t S(s)Ax_n ds - \int_0^t S(s)yds\| = 0,$$

therefore our claim is true. Using the previous claim and letting $n \to +\infty$ in (1.9.10) yields

$$S(t)x - x = \int_0^t S(s)ydy \tag{1.9.11}$$

Dividing (1.9.11) by $t > 0$ and letting $t \to 0$, we see, using part (a) of Theorem 1.9.3 that $x \in D(A)$ and $Ax = y$. □

Theorem 1.9.5. *A linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator.*

Proof. (a) It is known that the series $\sum_{n=0}^{\infty}\frac{(tA)^n}{n!}$ in norm for every $t \geq 0$ and defines for each such t a bounded linear operator $S(t)$. It is easy to see that

- $S(0) = I$,
- S(t+s)=S(t)S(s), for all $t, s \geq 0$,
- $$\begin{aligned} e^{tA} &= \sum_{n=0}^{\infty}\frac{(tA)^n}{n!} = I + \sum_{n=1}^{\infty}\frac{(tA)^n}{n!} \\ e^{tA} - I &= tA\sum_{n=1}^{\infty}\frac{(tA)^{n-1}}{(n-1)!}. \end{aligned}$$

Taking the norm of both side,one has

$$\begin{aligned} \|e^{tA} - I\| &\leq \|tA\|\|\sum_{n=1}^{\infty}\frac{(tA)^{n-1}}{(n-1)!}\| \\ &\leq |t|\|A\|\|\sum_{n=1}^{\infty}\frac{(tA)^{n-1}}{(n-1)!}\| \\ &\leq |t|\|A\|e^{t\|A\|}. \end{aligned}$$

$\|e^{tA} - I\| \leq t\|A\|e^{t\|A\|}$ which goes to 0 as t goes to 0. Now, we claim that A is the infinitesimal generator of $S(t)$. Let us prove our claim, let $t > 0$. We have

$$\begin{aligned} e^{tA} - I &= tA\sum_{n=1}^{\infty}\frac{(tA)^{n-1}}{(n-1)!} \\ \frac{e^{tA} - I}{t} &= A\sum_{n=1}^{\infty}\frac{(tA)^{n-1}}{(n-1)!} \\ \frac{e^{tA} - I}{t} - A &= A\Big[\sum_{n=1}^{\infty}\frac{(tA)^{n-1}}{(n-1)!} - I\Big]. \end{aligned}$$

Taking the norm of both side, one has

$$\begin{aligned} \|\frac{e^{tA} - I}{t} - A\| &\leq \|A\|\|\sum_{n=1}^{\infty}\frac{(tA)^{n-1}}{(n-1)!} - I\|. \\ &\leq \|A\|\|\sum_{n=1}^{\infty}\frac{(tA)^{n-1}}{(n-1)!} - I\| = \|A\|\|e^{tA} - I\|. \end{aligned}$$

That is $\|\frac{e^{tA}-I}{t} - A\| \leq \|A\|\|S(t) - I\|$. Which implies as $t \to 0^+$ that $\lim_{t\to 0^+} \dfrac{e^{tA}-I}{t} = A$. We have have established that $S(t)$ is a uniformly continuous semgroup of bounded linear operators on X and that A is its infinitesimal generator.
(b) Let $S(t)$ be a C_0 semigroup of bounded linear operator on X.
Fix $\rho > 0$, small enough such that $\|I - \rho^{-1}\displaystyle\int_0^\rho S(s)ds\| \leq 1$ this implies that $\rho^{-1}\displaystyle\int_0^\rho S(s)ds$ is invertible and therefore $\displaystyle\int_0^\rho S(s)ds$ is invertible.

Now, let $h > 0$,

$$h^{-1}(S(h) - I)\int_0^\rho S(s)ds = h^{-1}\int_0^\rho S(h+s)ds - h^{-1}\int_0^\rho S(s)ds$$

and therefore

$$\begin{aligned} h^{-1}\big(S(h) - I\big)\int_0^\rho S(s)ds &= h^{-1}\Big(\int_0^\rho S(s+h)ds - \int_0^\rho S(s)ds\Big) \\ &= h^{-1}\Big(\int_h^{h+\rho} S(s)ds - \int_0^\rho S(s)ds\Big) \end{aligned}$$

$h^{-1}(S(h) - I) = h^{-1}\Big(\displaystyle\int_h^{h+\rho} S(s)ds - \int_0^\rho S(s)ds\Big)\Big(\int_0^\rho S(s)ds\Big)^{-1}$ and letting $h \to 0$ it follows that $h^{-1}(S(h) - I)$ converges in norm to a bounded linear operator $(S(\rho - I)\Big(\displaystyle\int_0^\rho S(s)ds\Big)^{-1}$ which is the infinitesimal generator of $S(t)$. □

Theorem 1.9.6. *Let $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ be two C_0 semigroup on X, generated respectively by A and B. If $A = B$ then $T(t) = S(t)$, $t \geq 0$.*

Proof. Assume $A = B$ and let $x \in D(A) = D(B)$.
Define $\alpha(s) := T(t-s)S(s)x$, $s \in [0, t]$.
From Theorem 1.9.3 part (c) it follows that α is differentiable and that
$\alpha'(s) = \frac{d}{ds}T(t-s)S(s)x = -T(t-s)AS(s)x + T(t-s)BS(s)x = 0$, since $A = B$.
It follows $\alpha(s) = constant$. In particular, its values at $s = 0$ and $s = t$ are the same that is $T(t)x = S(s)x$ $\forall x \in D(A)$. By Corollary 1.9.4 $D(A)$ is dense in X and $T(t)$, $S(s)$ are closed;
therefore $T(t)x = S(s)x$; $\forall x \in X$.

□

For more details see [5, 9, 41, 43, 44, 55, 62, 98]

Chapter 2

The Basic Theory of Retarded Functional Differential Equations

2.1 Introdution

In applications, the future behavior of many phenomena are assumed to be described by the solutions of an ordinary differential equation. Implicit in this assumption is that the future behavior is uniquely determined by the present and independent of the past. In differential difference equations, or more generally functional differential equations, the past exerts its influence in a significant manner upon the future. Many models are better represented by functional differential equations, than by ordinary differential equations.

Example 2.1.1. *A retarded functional differential equation. Imagine a biological population composed of adult and juvenile individuals. Let $N(t)$ denote the density of adults at time t. Assume that the length of the juvenile period is exactly h units of time for each individual. Assume that adults produce offspring at a per capita rate α and that their probability per unit of time of dying is μ. Assume that a newborn survives the juvenile period with probability ρ and put $t = \alpha\rho$. Then the dynamics of N can be described by the differential equation*

$$\frac{dN}{dt}(t) = -\mu N(t) + rN(t-h) \tag{2.1.1}$$

which involves a nonlocal term, $rN(t-h)$ meaning that newborns become adults with some delay. So the time variation of the population density N involves the current as well as the past values of N. Such equations are called Retarded Functional Differential Equations (RFDE) or, alternatively, Delay Equations. Equation (2.1.1) describes the changes in N. To determine a solution past time $t = 0$, we need to prescribe the value of N at time $-h$ and we can see that it is not enough to give the value at the point $-h$, since the following example agree that this condition is not enough to determine completely the solution.

In this chapter, we introduce a general class of basic theory of retarded functional differential equations. The basic theory of existence, uniqueness, continuation and continuous dependence will be developed.

2.2 A General Initial Value Problem

Suppose $r \geq 0$ is a given real number, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^n$ is an $n-$dimensional linear vector space over the real with norm $|.|$, $C([a,b], \mathbb{R}^n)$ is the Banach space of continuous functions mapping the interval $[a,b]$ into $\mathbb{R}^n$ with the topology of uniform convergence. If $[a,b] = [-r, 0]$ we let $C = C([-r,0], \mathbb{R}^n)$ and designate the norm of an element φ in C by $\varphi = \sup_{-r \leq \theta \leq 0} |\varphi(\theta)|$. Even though single bares are used for norms in different spaces, no confusion should arise. If

$$\sigma \in \mathbb{R},\ A > 0 \text{ and } x \in C([\sigma - r, \sigma + A], \mathbb{R}^n),$$

then for any $t \in [\sigma, \sigma + A]$, we let $x_t \in C$ be defined by $x_t(\theta) = x(t+\theta)$, $-r \leq \theta \leq 0$. If D is a subset of $\mathbb{R} \times C$, $f : D \to \mathbb{R}^n$ is a given function and "$'$" represent the right-hand derivative, we say that the relation

$$x'(t) = f(t, x_t) \tag{2.2.1}$$

is a retarded functional differential equation on D and will denote this eqution **RFDE**. If we wish to emphasize that the equation is defined by f, we write **RFDE**(f).
Let $f : \mathbb{R} \times C \to \mathbb{R}^n$ be a given function. A functional differential equation is given by the following relation

$$\begin{cases} x'(t) = f(t, x_t), & \text{for } t \geq \sigma \\ x_\sigma = \varphi. \end{cases} \tag{2.2.2}$$

Definition 2.2.1. *x is said to be a solution of (2.2.2) if there are $\sigma \in \mathbb{R}$, $A > 0$ such that $x \in C([\sigma - r, \sigma + A], \mathbb{R}^n)$ and x satisfies (2.2.2) for $t \in [\sigma - r, \sigma + r]$. In such a case we say that x is a solution of (2.2.2) on $[\sigma - r, \sigma + r]$ for a given $\sigma \in \mathbb{R}$ and a given $\varphi \in C$ we say that $x = x(\sigma, \varphi)$, is a solution of (2.2.2) with initial value at σ or simply a solution of (2.2.2) through (σ, φ) if there is an $A > 0$ such that $x(\sigma, \varphi)$ is a solution of (2.2.2) on $[\sigma - r, \sigma + r]$ and $x_\sigma(\sigma, \varphi) = \varphi$.*

Equation (2.2.2) is a very general type of equation and includes differential difference equations of the type $(r = 0)$

$$x'(t) = f(t, x(t), x(t - r(t)) \text{ for } 0 \leq r(t) \leq r$$

as well as

$$x'(t) = \int_{-r}^{0} g(t, \theta, x(t+\theta))d\theta.$$

If

$$f(t,\varphi) = L(t,\varphi) + h(t),$$

in which L is linear in φ and $(t,\varphi) \to L(t,\varphi)$, we say that the equation is a linear delay differential equation, it is called homogeneous if $h = 0$.
If $f(t,\varphi) = g(\varphi)$, Equation (2.2.2) is an autonomous one.

Lemma 2.2.1. *Let $\sigma \in \mathbb{R}$ and $\varphi \in C$ be given and f be continuous on the product $\mathbb{R} \times C$. Then, finding a solution of Equation (2.2.2) through (σ,φ) is equivalent to solving:*

$$x(t) = \varphi(0) + \int_{\sigma}^{t} f(s, x_s)ds, \quad t \geq \sigma \text{ and } x_\sigma = \varphi.$$

2.2.1 Existence

Lemma 2.2.2. *If $x \in C([\sigma - r, \sigma + \alpha], \mathbb{R}^n)$, then, x_t is a continuous function of t for $t \in [\sigma - r, \sigma + \alpha]$.*

Proof. Since x is continuous on $[\sigma - r, \sigma + \alpha]$, it is uniformly continuous and thus $\forall \varepsilon > 0$, $\exists \delta > 0$, such that $|x(t) - x(s)| < \varepsilon$ if $|t - s| < \delta$.
Consequently for t, s in $[\sigma, \sigma + \alpha]$, $|t - s| < \delta$, we have $|x(t + \theta) - x(s + \theta)| < \varepsilon$, $\forall \theta \in [-r, 0]$. This proves the lemma. □

To bring out idea in the proof of existence as well as the results of subsequent sections, it is convenient to introduce some notation and to prove a few technical lemmas. For any $(\sigma,\varphi) \in \mathbb{R} \times C$, let $\widetilde{\varphi} \in C([\sigma - r, \infty], \mathbb{R}^n)$ be defined by

$$\widetilde{\varphi}_\sigma = \varphi, \quad \widetilde{\varphi}(t + \sigma) = \varphi(0), \quad t \geq 0. \tag{2.2.3}$$

Suppose x is a solution of Equation (2.2.1) through (σ,φ). If $x(t+\sigma) = \widetilde{\varphi}(t+\sigma)+y(t)$, $t \geq -r$, then Lemma 2.2.1 implies y satisfies

$$y(t) = \int_{0}^{t} f(\sigma + s, \widetilde{\varphi}_{s+\sigma} + y_s)ds, \quad t \geq 0 \text{ and } y_0 = 0. \tag{2.2.4}$$

Conversely, if y is a solution of this equation, then one obtains a solution x of Equation (2.2.1) by the above transformation. Therefore finding a solution of (2.2.1) is equivalent to finding an $\alpha > 0$ and a function $y \in C([-r, \alpha], \mathbb{R}^n)$ such that Equation (2.2.4) is satisfied for $0 \leq t \leq \alpha$.
If V is a subset of $\mathbb{R}\times C$, then $C(V, \mathbb{R}^n)$ is the class of all function $f : V \to \mathbb{R}^n$ which are continuous and $C^0(V, \mathbb{R}^n) \subseteq C(V, \mathbb{R}^n)$ is the subset of bounded, continuous functions from V to $\mathbb{R}^n$. The space $C^0(V, \mathbb{R}^n)$ becomes a Banach space with the norm

$$|f|_V = \sup_{(t,\varphi)\in V} |f(t,\varphi)|. \tag{2.2.5}$$

For any real α and β define

$$I_\alpha = [0,\alpha], \quad B_\beta = \{\psi \in C : |\psi| \le \beta\},$$
$$\mathcal{A}(\alpha,\beta) = \{y \in C([-r,\alpha],\mathbb{R}^n) : y_0 = 0, y_t \in B_\beta, t \in I_\alpha\}. \tag{2.2.6}$$

Lemma 2.2.3. *Suppose $\Omega \subseteq \mathbb{R} \times C$ is open, $W \subseteq \Omega$ is compact and $f^0 \in C(\Omega,\mathbb{R}^n)$ is given. Then there exists a neighborhood $V \subseteq \Omega$ of W such that $f^0 \in C^0(V,\mathbb{R}^n)$, there exists a neighborhood $U \subseteq C^0(V,\mathbb{R}^n)$ of f^0 and positive constants M, α, and β such that*

$$|f(\sigma,\varphi)| \le M \quad \text{for} \quad (\sigma,\varphi) \in V \ \text{and} \ f \in U. \tag{2.2.7}$$

Also, for any $(\sigma^0,\varphi^0) \in W$, we have $(\sigma^0+t, y_t+\widetilde{\varphi}^0_{\sigma^0+t}) \in V$ for $t \in I_\alpha$ and $y \in \mathcal{A}(\alpha,\beta)$.

Proof. Since W is compact and f^0 is continuous, there is a constant M such that $|f^0(\sigma^0,\varphi^0)| \le M$ for $(\sigma^0,\varphi^0) \in W$. For the same reason, there are positive constants $\bar{\alpha}$, $\bar{\beta}$, ε such that

$$|f^0(\sigma^0+t,\varphi^0+\psi)| < M-\varepsilon \text{ for } (\sigma^0,\varphi^0) \in W \text{ and } (t,\psi) \in I_{\bar{\alpha}} \times B_{\bar{\beta}}.$$

If $V = \{(\sigma^0+t,\varphi^0+\psi), (\sigma^0,\varphi^0) \in W, (t,\psi) \in I_{\bar{\alpha}} \times B_{\bar{\beta}}\}$, then $f^0 \in C^0(V,\mathbb{R}^n)$ and there is a neighborhood $U \subseteq C^0(V,\mathbb{R}^n)$ of f^0 such that Inequality (2.2.7) is satisfied. To prove the last assertion of the lemma, suppose $0 < \beta < \bar{\beta}$ and choose α so that $\beta < \bar{\beta}$ and $|\widetilde{\varphi}^0_{\sigma^0+t} - \varphi^0| < \bar{\beta}-\beta$ for all $(\sigma^0,\varphi^0) \in W$, $t \in I_\alpha$. Since W is compact, this last choose is possible. Therefore, $|\widetilde{\varphi}^0_{\sigma^0+t} - \varphi^0| < \beta+\bar{\beta}-\beta = \bar{\beta}$ for $y \in \mathcal{A}(\alpha,\beta)$. From the manner in which V was constructed, the proof of the lemma is complete. □

The next lemma will be used to apply fixed point theorems for existence and continuous dependence of solutions of Equation (2.2.2).

Lemma 2.2.4. *Suppose $\Omega \subseteq \mathbb{R} \times C$ is open, $W \subseteq \Omega$ is compact and $f^0 \in C(\Omega,\mathbb{R}^n)$ is given, and the neighborhoods U and V and constants M, α, and β are the ones obtained from Lemma 2.2.3. If*

$$T : W \times U \times \mathcal{A}(\alpha,\beta) \to C([-r,\alpha],\mathbb{R}^n)$$
$$T(\sigma,\varphi,f,y)(t) = 0, \quad t \in [-r,0],$$
$$T(\sigma,\varphi,f,y)(t) = \int_0^t f(\sigma+s,\widetilde{\varphi}_{\sigma+s}+y_s)ds, \quad t \in I_\alpha,$$

then T is continuous and there is a compact set K in $C([-r,\alpha],\mathbb{R}^n)$ such that

$$T : W \times U \times \mathcal{A}(\alpha,\beta) \to K.$$

Furthermore, if $M\alpha \le \beta$, then

$$T : W \times U \times \mathcal{A}(\alpha,\beta) \to \mathcal{A}(\alpha,\beta).$$

Proof. It is clear that $T : W \times U \times \mathcal{A}(\alpha,\beta) \to C([-r,\alpha],\mathbb{R}^n)$. Also, Relation (2.2.7) implies

$$|T(\sigma,\varphi,f,y)(t) - T(\sigma,\varphi,f,y)(\tau)| \leq M|t-\tau|$$
$$|T(\sigma,\varphi,f,y)(t)| \leq M\alpha$$

for all $t,\tau \in I_\alpha$. If

$$K = \{g \in C([-r,\alpha],\mathbb{R}^n) : |g(t)-g(\tau)| \leq M|t-\tau|, |g(t)| \leq M\alpha\},$$

then K is compact, $T : W \times U \times \mathcal{A}(\alpha,\beta) \to K$. If $M\alpha \leq \beta$, then $K \subseteq \mathcal{A}(\alpha,\beta)$.
It remains only to show that T is continuous. Suppose $(\sigma^k,\varphi^k,f^k,y^k) \in W \times U \times \mathcal{A}(\alpha,\beta)$ and $(\sigma^k,\varphi^k,f^k,y^k) \to (\sigma^0,\varphi^0,f^0,y^0) \in W \times U \times \mathcal{A}(\alpha,\beta)$ as $k \to \infty$. Since $T(\sigma^k,\varphi^k,f^k,y^k) \in K$ and K is compact, there is a subsequence which we designate with the same symbol and a $\gamma \in K$ such that

$$T(\sigma^k,\varphi^k,f^k,y^k) \to \gamma \quad \text{as } k \to \infty.$$

Since

$$f^k(\sigma^k+s,\varphi^k_{\sigma^k+s}+y^k_s) \to f^0(\sigma^0+s,\varphi^0_{\sigma^0+s}+y^0_s) \tag{2.2.8}$$

for all $s \in I_\alpha$ and all of these functions are uniformly bounded by Lemma 2.2.3, the Lebesgue dominated convergence theorem implies

$$\begin{aligned}\gamma(t) &= \lim_{k\to\infty}\int_0^t f^k(\sigma^k+s,\varphi^k_{\sigma^k+s}+y^k_s)ds \\ &= \int_0^t f^0(\sigma^0+s,\varphi^0_{\sigma^0+s}+y^0_s)ds \\ &= T(\sigma^0,\varphi^0,f^0,y^0)\end{aligned}$$

for all $t \in I_\alpha$. This implies of any convergent subsequence is independent of subsequence. But since every subsequence has a convergent subsequence, this obviously implies the sequence itself converges. Therefore, T is continuous and the lemma is proved. □

Lemma 2.2.5. *(Schauder fixed-point theorem)[49]. If U is a closed bounded convex subset of a Banach space X and $T : U \to U$ is completely continuous, then T has a fixed point in U.*

Theorem 2.2.6. *Let D be an open subset of $R \times C$ and $f : D \to \mathbb{R}^n$ be continuous. For any $(\sigma,\varphi) \in D$, there exists a solution of Equation (2.2.2) through (σ,φ).*
More generally, if $W \subseteq \Omega$ is compact and $f^0 \in C(\Omega,\mathbb{R}^n)$ is given. Then there is a neighborhood $V \subseteq \Omega$ of W such that $f^0 \in C^0(V,\mathbb{R}^n)$, there is a neighborhood $U \subseteq C^0(V,\mathbb{R}^n)$ of f^0 such that, for any $(\sigma,\varphi) \in W$, $f \in U$, there is a solution $x(\sigma,\varphi,f)$ of the ***RFDE***(f) *through (σ,φ) which exists on $[-\sigma-r,\sigma+\alpha]$.*

Proof. For the first part, take $W = \{(\sigma, \varphi)\}$, a single point. Lemma 2.2.4 and Schauder's fixed point theorem imply that $T(\sigma, \varphi, f^0, .)$ has a fixed point in $\mathcal{A}(\alpha, \beta)$ since $\mathcal{A}(\alpha, \beta)$ is a closed bounded convex set of $C([-r, \alpha], \mathbb{R}^n)$. This yields a solution of $\mathbf{RFDE}(f^0)$ by Formula (2.2.4) with f replaced by f^0. To obtain the last statement of the theorem, simply apply the same reasoning but use the general form of Lemma 2.2.4. □

Proposition 2.2.7. *If f is at most affine i.e. $|f(t, \phi| \leq a|\phi| + b$ with $a, b > 0$, then there exists a global solution i.e.$\forall \varphi$ the solution $x(\sigma, \varphi)$ is defined on $[\alpha, \infty)$.*

Proof. Let $\varphi \in C$, and assume that the solution is defined only on $[\alpha, \beta)$. By integrating the Equation (2.2.2), one has

$$x(t) = \varphi(0) + \int_0^t f(s, x_s)ds$$

which gives

$$x(t, \varphi) \leq |\varphi| + \int_\sigma^t (a|x_s| + b)ds$$

and

$$x(., \varphi) \leq |\varphi| + b\beta + a \int_\sigma^t (|x_s| + b)ds.$$

By the Gronwall lemma

$$x(., \varphi) \leq (|\varphi| + b\beta) \exp^{a\beta} < \infty.$$

On the other hand

$$\sup_{t \in [0, \beta)} |x'(t)| < \infty,$$

and gives that the solution is uniformly continuous on $[0, \beta)$ and this implies that $\lim\limits_{t \to \beta^-} |x_t(., \varphi)|$ exists and is finite, note it x_β.
Let us consider the following delay differential equation

$$\begin{cases} y'(t) = f(t, y_t), & \text{for } t \geq \beta \\ y_\beta = x_\beta \in C \end{cases}$$

this last equation has at least one solution on $[\beta, \beta + \varepsilon]$ for some $\varepsilon > 0$ and equation (2.2.2) has at least one solution defined on $[\beta, \beta + \varepsilon]$, which contradicts the maximality of the solution. □

