N° d'ordre :

REPUBLIQUE ALGERIENNE DEMOCRATIQUE & POPULAIRE

MINISTERE DE L'ENSEIGNEMENT SUPERIEUR & DE LA RECHERCHE SCIENTIFIQUE

UNIVERSITE DJILLALI LIABES
FACULTE DES SCIENCES EXACTES
SIDI BEL ABBES

THESE DE DOCTORAT EN SCIENCES

Présentée par

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Spécialité : Mathématiques
Option : Mathématiques Appliquées

Intitulée

Quelques contributions aux équations différentielles stochastiques d'ordre fractionnaire

Soutenue le............................
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Année universitaire : 2021/2022.

Remerciments

Je tiens à exprimer ma sincère reconnaissance et ma profonde gratitude à mon directeur de thèse, Adelghani Ouahab, Professeur à l'Université de Djillali Liabès de Sidi Bel Abbès, pour m'avoir proposé un sujet intéressant et passionnant et qui m'a beaucoup aidé durant la préparation de cette thèse. Il m'a permis grâce à ses conseils et ses encouragements de terminer ce travail dans de bonnes conditions.

Je remercie aussi monsieur le professeur Abderrahmane Yousfate d'avoir accepté d'être le président de jury de cette thèse.

J'exprime également toute ma gratitude à monsieur le professeur Mouffak Benchohra d'avoir accepté d'examiner cette thèse.

Mes remerciements vont également à messieurs les professeurs Sidi Mohammed Bouguima de l'Université Abou Bakr Belkaid de Tlemcen, Toufik Guendouzi de l'université de Docteur Moulay Tahar de Saïda et Tawfik Fawzi Mami de l'Université Belhadj Bouchaib de Ain Temouchent pour l'honneur qu'ils m'ont accordé en acceptant d'être membres de jury et d'avoir bien voulu évaluer ce travail de thèse.

To my Parents Mother and Father
My Wife
and
My two angels.
Iyad salahedine.
And
Younes chihebedine

Abbreviations and Notation

FODE :	Fractional ordinary differential equation
$FPDE$:	Fractional Partial differential equation IVP : Initial value problem
BVP :	Boundary value problem
$\Gamma(\cdot)$:	Gamma function
$\beta(\cdot,\cdot)$:	Beta function
$E_\alpha(\cdot)$:	Mittag-Leffler function
I_{0+}^{α} :	Right-fractional Riemann-Liouville integral
${}^{GL}D_{0+}^{\alpha}$:	Grunwald-Letnikov fractional derivative
${}^{RL}D_{0+}^{\alpha}$:	Right-fractional Riemann-Liouville derivative
${}^{C}D_{0+}^{\alpha}$:	Right-fractional Caputo derivative
CD_{0^+} :	Right-fractional Caputo derivative
$[\cdot]$:	Integer part of a real number
$\triangleq$:	Denoted by
$C(I,\mathbb{R})$:	Space of continuous functions on I
$C^n(I,\mathbb{R})$:	Space of $n-$ time continuously differentiable functions on I
$AC(I,\mathbb{R})$:	Space of absolutely continuous functions on I
$BC(\mathbb{R}_+,\mathbb{R})$:	Space of bounded continuous functions on I
$L^1(I,\mathbb{R})$:	space of Lebesgue integrable functions on I
$L^p(I,\mathbb{R})$:	space of measurable functions u with $\lvert u\rvert^p$ belongs to $L^1(I,\mathbb{R})$
$L^{p,\sigma}(I,\mathbb{R})$:	Weighted L^p- space with weighted function
$\sigma\, L^\infty((I,\mathbb{R})$:	space of measurable functions essentially bounded on I
$W^{m,p}(I,\mathbb{R})$:	$(m,p)-$ Sobolev space
$W_{RL}^{8,p}(I,\mathbb{R})$:	$(s,p)-$ Riemann-Liouville fractional Sobolev space
$\mathfrak{D}'(I)$:	Space of distributions
∇u :	Gradiant of u
Δu :	Laplacian of u
∂u :	Boundary of u
resp :	respectively $R-L$ Riemann-Liouville
a.e :	almost everywhere

Introduction

What if $n = \frac{1}{2}$?. It was the question raised in the year 1695 by Marquis de L'Hopital (1661-1704) to Gottfried Wilhelm Leibniz (1646-1716), the response was "This is an apparent paradox from which one day, useful consequences will be drawn "

The subject of fractional differential equations has