2.2.2 Uniqueness

Theorem 2.2.8. *[11] Let D be an open subset of $\mathbb{R} \times C$ and suppose that $f : D \to \mathbb{R}^n$ be continuous and $f(t, \varphi)$ be Lipschitzian with respect to φ in every compact subset of D. If $(\sigma, \varphi) \in D$, then Equation (2.2.2) has a unique solution passing through (σ, φ).*

Proof. Define I_α and B_β as in Equation (2.2.6) and suppose x and y be two solutions of (2.2.2) on $[\sigma - r, \sigma + \alpha]$ with $x_\alpha = \varphi = y_\alpha$. Then

$$\begin{aligned} x(t) - y(t) &= \int_\sigma^t (f(s, x_s) - f(s, y_s))ds. \quad t \geq \sigma \\ x_\sigma - y_\sigma &= 0. \end{aligned}$$

Let k be the Lipschitz constant of $f(t, \varphi)$ in a compact subset containing the trajectories (t, x_t) and (t, y_t), $t \in I_\alpha$. Choose $\bar{\alpha}$ such that $k\bar{\alpha} < 1$.
Then, for $t \in I_{\bar{\alpha}}$ one has:

$$|x(t) - y(t)| \leq \int_\sigma^t k|x_s - y_s|ds \leq k\bar{\alpha} \sup_{\sigma \leq s \leq t} |x_s - y_s|.$$

And this implies that $x(t) = y(t)$ for $t \in I_\alpha$. □

Remark 2.2.1. *The uniqueness may not hold if the function is not locally Lipschitzian.*

Example 2.2.1. *There may be two distinct solutions of (2.2.2) defined on* $(-\infty, +\infty)$ *and they coincide on* $(0, +\infty)$ *. The following example was given by A. Hausrath.*
Let $r = 1$, $f(s) = 0$, $0 \leq s \leq 1$, $f(s) = -3(\sqrt[3]{s} - 1)^2$, $s > 1$ *and consider the equation*

$$x'(t) = f(|x_t|).$$

The function $x \equiv 0$ *is a solution of this equation on* $(-\infty, +\infty)$ *. Also, the function* $x(t) = -t^3$, $t < 0$ *and* $x(t) =\equiv 0$, $t \geq 0$, $x'(t) = -3t^2$*. In fact, since* $x \leq 1$ *for* $t \geq -1$*, it is clear that* x *satisfies the equation for* $t \geq 0$*. Since* x *is monotone decreasing for* $t \leq 0$, $|x_t| = x(t-1) = -(t-1)^3$ *and* $x'(t) = -3t^2$*. It is easy to verify that*

$$-3t^2 = f((t-1)^3)$$

for $t < 0$*.*

2.2.3 Continuation of Solutions

Definition 2.2.2. *Suppose* f *in Equation (2.2.2) is continuous. If* x *is a solution of Equation (2.2.2) on an interval* $[\sigma, a]$, $a > \sigma$*, we say* $\widehat{x}$ *is a continuation of* x *if there is a* $b > a$ *such that* $\widehat{x}$ *is defined on* $[\sigma - r, b]$*, coincides with* x *on* $[\sigma - r, b]$*, and satisfies Equation (2.2.2) on* $[\sigma, b]$*. A solution* x *is noncontinuable if no such continuation exists; that is the interval* $[\sigma, b]$ *is the maximal interval of existence of the solution* x*.*

Theorem 2.2.9. *Furthermore on the hypotheses of the Theorem 2.2.8, if* f *is a bounded function, then Equation (2.2.2) has a maximal solution defined on* $[-r, \beta)$ *with*

$$\text{if } \beta < \infty \Rightarrow \overline{\lim_{t \to \beta^-}} |x_t(., \varphi)| = \infty.$$

Proof. By steps, we can integrate the Equation (2.2.2), let $[-r, \beta)$, be the maximal interval on which $x(., \varphi)$ is defined. Assume that $\lim_{t\to\beta^-} |x_t(., \varphi)| < \infty$ then there exists N such that . $|x_t(., \varphi)| \leq N$, $\forall t \in [0, \beta)$, with $x'(t) = f(t, x(t))$ and from the boundedness of f, we have

$$\sup_{t\in[0,\beta)} |x'(t)| < \infty,$$

then x is uniformly continuous on $[0, \beta)$. So $\lim_{t\to\beta^-} |x_t(., \varphi)|$ exists, which we denote by x_β. Let $\psi \in C([-r, \beta), \mathbb{R}^n)$ defined by $\psi = x_\beta$, under the existence theorem, there exists $\varepsilon > 0$ such that the equation

$$\begin{cases} y'(t) = f(t, y_t), & \text{for } t \geq \beta \\ y_\beta = x_\beta \in C \end{cases}$$

has at least one solution on $[\beta, \beta+\varepsilon]$, the recollement of x and y gives a solution defined on $[\alpha, \beta + \varepsilon]$, which contradicts the maximality of x. □

Theorem 2.2.10. *Let D be an open subset of $\mathbb{R} \times C$ and suppose that $f : D \to \mathbb{R}^n$ be continuous and $f(t, \varphi)$ be Lipschitzian with respect to φ. In every compact subset of D. If $(\sigma, \varphi) \in D$, then, the application $\varphi \to x_t(., \varphi)$ is continuous Lipschitz.*

Proof. From the Theorem 2.2.8 and the fact that f is Lipschitzian with respect to the second variable

$$|f(t, \varphi)| \leq k|\varphi| + |f(t, 0)|.$$

Let $\varphi_1, \varphi_2 \in C$ and $x(., \varphi_1)$, $x(., \varphi_2)$ the associated solutions, one has

$$\begin{aligned} |x(t, \varphi_1) - x(t, \varphi_2)| &\leq |\varphi_1 - \varphi_2| + k \int_0^t |f(s, x(s, \varphi_1) - f(s, x(s, \varphi_2)|ds. \\ & \quad |\varphi_1 - \varphi_2| + k \int_0^t |x(s, \varphi_1) - x(s, \varphi_2)|ds. \end{aligned}$$

By the Gronwalls lemma, one has

$$|x(t, \varphi_1) - x(t, \varphi_2)| \leq |\varphi_1 - \varphi_2| \exp(kt).$$

□

2.2.4 Differentiability of Solutions

In this subsection some results are given on the differentiability with respect to (σ, φ, f). If Ω is an open set in $\mathbb{R} \times C$, let $C^p(\Omega, \mathbb{R}^n)$, $p > 0$ designate the space of functions taking Ω into $\mathbb{R}^n$ that have bounded continuous derivatives up through order p with respect to φ in Ω.

Lemma 2.2.11. *[49] If U is a closed subset of a Banach space X, V is a subset of a Banach space Y, $T : U \times V \to U$ is a uniform contraction, and T is continuous, then the unique fixed point $x(v)$ of $T(.,v)$ in U is continuous in V.*
Furthermore, if U,V are the closure of open sets U^0, V^0 and $T(x,v)$ has continuous first derivative in x,v, then $x(v)$ has a continuous first derivative with respect to v. The same conclusion holds for higher derivatives.

Theorem 2.2.12. *If $f \in C^p(\Omega, \mathbb{R}^n)$, $p \geq 1$, then the solution (σ, φ, f) of the (2.2.2) through (σ, φ) is unique and continuously differentiable with respect to (φ, f) for t in any compact set in the domain of definition of (σ, φ, f). Furthermore, for each $t \geq \sigma$, the derivative of x with respect to φ, $D_\varphi x(\sigma, \varphi, f)(t)$ is a linear operator from C to $\mathbb{R}^n$, $D_\varphi x(\sigma, \varphi, f)(\sigma) = I$, the identity, and $D_\varphi x(\sigma, \varphi, f)\psi(t)$ for each ψ in C satisfies the linear variational equation*

$$y'(t) = D_\varphi f(t, x_t(\sigma, \varphi, f)y_t. \tag{2.2.9}$$

Also, for each $D_\varphi x(\sigma, \varphi, f)(t)$ is a linear operator from $C^p(\Omega, \mathbb{R}^n)$ to $\mathbb{R}^n$, $D_f x(\sigma, \varphi, f)(\sigma) = 0$ and $D_f x(\sigma, \varphi, f)(\sigma)g(t)$ for each g in $C^p(\Omega, \mathbb{R}^n)$ satisfies the nonhomogeneous variation equation

$$z'(t) = D_\varphi f(t,_t(\sigma, \varphi, f)z_t + g(t,_t(\sigma, \varphi, f)). \tag{2.2.10}$$

Proof. Since $p \geq 1$, it follows from Theorem 2.2.8 that the solution $x = x(\sigma, \varphi, f)$ of **RFDE**(f) through $(\sigma,, \varphi)$ is unique. Let the maximal interval of existence of x be $[\sigma - r, \sigma + \omega)$ and fix $b < \omega$.
Our first objective is to show that $x(\sigma, \varphi, f)(t)$ is continuously differentiable with respect to φ on $[\sigma - r, \sigma + b]$. There is an open neighborhood U of φ such that $x(\sigma, \psi, f)(t)$, $\psi \in U$, is defined for $t \in [\sigma - r, \sigma + b]$. If $W = \{(t, x_t) : t \in [\sigma, \sigma + b]\}$, then W is compact. Using the notation of Subsection 2.2.1, we can determine M, α, β, U and V as in Lemma 2.2.3. Chose α so that $M\alpha \leq \beta$ and $k\alpha < 1$, where k is a bound of the derivative of f with respect to φ on Ω. If $x(t + \sigma) = \widetilde{\varphi}(t + \sigma) + y(t)$, $t \in I_\alpha$, then y is a fixed point of the operator $T(\sigma, \varphi, f)$ of Lemma 2.2.4. On the other hand, the restriction on α, β implies that $T(\sigma, \varphi, f)$ takes $\mathcal{A}(\alpha, \beta)$ into itself for each α, β and is a contraction. Furthermore, the contraction constant is independent of $(\sigma, \varphi, f) \in V \times U$. Since the mapping $T(\sigma, \varphi, f)$ is easily shown to be continuously differentiable in Ω, it follows from Lemma 2.2.11 that the fixed point $y = y(\sigma, \varphi, f)$ is continuously differentiable in Ω. The same proof shows that $x(\sigma, \varphi, f)(t)$ is continuously differentiable in f for $[\sigma, \sigma + \alpha]$. Using the fact that the basic interval $[\sigma - r, \sigma + b]$ is compact, one completes the proof of the differentiability.
Knowing that $x(\sigma, \varphi, f)$ is continuously with respect to φ, f, one can use the integral equation for x to easily obtain Formulas (2.2.9) and (2.2.10). □

For more details see [[11], [49]]

Chapter 3

Neutral Differential Equations with State-Dependent Delays

3.1 Introduction

In this chapter, on $J := [0, T]$, we study the existence and uniqueness of solutions for neutral differential equations with state-dependent delays of the following form,

$$\frac{d}{dt}(x(t) - g(t, x(t - \eta(t)))) = A(x(t) - g(t, x(t - \eta(t)))) + f(t, x_t, x(t - \tau(t, x_t))),\ t \in J, \tag{3.1.1}$$

with initial condition

$$x(t) = \varphi(t),\ t \in [-r, 0], \tag{3.1.2}$$

where A generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ on a Banach space E, $f : J \times C([-r, 0], E) \times E \to E$, $g : J \times E \to E$ are given functions, and $\varphi : [-r, 0] \to E$, $\tau : [0, T] \times C([-r, 0], E) \to [0, r]$ and $\eta : J \to [0, r]$ are also given continuous functions.

This chapter is organized as follows: in Section 3.2, we give one of our main existence results for solutions of (3.1.1)-(3.1.2), with the proof based on Banach's fixed point theorem 1.6.1. In Section 3.3, we give two other existence results for solutions of (3.1.1)-(3.1.2). Their proofs involve the measure of noncompactness paired in one result with a Mönch fixed point theorem 1.6.4 and paired in the other result with a Darbo fixed point theorm 1.6.3.

3.2 Uniqueness of Mild Solutions

In this section we give our main existence result for problem (3.1.1)-(3.1.2). Before stating and proving this result, we give the definition of its mild solution.

Definition 3.2.1. *We say that a continuous function $x : [-r, T] \to E$ is a mild solution of problem (3.1.1), (3.1.2) if $x(t) = \varphi(t),\ t \in [-r, 0]$ and*

$$\begin{aligned} x(t) &= S(t)[\varphi(0) - g(0, x(-\eta(0)))] + g(t, x(t - \eta(t))) \\ &\quad + \int_0^t S(t-s) f(s, x_s, x(s - \tau(s, x_s)))ds,\ t \in J. \end{aligned}$$

Set $C := C([-r, 0], E)$.

Lemma 3.2.1. *(See [54]) Let $a > 0,\ b \geq 0,\ r_1 > 0,\ r_2 \geq 0,\ r = \max\{r_1, r_2\}$, and $v : [0, \sigma] \to [0, \infty)$ be continuous and nondecreasing. Let $u : [-r, \sigma] \to [0, \infty)$ be continuous and satisfy the inequality*

$$u(t) \leq v(t) + bu(t - r_1) + a \int_0^t u(s - r_2)ds,\ t \in [0, \sigma].$$

Then $u(t) \leq d(t)e^{ct}$ for $t \in [0, \sigma]$, where c is the unique positive solution of $cbe^{-cr_1} + ae^{-cr_2} = c$, and

$$d(t) = \max\Big\{\frac{v(t)}{1 - be^{-cr_1}}, \max_{-r \leq s \leq 0} e^{-cs}u(s)\Big\},\ t \in [0, \sigma].$$

Let $\Omega_1 \in C$, $\Omega_2 \in E$ and $\Omega_3 \in E$ be open subsets of their respective spaces. Let $T > 0$ be finite, or $T = \infty$, in which case $[0, T]$ denotes the interval $[0, \infty)$.
We define the set

$$\Pi = \{\varphi \in C :\ \varphi \in \Omega_1,\ \varphi(-\tau(0, \varphi)) \in \Omega_2,\ \varphi(-\eta(0)) \in \Omega_3\}.$$

Let us introduce the following hypotheses:

(H_1) A is the generator of a strongly continuous semigroup $S(t), t \in J$ which is compact for $t > 0$ in the Banach space E. Let $M > 0$ be such that

$$\|S(t)\| \leq M \quad \text{for all } t \in J.$$

(H_2) (i) $f : J \times \Omega_1 \times \Omega_2 \to E$ is continuous;
(ii) $f(t, \psi, u)$ is locally Lipschitz continuous in ψ and u in the following sense: for every finite $\sigma \in (0, T]$, for every closed and bounded subset $M_1 \subset \Omega_1$ of C and closed and bounded subset $M_2 \subset \Omega_2$ of E, there exists a constant $L_1 > 0$ such that

$$\|f(t, \psi_1, u_1)) - f(t, \psi_2, u_2))\| \leq L_1(\sup_{\zeta \in [-r, -r_0]} \|\psi_1(\zeta) - \psi_2(\zeta)\| + \|u_1 - u_2\|),$$

for every $t \in [0, \sigma]$, $\psi_1,\ \psi_2 \in M_1$ and $u_1,\ u_2 \in M_2$,

(H_3) (i) $g : J \times \Omega_3 \to E$ is continuous;
(ii) $g(t,u)$ is locally Lipschitz continuous in u in the following sense: for every finite $\sigma \in (0,T]$, for every closed and bounded subset $M_3 \subset \Omega_3$ of E , there exists a constant $0 < L_2 < 1$ such that

$$\|g(t,u_1) - g(t,u_2)\| \leq L_2 \|u_1 - u_2\|,$$

for every $t \in [0,\sigma]$ and $u_1,\, u_2 \in M_3$,

(H_4) there exists a constant $r_0 > 0$, such that

$$r_0 \leq \tau(t,\psi) \leq r,\ t \in [0,T], \text{and } \psi \in \Omega_1,$$

(H_5) there exists a constant $L_3 > 0$, such that

$$\|\varphi(\zeta) - \varphi(\bar{\zeta})\| \leq L_3 \|\zeta - \bar{\zeta}\|,$$

for $\zeta,\, \bar{\zeta} \in [-r,0]$.

Theorem 3.2.2. *Assume that assumptions* $(H_1)-(H_4)$ *hold and let* $\gamma \in \Pi$. *Then, there exist* $\delta > 0$ *and* $0 < \sigma \leq T$ *finite numbers such that*

(i) $P = \bar{B}_C(\gamma,\delta) \subset \Pi$;

(ii) the problem (3.1.1)-(3.1.2) has a unique mild solution on a maximal interval of existence $[-r,T)$ *for all* $\gamma \in P$.

Proof. (*i*) Let $\gamma := \widehat{\varphi} \in \Pi$. Since Ω_1, Ω_2 and Ω_3 are open subsets of their respective spaces, there exists $\delta_1 > 0$ such that $B_C(\widehat{\varphi},\delta_1) \subset \Omega_1$. Introduce the vectors
$w_1 = \widehat{\varphi}(-\tau(0,\widehat{\varphi}))$ and $w_2 = \widehat{\varphi}(-\eta(0))$. Let ε_1 be such that $\bar{B}_E(w_1,\varepsilon_1) \subset \Omega_2$ and $\bar{B}_E(w_2,\varepsilon_1) \subset \Omega_3$.
The map

$$[0,T] \times C \to E, \quad (t,\psi) \mapsto \psi(-\tau(t,\psi))$$

is continuous, since

$$\begin{aligned}
\|\psi(-\tau(t,\psi)) - \bar{\psi}(-\tau(\bar{t},\bar{\psi}))\| &\leq \|\psi(-\tau(t,\psi)) - \bar{\psi}(-\tau(t,\psi))\| \\
&\quad + \|\bar{\psi}(-\tau(t,\psi)) - \bar{\psi}(-\tau(\bar{t},\bar{\psi}))\| \\
&\leq \|\psi - \bar{\psi}\| + \|\bar{\psi}(-\tau(t,\psi)) - \bar{\psi}(-\tau(\bar{t},\bar{\psi}))\| \\
&\to 0, \text{ as } t \to \bar{t},\ \psi \to \bar{\psi}.
\end{aligned}$$

Similarly, the map

$$[0,T] \times C \to E, \quad (t,\psi) \mapsto \psi(-\eta(t))$$

is also continuous; therefore, there exist $\delta_2 \in (0, \delta_1]$ and $T_1 \in (0, T]$ such that

$$\|\psi(-\tau(t,\psi)) - w_1\| < \varepsilon_1, \ \|\psi(-\eta(t)) - w_2\| < \varepsilon_1 \text{ for } t \in [0, T_1] \text{ and } \psi \in B_C(\widehat{\varphi}, \delta_2). \tag{3.2.1}$$

In particular, we get that for $\varphi \in B_C(\widehat{\varphi}, \delta_2)$, it follows $\varphi \in \Omega_1$, $\varphi(-\tau(0,\varphi)) \in \Omega_2$, $\varphi(-\eta(0)) \in \Omega_3$. Therefore, part (i) of the theorem holds for any $0 < \delta \leq \delta_2$.

Fix $\varepsilon_0 > 0$. The continuity of the map $(t, \psi) \mapsto f(t, \psi, \psi(-\tau(t, \psi)))$ yields that there exist $\delta_3 \in (0, \delta_2]$ and $T_2 \in (0, T_1]$ such that

$$\|f(t, \psi, \psi(-\tau(t, \psi))) - f(0, \varphi, \varphi(-\tau(t, \varphi)))\| < \varepsilon_0$$

for $t \in [0, T_2]$, $\psi \in B_C(\widehat{\varphi}, \delta_3)$.
Similarly,

$$\|g(t, \psi(-\eta(t))) - g(0, \varphi(-\eta(0)))\| < \varepsilon_0$$

for $t \in [0, T_2]$, $\psi \in B_C(\widehat{\varphi}, \delta_3)$.
Define the sets

$$M_2 = \bar{B}_E(w_1, \varepsilon_1), \quad M_3 = \bar{B}_E(w_2, \varepsilon_1).$$

The extension of the function $\psi \in C$ to the interval $[-r, \infty)$ by the constant value $\psi(0)$ will be denoted by

$$\widetilde{\psi}(t) = \begin{cases} \psi(t), & \text{t} \in [-r, 0], \\ \psi(0), & \text{t} \geq 0. \end{cases}$$

We define the following constants

$$K_1 = \|f(0, \varphi, \varphi(-\tau(t, \varphi)))\| + \varepsilon_0,$$

$$K_2 = \|g(0, \varphi(-\eta(0)))\| + \varepsilon_0,$$

$$K_3 = \|\varphi(0)\|,$$

$$\beta = \max\{(M+1)K_3, (M+1)K_2, \sigma M K_1\},$$

$$\delta = \frac{\delta_3}{3},$$

$$\sigma = \min\{T_2, r_0\}.$$

Then, let

$$\max\{3\beta, rL_3\} \leq \delta$$

and

$$E_0 = \{u \in C([-r, \sigma], E), \ u(t) = \varphi(t) \text{ if } t \in [-r, 0] \text{ and } \sup_{t \in [0,\sigma]} \|u(t) - \varphi(0)\| \leq \delta\}.$$

It is clear that E_0 is a closed set of $C([-r,\sigma],E)$. For $u\in E_0$, $\varphi\in B_C(\widehat{\varphi},\delta)$, $t\in[0,\sigma]$ and $\zeta\in[-r,0]$, we have

$$\begin{aligned}
\|u(t+\zeta)-\widehat{\varphi}(\zeta)\| &\leq \|u(t+\zeta)-\widetilde{\varphi}(t+\zeta)\|+\|\widetilde{\varphi}(t+\zeta)-\widetilde{\varphi}(\zeta)\|+\|\varphi(\zeta)-\widehat{\varphi}(\zeta)\| \\
&< \delta+rL_3+\delta \\
&\leq \delta+\delta+\delta \\
&\leq \delta_3,
\end{aligned}$$

and hence $\|u_t-\widehat{\varphi}\|_C<\delta_3$. Consequently, $u_t\in B_C(\widehat{\varphi},\delta_3)\subset\Omega_1$, and so

$$\|f(t,u_t,u(t-\tau(t,u_t)))\|\leq K_1,$$

$$\|g(t,u(t-\eta(t)))\|\leq K_2,$$

and $\psi=u_t$ satisfies (3.2.1) for $u\in E_0$, $\varphi\in B_C(\widehat{\varphi},\delta)$, and $t\in[0,\sigma]$. Therefore the definitions of M_2, M_3 and (3.2.1) yield

$$u_t(-\tau(u_t))\in M_2,\ \ u_t(-\eta(t))\in M_3$$

for $t\in[0,\sigma]$, $u\in E_0$, and $\varphi\in B_C(\widehat{\varphi},\delta)$. Transform the problem (3.1.1)-(3.1.2) into a fixed point problem. Consider the operator

$$N:E_0\to C([-r,\sigma],E)$$

defined by

$$Nx(t)=\begin{cases}\varphi(t), & \text{t}\in[-r,0]\\ S(t)[\varphi(0)-g(0,x(-\eta(0)))]+g(t,x(t-\eta(t))) & \\ +\displaystyle\int_0^t S(t-s)f(s,x_s,x(s-\tau(s,x_s)))ds\ , & \text{t}\in J.\end{cases}\tag{3.2.2}$$

Note that a fixed point of N is a mild solution of (3.1.1)-(3.1.2). We will show that

$$N(E_0)\subseteq E_0.$$

Let $v\in E_0$ and $t\in[0,\sigma]$. We have

$$\begin{aligned}
\|N(v)(t)-\varphi(0)\| &\leq \|S(t)[\varphi(0)-g(0,v(-\eta(0)))]-\varphi(0)\|+\|g(t,v(t-\eta(t)))\| \\
&\quad +\|\int_0^t S(t-s)f(s,v_s,v(s-\tau(s,v_s)))ds\| \\
&\leq (M+1)\|\varphi(0)\|+M\|g(0,v(-\eta(0)))\|+\|g(t,v(t-\eta(t)))\| \\
&\quad +M\|\int_0^t f(s,v_s,v(s-\tau(s,v_s)))ds\| \\
&\leq (M+1)K_3+MK_2+K_2+MK_1\int_0^t ds \\
&\leq (M+1)K_3+(M+1)K_2+M\sigma K_1 \\
&\leq 3\beta\leq\delta.
\end{aligned}$$

Hence,

$$N(E_0) \subseteq E_0.$$

On the other hand, let $v,\, w \in E_0$. Then for $t \in [0,\sigma]$, we have

$$\begin{aligned}
\|N(v)(t) - N(w)(t)\| &\leq \|S(t)[g(0, v(-\eta(0))) - g(0, w(-\eta(0)))]\| \\
&\quad + \|g(t, v(t-\eta(t))) - g(t, w(t-\eta(t)))\| \\
&\quad + \|\int_0^t S(t-s)[f(s, v_s, v(s-\tau(s,v_s))) \\
&\quad - f(s, w_s, w(s-\tau(s,w_s)))]ds\| \\
&\leq ML_2\|v(-\eta(0)) - w(-\eta(0))\| \\
&\quad + L_2\|v(t-\eta(t)) - w(t-\eta(t))\| \\
&\quad + ML_1\int_0^t \sup_{\zeta\in[-r,-r_0]} \|v_s(\zeta) - w_s(\zeta)\| \\
&\quad + ML_1\int_0^t \|v(s-\tau(s,v_s)) - w(s-\tau(s,w_s))\|ds \\
&\leq ML_2\|v(-\eta(0)) - w(-\eta(0))\| \\
&\quad + L_2\|v(t-\eta(t)) - w(t-\eta(t))\| \\
&\quad + ML_1\int_0^t \sup_{\zeta\in[-r,-\sigma]} \|v_s(\zeta) - w_s(\zeta)\| \\
&\quad + ML_1\int_0^t \|v(s-\tau(s,v_s)) - w(s-\tau(s,w_s))\|ds.
\end{aligned}$$

Since $u_t(\zeta) = u(t+\zeta) = \varphi(t+\zeta) = \varphi_t(\zeta)$ for $t \in [0,\sigma]$ and $\zeta \in [-r,-\sigma]$. We have $t - \tau(t,\varphi_t) \leq t - r_0 \leq t - \sigma \leq 0$ for $t \in [0,\sigma]$, so $u_t(-\tau(t,\varphi_t)) = \varphi_t$ for $t \in [0,\sigma]$, and $v(-\eta(0)) = w(-\eta(0)) = \varphi(-\eta(0))$. Then

$$\begin{aligned}
\|N(v)(t) - N(w)(t)\| &\leq ML_2\|\varphi(-\eta(0)) - \varphi(-\eta(0))\| \\
&\quad + L_2\|v(t-\eta(t)) - w(t-\eta(t))\| \\
&\quad + ML_1\int_0^t [\|\varphi_s - \varphi_s\| + \|\varphi(s-\tau(s,\varphi_s)) - \varphi(s-\tau(s,\varphi_s))\|]ds. \\
&\leq L_2\|v(t-\eta(t)) - w(t-\eta(t))\| \\
&\leq L_2 \sup_{\theta\in[-r,0]} \sup_{t\in[0,\sigma]} \|v(t+\theta) - w(t+\theta)\| \\
&\leq L_2\|v - w\|_\infty.
\end{aligned}$$

Consequently,

$$\|N(v) - N(w)\|_\infty \leq L_2\|v - w\|_\infty.$$

Since $L_2 < 1$, N is a contraction. By the Banach fixed point theorem 1.6.1 we conclude that N has a unique fixed point in E_0 and the problem (3.1.1)-(3.1.2) has a unique mild solution on $[-r,\sigma]$.

Let $u(t)$ be the unique mild solution of problem (3.1.1)-(3.1.2) defined on its maximal interval of existence $[0,T)$, $T>0$. Assume that $T<\infty$ and

$$\lim_{t\to T^-}\|u(t)\|<\infty.$$

Then, there exists a constant $\rho>0$ such that $\|u(t)\|\le\rho$, for $t\in[-r,T)$.

Note that (H_2) and (H_3) imply that

$$\begin{aligned}\|f(t,\psi,\psi(-\tau(t,\psi)))-f(0,\varphi,\varphi(-\tau(0,\varphi)))\| \ &\le\ L_1(\|\psi-\varphi\|+\|\psi(-\tau(t,\psi))\\ &\qquad -\varphi(-\tau(0,\varphi))\|)\end{aligned}$$

for $t\in[0,\sigma]$, $\psi\in\bar{B}_C(\widehat{\varphi},\delta)$. Similarly,

$$\|g(t,\psi(-\eta(t)))-g(0,\varphi(-\eta(0)))\|\le L_2\|\psi(-\eta(t))-\varphi(-\eta(0))\|$$

for $t\in[0,\sigma]$, $\psi\in\bar{B}_C(\widehat{\varphi},\delta)$.
We define the following constants

$$c_1=\|f(0,\varphi,\varphi(-\tau(0,\varphi)))\|+L_1(\|\varphi\|+\|\varphi(-\tau(0,\varphi))\|),$$

$$c_2=\|g(0,\varphi(-\eta(0)))\|+L_2\|\varphi(-\eta(0))\|.$$

Let $t\in[0,T)$. We obtain

$$\begin{aligned}\|u(t)\| \ &\le\ \|S(t)[\varphi(0)-g(0,u(-\eta(0)))]\|\\ &\quad +\|g(t,u(t-\eta(t)))\|\\ &\quad +\|\int_0^t S(s)f(s,u_s,u(s-\tau(s,u_s)))ds\|\\ &\le\ M[\|\varphi(0)\|+\|g(0,u(-\eta(0)))\|]+L_2\|u(t-\eta(t))\|+c_2+Mc_1t\\ &\quad +ML_1\int_0^t[\|u_s\|+\|u(s-\tau(s,u_s))\|]ds\\ &\le\ M[\|\varphi(0)\|+\|g(0,u(-\eta(0)))\|]+L_2\|u(t-\eta(t))\|+c_2+tMc_1\\ &\quad +tML_1\|u\|_\infty+ML_1\int_0^t\|u(s-\tau(s,u_s))\|ds\\ &\le\ v(t)+L_2\|u(t-r_1)\|+ML_1\int_0^t\|u(s-r_2)\|ds\end{aligned}$$

where $r_1=\eta$, $r_2=\tau$ and

$$v(t)=M[\|\varphi(0)\|+\|g(0,u(-\eta(0)))\|]+c_2+tMc_1+tML_1\|u\|_\infty.$$

By Lemma 3.2.1, it follows that

$$\|u(t)\| \ \le\ d(t)e^{ct}$$

for $t \in [0,T)$, where c is the unique positive solution of $cL_2e^{-cr_1} + ML_1e^{-cr_2} = c$, and

$$d(t) = \max\left\{ \tfrac{v(t)}{1-L_2e{-cr_1}}, \ \max_{-r\leq s\leq 0} e^{-cs}u(s), \right\}, \ t \in [0,T).$$

It follows that $\lim_{t\to T^-} u(t)$ exists. Consequently, $u(t)$ can be extended to T, which contradicts the maximality of $[0,T)$. □

3.3 Existence of Mild Solutions

In this section we apply a technique based on noncompactness measure assumption on the nonlinear term in proving an existence result for problem (3.1.1)-(3.1.2).

We introduce some additional hypotheses:

(H_5) The function $f : J \times C \times E \to E$ is continuous.

(H_6) (i) There exist constants $c_1 \geq 0$ and $c_2 \geq 0$ such that

$$\|g(t,u)\| \leq c_1\|u\| + c_2, \text{ a.e. } t \in J, \ u \in E;$$

(ii) the function g is completely continuous and for any bounded set B in Ω, the set $\left\{ \ t \to g(t, x(t-\eta(t))) : \ \ \mathrm{x} \in B \ \right\}$ is equicontinuous in Ω.

(H_7) There exist $c_3 > 0$, $p \in L^1(J,\mathbb{R}_+)$ and a continuous nondecreasing function $\psi : [0,\infty) \to [0,\infty)$ such that

$$\|f(t,u,v)\| \leq p(t)\psi(\|u\|) + c_3\|v\|, \text{ for each } u \in C, v \in E \text{ and } t \in J.$$

(H_8) For each bounded $B \subset E$, $B' \subset E$ and $t \in J$ we have

$$\alpha(f(t,B,B')) \leq p(t)\alpha(B) + c_3\alpha(B').$$

(H_9) For each $t \in J$ and bounded $B \subset E$ we have

$$\alpha(g(t,B)) \leq c_1\alpha(B).$$

(H_{10}) There exists $q > 0$ such that

$$M\|\varphi\|_\infty + (M+1)[c_1q + c_2] + M[\|p\|_{L^1}\psi(q) + Tc_3q] \leq q.$$

Theorem 3.3.1. *Assume that* (H_1), (H_5), (H_6), (H_7), (H_8), (H_9) *and* (H_{10}) *hold. Suppose that*

$$\left[c_1 + M\big(c_1 + \|p\|_{L^1} + c_3T\big)\right] < 1. \tag{3.3.1}$$

Then the problem (3.1.1)-(3.1.2) has at least one mild solution on $[-r,T]$.

Proof. Transform the problem (3.1.1)-(3.1.2) into a fixed point problem. Consider the operator

$$N : \Omega \to \Omega$$

defined by

$$(Nx)(t) = \begin{cases} \varphi(t), & \text{t} \in [-r, 0] \\ S(t)[\varphi(0) - g(0, x(-\eta(0)))] + g(t, x(t - \eta(t))) & \\ + \displaystyle\int_0^t S(t-s) f(s, x_s, x(s - \tau(s, x_s))) ds \ , & \text{t} \in J. \end{cases} \tag{3.3.2}$$

Note that a fixed point of N is a mild solution of (3.1.1)-(3.1.2).

We will show that N satisfies the assumptions of the Mönch fixed point theorem 1.6.4.
Consider the set

$$B_q = \{u \in \Omega : \|u\|_\infty \le q\},$$

where q is the constant defined in (H_{10}). Clearly, the subset B_q is closed, bounded, and convex.

The proof will be given in several steps.

Step 1: N is continuous.
Using (H_6), it suffices to show that the operator $N_1 : \Omega \to \Omega$ defined by

$$N_1(x)(t) = \begin{cases} \varphi(t), & \text{t} \in [-r, 0] \\ S(t)\varphi(0) + \displaystyle\int_0^t S(t-s) f(s, x_s, x(s - \tau(s, x_s))) ds \ , & \text{t} \in J, \end{cases} \tag{3.3.3}$$

is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in Ω. Then

$$\begin{aligned} \|N_1(u_n)(t) - N_1(u)(t)\| &\le \| \int_0^t S(t-s)[f(s, u_{ns}, u_n(s - \tau(s, u_{ns}))) \\ &\quad - f(s, u_s, u(s - \tau(s, u_s)))] ds\| \\ &\le M \int_0^t \|f(s, u_{ns}, u_n(s - \tau(s, u_{ns}))) \\ &\quad - f(s, u_s, u(s - \tau(s, u_s)))\| ds \\ &\le M \int_0^t \sup_{\theta \in [-r,0]} \sup_{s \in [0,T]} \|f(s, u_{ns}, u_n(s + \theta)) \\ &\quad - f(s, u_s, u(s + \theta))\| ds \\ &\le MT \|f(\cdot, u_{n.}, u_n(\cdot)) - f(\cdot, u_., u(\cdot))\|_\infty. \end{aligned}$$

Since f is a continuous function, we have

$$\|N_1(u_n) - N_1(u)\|_\infty \leq MT\|f(\cdot, u_{n\cdot}, u_n(\cdot)) - f(\cdot, u_\cdot, u(\cdot))\|_\infty \to 0 \text{ as } n \to \infty \,.$$

Thus N_1 is continuous.

Step 2: N maps B_q into itself.

For each $u \in B_q$, by (H_6), (H_7) and (H_{10}), we have for each $t \in [0, T]$

$$\begin{aligned}
\|N(u)(t)\| &\leq \|S(t)[\varphi(0) - g(0, u(-\eta(0)))]\| + \|g(t, u(-\eta(0)))\| \\
&\quad + \|\int_0^t S(t-s) f(s, f(s, u_s, u(s - \tau(s, u_s)))ds\| \\
&\leq M\|\varphi(0)\| + (M+1)(c_1 q + c_2) + M[\psi(q)\int_0^t p(s)ds + c_3 q\int_0^t ds] \\
&\leq M\|\varphi\|_\infty + (M+1)[c_1 q + c_2] + M\psi(q)\|p\|_{L^1} + MTc_3 q.
\end{aligned}$$

Thus

$$\|N(u)\|_\infty \;\leq\; M\|\varphi\|_\infty + (M+1)[c_1 q + c_2] + M[\psi(q)\|p\|_{L^1} + Tc_3 q] \leq q.$$

Step 3: $N(B_q)$ is bounded and equicontinuous.

By Step 2, it is obvious that $N(B_q) \subset B_q$ is bounded. Using (H_6), it suffices to show that the operator N_1 defined in (3.2.2) is equicontinuous.

Let $0 < \tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$ and B_q be a bounded set of Ω as in Step 2. Let $u \in B_q$ then for each $t \in J$ we have

$$\begin{aligned}
\|N_1(u)(\tau_2) - N_1(u)(\tau_1)\| &\leq \|S(\tau_2)\varphi(0) - S(\tau_1)\varphi(0)\| \\
&\quad + \int_0^{\tau_1 - \epsilon} \|S(\tau_2 - s) - S(\tau_1 - s)\|[p(s)\psi(q) + c_3 q]ds \\
&\quad + \int_{\tau_1}^{\tau_1 - \epsilon} \|S(\tau_2 - s) - S(\tau_1 - s)\|[p(s)\psi(q) + c_3 q]ds \\
&\quad + \int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|[p(s)\psi(q) + c_3 q]ds.
\end{aligned}$$

The right-hand side tends to zero as $\tau_2 - \tau_1 \to 0$, and ϵ sufficiently small, since $S(t)$ is a strongly continuous operator and the compactness of $S(t)$ for $t > 0$ implies the continuity in the uniform operator topology.

Now let V be a subset of B_q such that $V \subset \overline{conv}(N(V) \cup \{0\})$. V is bounded and equicontinuous and therefore the function $t \to v(t) = \alpha(V(t))$ is continuous on J. By

(H_8), (H_9), and the properties of the measure α we have for each $t \in J$,

$$\begin{aligned} v(t) &\leq \alpha(N(V)(t) \cup \{0\}) \\ &\leq \alpha(N(V)(t)) \\ &\leq c_1\big[M\alpha(V(-\eta(0))) + \alpha(V(t-\eta(t)))\big] \\ &\quad + M\int_0^t \big[p(s)\alpha(V_s) + c_3\alpha(V(s-\tau(s,V_s)))\big]ds \\ &\leq c_1\big[Mv(-\eta(0)) + v(t-\eta(t))\big] + M\int_0^t \big[p(s)v_s + c_3 v(s-\tau(s,V_s))\big]ds \\ &\leq c_1(M+1)\|v\|_\infty + M\big[\|p\|_{L^1}\|v\|_\infty + c_3T\|v\|_\infty\big] \\ &\leq \big[c_1 + M(c_1 + \|p\|_{L^1} + c_3T\big]\|v\|_\infty. \end{aligned}$$

Then

$$\|v\|_\infty\big(1 - \big[c_1 + M\big(c_1 + \|p\|_{L^1} + c_3T\big)\big]\big) \leq 0.$$

Since $\big[c_1 + M\big(c_1 + \|p\|_{L^1} + c_3T\big)\big] < 1$ it follows that $v(t) = 0$ for each $t \in J$, and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzelà theorem, V is relatively compact in B_q. As a consequence of the Mönch fixed theorem 1.6.4 we deduce that N has a fixed point which is a mild solution of problem (3.1.1)-(3.1.2). □

□

For the next theorem we replace the condition (3.3.1) by

$$c_1(M+1) < 1. \tag{3.3.4}$$

Now, consider the Kuratowski measure of noncompactness α_C defined on the family of bounded subsets of the space $C(J,E)$ by

$$\alpha_C(H) = \sup_{\theta\in[-r,0]} \sup_{t\in J} e^{-\tau L(t)}\alpha(H(t+\theta)),$$

where $L(t) = \displaystyle\int_0^t \tilde{l}(s)ds, \quad \tilde{l}(t) = M(p(t)+c_3), \quad \tau > \frac{1}{1-c_1(M+1)}$.

Our next result is based on the Darbo fixed point theorem 1.6.3.

Theorem 3.3.2. *Assume that* $(H_1), (H_5), (H_6), (H_8), (H_9)$ *and (3.3.4) are satisfied. Then the problem (3.1.1)-(3.1.2) has at least one mild solution on* $[-r, T]$.

Proof. As in Theorem 3.3.1, we can prove that the operator $N : B_q \to B_q$ defined in that theorem is continuous and $N(B_q)$ is bounded.

Now we show that the operator $N : B_q \to B_q$ is a strict set contraction, i.e., there is a constant $0 \leq \lambda < 1$ such that $\alpha(N(H)) \leq \lambda\alpha(H)$ for any $H \subset B_q$. In particular, we need to prove that there exists a constant $0 \leq \lambda < 1$ such that $\alpha_C(N(H)) \leq \lambda\alpha_C(H)$ for any $H \subset B_q$. For each $t \in J$ we have

$$\begin{aligned}
\alpha((N(H)(t)) &\leq c_1[M\alpha(H(-\eta(0))) + \alpha(H(t-\eta(t)))] \\
&\quad + M\int_0^t [p(s)\alpha(H_s) + c_3\alpha(H(s-\tau(s,H_s)))]ds \\
&\leq c_1[M\alpha(H(-\eta(0))) + \alpha(H(t-\eta(t)))] \\
&\quad + M\int_0^t e^{\tau L(s)}e^{-\tau L(s)}[p(s)\alpha(H_s) + c_3\alpha(H(s-\tau(s,H_s)))]ds \\
&\leq c_1[M\alpha(H(-\eta(0))) + \alpha(H(t-\eta(t)))] \\
&\quad + M\sup_{\theta\in[-r,0]}\sup_{s\in J} e^{-\tau L(s)}\alpha(H(s+\theta))\int_0^t e^{\tau L(s)}[p(s)+c_3]ds \\
&\leq c_1[M\alpha(H(-\eta(0))) + \alpha(H(t-\eta(t)))] + \alpha_C(H)\int_0^t \tilde{l}(s)e^{\tau L(s)}ds \\
&\leq c_1[M\alpha(H(-\eta(0))) + \alpha(H(t-\eta(t)))] + \alpha_C(H)\int_0^t \Big(\frac{e^{\tau L(s)}}{\tau}\Big)' ds \\
&\leq c_1[M\alpha(H(-\eta(0))) + \alpha(H(t-\eta(t)))] + \alpha_C(H)\frac{1}{\tau}e^{\tau L(t)}.
\end{aligned}$$

Then

$$\begin{aligned}
e^{-\tau L(t)}\alpha((N(H)(t)) &\leq c_1 e^{-\tau L(t)}[M\alpha(H(-\eta(0))) + \alpha(H(t-\eta(t)))] + \alpha_C(H)\frac{1}{\tau} \\
&\leq c_1[M+1]\sup_{\theta\in[-r,0]}\sup_{s\in J} e^{-\tau L(s)}\alpha(H(s+\theta)) + \alpha_C(H)\frac{1}{\tau}.
\end{aligned}$$

Consequently,

$$\alpha_C(NH) \leq \big[c_1(M+1) + \frac{1}{\tau}\big]\alpha_C(H).$$

So, the operator N is a set contraction. By the Darbo fixed point theorem 1.6.3 we deduce that N has a fixed point which is a mild solution of problem (3.1.1)-(3.1.2). □ □

Chapter 4

Functional Differential Equations with Periodic Conditions

In this chapter, we consider the following boundary value problem,

$$x'(t) = Ax(t) + f(t, x_t, x(t - \tau(t, x_t))),\ t \in J := [0, b], \tag{4.0.1}$$

$$x_0(\theta) = x_b(\theta),\ \theta \in [-r, 0]. \tag{4.0.2}$$

Where A generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Hilbert space $(H, \|.\|)$,
$f : J \times C([-r, 0], H) \times H \to H$, is given function and
$\tau : [0, b] \times C([-r, 0], H) \to [0, r]$ is given continuous function.
Let the norm

$$\|x\|_1 = \|x\|_\infty + \|x'\|_\infty.$$

4.1 History and Motivations

The so called theory "Brouwer degree" develop the topological theory of continuous mappings in finite dimensional spaces. This theory in infinite dimensional spaces has been initiated in a celebrated paper published in 1922 by G.D. Birkhoff and O.D. Kellogg [22]. An important period in infinite dimensional topology started from the collaboration between Leray and Schauder who realized that the topology of completely continuous perturbations of identity in a Banach space was the setting to develop Leray's continuation method introduced in his thesis.

Leray and Schauder extend the Brouwer degree to compact perturbations of the identity in a Banach space. The Leray-Schauder degree was applied to various problems for partial differential equations and the second half of the past century will see a tremendous development of the applications of the Leray-Schauder degree to nonlinear problems. For an introduction in the Brouwer degree and Leray-Schauder degree theory see ([71],[31]).

Based on Cesari's paper [26] and Cronin's monograph [27], J.Mawhin realized in october 1967 that combining Brouwer degree with Cesari's method could provide a link between the small parameter results and the study of strongly nonlinear systems (Theorem 1 in [75]). Between 1964 and 1966, Reissig ([87],[88],[89]) showed in successive steps, using Leray-Schauder theory an important existence result for weakly non linear differential equation (Theorem 2 in [75]). In 1969, J.Mawhin develop the machinery which allow reducing periodic systems solutions to a fixed point problem, leading to an improved version of Theorem 1 (Theorem 3 in [75]). Results of the type of Theorems 1 to 3 are generally refered as continuation theorems. Three years later, this theorems were extended By J.Mawhin to $L-$compact perturbations N of a Fredholm linear operator L of index zero in a normed space, within the frame of coincidence degree, an extension of Leray-Schauder degree to mappings of the form $L-N$ [72]. In the hands of many mathematicians, this theory, presented in book form in ([38],[76]), has provided a large number of new existence and multiplicity theorems for various boundary value problems associated to ordinary, functional or partial differential equations.

A continuation theorem of coincidence degree theory introduced by Gaines and Mawhin in 1970s in analyzing functional and differential equations ([38],[72]) and developed by Mawhin [73] later on, who made some important contributions on this subject has been one of the most powerful technique to study the existence of solutions for nonlinear equations.

Coincidence degree theory , also known as Mawhin's coincidence degree theory has especially so broad applications in the existence of periodic solutions of nonlinear differential equations so that many researchers have used it for their investigations (see [20],[74],[70], [81],[83],[86] and references therein).

The main goal in the coincidence degree theory is to search the existence of a solutions of the abstract equation

$$Lx = Nx$$

in some bounded and open set Ω in some Banach space for L being a noninvertible linear operator and N nonlinear operator using Leray-Schauder degree theory. Continuation theorems turn out to be specially suitable for the existence problem and became an effective tool in finding solutions to BVPs including periodic BVPs. Basically, the "continuation" is performed through an admissible homotopy carrying the given problem to a simpler one.

4.2 Fixed Point Formulation

Let X and Z be normed Banach spaces with
$X = \{y \in C^1([-r,b],H) | y_0(\theta) = y_b(\theta),\ \theta \in [-r,0]\}$ and
$Z = \{y \in C([-r,b],H) | y_0(\theta) = y_b(\theta),\ \theta \in [-r,0]\}$.

4.2 Fixed Point Formulation

Let $L : domL \subseteq X \to Z$, $y \to y'$ a linear mapping.

$$KerL = \{y \in domL : y' = 0,\, t \in J\ \}.$$

$ImL = \{x \in Z : y'(t) = x(t),\, t \in J\}$ is closed in Z.
We have

$$y(t) = y_0 + \int_0^t x(s)ds.$$

Since $y(0) = y(b)$, then

$$\int_0^b x(s)ds = 0,$$

and then

$$ImL = \{x \in Z : \int_0^b x(s)ds = 0\}.$$

Let us choose continuous projections $P : X \to X, \quad Q : Z \to Z$ such that $ImP = KerL$, $ImL = KerQ$. For example

$$P(y) = \frac{1}{b}\int_0^b y(s)ds, \quad \forall y \in X$$

and

$$Q(x) = \frac{1}{b}\int_0^b x(s)ds, \quad \forall x \in Z.$$

Let the homeomorphism

$$J : ImQ \to kerL \quad \text{be an given by } x \mapsto J(x) = x_0.$$

For any $\Omega \subset domL$ define

$$N : \overline{\Omega} \to Z,\ \ y \mapsto Ax(t) + f(t, x_t, x(t - \tau(t, x_t))).$$

The mapping $L : domL \cap KerP \to ImL$ is invertible; denote its inverse by

$$K = L^{-1} : ImL \to domL \cap KerP,\ \ z \mapsto \int_0^t z(s)ds - \frac{1}{b}\int_0^b \int_0^t z(s)dsdt$$

and $K(I - Q)N : X \to X$,

$$K(I-Q)Nx(t) = \int_0^t Nx(s)ds - \frac{1}{b}\int_0^b \int_0^t Nx(s)dsdt - (\frac{t}{b} - \frac{1}{2})\int_0^b Nx(s)ds.$$

Now (4.0.1), (4.0.2) is equivalent to the operator equation

$$Ly = Ny. \tag{4.2.1}$$

We will prove that N is $L-$compact in $\overline{\Omega}$.

We have the following result.

Definition 4.2.1. *Let $\Omega \subset X$ an open bounded set and $N : \overline{\Omega} \to Z$. We say that N is $L-$compcat if $K(I-Q)$ is compact, QN is continuous and $QN(\overline{\Omega})$ is a bounded set in Z.*

Lemma 4.2.1. *Suppose that $\Omega \subset X$ is bounded open set, then N is $L-$compact in $\overline{\Omega}$.*

Proof. we will prove that

- $QN : X \to Z$ is continuous and sends bounded sets into bounded sets.
- $K(I-Q)N : \overline{\Omega} \to X$ is completely continuous.

Step 1: QN sends bounded sets into bounded sets and it is continuous.

$$QN(y)(t) = \frac{1}{b}\big[A\int_0^b y(t)dt + \int_0^b f(t, y_t, y(t-\tau(t-y_t)))dt\big]$$

- ○ QN sends bounded sets into bounded sets in X.

Let $y \in \overline{\Omega} = \{y \in X : \|y\|_1 \leq r\}$.

$$\begin{aligned}\|QN(y)(t)\| &\leq \frac{1}{b}\big[\|A\int_0^b y(t)dt\| + \int_0^b \|f(t, y_t, y(t-\tau(t, y_t)))\|dt\big] \\ &\leq \|A\|r + \|f(., y_., y(. - \tau(., y_.)))\|_\infty := l.\end{aligned}$$

Thus

$$\|QN(y)\|_1 \leq l.$$

- ○ QN is continuous.

Let $(y_n)_{n\in\mathbb{N}}$ be a sequence such that $y_n \to y$ in X. Then ther exists $r > 0$ such that

$$\|y_n\|_1 \leq r \text{ for all } n \in \mathbb{N}.$$

We have then by the dominated convergence theorem

$$\begin{aligned}\|QN(y_n)(t) - QN(y)(t)\| &\leq \frac{1}{b}\|A\int_0^b (y_n(t) - y(t))dt\| \\ &\quad + \frac{1}{b} \\ &\quad \int_0^b \|f(t, y_{nt}, y_n(t-\tau(t, y_{nt}))) - f(t, y_t, y(t-\tau(t, y_t)))\|dt \\ &\leq \|A\|\|y_n - y\|_1 + \\ &\quad \|f(., y_{n.}, y_n(. - \tau(., y_{n.}))) - f(., y_., y(. - \tau(., y_.)))\|_\infty,\end{aligned}$$

then

$$\begin{aligned}\|QN(y_n) - QN(y)\|_1 &\leq \|A\|\|y_n - y\|_1 + \\ &\quad \|f(., y_{n.}, y_n(. - \tau(., y_{n.}))) - f(., y_., y(. - \tau(., y_.)))\|_1 \to 0\end{aligned}$$

as $n \to \infty$. Thus N is continuous.
Step 2: $K(I-Q)N$ is completely continuous.

$$K(I-Q)Nx(t) = \int_0^t Nx(s)ds - \frac{1}{b}\int_0^b \int_0^t Nx(s)dsdt - (\frac{t}{b} - \frac{1}{2})\int_0^b Nx(s)ds.$$

By using the generalized Arzela-Ascoli theorem, it is not difficult to prove that $K(I-Q)N(\overline{\Omega})$ is relatively compact in the space $(X, \|.\|_1)$. The proof of this lemma is complete. □

Lemma 4.2.2. *[19] If $N : \Delta \subset X \to Z$ is a mapping, the problem*

$$y \in D(L) \cap \Delta, Ly = Ny$$

is equivalent to the fixed point problem

$$x \in \Delta, y = Py + JQNy + K(I-Q)Ny.$$

Fot $\lambda \in [0,1]$, Let us the abstract differential periodic problem

$$Ly = \lambda Ny, \;\; y \in DomL \cap \overline{\Omega} \tag{4.2.2}$$

and define

$$\mathcal{M}(y,\lambda) = Py + JQNy + \lambda K(I-Q)Ny, \quad y \in \overline{\Omega}.$$

Using Lemma 4.2.2 we have that the problem (4.2.2) is equivalent to the fixed point problem $y \in \overline{\Omega}$ and

$$y = \mathcal{M}(y,\lambda), \quad y \in X.$$

4.3 Existence Result

Theorem 4.3.1. *Let $\Omega \subset X$ be an open bounded set and let $N : \overline{\Omega} \to Y$ be a L-compact operator. Assume*

(H_1) *For each $\lambda \in (0,1)$ the problem*

$$\begin{cases} y'(t) = \lambda Ay(t) + \lambda f(t, y_t, y(t - \tau(t, y_t))), & t \in J := [0,b], \\ y_0(\theta) = y_b(\theta), & \theta \in [-r, 0]. \end{cases} \tag{4.3.1}$$

has no solution on $\partial\Omega \cap (D(L) \setminus \ker(L))$.

(H_2) $QN(y)(t) = \frac{1}{b}\big[A\int_0^b y(t)dt + \int_0^b f(t, y_t, y(t-\tau(t-y_t)))dt\big] = 0$
has no solution on $\partial\Omega \cap \ker(L)$ *and the Brouwer degree*

$$d_B[JQN, \Omega \cap \ker L, 0] \neq 0.$$

Then the problem (4.0.1), (4.0.2) has a solution in $\overline{\Omega} \cap DomL$.

Proof. For $\lambda \in [0,1]$ consider the family of problems

$$\begin{cases} y'(t) = \lambda\big[Ay(t) + f(t, y_t, y(t-\tau(t, y_t)))\big] + \\ (1-\lambda)\frac{1}{b}\big[A\int_0^b y(t)dt + \int_0^b f(t, y_t, y(t-\tau(t-y_t)))dt\big], \\ x_0 = x_b. \end{cases} \tag{4.3.2}$$

$y \in D(L) \cap \overline{\Omega}$.
Let $\mathcal{M} : \overline{\Omega} \times [0,1] \to Z$ be the homotopy defined by

$$\mathcal{M}(y, \lambda) = P(y) + JQN_f(y) + \lambda K(I-Q)N(y).$$

Using Lemma 4.2.2, we have that the problem (4.3.2) is equivalent to the fixed point problem $y \in \overline{\Omega}$ and

$$y = \mathcal{M}(\lambda, y). \tag{4.3.3}$$

If there exists $y \in \partial\Omega$ such that

$$\begin{cases} y'(t) = Ay(t) + f(t, y_t, y(t-\tau(t, y_t))), & t \in J := [0, b], \\ y_0(\theta) = y_b(\theta), & \theta \in [-r, 0]. \end{cases} \tag{4.3.4}$$

then we are done. Now assume that

$$\begin{cases} y'(t) = Ay(t) + f(t, y_t, y(t-\tau(t, y_t))), & t \in J := [0, b], \\ y_0(\theta) = y_b(\theta), & \theta \in [-r, 0] \end{cases} \tag{4.3.5}$$

has no solution for each $y \in D(L) \cap \partial\Omega$.
On the other hand

$$\begin{cases} y'(t) = \lambda\big[Ay(t) + f(t, y_t, y(t-\tau(t, y_t)))\big] + \\ (1-\lambda)\frac{1}{b}\big[A\int_0^b y(t)dt + \int_0^b f(t, y_t, y(t-\tau(t-y_t)))dt\big], \\ x_0 = x_b. \end{cases} \tag{4.3.6}$$

has no solution for each $(\lambda, y) \in (0,1) \times (D(L) \cap \partial\Omega$. If

$$\begin{cases} y'(t) = \lambda\big[Ay(t) + f(t, y_t, y(t-\tau(t, y_t)))\big] + \\ (1-\lambda)\frac{1}{b}\big[A\int_0^b y(t)dt + \int_0^b f(t, y_t, y(t-\tau(t-y_t)))dt\big], \\ x_0 = x_b. \end{cases} \tag{4.3.7}$$

has a solution for some $(\lambda, y) \in (0,1) \times (D(L) \cap \partial\Omega)$, we obtain, by applying Q to both members of the preceding equality,

$$QN(y)(t) = \frac{1}{b}\big[A\int_0^b y(t)dt + \int_0^b f(t, y_t, y(t-\tau(t-y_t)))dt\big] = 0 \tag{4.3.8}$$

has a solution, which together with hypothesis (H_2) implies that $y \in \partial\Omega \cap \ker(L)$ i. e. $y \in \partial\Omega \cap D(L) \setminus \ker(L)$ and hence the second contradicts (H_2). Using again (H_2) it follows that

$$\begin{cases} y'(t) = \frac{1}{b}\big[A\int_0^b y(t)dt + \int_0^b f(t, y_t, y(t-\tau(t-y_t)))dt\big], \\ x_0 = x_b. \end{cases} \tag{4.3.9}$$

has no solution for each $y \in D(L) \cap \partial\Omega$.
Using (4.3.7), (4.3.8) and 4.3.9 we deduce that

$$y \neq \mathcal{M}(\lambda, y). \tag{4.3.10}$$

for each $(\lambda, y) \in [0,1] \times \partial\Omega$.
Because N is L-compact it follows $\mathcal{M}$ compact, hence using (4.3.10) and the invariance property under compact homotopy of Leray-Schauder degree, we have

$$d_{LS}[I - \mathcal{M}(.,1), \Omega, 0] = d_{LS}[I - \mathcal{M}(.,0), \Omega, 0]. \tag{4.3.11}$$

We have that

$$\mathcal{M}(.,0) = Py + JQN(y),$$

then

$$d_{LS}[I - \mathcal{M}(.,0), \Omega, 0] = d_{LS}[I - Py + JQN(y)(.,0), \Omega, 0] \tag{4.3.12}$$

The range of $P + JQN$ is contained in $ker(L)$, so, using the reduction property of the Leray-Schauder degree and the fact that $P|_{kerL} = I|_{kerL}$ it follows that

$$\begin{aligned} d_{LS}[I - (P + JQN), \Omega, 0] &= d_B[I - (P + JQN), \Omega \cap kerL, 0] \\ &= d_B[-JQN, \Omega \cap kerL, 0]. \end{aligned} \tag{4.3.13}$$

Using (4.3.11), (4.3.12) and (4.3.13) it follows that $d_{LS}[I - \mathcal{M}(.,1), \Omega, 0] \neq 0$, hence the existence property of Leray-Schauder degree implies that there exists $y \in \Omega$ such that $y = \mathcal{M}(y, 1)$ i. e. $y \in D(L) \cap \Omega$,

$$\begin{cases} y'(t) = Ay(t) + f(t, y_t, y(t - \tau(t, y_t))), & t \in J := [0, b], \\ y_0(\theta) = y_b(\theta), & \theta \in [-r, 0] \end{cases} \tag{4.3.14}$$

the theorem is proved. □

Chapter 5

Random Functional Differential Equations System

In this chapter, we give some existence results for functional differential equations, we study the following systems

$$\begin{cases} x'(t,\omega) = f(t, x_t(.,\omega), y_t(.,\omega), \omega), & t \in J := [0,T] \\ y'(t,\omega) = g(t, x_t(.,\omega), y_t(.,\omega), \omega), & t \in J := [0,T] \\ x_0(\theta) = \varphi(\theta,\omega), & \theta \in [-r,0] \\ y_0(\theta) = \psi(\theta,\omega), & \theta \in [-r,0]. \end{cases} \tag{5.0.1}$$

where $f, g : J \times C([-r,0],\mathbb{R}) \times C([-r,0],\mathbb{R}) \times \Omega \to \mathbb{R}$, $(\Omega, \mathcal{A})$ is a measurable space and $x_0, y_0 : \Omega \to \mathbb{R}$ are a random variable.

Definition 5.0.1. *The map $f : J \times B \times \Omega \to \mathbb{R}$ is called random Carathéodory if*

(i) the map $(t,\omega) \to f(t,x,\omega)$ is jointly measurable for each $x \in B$;

(ii) the map $x \to f(t,x,\omega)$ is continuous for all $t \in J$ and $\omega \in \Omega$.

Definition 5.0.2. *A Carathéodory function $f : J \times B \times \Omega \to \mathbb{R}$ is called random L^1-Carathéodory for each $q > 0$, there exists $h_q \in L^1(J \times \Omega, \mathbb{R}^+)$ such that*

$$|f(t,x)| \leq h_q(t,\omega), \quad a.e. \ t \in J$$

for all $\|x\|_B \leq q$ and $\omega \in \Omega$.

Theorem 5.0.2. *[80] Let $(\Omega, \mathcal{F})$ be a measurable space, X be a real separable generalized Banach space and $F : \Omega \times X \to X$ be a continuous random operator, and let $M(\omega) \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ be a random variable matric such that for every $\omega \in \Omega$ the matrix, $M(\omega)$ converge to 0 and*

$$d(F(\omega,x_1), F(\omega,x_2)) \leq M(\omega) d(x_1,x_2) \text{ for each } x_1, x_2 \in X, \ \omega \in \Omega.$$

then there exists any random variable $x : \Omega \to X$ which is the unique random fixed point of F.

Theorem 5.0.3. *[80] Let X be a separable generalized Banach space and let $F : \Omega \times X \to X$ be a completely continuous random operator. Then, either*

(i) the random equation $F(\omega, x) = x$ has a random solution, i.e., there is a measurable function $x : \Omega \to X$ such that $F(\omega, x(\omega)) = x(\omega)$ for all $\omega \in \Omega$, or

(ii) the set $\mathcal{M} = \{x : \Omega \to X \text{ is measurable} | \lambda(\omega) F(\omega, x) = x\}$ is unbounded for some measurable $\lambda : \Omega \to X$ with $0 < \lambda(\omega) < 1$ on Ω.

Theorem 5.0.4. *(Carathéodory)[80] Let X be a separable metric space and $G : \Omega \times X \to X$ be a mapping such that $G(., x)$ is measurable for all $x \in X$ and $G(\omega, .)$ is continuous for all $\omega \in \Omega$. Then the map $(\omega, x) \to G(\omega, x)$ is jointly measurable.*

As consequence of above theorem we can easily prove the following result.

Lemma 5.0.5. *[80] Let X be a separable generalized metric space and $G : \Omega \times X \to X$ be a mapping such that $G(., x)$ is measurable for all $x \in X$ and $G(\omega, .)$ is continuous for all $\omega \in \Omega$. Then the map $(\omega, x) \to G(\omega, x)$ is jointly measurable.*

5.1 Existence and Uniqueness Results

In this section we shall use a random version of the Perov type and study the nonlinear initial value problems of random functional differential equations.

Set $C_r := C([-r, 0] \times \Omega, \mathbb{R})$ and $C := C([-r, T] \times \Omega, \mathbb{R})$.

Theorem 5.1.1. *$f, g : J \times C_r \times C_r \times \Omega \to \mathbb{R}$ are two Carathéodory functions. Assume that the following condition*

(H_1) There exist $p_1, p_2, p_3, p_4 : \Omega \to \mathbb{R}_+$ are random variable such that

$$|f(t, x, y, \omega) - f(t, \widetilde{x}, \widetilde{y}, \omega)| \leq p_1(\omega)|x - \widetilde{x}| + p_2(\omega)|y - \widetilde{y}|$$

and

$$|g(t, x, y, \omega) - g(t, \widetilde{x}, \widetilde{y}, \omega)| \leq p_3(\omega)|x - \widetilde{x}| + p_4(\omega)|y - \widetilde{y}|$$

where

$$M(\omega) = \begin{pmatrix} Tp_1(\omega) & Tp_2(\omega) \\ Tp_3(\omega) & Tp_4(\omega) \end{pmatrix}$$

holds. If $M(\omega)$ converge to 0, then the problem (5.0.1) has a unique random solution.

Proof. Consider the operator $N : C\times C\times\Omega \to C\times C$, $(x, y, \omega) \to (L_1(x, y, \omega), L_2(x, y, \omega))$ where

$$L_1(x(t,\omega), y(t,\omega), \omega) = \varphi(\theta,\omega) + \int_0^t f(s, x_s(.,\omega), y_s(.,\omega), \omega)ds$$

and

$$L_2(x(t,\omega), y(t,\omega), \omega) = \psi(\theta,\omega) + \int_0^t g(s, x_s(.,\omega), y_s(.,\omega), \omega)ds.$$

First we show that N is a random operator on $C \times C \times \Omega$. Since f and g are Carathéodory functions, then $\omega \to f(t, x, y, \omega)$ and $\omega \to g(t, x, y, \omega)$ are measurable maps in view of Lemma 5.0.5. Further, the integral is a limit of a finite sum of measurable functions, therefore, the maps

$$\omega \to L_1(x(t,\omega), y(t,\omega), \omega), \quad \omega \to L_2(x(t,\omega), y(t,\omega), \omega)$$

are measurable. As a result, N is a random operator on $N : C \times C \times \Omega$ into $C \times C$.
We show that N satisfies all the conditions of Theorem on $C \times C \times \Omega$.
Let $(x, y), (\widetilde{x}, \widetilde{y}) \in C \times C$ then

$$\begin{aligned}
|L_1(x(t,\omega), y(t,\omega), \omega) - L_1(\widetilde{x}(t,\omega), \widetilde{y}(t,\omega), \omega)| &= \\
&\left|\int_0^t (f(s, x_s(.,\omega), y_s(.,\omega), \omega) - f(s, \widetilde{x}_s(.,\omega), \widetilde{y}_s(.,\omega), \omega))ds\right| \\
&\leq \int_0^t |f(s, x_s(.,\omega), y_s(.,\omega), \omega) - f(s, \widetilde{x}_s(.,\omega), \widetilde{y}_s(.,\omega), \omega)|ds \\
&\leq \int_0^t p_1(\omega)|x_s(.,\omega) - \widetilde{x}_s(.,\omega)|ds \\
&\quad + \int_0^t p_2(\omega)|y_s(.,\omega) - \widetilde{y}_s(.,\omega)|ds.
\end{aligned}$$

Then

$$\|L_1(x, y, \omega) - L_1(\widetilde{x}, \widetilde{y}, \omega)\|_\infty \leq Tp_1(\omega)\|x - \widetilde{x}\|_\infty + Tp_2(\omega)\|y - \widetilde{y}\|_\infty.$$

Similarly, we obtains

$$\|L_2(x, y, \omega) - L_2(\widetilde{x}, \widetilde{y}, \omega)\|_\infty \leq Tp_3(\omega)\|x - \widetilde{x}\|_\infty + Tp_4(\omega)\|y - \widetilde{y}\|_\infty.$$

Hence

$$d(N(x, y, \omega), N(\widetilde{x}, \widetilde{y}, \omega)) \leq M(\omega)d((x, y), (\widetilde{x}, \widetilde{y})),$$

where

$$d((x, y), (\widetilde{x}, \widetilde{y})) = \begin{pmatrix} \|x - \widetilde{x}\|_\infty \\ \|y - \widetilde{y}\|_\infty \end{pmatrix}.$$

From Theorem 5.0.2 there exists unique random solution of problem (5.0.1). □

Lemma 5.1.2. *(Grönwall-Bihari)[21] Let $I = [p, q]$ and let $u, g : I \to \mathbb{R}$ be positive continuous functions. Assume there exist $c > 0$ and a continuous nondecreasing function $h : [0, \infty) \to (0, +\infty)$ such that*

$$u(t) \leq c + \int_p^t g(s)h(u(s))ds, \quad \forall t \in I.$$

Then

$$u(t) \leq H^{-1}\Big(\int_p^t g(s)ds\Big), \quad \forall t \in I,$$

provided

$$\int_c^{+\infty} \frac{dy}{h(y)} > \int_p^q g(s)ds,$$

where H^{-1} refers to inverse of the function $H(u) = \int_c^u \frac{dy}{h(y)}$ for $u \geq c$.

We consider the following set of hypotheses in what follows:

(H_2) The functions f and g are random Carathéodory on $[0, T] \times C_r \times C_r \times \Omega$.

(H_3) There exist a measurable and bounded functions $\gamma_1, \gamma_2 : \Omega \to L^1([0, T], \mathbb{R}_+)$ and a continuous and nondecreasing function $\psi_1, \psi_2 : \mathbb{R}_+ \to (0, \infty)$ such that

$$|f(t, x, y)| \leq \gamma_1(t, \omega)\psi_1(|x| + |y|), \quad |g(t, x, y)| \leq \gamma_2(t, \omega)\psi_2(|x| + |y|) \quad \text{a.e. } t \in J$$

for all $\omega \in \Omega$ and $x, y \in C_r$.

Now, we give prove of the existence result of problem (5.0.1) by using Leary-Schauder random fixed point theorem type in generalized Banach space.

Theorem 5.1.3. *Assume that the hypotheses (H_2) and (H_3) hold. If*

$$\int_0^T (\gamma_1(s, \omega) + \gamma_2(s, \omega))ds < \int_{\|\varphi(.,\omega)\|_\infty + \|\psi(.,\omega)\|_\infty}^\infty \frac{du}{\psi_1(u) + \psi_2(u)}, \quad \text{for all } \omega \in \Omega$$

Then the problem (5.0.1) has a random solution.
moreover the set

$$S = \{(x; y) \in C \times C : \text{(x,y) is solution of the problem (5.0.1) }\}$$

is compact.

Proof. Let $N : C \times C \times \Omega \to C \times C$ a random operator defined in Theorem 5.1.1. Clearly, the random fixe point of N are solutions to (5.0.1), where N is defined in Theorem 5.1.1. In order to apply Theorem 5.0.3, we first show that N is completely continuous. The proof will be given in several steps.

Step 1: $N(.,.,\omega) = (L_1(.,.,\omega), L_2(.,.,\omega))$ is continuous.

Let (x_n, y_n) be a sequence such that $(x_n, y_n) \to (x, y)$ in $C \times C$ as $n \to \infty$. Then

$$
\begin{aligned}
& |L_1(x_n(t,\omega), y_n(t,\omega), \omega) - L_1(x(t,\omega), y(t,\omega), \omega)| \\
\leq & \int_0^t |f(s, x_{ns}(.,\omega), y_{ns}(.,\omega), \omega) - f(s, x_s(.,\omega), y_s(.,\omega), \omega)| ds,
\end{aligned}
$$

and so

$$
\begin{aligned}
& \|L_1(x_n(.,\omega), y_n(.,\omega), \omega) - L_1(x(.,\omega), y(.,\omega), \omega)\|_\infty \\
\leq & \int_0^T |f(s, x_{ns}(.,\omega), y_{ns}(.,\omega), \omega) - f(s, x_s(.,\omega), y_s(.,\omega), \omega)| ds.
\end{aligned}
$$

Since f is an L^1-Carath*é*odory function, we have by the Lebesgue dominated convergence theorem, we have

$$\|L_1(x_n(.,\omega), y_n(.,\omega), \omega) - L_1(x(.,\omega), y(.,\omega), \omega)\|_\infty \to 0 \quad \text{as} \quad n \to \infty.$$

Similarly

$$\|L_2(x_n(.,\omega), y_n(.,\omega), \omega) - L_2(x(.,\omega), y(.,\omega), \omega)\|_\infty \to 0 \quad \text{as} \quad n \to \infty.$$

Thus N is continuous.
Step 2: N maps bounded sets into bounded sets in $C \times C$. Indeed, it is enough to show that for any $q > 0$ there exists a positive constant l such that for each $(x, y) \in B_q = \{(x, y) \in C \times C : \|x\|_\infty \leq q, \|y\|_\infty \leq q\}$, we have

$$\|N(x, y, \omega)\|_\infty \leq l = (l_1, l_2).$$

Then for each $t \in [0, T]$, we get

$$
\begin{aligned}
|L_1(x(t,\omega), y(t,\omega), \omega)| &= |\varphi(\theta,\omega) + \int_0^t f(s, x_s(.,\omega), y_s(.,\omega), \omega) ds| \\
&\leq |\varphi(\theta,\omega)| + \int_0^t |f(s, x_s(.,\omega), y_s(.,\omega), \omega)| ds.
\end{aligned}
$$

From (H_3), we have

$$\|L_1(x(.,\omega),y(.,\omega),\omega)\|_\infty \;\leq\; \|\varphi(\theta,\omega)\|+\psi_1(2q)\int_0^T \gamma_1(s,\omega)ds := l_1.$$

Similarly, we have

$$\|L_2(x(.,\omega),y(.,\omega),\omega)\|_\infty \;\leq\; \|\psi(\theta,\omega)\|+\psi_2(2q)\int_0^T \gamma_2(s,\omega)ds := l_2.$$

Step 3: N maps bounded sets into equicontinuous sets of $C\times C$.

Let $0<\tau_1,\tau_2\in J$, $\tau_1<\tau_2$ and B_q be a bounded set of $C\times C$ as in Step 2. Let $(x,y)\in B_q$ then for each $t\in J$ we have

$$|L_1(x(\tau_2,\omega),y(\tau_2,\omega),\omega)-L_1(x(\tau_1,\omega),y(\tau_1,\omega),\omega)| \;\leq\; \int_{\tau_1}^{\tau_2}|f(s,x_s(.,\omega),y_s(.,\omega),\omega)|ds.$$

Hence

$$|L_1(x(\tau_2,\omega),y(\tau_2,\omega),\omega)-L_1(x(\tau_1,\omega),y(\tau_1,\omega),\omega)| \;\leq\; \psi_1(2q)\int_{\tau_1}^{\tau_2}\gamma_1(s,\omega)ds$$

and

$$|L_2(x(\tau_2,\omega),y(\tau_2),\omega)-L_2(x(\tau_1),y(\tau_1,\omega),\omega)| \;\leq\; \psi_2(2q)\int_{\tau_1}^{\tau_2}\gamma_2(s,\omega)ds.$$

he right-hand side tends to zero as $\tau_2-\tau_1\to 0$.

As a consequence of Steps 2, 3 and the Arzelá -Ascoli theorem we can conclude that we conclude that N maps B_q into a precompact set in $C\times C$.

Step 4: *(A priori bounds on solutions.)*

Now, it remains to show that the set

$$\Sigma=\Big\{\;(x,y)\in C\times C:(x,y)=\lambda(\omega)N(x,y),\lambda(\omega)\in(0,1)\;\Big\}\text{ is bounded.}$$

Let $(x,y)\in\Sigma$. Then $x=\lambda(\omega)L_1(x,y)$ and $y=\lambda(\omega)L_2(x,y)$ for some $0<\lambda(\omega)<1$. Thus, for $t\in[0,T]$, we have

$$|x(t,\omega)| \;\leq\; |\varphi(\theta,\omega)|+\int_0^t|\gamma_1(t,\omega)\psi_1(|x_s(.,\omega)|+|y_s(.,\omega)|)|ds$$

and

$$|y(t,\omega)| \leq |\psi(\theta,\omega)| + \int_0^t |\gamma_2(t,\omega)\psi_2(|x_s(.,\omega)| + |y_s(.,\omega)|)|ds.$$

Therefore

$$|x(t,\omega)| + |y(t,\omega)| \leq c + \int_0^t p(s)\phi(|x_s(.,\omega)| + |y_s(.,\omega)|)ds,$$

where

$$c = |\varphi(\theta,\omega)| + |\psi(\theta,\omega)|,\ \phi = \psi_1 + \psi_2 \text{ and } p = \gamma_1 + \gamma_2.$$

By Lemma 5.1.2, we have

$$|x(t,\omega)| + |y(t,\omega)| \leq \Gamma^{-1}\big(\int_0^t p(s)ds\big) := K_*, \text{ for each } t \in [0,T],$$

where

$$\Gamma(z) = \int_c^z \frac{du}{\phi(u)}.$$

Consequently

$$\|x\|_\infty \leq K_* \text{ and } \|y\|_\infty \leq K_*.$$

This shows that Σ is bounded. As a consequence of Theorem 5.0.3 we deduce that N has a random fixed point (x,y) which is a solution to the problem (5.0.1).
Step 5: It remains to show that the set S is compact.

Let the sequence $(x_n,y_n)_{n\in\mathbb{N}} \subset S$, then

$$x_n(t,\Omega) = \begin{cases} \varphi(t,\omega), & \text{t} \in [-r,0] \\ \varphi(0,\omega) + \displaystyle\int_0^t f(s,x_{ns}(.,\omega),y_{ns}(.,\omega),\omega)ds\ , & \text{t} \in J \end{cases}$$

and

$$y_n(t,\omega) = \begin{cases} \psi(t,\omega), & \text{t} \in [-r,0] \\ \psi(0,\omega) + \displaystyle\int_0^t f(s,x_{ns}(.,\omega),y_{ns}(.,\omega),\omega)ds\ , & \text{t} \in J. \end{cases}$$

Let $B = \{(x_n,y_n) : n \in \mathbb{N}\} \subseteq C \times C$.
Then from earlier parts of the proof of this theorem, we conclude that B is bounded and equicontinuous. Then from the Ascoli-Arzelà theorem we can conclude that B is compact, then there exists a subsequence $(x_{nm},y_{nm}) \subset S$; $(x_{nm},y_{nm}) \to (x,y)$ as $n_m \to \infty$. Consider

$$z(t,\omega) = \begin{cases} \varphi(t,\omega), & \text{t} \in [-r,0] \\ \varphi(0,\omega) + \displaystyle\int_0^t f(s,z_s(.,\omega),j_s(.,\omega),\omega)ds\ , & \text{t} \in J \end{cases}$$

and

$$j(t,\omega) = \begin{cases} \psi(t,\omega), & \text{t} \in [-r,0] \\ \psi(0,\omega) + \displaystyle\int_0^t f(s, z_s(.,\omega), j_s(.,\omega), \omega)ds \ , & \text{t} \in J, \end{cases}$$

then

$$|x_{nm}(t,\omega) - z(t,\omega)| \ \leq \ \int_0^t |f(s, x_{ns}(.,\omega), y_{ns}(.,\omega), \omega) - f(s, z_s(.,\omega), j_s(.,\omega), \omega)|ds$$

and

$$|y_{nm}(t,\omega) - j(t,\omega)| \ \leq \ \int_0^t |f(s, x_{ns}(.,\omega), y_{ns}(.,\omega), \omega) - f(s, z_s(.,\omega), j_s(.,\omega), \omega)|ds$$

$(x_{nm}(t,\omega), y_{nm}(t,\omega)) \to (z(t,\omega), j(t,\omega))$ as $n_m \to \infty$. Thus

$$x(t,\omega) = \begin{cases} \varphi(t,\omega), & \text{t} \in [-r,0] \\ \varphi(0,\omega) + \displaystyle\int_0^t f(s, x_s(.,\omega), y_s(.,\omega), \omega)ds \ , & \text{t} \in J \end{cases}$$

and

$$j(t,\omega) = \begin{cases} \psi(t,\omega), & \text{t} \in [-r,0] \\ \psi(0,\omega) + \displaystyle\int_0^t f(s, x_s(.,\omega), y_s(.,\omega), \omega)ds \ , & \text{t} \in J, \end{cases}$$

□

Chapter 6

Functional Differential Inclusions with State-Dependent Delays

This chapter is organized as follows: in Section 6.1, we prove the existence of solutions for functional differential inclusions system with state-dependent delays. In Section 6.2, we prove the existence of solutions for functional differential inclusions system with state-dependent delays with periodic conditions.

6.1 System of Functional Differential Inclusions

We consider the system with state-dependent delays of the following form:

$$\begin{cases} x'(t) - A_1 x(t) \in F_1(t, x_t, x(t-\tau_1(t,x_t)), y_t, y(t-\tau_2(t,y_t))), & t \in J := [0,b] \\ y'(t) - A_2 y(t) \in F_2(t, x_t, x(t-\tau_1(t,x_t)), y_t, y(t-\tau_2(t,y_t))), & t \in J := [0,b] \\ x(t) = \varphi(t), & t \in [-r,0] \\ y(t) = \psi(t), & t \in [-r,0]. \end{cases} \tag{6.1.1}$$

where the operators $A_i,\ i = 1,2$ are infinitesimal generator of a C_0-semigroup $T_i(t)_{t\geq 0}$ on a Banach space E,
$F_1, F_2 : J \times C([-r,0],E) \times E \times C([-r,0],E) \times E \to \mathcal{P}(E)$ are a multifunctions,
$\varphi, \psi : [-r,0] \to E$ and $\tau_1, \tau_2 : [0,b] \times C([-r,0],E) \to [0,r]$ are given continuous functions.
Throughout this section the operators $A_i,\ i = 1,2$ are infinitesimal generator of a C_0-semigroup $T_i(t)_{t\geq 0}$ and there exists $M > 0$ such that

$$\|T(t)\| \leq M \quad \text{for all } t \in J.$$

Set $C_r := C([-r,0],E)$ and $C := C([-r,b],E)$.

6.1.1 Existence Result

Definition 6.1.1. *A function $(x, y) \in C \times C$ is said to be a mild solution of problem (6.1.1) if $x(t) = \varphi(t), y(t) = \psi(t),\ t \in [-r, 0]$ and there exists $v_1, v_2 \in L^1(J, E)$ such that $v_i \in F_i(t, x_t, x(t - \tau_1(t, x_t)), y_t, y(t - \tau_2(t, y_t)))$ a.e. on J such that*

$$x(t) = T_1(t)\varphi(0) + \int_0^t T_1(t-s)v_1(s)ds$$

and

$$y(t) = T_2(t)\psi(0) + \int_0^t T_2(t-s)v_2(s)ds.$$

Let $(E, |.|)$ be a separable Banach space and
$F_i : J \times C([-r, 0], E) \times E \times C([-r, 0], E) \times E \to \mathcal{P}_{cp,cv}(E)$, $i = 1, 2$ are Carathéodory multimap which satisfies some of the following assumptions:

(H_1) There exists $c_i > 0$, $p_i \in L^1([0, b], \mathbb{R}_+)$ and a continuous and nondecreasing function $\psi_i : \mathbb{R}_+ \to (0, \infty)$, $i = 1, 2$ such that

$$\|F_i(t, u_1, v_1, u_2, v_2)\| \leq p_i(t)\psi_i(\|u_1\| + \|u_2\|) + c_i(\|v_1\| + \|v_2\|),$$

for each $u_1, u_2 \in C, v_1, v_2 \in E$ and $t \in J$
with

$$\int_0^b \overline{m}(s)ds < \int_c^\infty \frac{du}{u + \psi_1(u) + \psi_2(u)},$$

where

$$\overline{m}(t) = \max\{2M(p_1(t) + p_2(t)), M(c_1 + c_2)\}$$

and

$$c = M(\|\psi(0)\| + \|\varphi(0)\|).$$

(H_2) The semigroup $T_i(.)$, $i = 1, 2$ is compact for $t > 0$.

Theorem 6.1.1. *Assume that the hypotheses (H_1) and (H_2) hold. Then the set of solutions for problem (6.1.1) is nonempty and compact.*

Proof. Consider the operator $N : C \times C \to \mathcal{P}(C \times C)$ defined for $(x, y) \in C \times C$ by

$$N(x,y) = \left\{ (h_1, h_2) \in C \times C : (h_1(t), h_2(t)) = \left\{ \begin{array}{l} T_1(t)\varphi(0) \\ + \displaystyle\int_0^t T_1(t-s)v_1(s)ds, \quad t \in J := [0, b] \\ T_2(t)\psi(0) \\ + \displaystyle\int_0^t T_2(t-s)v_2(s)ds, \quad t \in J := [0, b] \end{array} \right. \right\},$$

where $v_i \in S_{F_i,x,y} = \{f \in L^1(J,E) : f(t) \in F_i(t, x_t, x(t-\tau_1(t,x_t)), y_t, y(t-\tau_2(t,y_t))), a.e.\ t \in J\}$. Clearly, fixed points of the operator N are solutions of problem (6.1.1). Let

$$N_1(x,y) = \left\{ h_1 \in C : h_1(t) = \left\{ \begin{array}{l} T_1(t)\varphi(0) \\ + \displaystyle\int_0^t T_1(t-s)v_1(s)ds, \quad t \in J := [0,b] \end{array} \right. \right\}$$

and

$$N_2(x,y) = \left\{ h_2 \in C : h_2(t) = \left\{ \begin{array}{l} T_2(t)\psi(0) \\ + \displaystyle\int_0^t T_2(t-s)v_2(s)ds, \quad t \in J := [0,b] \end{array} \right. \right\}.$$

Hence

$$N(x,y) = (N_1(x,y), N_2(x,y)) \text{ for every } (x,y) \in C \times C.$$

Since, for each $(x,y) \in C \times C$, the nonlinearity F_i takes convex values, the selection set $S_{F_i,x,y}$ is convex, then N has convex values. From (H_1) and (H_2), we can prove that N is completely continuous. The proof will be given in several steps.

Step 1: N maps bounded sets into bounded sets in $C \times C$. Indeed, it is enough to show that for any $q > 0$ such that for each $h = (h_1, h_2) \in N(x,y)$ there exists a positive constant l such that for each $(x,y) \in B_q = \{(x,y) \in C \times C : \|x\|_\infty \leq q, \|y\|_\infty \leq q\}$, we have

$$\|h\|_\infty \leq l = (l_1, l_2).$$

If $h = (h_1, h_2) \in N(x,y)$ then there exists $v_i \in S_{F_i,x,y}$, $i = 1,2$ such that for each $t \in [0,b]$, we get

$$h_1(t) = T_1(t)\varphi(0) + \int_0^t T_1(t-s)v_1(s)ds$$

and

$$h_2(t) = T_2(t)\psi(0) + \int_0^t T_2(t-s)v_2(s)ds.$$

By (H_1) we have for each $t \in J$

$$\begin{aligned} |h_1(t)| &\leq \|T_1(t)\varphi(0)\| + \int_0^t \|T_2(t-s)v_2(s)\|ds \\ &\leq M\|\varphi\|_\infty + 2qc_1b + M\psi_1(2q)\int_0^t p_1(s)ds. \end{aligned}$$

Then

$$\|h_1\|_\infty \leq M\|\varphi\|_\infty + 2qc_1b + M\psi_1(2q)\int_0^b p_1(s)ds := l_1.$$

Similarly, we have

$$\|h_2\|_\infty \leq M\|\psi\|_\infty + 2qc_2b + M\psi_2(2q)\int_0^b p_2(s)ds := l_2.$$

Step 2: N maps bounded sets into equicontinuous sets of $C \times C$. Let $0 < \tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$ and B_q be a bounded set of $C \times C$ as in Step 1. Let $(x, y) \in B_q$ and $h = (h_1, h_2) \in N(x, y)$, then there exists $v_i \in S_{F_i,x,y}$, $i = 1, 2$ such that

$$\begin{aligned}\|h_1(\tau_2) - h_1(\tau_1)\| \leq\ & \|T_1(\tau_2)\varphi(0) - T_1(\tau_1)\varphi(0)\| \\ & + \int_0^{\tau_1-\epsilon} \|T_1(\tau_2 - s) - T_1(\tau_1 - s)\|[p_1(s)\psi_1(q) + c_1q]ds \\ & + \int_{\tau_1}^{\tau_1-\epsilon} \|T_1(\tau_2 - s) - T_1(\tau_1 - s)\|[p_1(s)\psi_1(q) + c_1q]ds \\ & + \int_{\tau_1}^{\tau_2} \|T_1(\tau_2 - s)\|[p_1(s)\psi_1(q) + c_1q]ds.\end{aligned}$$

The right-hand side tends to zero as $\tau_2 - \tau_1 \to 0$, and ϵ sufficiently small, since $T(t)$ is a strongly continuous operator and the compactness of $T(t)$ for $t > 0$ implies the continuity in the uniform operator topology.
As a consequence of Steps 1, 2 and the Arzelá -Ascoli theorem we can conclude that $N_i : C \times C \to \mathcal{P}_{cp,cv}(C)$ are a completely continuous operator.

Step 3: N has a closed graph.

Let (x_n, y_n) be a sequence such that $(x_n, y_n) \to (x_*, y_*)$, $h_n \in N(x_n, y_n)$ and $h_n := (h_n^1, h_n^2) \to h_* := (h_*^1, h_*^2)$ as $n \to \infty$. We shall prove that $h_* \in N(x_*, y_*)$. Now $h_n \in N(x_n, y_n)$ means there exist $v_n^i \in S_{F_i,x_n,y_n}$, $i = 1, 2$ such that

$$h_n^1(t) = T_1(t)\varphi(0) + \int_0^t T_1(t - s)v_n^1(s)ds,\ t \in J$$

and

$$h_n^2(t) = T_2(t)\psi(0) + \int_0^t T_2(t - s)v_n^2(s)ds,\ t \in J.$$

Consider the linear continuous operator

$$\Gamma_1 : L^1(J, E) \to C(J, E)$$

defined by

$$v \to \Gamma_1(v_1)(t) = \int_0^t T_1(t - s)v_1(s)ds.$$

From Lemma 1.8.4 it follows that $\Gamma_1 \circ S_{F_1}$ is closed graph. Moreover, we have that

$$\big(h_n^1(t) - T_1(t)\varphi(0)\big) \in \Gamma_1 \circ S_{F_1,x_n,y_n}.$$

Then there exists $v_*^1 \in S_{F_1,x_*,y_*}$

$$h_*^1(t) = T_1(t)\varphi(0) + \int_0^t T_1(t-s)v_*^1(s)ds.$$

Similarly we can prove that there exist $v_*^2 \in S_{F_2,x_*,y_*}$ such that

$$h_*^2(t) = T_2(t)\psi(0) + \int_0^t T_1(t-s)v_*^2(s)ds.$$

Then we have $(x_*, y_*, h_*) \in Graph(N)$.

Step 4: *(A priori bounds on solutions.)*
Now, it remains to show that the set

$$\Sigma = \Big\{ \ (x,y) \in C \times C : (x,y) = \lambda N(x,y), \lambda \in (0,1) \ \Big\} \text{ is bounded.}$$

We consider the norm

$$\|u\|_C = \sup_{\theta \in [-r,0]} \|u(\theta)\|.$$

Let $(x,y) \in \Sigma$. Then there exists $(v_1, v_2) \in S_{F_1,x,y} \times S_{F_2,x,y}$ such that

$$x(t) = \lambda T_1(t)\varphi(0) + \lambda \int_0^t T_1(t-s)v_1(s)ds, \ t \in J$$

and

$$y(t) = \lambda T_2(t)\psi(0) + \lambda \int_0^t T_2(t-s)v_2(s)ds, \ t \in J.$$

Then

$$\begin{aligned}
\|x(t)\| &\leq M\|\varphi(0)\| + M\int_0^t p_1(s)\psi_1(\|x_s\| + \|y_s\|)ds \\
&\quad + Mc_1 \int_0^t (\|x(s-\tau_1(s,x_s))\| + \|y(s-\tau_2(s,y_s))\|)ds \\
&\leq M\|\varphi(0)\| + Mc_1\int_0^t (\|x(t)\|_C + \|y(t)\|_C)ds + M\int_0^t p_1(s)\psi_1(\|x_s\| + \|y_s\|)ds
\end{aligned}$$

and

$$\|y(t)\| \leq M\|\psi(0)\| + Mc_2\int_0^t (\|x(t)\|_C + \|y(t)\|_C)ds + M\int_0^t p_2(s)\psi_2(\|x_s\| + \|y_s\|)ds.$$

Therefore

$$\begin{aligned}
\|x(t)\| + \|y(t)\| &\leq c + M((c_1 + c_2))\int_0^t (\|x(t)\|_C + \|y(t)\|_C)ds \\
&\quad + M\int_0^t (p_1(s)\psi_1(\|x_s\| + \|y_s\|)ds + M\int_0^t p_2\psi_2(\|x_s\| + \|y_s\|)ds,
\end{aligned}$$

where

$$c = M(\|\psi(0)\| + \|\varphi(0)\|) \text{ and } \phi = \psi_1 + \psi_2.$$

We consider the function μ defined by

$$\mu(t) := \sup\{\|x(s)\| + \|y(s)\| : -r \leq s \leq t\},\ t \in J.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = \|x(t^*)\| + \|y(t^*)\|$. If $t^* \in J$, by the previous inequality we have for $t \in J$

$$\mu(t) \quad \leq \quad c + \int_0^t \overline{m}(s)[(\phi(\mu(s)) + \mu(s)]ds.$$

If $t^* \in [-r, 0]$ then $\mu(t) = c$ and the inequality holds. Let us take the right-hand side of the above inequality as $v(t)$, then we have $v(0) = c$ and

$$v'(t) = \overline{m}(t)[\phi(\mu(t)) + \mu(t)].$$

Then for each $t \in J$ we have

$$v'(t) \quad \leq \quad \overline{m}(t)[\phi(v(t)) + v(t)], t \in J.$$

By using (H_1) we then have

$$\int_{v(0)}^{v(t)} \frac{du}{u + \phi(u)} \leq \int_0^t \overline{m}(s)ds \leq \int_0^b \overline{m}(s)ds < \int_{v(0)}^{\infty} \frac{du}{u + \phi(u)}.$$

This inequality implies that there exists a constant d such that

$$\|x\|_\infty \leq d \text{ and } \|y\|_\infty \leq d.$$

This shows that Σ is bounded. As a consequence of Theorem 1.8.11 we deduce that N has a fixed point $(x, y) \in C \times C$ which is a solution to the problem (6.1.1). □

6.2 Functional Differential Inclusions with Periodic Conditions

We consider the system with with periodic conditions of the following form:

$$\begin{cases} x'(t) - A_1x(t) \in F_1(t, x_t, x(t - \tau_1(t, x_t)), y_t, y(t - \tau_2(t, y_t))), & t \in J := [0, b] \\ y'(t) - A_2y(t) \in F_2(t, x_t, x(t - \tau_1(t, x_t)), y_t, y(t - \tau_2(t, y_t))), & t \in J := [0, b] \\ x_0(\theta) = x_b(\theta), & \theta \in [-r, 0] \\ y_0(\theta) = y_b(\theta), & \theta \in [-r, 0]. \end{cases} \tag{6.2.1}$$

where the operators A_i, $i = 1, 2$ are infinitesimal generator of a C_0-semigroup $T_i(t)_{t\geq 0}$ on a Banach space E,
$F_1, F_2 : J \times C([-r, 0], E) \times E \times C([-r, 0], E) \times E \to \mathcal{P}(E)$ are a multifunctions and

$\tau_i : [0,b] \times C([-r,0],E) \to [0,r]$, $i = 1,2$ are given continuous functions. Throughout this section the operators A_i, $i = 1,2$ are infinitesimal generator of a C_0-semigroup $T_i(t)_{t\geq 0}$ and there exists $M > 0$ such that

$$\|T(t)\| \leq M \quad \text{for all } t \in J.$$

Set $C_r := C([-r,0],E)$ and $C := C([-r,b],E)$.

6.2.1 Existence Results: $1 \in \rho(T(b))$

We give our main existence and uniqueness result for problem (6.2.1). Before starting and proving this result, we give the definition of its mild solution.

Definition 6.2.1. *A function $(x,y) \in C \times C$ is said to be a mild solution of problem (6.2.1) if there exists $v_1, v_2 \in L^1(J,E)$ such that $v_i \in F_i(t, x_t, x(t-\tau_1(t,x_t)), y_t, y(t-\tau_2(t,y_t)))$ a.e. on J such that*

$$x(t) = T_1(t)(I - T_1(b))^{-1} \int_0^b T_1(b-s)v_1(s)ds + \int_0^t T_1(t-s)v_1(s)ds,$$

$$y(t) = T_2(t)(I - T_2(b))^{-1} \int_0^b T_2(b-s)v_2(s)ds + \int_0^t T_2(t-s)v_2(s)ds,$$

$x_0 = x_b$ and $y_0 = y_b$.

Assume that the following conditions

(G_1) $F_i : J \times C([-r,b],E) \times E \times C([-r,b],E) \times E \to \mathcal{P}_{cp,cv}(E); t \mapsto F_i(t,u,\bar{u},v,\bar{v})$, $i = 1,2$ are measurable for each $u, v \in C$ and $\bar{u}, \bar{v} \in E$.

(G_2) There exist a functions $l_i, \bar{l}_i \in L^1(J,\mathbb{R}) \to \mathbb{R}_+$, $i = 1,2$ such that

$$\begin{aligned} H_d(F_i(t,u_1,u_2,v_1,v_2), F_i(t,\widetilde{u}_1,\widetilde{u}_2,\widetilde{v}_1,\widetilde{v}_2)) \quad \leq \quad & l_i(t)(\|u_1 - \widetilde{u}_1\| + \|v_1 - \widetilde{v}_1\|) \\ & + \bar{l}_i(t)(\|u_2 - \widetilde{u}_2\| + \|v_2 - \widetilde{v}_2\|), \end{aligned}$$

for every $t \in J$, $u_1, \widetilde{u}_1, v_1, \widetilde{v}_1 \in C$ and $u_2, \widetilde{u}_2, v_2, \widetilde{v}_2 \in E$

and

$$H_d(0, F_i(t,0,0,0,0)) \leq l_i(t), \text{ for all a.e. } t \in J \text{ and } i = 1,2.$$

Theorem 6.2.1. *Assume that (G_1) and (G_2) are satisfied and the matrix*

$$\overline{M} = (\tfrac{M^2}{\|I-T(b)\|_\infty} + M) \begin{pmatrix} \|l_1\|_{L^1} + \|\bar{l}_1\|_{L^1} & \|l_1\|_{L^1} + \|\bar{l}_1\|_{L^1} \\ \|l_2\|_{L^1} + \|\bar{l}_2\|_{L^1} & \|l_2\|_{L^1} + \|\bar{l}_2\|_{L^1} \end{pmatrix}$$

converge to zero, then the problem (6.2.1) has at least one solution.

Proof. Consider the operator $N : C \times C \to \mathcal{P}(C \times C)$ defined for $(x, y) \in C \times C$ by $N(x, y) =$

$$\left\{ (h_1, h_2) \in C \times C : \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} = \begin{pmatrix} T_1(t)(I - T_1(b))^{-1} \int_0^b T_1(b-s)v_1(s)ds \\ + \int_0^t T_1(t-s)v_1(s)ds \\ T_2(t)(I - T_2(b))^{-1} \int_0^b T_2(b-s)v_2(s)ds \\ + \int_0^t T_2(t-s)v_2(s)ds \end{pmatrix}, \text{for } t \in [0, b] \right\}$$

where $v_i \in S_{F_i,x,y} = \{f \in L^1(J, E) : f(t) \in F_i(t, x_t, x(t - \tau_1(t, x_t)), y_t, y(t - \tau_2(t, y_t))), a.e.$ $t \in J\}$. Clearly, fixed points of the operator N are solutions of problem (6.2.1). Let

$$N_1(x, y) = \left\{ h_1 \in C : h_1(t) = \left\{ \begin{array}{ll} T_1(t)(I - T_1(b))^{-1} \int_0^b T_1(b-s)v_1(s)ds & \\ + \int_0^t T_1(t-s)v_1(s)ds, & t \in J := [0, b] \end{array} \right. \right\}$$

and

$$N_2(x, y) = \left\{ h_2 \in C : h_2(t) = \left\{ \begin{array}{ll} T_2(t)(I - T_2(b))^{-1} \int_0^b T_2(b-s)v_2(s)ds & \\ + \int_0^t T_2(t-s)v_2(s)ds, & t \in J := [0, b] \end{array} \right. \right\}.$$

Hence

$$N(x, y) = (N_1(x, y), N_2(x, y)) \text{ for every } (x, y) \in C \times C.$$

Since, for each $(x, y) \in C \times C$, the nonlinearity F_i takes convex values, the selection set $S_{F_i,x,y}$ is convex, then N has convex values. We show N satisfies the assumptions of Theorem 1.8.10. Let $(x, y), (\widetilde{x}, \widetilde{y}) \in C^2 \times C^2$ and $(h_1, h_2) \in N(x, y)$. Then there exists $v_i \in F_i(t, x_t, x(t - \tau_1(t, x_t)), y_t, y(t - \tau_2(t, y_t)))$, $i = 1, 2$ such that

$$\begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} = \begin{pmatrix} T_1(t)(I - T_1(b))^{-1} \int_0^b T_1(b-s)v_1(s)ds \\ + \int_0^t T_1(t-s)v_1(s)ds \\ T_2(t)(I - T_2(b))^{-1} \int_0^b T_2(b-s)v_2(s)ds \\ + \int_0^t T_2(t-s)v_2(s)ds \end{pmatrix}, \text{for } t \in [0, b].$$

(G_2) implies that
$H_{d_1}(F_1(t,x_t,x(t-\tau_1(t,x_t)),y_t,y(t-\tau_2(t,y_t))),F_1(t,\widetilde{x}_t,\widetilde{x}(t-\tau_1(t,\widetilde{x}_t)),\widetilde{y}_t,\widetilde{y}(t-\tau_2(t,\widetilde{y}_t))))) \leq$

$$l_1(t)(\|x_t-\widetilde{x}_t\|+\|y_t-\widetilde{y}_t\|)+$$
$$\bar{l_1}(t)(\|x(t-\tau_1(t,x_t))-\widetilde{x}(t-\tau_1(t,\widetilde{x}_t))\|+\|y(t-\tau_2(t,y_t))-\widetilde{y}(t-\tau_2(t,\widetilde{y}_t))\|), t\in J$$

and
$H_{d_2}(F_2(t,x_t,x(t-\tau_1(t,x_t)),y_t,y(t-\tau_2(t,y_t))),F_2(t,\widetilde{x}_t,\widetilde{x}(t-\tau_1(t,\widetilde{x}_t)),\widetilde{y}_t,\widetilde{y}(t-\tau_2(t,\widetilde{y}_t))))) \leq$

$$l_2(t)(\|x_t-\widetilde{x}_t\|+\|y_t-\widetilde{y}_t\|)+$$
$$\bar{l_2}(t)(\|x(t-\tau_1(t,x_t))-\widetilde{x}(t-\tau_1(t,\widetilde{x}_t))\|+\|y(t-\tau_2(t,y_t))-\widetilde{y}(t-\tau_2(t,\widetilde{y}_t))\|), t\in J.$$

Hence, there is some

$$(w,\widetilde{w})\in F_1(t,\widetilde{x}_t,\widetilde{x}(t-\tau_1(t,\widetilde{x}_t)),\widetilde{y}_t,\widetilde{y}(t-\tau_2(t,\widetilde{y}_t)))\times F_2(t,\widetilde{x}_t,\widetilde{x}(t-\tau_1(t,\widetilde{x}_t)),\widetilde{y}_t,\widetilde{y}(t-\tau_2(t,\widetilde{y}_t)))$$

such that for each $t\in J$

$$\begin{aligned}\|v_1(t)-w(t)\| &\leq l_1(t)(\|x_t-\widetilde{x}_t\|+\|y_t-\widetilde{y}_t\|)+\\ &\bar{l_1}(t)(\|x(t-\tau_1(t,x_t))-\widetilde{x}(t-\tau_1(t,\widetilde{x}_t))\|+\|y(t-\tau_2(t,y_t))-\widetilde{y}(t-\tau_2(t,\widetilde{y}_t))\|)\end{aligned}$$

and

$$\begin{aligned}\|v_2(t)-\widetilde{w}(t)\| &\leq l_2(t)(\|x_t-\widetilde{x}_t\|+\|y_t-\widetilde{y}_t\|)+\\ &\bar{l_2}(t)(\|x(t-\tau_1(t,x_t))-\widetilde{x}(t-\tau_1(t,\widetilde{x}_t))\|+\|y(t-\tau_2(t,y_t))-\widetilde{y}(t-\tau_2(t,\widetilde{y}_t))\|).\end{aligned}$$

Consider the multi-valued maps $U_i: J\to\mathbb{R}$, $i=1,2$ defined by
$U_1(t)=\big\{f\in F_1(t,x_t,x(t-\tau_1(t,x_t)),y_t,y(t-\tau_2(t,y_t))):$

$$\begin{aligned}\|v_1(t)-w(t)\| &\leq l_1(t)(\|x_t-\widetilde{x}_t\|+\|y_t-\widetilde{y}_t\|)+\\ &\bar{l_1}(t)\|x(t-\tau_1(t,x_t))-\widetilde{x}(t-\tau_1(t,\widetilde{x}_t))\|+\\ &\bar{l_1}(t)\|y(t-\tau_2(t,y_t))-\widetilde{y}(t-\tau_2(t,\widetilde{y}_t))\|\big\}\end{aligned}$$

and

$U_2(t)=\big\{f\in F_2(t,\widetilde{x}_t,\widetilde{x}(t-\tau_1(t,\widetilde{x}_t)),\widetilde{y}_t,\widetilde{y}(t-\tau_2(t,\widetilde{y}_t))):$

$$\begin{aligned}\|v_2(t)-w(t)\| &\leq l_2(t)(\|x_t-\widetilde{x}_t\|+\|y_t-\widetilde{y}_t\|)+\\ &\bar{l_2}(t)\|x(t-\tau_1(t,x_t))-\widetilde{x}(t-\tau_1(t,\widetilde{x}_t))\|+\\ &\bar{l_2}(t)\|y(t-\tau_2(t,y_t))-\widetilde{y}(t-\tau_2(t,\widetilde{y}_t))\|\big\}.\end{aligned}$$

Then $U_i(t)$ are a nonempty set and Theorem III,4.1 in [25] tells us that U_i are measurable. Moreover, the multi-valued intersection operator
$V_i(.)=U_i(.)\bigcap F_i(.,x_.,x(.-\tau_1(.,x_.)),y_.,y(.-\tau_2(.,y_.)))$ are s measurable. Therefore, by

Lemma 1.8.2, there exists a function $t \mapsto \bar{v}_i$, which are a measurable selection for V_i, that is
$\widetilde{v}_i(t) \in F_2(t, \widetilde{x}_t, \widetilde{x}(t-\tau_1(t,\widetilde{x}_t)), \widetilde{y}_t, \widetilde{y}(t-\tau_2(t,\widetilde{y}_t)))$ and

$$\begin{aligned}\|v_1(t)-\widetilde{v}_1(t)\| \leq\ & l_1(t)(\|x_t-\widetilde{x}_t\|+\|y_t-\widetilde{y}_t\|)+\\ & \bar{l_1}(t)(\|x(t-\tau_1(t,x_t))-\widetilde{x}(t-\tau_1(t,\widetilde{x}_t))\|+\|y(t-\tau_2(t,y_t))-\widetilde{y}(t-\tau_2(t,\widetilde{y}_t))\|)\end{aligned}$$

and

$$\begin{aligned}\|v_2(t)-\widetilde{v}_2(t)\| \leq\ & l_2(t)(\|x_t-\widetilde{x}_t\|+\|y_t-\widetilde{y}_t\|)+\\ & \bar{l_2}(t)(\|x(t-\tau_1(t,x_t))-\widetilde{x}(t-\tau_1(t,\widetilde{x}_t))\|+\|y(t-\tau_2(t,y_t))-\widetilde{y}(t-\tau_2(t,\widetilde{y}_t))\|).\end{aligned}$$

Define $\widetilde{h}_1, \widetilde{h}_2$ by

$$\widetilde{h}_1(t) = T_1(t)(I-T_1(b))^{-1}\int_0^b T_1(b-s)\widetilde{v}_1(s)ds + \int_0^t T_1(t-s)\widetilde{v}_1(s)ds, t \in J$$

and

$$\widetilde{h}_2(t) = T_2(t)(I-T_2(b))^{-1}\int_0^b T_2(b-s)\widetilde{v}_2(s)ds + \int_0^t T_2(t-s)\widetilde{v}_2(s)ds, t \in J.$$

Then for $t \in J$, we have

$$\begin{aligned}\|h_1(t)-\widetilde{h}_1(t)\| \leq\ & \frac{M^2}{\|I-T(b)\|}\Big(\int_0^b l_1(s)(\|x_s-\widetilde{x}_s\|+\|y_s-\widetilde{y}_s\|)ds+\\ & \int_0^b \bar{l_1}(s)\|x(s-\tau_1(s,x_s))-\widetilde{x}(s-\tau_1(s,\widetilde{x}_s))\|ds+\\ & \int_0^b \bar{l_1}(s)\|y(s-\tau_2(s,y_s))-\widetilde{y}(s-\tau_2(s,\widetilde{y}_s))\|ds\Big)+\\ & M\Big(\int_0^t l_1(s)(\|x_s-\widetilde{x}_s\|+\|y_s-\widetilde{y}_s\|)ds+\\ & \int_0^t \bar{l_1}(s)\|x(s-\tau_1(s,x_s))-\widetilde{x}(s-\tau_1(s,\widetilde{x}_s))\|ds+\\ & \int_0^t \bar{l_1}(s)\|y(s-\tau_2(s,y_s))-\widetilde{y}(s-\tau_2(s,\widetilde{y}_s))\|ds\Big)\\ \leq\ & \Big(\frac{M^2}{\|I-T(b)\|}+M\Big)\Big(\int_0^b l_1(s)(\|x_s-\widetilde{x}_s\|+\|y_s-\widetilde{y}_s\|)ds+\\ & \int_0^b \bar{l_1}(s)\|x(s-\tau_1(s,x_s))-\widetilde{x}(s-\tau_1(s,\widetilde{x}_s))\|ds+\\ & \int_0^b \bar{l_1}(s)\|y(s-\tau_2(s,y_s))-\widetilde{y}(s-\tau_2(s,\widetilde{y}_s))\|ds\Big)\\ \leq\ & \Big(\frac{M^2}{\|I-T(b)\|}+M\Big)\big(\|l_1\|_{L^1}+\|\bar{l_1}\|_{L^1})(\|x-\widetilde{x}\|_\infty+\|y-\widetilde{y}\|_\infty).\end{aligned}$$

Thus

$$\|h_1 - \widetilde{h}_1\|_\infty \;\leq\; \big(\frac{M^2}{\|I-T(b)\|} + M\big)\big(\|l_1\|_{L^1} + \|\bar{l_1}\|_{L^1})(\|x-\widetilde{x}\|_\infty + \|y-\widetilde{y}\|_\infty).$$

By an analogous relation, we finally arrive at the estimate

$$H_{d_1}(N_1(x,y), N_1(\widetilde{x},\widetilde{y})) \leq \big(\frac{M^2}{\|I-T(b)\|} + M\big)\big(\|l_1\|_{L^1} + \|\bar{l_1}\|_{L^1})(\|x-\widetilde{x}\|_\infty + \|y-\widetilde{y}\|_\infty).$$

Similarly we have

$$H_{d_2}(N_2(x,y), N_2(\widetilde{x},\widetilde{y})) \leq \big(\frac{M^2}{\|I-T(b)\|} + M\big)\big(\|l_2\|_{L^1} + \|\bar{l_2}\|_{L^2})(\|x-\widetilde{x}\|_\infty + \|y-\widetilde{y}\|_\infty).$$

Therefore

$$H_d(N(x,y), N(\widetilde{x},\widetilde{y})) \leq \overline{M}\begin{pmatrix} \|x-\widetilde{x}\|_\infty \\ \|y-\widetilde{y}\|_\infty \end{pmatrix}, \text{ for all } (x,y),(\widetilde{x},\widetilde{y}) \in C^2 \times C^2.$$

Hence, by Theorem 1.8.10, the operator N has at least one fixed point which is solution of (6.2.1). □

Let $F_i : J \times C([-r,0],E) \times E \times C([-r,0],E) \times E \to \mathcal{P}_{cp,cv}(E)$, $i = 1,2$ are Carathéodory multimap which satisfies some of the following assumptions:

(H_1) There exists $p_i \in L^1([0,b],\mathbb{R}_+)$, $i = 1,2$ such that

$$\|F_i(t,u_1,v_1,u_2,v_2)\| \leq p_i(t)$$

for each $u_1, u_2 \in C, v_1, v_2 \in E$ and $t \in J$.

(H_2) The semigroup $T_i(.)$, $i = 1,2$ is compact for $t > 0$.

Theorem 6.2.2. *Assume that the hypotheses (H_1) and (H_2) hold. Then the set of solutions for problem (6.2.1) is nonempty and compact.*

Proof. Part 1. Consider the operator $N : C\times C \to \mathcal{P}(C\times C)$ defined for $(x,y) \in C\times C$ by $N(x,y) =$

$$\left\{ (h_1,h_2) \in C\times C : \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} = \begin{pmatrix} T_1(t)(I-T_1(b))^{-1}\displaystyle\int_0^b T_1(b-s)v_1(s)ds \\ +\displaystyle\int_0^t T_1(t-s)v_1(s)ds \\ T_2(t)(I-T_2(b))^{-1}\displaystyle\int_0^b T_2(b-s)v_2(s)ds \\ +\displaystyle\int_0^t T_2(t-s)v_2(s)ds \end{pmatrix}, \text{for } t \in [0,b] \right\}$$

where $v_i \in S_{F_i,x,y} = \{f \in L^1(J,E) : f(t) \in F_i(t, x_t, x(t-\tau_1(t,x_t)), y_t, y(t-\tau_2(t,y_t))), a.e.\ t \in J\}$. Clearly, fixed points of the operator N are solutions of problem (6.2.1). Let

$$N_1(x,y) = \left\{ h_1 \in C : h_1(t) = \left\{ \begin{array}{ll} T_1(t)(I - T_1(b))^{-1} \displaystyle\int_0^b T_1(b-s)v_1(s)ds & \\ + \displaystyle\int_0^t T_1(t-s)v_1(s)ds, & t \in J := [0,b] \end{array} \right. \right\}$$

and

$$N_2(x,y) = \left\{ h_2 \in C : h_2(t) = \left\{ \begin{array}{ll} T_2(t)(I - T_2(b))^{-1} \displaystyle\int_0^b T_2(b-s)v_2(s)ds & \\ + \displaystyle\int_0^t T_2(t-s)v_2(s)ds, & t \in J := [0,b] \end{array} \right. \right\}.$$

Hence

$$N(x,y) = (N_1(x,y), N_2(x,y)) \text{ for every } (x,y) \in C \times C.$$

Since, for each $(x,y) \in C \times C$, the nonlinearity F_i takes convex values, the selection set $S_{F_i,x,y}$ is convex, then N has convex values. From (H_1) and (H_2), we can prove that N is completely continuous. The proof will be given in several steps.

Step 1: N maps bounded sets into bounded sets in $C \times C$. Indeed, it is enough to show that for any $q > 0$ such that for each $h = (h_1, h_2) \in N(x,y)$ there exists a positive constant l such that for each $(x,y) \in B_q = \{(x,y) \in C \times C : \|x\|_\infty \le q, \|y\|_\infty \le q\}$, we have

$$\|h\|_\infty \le l = (l_1, l_2).$$

If $h = (h_1, h_2) \in N(x,y)$ then there exists $v_i \in S_{F_i,x,y}$, $i = 1,2$ such that for each $t \in [0,b]$, we get

$$h_1(t) = T_1(t)(I - T_1(b))^{-1} \int_0^b T_1(b-s)v_1(s)ds + \int_0^t T_1(t-s)v_1(s)ds$$

and

$$h_2(t) = T_2(t)(I - T_2(b))^{-1} \int_0^b T_2(b-s)v_2(s)ds + \int_0^t T_2(t-s)v_2(s)ds.$$

By (H_1) we have for each $t \in J$

$$\begin{aligned} \|h_1(t)\| &\le \|T_1(t)(I - T_1(b))^{-1} \int_0^b T_1(b-s)v_1(s)ds\| + \|\int_0^t T_1(t-s)v_1(s)ds\| \\ &\le M^2 \|(I - T_1(b))^{-1}\|_\infty \int_0^b p_1(s)ds + M \int_0^t p_1(s)ds. \end{aligned}$$

Then

$$\|h_1\|_\infty \leq (M^2\|(I-T_1(b))^{-1}\|_\infty + M)\int_0^b p_1(s)ds := l_1.$$

Similarly, we have

$$\|h_2\|_\infty \leq (M^2\|(I-T_2(b))^{-1}\|_\infty + M)\int_0^b p_2(s)ds := l_2.$$

Step 2: N maps bounded sets into equicontinuous sets of $C\times C$. Let $0<\tau_1,\tau_2\in J$, $\tau_1<\tau_2$ and B_q be a bounded set of $C\times C$ as in Step 1. Let $(x,y)\in B_q$ and $h=(h_1,h_2)\in N(x,y)$, then there exists $v_i\in S_{F_i,x,y}$, $i=1,2$ such that

$$\begin{aligned}\|h_1(\tau_2)-h_1(\tau_1)\| \leq\ & \|T_1(\tau_2)-T_1(\tau_1)\|(I-T_1(b))^{-1}\int_0^b T_1(b-s)p_1(s)ds\\ &+\int_0^{\tau_1-\epsilon}\|T_1(\tau_2-s)-T_1(\tau_1-s)\|p_1(s)ds\\ &+\int_{\tau_1}^{\tau_1-\epsilon}\|T_1(\tau_2-s)-T_1(\tau_1-s)\|p_1(s)ds\\ &+\int_{\tau_1}^{\tau_2}\|T_1(\tau_2-s)\|p_1(s)ds.\end{aligned}$$

The right-hand side tends to zero as $\tau_2-\tau_1\to 0$, and ϵ sufficiently small, since $T(t)$ is a strongly continuous operator and the compactness of $T(t)$ for $t>0$ implies the continuity in the uniform operator topology.
As a consequence of Steps 1, 2 and the Arzelá -Ascoli theorem we can conclude that $N_i : C\times C\to \mathcal{P}_{cp,cv}(C)$ are a completely continuous operator.

Step 3: N has a closed graph.

Let (x_n,y_n) be a sequence such that $(x_n,y_n)\to(x_*,y_*)$, $h_n\in N(x_n,y_n)$ and $h_n:=(h_n^1,h_n^2)\to h_*:=(h_*^1,h_*^2)$ as $n\to\infty$. We shall prove that $h_*\in N(x_*,y_*)$. Now $h_n\in N(x_n,y_n)$ means there exist $v_n^i\in S_{F_i,x_n,y_n}$, $i=1,2$ such that

$$h_n^1(t)=T_1(t)(I-T_1(b))^{-1}\int_0^b T_1(b-s)v_n^1(s)ds+\int_0^t T_1(t-s)v_n^1(s)ds,\ t\in J$$

and

$$h_n^2(t)=T_2(t)(I-T_2(b))^{-1}\int_0^b T_2(b-s)v_n^2(s)ds+\int_0^t T_2(t-s)v_n^2(s)ds,\ t\in J.$$

Consider the linear continuous operator

$$\Gamma_1 : L^1(J,E)\to C(J,E)$$

defined by

$$v \to \Gamma_1(v)(t) = T_1(t)(I - T_1(b))^{-1} \int_0^b T_1(b-s)v(s)ds + \int_0^t T_1(t-s)v(s)ds.$$

From Lemma 1.8.4 it follows that $\Gamma_1 \circ S_{F_1}$ is closed graph. Moreover, we have that

$$\left(h_n^1(t) - T_1(t)(I - T_1(b))^{-1} \int_0^b T_1(b-s)v_1(s)ds\right) \in \Gamma_1 \circ S_{F_1,x_n,y_n}.$$

Then there exists $v_*^1 \in S_{F_1,x_*,y_*}$

$$h_*^1(t) = T_1(t)(I - T_1(b))^{-1} \int_0^b T_1(b-s)v_*^2(s)ds + \int_0^t T_1(t-s)v_*^2(s)ds.$$

Similarly we can prove that there exist $v_*^2 \in S_{F_2,x_*,y_*}$ such that

$$h_*^2(t) = T_2(t)(I - T_2(b))^{-1} \int_0^b T_2(b-s)v_*^2(s)ds + \int_0^t T_2(t-s)v_*^2(s)ds.$$

Then we have $(x_*, y_*, h_*) \in Graph(N)$.

Step 4: *(A priori bounds on solutions.)*
Now, it remains to show that the set

$$\Sigma = \left\{ \ (x,y) \in C \times C : (x,y) = \lambda N(x,y), \lambda \in (0,1) \ \right\} \text{is bounded.}$$

We consider the norm

$$\|u\|_C = \sup_{\theta \in [-r,0]} \|u(\theta)\|.$$

Let $(x,y) \in \Sigma$. Then there exists $(v_1, v_2) \in S_{F_1,x,y} \times S_{F_2,x,y}$ such that

$$x(t) = \lambda T_1(t)(I - T_1(b))^{-1} \int_0^b T_1(b-s)v_1(s)ds + \lambda \int_0^t T_1(t-s)v_1(s)ds, \ t \in J$$

and

$$y(t) = \lambda T_2(t)(I - T_2(b))^{-1} \int_0^b T_2(b-s)v_2(s)ds + \lambda \int_0^t T_2(t-s)v_2(s)ds, \ t \in J.$$

Then

$$\begin{aligned}
\|x(t)\| &\leq \|T_1(t)(I - T_1(b))^{-1} \int_0^b T_1(b-s)v_1(s)ds\| + \|\int_0^t T_1(t-s)v_1(s)ds\| \\
&\leq M^2\|(I - T_1(b))^{-1}\|_\infty \int_0^b p_1(s)ds + M \int_0^t p_1(s)ds \\
&\leq (M^2\|(I - T_1(b))^{-1}\|_\infty + M) \int_0^b p_1(s)ds := K_1
\end{aligned}$$

and

$$\begin{aligned}\|y(t)\| &\leq \|T_2(t)(I-T_2(T))^{-1}\int_0^b T_2(b-s)v_2(s)ds\| + \|\int_0^t T_2(t-s)v_2(s)ds\| \\ &\leq M^2\|(I-T_2(b))^{-1}\|_\infty \int_0^b p_2(s)ds + M\int_0^t p_2(s)ds \\ &\leq (M^2\|(I-T_2(b))^{-1}\|_\infty + M)\int_0^b p_2(s)ds := K_2.\end{aligned}$$

Then

$$\|x(t)\| + \|y(t)\| \leq K,$$

where

$$K = K_1 + K_2.$$

This shows that Σ is bounded. As a consequence of Theorem 1.8.11 we deduce that N has a fixed point $(x,y) \in C \times C$ which is a solution to the problem (6.2.1).

Part 2. Compactness of the solution set. Let

$$S = \{(x,y) \in C \times C : (x,y) \text{ is a solution of } (6.2.1)\}$$

is compact. From Part 1, $S \neq \emptyset$ and there exists l such that for every $(x,y) \in S$, $\|(x,y)\| \leq l$. Since N is completely continuous, then $N(S)$ is relatively compact in C. Let $(x,y) \in S$; then $(x,y) \in N((x,y))$ and $S \subset \overline{N(S)}$. It remains to prove that S is a closed set in C. Let $(x_n, y_n) \in S$ such that (x_n, y_n) converge to (x,y). For every $n \in N$, there exists $v_n^i(t) \in F_i(t, x_{nt}, x_n(t-\tau_1(t,x_{nt})), y_{nt}, y_n(t-\tau_2(t,y_{nt})))$, a.e. $t \in J$ such that

$$x_n(t) = T_1(t)(I-T_1(b))^{-1}\int_0^b T_1(b-s)v^1(s)ds + \int_0^t T_1(t-s)v^1(s)ds$$

and

$$y_n(t) = T_2(t)(I-T_2(b))^{-1}\int_0^b T_2(b-s)v^2(s)ds + \int_0^t T_2(t-s)v^2(s)ds.$$

implies that for a.e. $t \in J$ $v_n^i(t) \in B(0, p_i(t))$. Since E is reflexive, $(v_n^i)_{n\in\mathbb{N}}$ is semi-compact. By Lemma (1.8.7), there exists a subsequence, still denoted by $(v_n^i)_{n\in\mathbb{N}}$, which converges weakly to some limit $v^i \in L^1(J,E)$. Moreover, the mapping $\Gamma : L^1(J,E) \to C(J,E)$ defined by

$$\Gamma(g^i)(t) = \int_0^t T(t-s)g^i(s)ds$$

is a continuous linear operator. Then it remains continuous if these spaces are endowed with their weak topologies [23]. Therefore for a.e. $t \in J$, the sequence $(x_n(t), y_n(t))$ converge to $(x(t), y(t))$, it follows that

$$x(t) = T_1(t)(I - T_1(b))^{-1} \int_0^b T_1(b-s)v^1(s)ds + \int_0^t T_1(t-s)v^1(s)ds,$$

and

$$y(t) = T_2(t)(I - T_2(b))^{-1} \int_0^b T_2(b-s)v^2(s)ds + \int_0^t T_2(t-s)v^2(s)ds$$

It remains to prove that $v^i \in L^1(J, E)$, for a.e. $t \in J$. Lemma (1.8.8) yields the existence of constants $\alpha^n_{i'} \geq 0$, $i' = n, ..., k(n)$ such that $\Sigma_{i'=1}^{k(n)} \alpha^n_{i'} = 1$ and the sequence of convex combinations $g^i_n(.) = \Sigma_{i'=1}^{k(n)} \alpha^n_{i'} v^i_{i'}(.)$ converges strongly to some limit v^i in L^1. Since F_i takes convex values, using Lemma (1.8.6), we obtain that

$$\begin{aligned} v^i(t) &\in \bigcap_{n\geq 1} \overline{\{g^i_k(t), k \geq n\}}, \text{ a.e } t \in J \\ &\subset \bigcap_{n\geq 1} \overline{co}\{v^i_k(t), k \geq n\} \\ &\subset \bigcap_{n\geq 1} \overline{co}\{\bigcup_{k\geq n} F_i(t, x_{kt}, x_k(t - \tau_1(t, x_{kt})), y_{kt}, y_k(t - \tau_2(t, y_{kt})))\} \\ &= \overline{co}(\lim_{k\to\infty} \sup F_i(t, x_{kt}, x_k(t - \tau_1(t, x_{kt})), y_{kt}, y_k(t - \tau_2(t, y_{kt})))). \quad (6.2.2) \end{aligned}$$

Since F_i is u.s.c. and has compact values, then by Lemma (1.8.5), we have

$$\lim_{n\to\infty} \sup F_i(t, x_{nt}, x_n(t - \tau_1(t, x_{nt})), y_{nt}, y_n(t - \tau_2(t, y_{nt}))) =$$
$F_i(t, x_t, x(t - \tau_1(t, x_t)), y_t, y(t - \tau_2(t, y_t)))$, for a.e $t \in J$.

This with (6.2.2) imply that

$$v^i(t) \in \overline{co} F_i(t, x_t, x(t - \tau_1(t, x_t)), y_t, y(t - \tau_2(t, y_t))).$$

Since $F_i(., ., ., .)$ has closed, convex values, we deduce that

$$v^i(t) \in F_i(t, x_t, x(t - \tau_1(t, x_t)), y_t, y(t - \tau_2(t, y_t))), \text{ for a.e } t \in J,$$

as claimed. Hence $(x, y) \in S$ which proves that S is closed, hence compact in $C \times C$. □

Chapter 7

Impulsive Boundary Value Problem with Parameter

7.1 Introduction

In this chapter, we prove the existence, uniqueness and multiplicity of solutions for second order impulsive differential equations with parameter. We consider the impulsive periodic boundary value problem with a parameter

$$y'' - \rho^2 y = -f(t, y, \lambda),\ t \in J := [0, 2\pi],\ t \neq t_k,\ k = 1, ..., m, \tag{7.1.1}$$

$$y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \quad t = 1, ..., m, \tag{7.1.2}$$

$$y'(t_k^+) - y'(t_k^-) = \bar{I}_k(y(t_k^-)), \quad t = 1, ..., m, \tag{7.1.3}$$

$$y(0) = y(2\pi), \quad y'(0) = y'(2\pi). \tag{7.1.4}$$

where $\rho \in \mathbb{R}^*$, λ is a real parameter, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, I_k, $\bar{I}_k \in C(\mathbb{R}, \mathbb{R})$, $t_k \in [0, 2\pi]$, $0 = t_0 < t_1 < ... < t_m < t_{m+1} = 2\pi$, $y(t_k^+) = \lim_{h \to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h \to 0^+} y(t_k - h)$ represent the right and left limits of $y(t)$ at $t = t_k$.

7.2 Uniqueness of Solution

In this section we give our main existence and uniqueness result for problem (7.1.1)-(7.1.4). We begin by giving the definition of the solution of this problem.

Definition 7.2.1. *A function $y \in PC(J, \mathbb{R}) \cap AC^1((t_k, t_{k+1}), \mathbb{R})$, $k = 0, ..., m$, is said to be a solution of (7.1.1)-(7.1.4) if y satisfies the equation $y'' - \rho^2 y = -f(t, y, \lambda)$*

a.e, on $J\backslash\{t_1,...,t_m\}$ *and the conditions* $\Delta y|_{t=t_k} = I_k(y(t_k^-))$ *and* $\Delta y'|_{t=t_k} = \bar{I}_k(y(t_k^-))$, $k = 1, 2, ..., m$ *and*
$y(0) = y(2\pi), \quad y'(0) = y'(2\pi)$.

Lemma 7.2.1. [99] $y \in PC(J,\mathbb{R}) \cap AC^1((t_k, t_{k+1}),\ k = 0, ..., m$, *is a solution of (7.1.1)-(7.1.4)), if and only if* $y \in PC(J,\mathbb{R})$ *is a solution of the following impulsive integral equation*

$$y(t) = \begin{cases} \int_0^{2\pi} G(t,s) f(t, y(s), \lambda) ds \\ -\sum_{k=1}^{m} \left[G(t,t_k) I_k(y(t_k)) + L(t,t_k)\bar{I}_k(y(t_k))\right], \qquad t \in J, \end{cases} \tag{7.2.1}$$

where

$$G(t,s) = \frac{1}{2\rho(e^{2\rho\pi}-1)} \begin{cases} e^{\rho(t-s)} + e^{\rho(2\pi-t+s)}, & 0 \le s \le t \le 2\pi, \\ e^{\rho(s-t)} + e^{\rho(2\pi-s+t)}, & 0 \le t \le s \le 2\pi, \end{cases} \tag{7.2.2}$$

and

$$L(t,s) = \frac{\partial}{\partial t} G(t,s) = \frac{1}{2(e^{2\rho\pi}-1)} \begin{cases} e^{\rho(2\pi-t+s)} - e^{\rho(t-s)}, & 0 \le s \le t \le 2\pi, \\ e^{\rho(s-t)} - e^{\rho(2\pi-s+t)}, & 0 \le t \le s \le 2\pi. \end{cases} \tag{7.2.3}$$

We now give conditions that will be needed in our main theorem in this section.

(H_1) There exists a constant $d \ge 0$ such that

$$|f(t,y,\lambda) - f(t,\bar{y},\lambda)| \le d|y - \bar{y}|,$$

for each $t \in J$, $\forall\ y, \bar{y}, \lambda \in \mathbb{R}$.

(H_2) There exist constants $c_k \ge 0$ such that

$$|I_k(y) - I_k(\bar{y})| \le c_k|y - \bar{y}|, \quad \text{for each } k = 1, ..., m,\ \forall\ y, \bar{y} \in \mathbb{R}.$$

(H_3) There exist constants $\bar{c}_k \ge 0$ such that

$$|\bar{I}_k(y) - \bar{I}_k(\bar{y})| \le \bar{c}_k|y - \bar{y}|, \quad \text{for each } k = 1, ..., m,\ \forall\ y, \bar{y} \in \mathbb{R}.$$

Theorem 7.2.2. *Assume that conditions* (H_1)*-*(H_3) *are satisfied and*

$$2\pi d \sup_{(t,s)\in J\times J} |G(t,s)| + \sum_{k=1}^{m} \left(c_k \sup_{t\in J} |G(t,t_k)| + \bar{c}_k \sup_{t\in J} |L(t,t_k)|\right) < 1. \tag{7.2.4}$$

Then the problem (7.1.1)-(7.1.4) has a unique solution on J.

Proof. We transform the problem (7.1.1)-(7.1.4) into a fixed point problem. Consider the operator $N : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ defined by,

$$N(y)(t) = \begin{cases} \displaystyle\int_0^{2\pi} G(t,s)f(t,y(s),\lambda)ds \\ \displaystyle -\sum_{k=1}^{m} \big[G(t,t_k)I_k(y(t_k)) + L(t,t_k)\bar{I}_k(y(t_k))\big], \qquad t \in J, \end{cases} \tag{7.2.5}$$

As we saw in Lemma 7.2.1 the fixed points of N are solutions to (7.1.1)-(7.1.4). We will show that N is a contraction, so let $y, \bar{y} \in PC(J,\mathbb{R})$. Then, for $t \in J$ and $\lambda \in \mathbb{R}$, we have

$$\begin{aligned}
|N(y)(t) - N(\bar{y})(t)| &\leq \int_0^{2\pi} |G(t,s)||f(t,y(s),\lambda) - f(t,\bar{y}(s),\lambda)|ds \\
&+ \sum_{k=1}^{m} |G(t,t_k)||I_k(y(t_k)) - I_k(\bar{y}(t_k))| \\
&+ \sum_{k=1}^{m} |L(t,t_k)||\bar{I}_k(y(t_k)) - \bar{I}_k(\bar{y}(t_k))| \\
&\leq d \sup_{(t,s)\in J\times J} |G(t,s)| \int_0^{2\pi} |y(s) - \bar{y}(s)|ds \\
&\quad + \sum_{k=1}^{m} c_k \sup_{t\in J} |G(t,t_k)||y(t_k) - \bar{y}(t_k)| \\
&\quad + \sum_{k=1}^{m} \bar{c}_k \sup_{t\in J} |L(t,t_k)||y(t_k) - \bar{y}(t_k)| \\
&\leq 2\pi d \sup_{(t,s)\in J\times J} |G(t,s)|\|y - \bar{y}\|_{PC} \\
&\quad + \sum_{k=1}^{m} \big[c_k \sup_{t\in J} |G(t,t_k)| + \bar{c}_k \sup_{t\in J} |L(t,t_k)|\big] \|y - \bar{y}\|_{PC} \\
&\leq \Big[2\pi d \sup_{(t,s)\in J\times J} |G(t,s)| + \sum_{k=1}^{m} \big(c_k \sup_{t\in J} |G(t,t_k)| + \bar{c}_k \sup_{t\in J} |L(t,t_k)|\big)\Big] \|y - \bar{y}\|_{PC} \\
&\leq \theta\|y - \bar{y}\|_{PC}.
\end{aligned}$$

Thus

$$\|N(y) - N(\bar{y})\|_{PC} \leq \theta\|y - \bar{y}\|_{PC},$$

where

$$\theta = \Big[2\pi d \sup_{(t,s)\in J\times J} |G(t,s)| + \sum_{k=1}^{m} \big(c_k \sup_{t\in J} |G(t,t_k)| + \bar{c}_k \sup_{t\in J} |L(t,t_k)|\big)\Big]$$

Since $\theta < 1$, N is a contraction. By the Banach fixed point theorem 1.6.1 we conclude that N has a unique fixed point in $PC(J,\mathbb{R})$ and the problem (7.1.1), (7.1.4) has a unique solution on $[0, 2\pi]$. □

7.3 Existence and Compactness of the Solution Set

In this section, we give conditions under which the problem (7.1.1)-(7.1.4) has a solution and the set of solutions to the problem form a compact set.

Theorem 7.3.1. *Assume that the following conditions hold:*

(H_4) $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is an Carathéodory function and* I_k, $\bar{I}_k \in C(\mathbb{R}, \mathbb{R})$.

(H_5) *There exists* $P \in L^1(J, \mathbb{R}_+)$ *and a constants* $\alpha \in [0, 1)$ *such that*

$$|f(t, y, \lambda)| \leq P(t)|y|^{\alpha}, \text{ for each } y, \lambda \in \mathbb{R} \text{ and } t \in J.$$

(H_6) *There exist constants* $d_k > 0$ *and* $\alpha_k \in [0, 1)$ *such that*

$$|I_k(y)| \leq d_k|y|^{\alpha_k}, \quad \text{for each } y \in \mathbb{R},\ k = 1, ..., m.$$

(H_7) *There exist constants* $\bar{d}_k > 0$ *and* $\bar{\alpha}_k \in [0, 1)$ *such that*

$$|\bar{I}_k(y)| \leq \bar{d}_k|y|^{\bar{\alpha}_k}, \quad \text{for each } y \in \mathbb{R},\ k = 1, ..., m.$$

Then the problem (7.1.1)-(7.1.4) has at least one solution and the solutions set

$$S = \{y \in PC(J, \mathbb{R}) : y \text{ is a solution of (7.1.1)-(7.1.4)}\}$$

is compact.

Proof. We again transform the problem (7.1.1)-(7.1.4) into a fixed point problem. Consider the operator N defined in the proof of Theorem 7.2.2. In order to apply Schaefers fixed point theorem 1.6.2, we first show that N is completely continuous. The proof of this fact will be given in three steps.

Step 1: N maps bounded sets into bounded sets in $PC(J, \mathbb{R})$.

It suffices to show that for any $q > 0$ there exists a positive constant ℓ such that for each $y \in B_q = \{u \in PC(J, \mathbb{R}) : |u|_{PC} \leq q\}$ we have $\|N(y)\|_{PC} \leq \ell$. For each $t \in [0, b]$, we have

$$
\begin{aligned}
|N(y)(t)| &\leq \int_0^{2\pi} |G(t,s)||f(t,y(s),\lambda)|ds \\
&+ \sum_{k=1}^{m} |G(t,t_k)||I_k(y(t_k))| \\
&+ \sum_{k=1}^{m} |L(t,t_k)||\bar{I}_k(y(t_k))| \\
&\leq \sup_{(t,s)\in J\times J} |G(t,s)| q^{\alpha} \int_0^{2\pi} p(s)ds + \sum_{k=1}^{m} \sup_{t\in J} |G(t,t_k)| d_k q^{\alpha_k} \\
&\quad + \sum_{k=1}^{m} \sup_{t\in J} |L(t,t_k)| \bar{d}_k q^{\bar{\alpha}_k}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|N(y)\|_{PC} &\leq \sup_{(t,s)\in J\times J} |G(t,s)| q^{\alpha} \|p\|_{L^1} + \sum_{k=1}^{m} \sup_{t\in J} |G(t,t_k)| d_k q^{\alpha_k} \\
&\quad + \sum_{k=1}^{m} \sup_{t\in J} |L(t,t_k)| \bar{d}_k q^{\bar{\alpha}_k} := \ell
\end{aligned}
$$

as desired. **Step 2:** N maps bounded sets into equicontinuous sets of $PC(J,\mathbb{R})$.

Let $0 < \tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$ and B_q be a bounded set of $PC(J,\mathbb{R})$ as in Step 1. Let $y \in B_q$ then for each $t \in J$ we have

$$
\begin{aligned}
|N(y)(\tau_2) - N(y)(\tau_1)| &\leq q^{\alpha} \int_0^{\tau_1} \big[|G(\tau_2,s) - G(\tau_1,s)|\big] p(s)ds \\
&\quad + q^{\alpha} \int_{\tau_1}^{\tau_2} |G(\tau_2,s)| p(s)ds \\
&\quad + \sum_{\tau_1<t<\tau_2} |G(\tau_2,t_k) - G(\tau_1,t_k)| d_k q^{\alpha_k} \\
&\quad + \sum_{\tau_1<t<\tau_2} |L(\tau_2,t_k) - L(\tau_1,t_k)| \bar{d}_k q^{\bar{\alpha}_k}.
\end{aligned}
$$

The right-hand side tends to zero as $\tau_2 - \tau_1 \to 0$, we have the equicontinuity in case where $t \neq t_i$, $i = 1, ..., m+1$. It remains to examine the equicontinuity at $t = t_i$. First we prove equicontinuity at $t = t_i^-$.

Fix $\delta_1 > 0$ such that $\{t_k : k \neq i\} \cap [t_i - \delta_1, t_i + \delta_1] = \varnothing$.

For $0 < h < \delta_1$, we have that

$$\begin{aligned}
|N(y)(t_i) - N(y)(t_i - h)| &\leq q^\alpha \int_0^{t_i - h} \big[|G(t_i, s) - G(t_i - h, s)|\big] p(s) ds \\
&+ q^\alpha \int_{t_i - h}^{t_i} |G(t_i, s)| p(s) ds \\
&+ \sum_{k=1}^{i-1} |G(t_i, t_k) - G(t_i - h, t_k)| d_k q^{\alpha_k} \\
&+ \sum_{k=1}^{i-1} |L(t_i, t_k) - L(t_i - h, t_k)| \bar{d}_k q^{\bar{\alpha}_k}.
\end{aligned}$$

and we see that the right-hand side tends to zero as $h \to 0$.
To prove equicontinuity at $t = t_i^+$. Fix $\delta_2 > 0$ such that $\{t_k : k \neq i\} \cap [t_i - \delta_2, t_i + \delta_2] = \varnothing$.
For $0 < h < \delta_2$, we have that

$$\begin{aligned}
|N(y)(t_i + h) - N(y)(t_i)| &\leq q^\alpha \int_0^{t_i} \big[|G(t_i + h, s) - G(t_i, s)|\big] p(s) ds \\
&+ q^\alpha \int_{t_i}^{t_i + h} |G(t_i + h, s)| p(s) ds \\
&+ \sum_{0 < t_k \leq t_i} |G(t_i + h, t_k) - G(t_i, t_k)| d_k q^{\alpha_k} \\
&+ \sum_{t_i < t \leq t_i + h} |G(t_i + h, t_k)| d_k q^{\alpha_k} \\
&+ \sum_{0 < t_k \leq t_i} |L(t_i + h, t_k) - L(t_i, t_k)| \bar{d}_k q^{\bar{\alpha}_k} \\
&+ \sum_{t_i < t \leq t_i + h} |L(t_i + h, t_k)| \bar{d}_k q^{\bar{\alpha}_k},
\end{aligned}$$

and again the right-hand side tends to zero as $h \to 0$.
Step 3: N is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $PC(J, \mathbb{R})$. Then there is an integer q such that $|y_n| \leq q$ for all $n \in \mathbb{N}$ and $|y| \leq q$. Hence, $y_n \in B_q$ and $y \in B_q$.

We have then by the dominated convergence theorem

$$\begin{aligned}|N(y_n)(t)-N(y)(t)| &\leq \int_0^{2\pi}|G(t,s)||f(t,y_n(s),\lambda)-f(t,y(s),\lambda)|ds\\ &\quad +\sum_{k=1}^{m}|G(t,t_k)||I_k(y_n(t_k))-I_k(y(t_k))|\\ &\quad +\sum_{k=1}^{m}|L(t,t_k)||\bar{I}_k(y_n(t_k))-\bar{I}_k(y(t_k))|\\ &\leq \sup_{(t,s)\in J\times J}|G(t,s)|\int_0^{2\pi}|f(t,y_n(s),\lambda)-f(t,y(s),\lambda)|ds\\ &\quad +\sum_{k=1}^{m}c_k\sup_{t\in J}|G(t,t_k)||I_k(y_n(t_k))-I_k(y(t_k))|\\ &\quad +\sum_{k=1}^{m}\bar{c}_k\sup_{t\in J}|L(t,t_k)||\bar{I}_k(y_n(t_k))-\bar{I}_k(y(t_k))|\to 0,\end{aligned}$$

as $n\to\infty$. Thus, N is continuous.
As a consequence of Steps 1 to 3 and the Arzel-Ascoli theorem we can conclude that $N: PC(J,\mathbb{R})\to PC(J,\mathbb{R})$ is a completely continuous operator.

Step 4: *(Now, it remains to show that the set)*

$$\Gamma(N)=\left\{\ y\in PC(J,\mathbb{R}): y=\bar{\lambda}N(y)\text{for some } 0<\bar{\lambda}<1\ \right\}$$

is bounded. Let $y\in\Gamma(N)$. Then $y=\bar{\lambda}N(y)$ for some $0<\bar{\lambda}<1$, Thus, for each $t\in J$,

$$\begin{aligned}|y(t)| &\leq \int_0^{2\pi}|G(t,s)||f(t,y(s),\lambda)|ds\\ &+ \sum_{k=1}^{m}|G(t,t_k)||I_k(y(t_k))|\\ &+ \sum_{k=1}^{m}|L(t,t_k)||\bar{I}_k(y(t_k))|\\ &\leq \int_0^{2\pi}\sup_{(t,s)\in J\times J}|G(t,s)|p(s)|y(s)|^{\alpha}ds+\sum_{k=1}^{m}\sup_{t\in J}|G(t,t_k)|d_k|y(t_k)|^{\alpha_k}\\ &\quad +\sum_{k=1}^{m}\sup_{t\in J}|L(t,t_k)|\bar{d}_k|y(t_k)|^{\bar{\alpha}_k}.\end{aligned}$$

This implies by (H_5)-(H_7) that for each $t \in J$ and $y \in \Gamma$, we have

$$\begin{aligned}
\|y\|_{PC} &\leq \sup_{(t,s)\in J\times J} |G(t,s)\|p\|_{L^1}\|y\|_{PC}^{\alpha} + \sum_{k=1}^{m}\sup_{t\in J}|G(t,t_k)|d_k\|y\|_{PC}^{\alpha_k} \\
&\quad + \sum_{k=1}^{m}\sup_{t\in J}|L(t,t_k)|\bar{d}_k\|y\|_{PC}^{\bar{\alpha}_k} \\
&\leq \sup_{(t,s)\in J\times J} |G(t,s)\|p\|_{L^1}\|y\|_{PC}^{\beta} + \sum_{k=1}^{m}\sup_{t\in J}|G(t,t_k)|d_k\|y\|_{PC}^{\beta} \\
&\quad + \sum_{k=1}^{m}\sup_{t\in J}|L(t,t_k)|\bar{d}_k\|y\|_{PC}^{\beta},
\end{aligned}$$

where $\beta = \max\{\alpha, \alpha_k, \bar{\alpha}_k\}$, $k = 1, ..., m$.
If

$$\|y\|_{PC} > 1,$$

then we have

$$\begin{aligned}
\|y\|_{PC}^{1-\beta} &\leq \sup_{(t,s)\in J\times J} |G(t,s)\|p\|_{L^1} + \sum_{k=1}^{m}\sup_{t\in J}|G(t,t_k)|d_k \\
&\quad + \sum_{k=1}^{m}\sup_{t\in J}|L(t,t_k)|\bar{d}_k,
\end{aligned}$$

and, we obtain

$$\|y\|_{PC} \leq \Big(\sup_{(t,s)\in J\times J} |G(t,s)\|p\|_{L^1} + \sum_{k=1}^{m}\sup_{t\in J}|G(t,t_k)|d_k + \sum_{k=1}^{m}\sup_{t\in J}|L(t,t_k)|\bar{d}_k\Big)^{\frac{1}{1-\beta}} := \psi_*.$$

Therefore,

$$\|y\|_{PC} \leq \max\{1, \psi_*\} := \bar{d},$$

so $\Gamma(N)$ is bounded. By Schaefer's fixed point theorem 1.6.2, N has a fixed point which in turn is a solution of problem (7.1.1)-(7.1.4).

Step 5: Now we show that the set

$$S = \{y \in PC(J,\mathbb{R}) : y \text{ is a solution of (7.1.1)-(7.1.4)}\}$$

is compact. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in S, We put $B = \{y_n : n \in \mathbb{N}\} \subseteq PC(J,\mathbb{R})$. Then from earlier parts of the proof of this theorem, we see that B is bounded and equicontinuous. Then from the Ascoli-Arzela theorem, we can conclude that B is compact. Hence, $(y_n)_{n\in\mathbb{N}}$ has a subsequence $(y_{n_m})_{n_m\in\mathbb{N}} \subseteq S$ such that y_{n_m} converges to y. Let

$$z_0(t) = \int_0^{2\pi} G(t,s)f(t,y(s),\lambda)ds + \sum_{k=1}^{m} G(t,t_k)I_k(y(t_k)) + \sum_{k=1}^{m} L(t,t_k)\bar{I}_k(y(t_k)).$$

Then

$$\begin{aligned} |y_{n_m} - z_0(t)| \;\; &\leq \;\; \int_0^{2\pi} |G(t,s)||f(t,y_{n_m}(s),\lambda) - f(t,y(s),\lambda)|ds \\ &\quad + \sum_{k=1}^{m} |G(t,t_k)||I_k(y_{n_m}(t_k)) - I_k(y(t_k))| \\ &\quad + \sum_{k=1}^{m} |L(t,t_k)||\bar{I}_k(y_{n_m}(t_k)) - \bar{I}_k(y(t_k))|. \end{aligned}$$

As $n_m \to \infty$, $y_{n_m} \to z_0(t)$ and then

$$\begin{aligned} y(t) \;\; &= \;\; \int_0^{2\pi} G(t,s)f(t,y(s),\lambda)ds \\ &\quad + \sum_{k=1}^{m} G(t,t_k)I_k(y(t_k)) \\ &\quad + \sum_{k=1}^{m} L(t,t_k)\bar{I}_k(y(t_k)). \end{aligned}$$

Hence S is compact. □

7.4 Dependence on the Parameter

In addition to determining additional conditions for the existence of positive solutions, in this section we also wish to examine their dependence on the parameter λ (see Theorem 7.4.2 below). Difficulties arise in this analysis due to the fact that, although the Greens function $G(t,s)$ is positive, $L(t,s) = \frac{\partial}{\partial t}G(t,s)$ does not have fixed sign and is not continuous at $t = s$.

Hence, in this section we consider a special case of problem (7.1.1)(7.1.4), namely,

$$y'' - \rho^2 y = -g(t)h(y), \;\; t \in J := [0,2\pi], \;\; t \neq t_k, \;\; k = 1,...,m, \tag{7.4.1}$$

$$y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \quad t = 1,...,m, \tag{7.4.2}$$

$$y(0) = y(2\pi), \quad y'(0) = y'(2\pi). \tag{7.4.3}$$

where $g : J \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ are continuous functions.

We begin with a lemma that ensures the existence of a unique positive solution to our problem.

Lemma 7.4.1. *In addition to conditions* $(H_1) - (H_3)$ *and (7.2.4) assume that:*

(H_8) $g(t) \geq 0$ *for* $t \in J$ *and* $h(y) \geq 0$ *for* $y \in \mathbb{R}$.

(H_9) $I_k(y) \leq 0$, *for each* $y \in \mathbb{R}$ *and* $k = 1, ..., m$.

Then the problem (7.4.1)-(7.4.3) has a unique positive solution on J.

Proof. We transform problem (7.4.1)-(7.4.3) into a fixed point problem by considering the operator $N : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ defined by,

$$N(y)(t) = \lambda \int_0^{2\pi} G(t,s)g(s)h(y(s))ds - \sum_{k=1}^{m} \big[G(t,t_k)I_k(y(t_k))\big], \quad t \in J. \tag{7.4.4}$$

To show that the fixed points of N are positive solutions to (7.4.1)-(7.4.3), let $y \in PC(J, \mathbb{R})$ is a fixed point of N. It is clear that

$$y(t) = \lambda \int_0^{2\pi} G(t,s)g(s)h(y(s))ds - \sum_{k=1}^{m} \big[G(t,t_k)I_k(y(t_k))\big], \quad t \in J, \tag{7.4.5}$$

(see (7.2.1)) which implies that y is a solution of(7.4.1)-(7.4.3).

If y is a fixed point of N, then (H_8)-(H_9) imply that $y(t) \geq 0$ for each $t \in J$.

As in the proof of Theorem 7.2.2, we can show that N is contraction. So by the Banachs fixed point theorem 1.6.1, N has a unique fixed point y that is a positive solution of (7.4.1)-(7.4.3). □

Theorem 7.4.2. *In addition to conditions* (H_8)-(H_9) *assume that the following conditions are satisfied:*

(i) h *is a nondecreasing;*

(ii) I_k, $k = 1, ..., m$, *are decreasing functions;*

(iii) $f(\eta u) \geq \eta^{\alpha} f(u)$ *and* $I_k(\eta u) \leq \eta^{\alpha} I_k(u)$ *for any* $0 < \eta < 1$, *where* $0 \leq \alpha < 1$.

Then the problem (7.4.1)-(7.4.3) has a unique positive solution $y_\lambda(t)$. *Furthermore, such a solution* $y_\lambda(t)$ *satisfies the following properties:*

(j) $y_\lambda(t)$ *is increasing in* λ, *i.e*, $y_{\lambda_1} > y_{\lambda_2}$ *for* $t \in J$;

(jj) $\lim\limits_{\lambda \to 0^+} \|y_\lambda(t)\| = 0$ *and* $\lim\limits_{\lambda \to +\infty} \|y_\lambda(t)\| = +\infty$ *for any* $t \in J$;

(jjj) $y_\lambda(t)$ *is continuous with respect to* λ, *i.e., if* $\lambda \to \lambda_0 > 0$, *then* $\|y_\lambda(t) - y_{\lambda_0}(t)\| \to 0$ *for any* $t \in J$.

Proof. Let
$$C = \{y \in PC(J, \mathbb{R}) : y(t) \geq 0 \text{ for } t \in J\} \tag{7.4.6}$$
be a cone in $PC(J, \mathbb{R})$. Then (H_8) through (H_{10}) imply that $N(C) \subset C$, it is easy to see that $N : C^\circ \to C^\circ$. We assert that $N : C^\circ \to C^\circ$ is an α-concave increasing operator. Indeed

$$\begin{aligned} N(\eta y) &= \lambda \int_0^{2\pi} G(t,s)g(s)h(\eta y(s))ds - \sum_{k=1}^{m} \big[G(t,t_k)I_k(\eta y(t_k))\big] \\ &\geq \eta^\alpha \big(\lambda \int_0^{2\pi} G(t,s)g(s)h(y(s))ds - \sum_{k=1}^{m} \big[G(t,t_k)I_k(y(t_k))\big]\big) \\ &\geq \eta^\alpha N(y) \quad \text{for any } 0 < \eta < 1, \end{aligned}$$

where $0 \leq \alpha < 1$. By (i) and (ii),

$$\begin{aligned} N(y_*)(t) &= \lambda \int_0^{2\pi} G(t,s)g(s)h(y_*(s))ds - \sum_{k=1}^{m} \big[G(t,t_k)I_k(y_*(t_k))\big] \\ &\leq \lambda \int_0^{2\pi} G(t,s)g(s)h(y_{**}(s))ds - \sum_{k=1}^{m} \big[G(t,t_k)I_k(y_{**}(t_k))\big] \\ &= N(y_{**})(t) \quad \text{for} \quad y_*, y_{**} \in C \text{ with } \quad y_* \leq y_{**}. \end{aligned}$$

In view of Lemma 1.6.5, N has a unique fixed point $y_\lambda \in C^\circ$. Using exactly the same argument as in the second part of the proof of [[44], Theorem 6], we can show that (j), (jj), and (jjj) hold, and we omit the details. This completes the proof of the theorem. □

Theorem 7.4.3. *Assume that conditions* (H_4)-(H_6) *and* (H_8)-(H_9) *hold. Then the problem (7.4.1)-(7.4.3) has at least one positive solution on* J.

Proof. First we transform problem (7.4.1)-(7.4.3) into a fixed point problem using the operator N defined in (7.4.4). As in the proof of Lemma 7.4.1, we can show that the fixed points of N are positive solutions of (7.4.1)-(7.4.3). As in the proof of Theorem 7.2.2, we can use Schaefers fixed point theorem 1.6.2 to show that N does in fact have a fixed point. This proves the theorem. □

Next, we obtain two solutions of problem (7.1.1)-(7.1.4) using Krasnoselskiis twin fixed point theorem 1.6.6. This will employ the use of an appropriate cone.

Theorem 7.4.4. *In addition to* (H_4)-(H_6) *and* (H_8)-(H_9)*, assume that:*

(H_{10}) *there exist* $R>0$ *and* $r>0$*, with* $r<R$ *such that*

$$\lambda \sup_{(t,s)\in J\times J}|G(t,s)|\|g\|_{\infty}h^*(r)+\sum_{k=1}^{m}\sup_{t\in J}|G(t,t_k)|d_k r^{\alpha_k}<r,$$

where $h^*(r)=\sup_{u\in(0,r]}|h(u)|$ *and*

$$\min_{t\in[0,2\pi]}\left(\lambda\int_0^{2\pi}G(t,s)g(s)h(w(s))ds-\sum_{k=1}^{m}G(t,t_k)I_k(w(t_k))\right)>R \text{ if } w>r.$$

Then the problem (7.4.1)-(7.4.3) has at least has at least two positive solutions y_1,y_2 *such that* $\|y_1\|<r$ *and* $r<\|y_2\|\le R$*.*

Proof. Let C be a cone defined in (7.4.6). Then (H_8)-(H_9) imply that $N(C)\subset C$. for any $R>0$,

$$C_R=\{y\in C:\|y\|<R\}.$$

Using (H_4)-(H_6), we can show that $N:C_R\to C$ is a completely continuous operator.

Now it remains to show that the hypotheses of the Krasnosel'skii twin fixed point theorem 1.6.6 are satisfied.

Claim 1: $\|N(y)\|_{PC}<\|y\|_{PC}$ for all $y\in\partial C_r$, where $C_r=\{y\in C:\|y\|<r\}$.

Now for $y\in\partial C_r$, we have $\|y\|_{PC}=r$, and from (H_6) and (H_{10}),

$$\begin{aligned}|N(y)(t)| &\le \lambda\int_0^{2\pi}|G(t,s)||g(s)h(u(s))|ds+\sum_{k=1}^{m}|G(t,t_k)||I_k(y(t_k))|\\ &\le \lambda\sup_{(t,s)\in J\times J}|G(t,s)|\|g\|_{\infty}h^*(r)+\sum_{k=1}^{m}\sup_{t\in J}|G(t,t_k)|d_k r^{\alpha_k}\\ &< r=\|y\|_{PC}.\end{aligned}$$

Hence

$$\|N(y)\|_{PC}<\|y\|_{PC},\quad \text{for each } y\in\partial C_r.$$

Claim 2: $\|N(y)\|_{PC}>\|y\|_{PC}$ for all $y\in\partial C_R$.
For $y\in\partial C_R$, we have $\|y\|_{PC}=R$, so $r<\|y\|_{PC}=R$, and for $w>r$, from (H_{10}) we have

$$\begin{aligned}N(y)(t) &\ge \min_{t\in[0,2\pi]}\left(\lambda\int_0^{2\pi}G(t,s)g(s)h(w(s))ds-\sum_{k=1}^{m}\big[G(t,t_k)I_k(w(t_k))\right)\\ &> R=\|y\|_{PC}.\end{aligned}$$

Therefore,

$$\|N(y)\|_{PC} > \|y\|_{PC}, \quad \text{for each } y \in \partial C_R.$$

Thus, the problem (7.4.1)-(7.4.3) has at least has at least two positive solutions y_1, y_2 such that $\|y_1\| < r, \quad r < \|y_2\| \leq R$. □

Conclusion

The main goals of this thesis is to investigate the existence and uniqueness of solutions for some differential functional equations and inclusions with state-dependent delays.

In chapters 2, 3 and 4, we have considered the problem of the existence of solutions for different classes of initial and boundary value problems for differential equations and differential inclusions. In most of these works sufficient conditions were considered to get the existence of solution by reducing the research to the search of the existence of fixed points of appropriate operators by applying different fixed point theorems. Existence results were given for some classes by Banach's, the nonlinear alternative of Leray, schaefer's, Darbo's, Mönch's, Manasevich and Mawhin continuation theorem.

In chapters 5 and 6, we have considered the problem of the existence of solutions for different classes of initial and boundary value problems for some systems of differential equations and differential inclusions.

In chapter 7, we have proved the existence, uniqueness, compactness of the solution set the dependence of the solutions on the parameter. Various fixed point theorems are used.

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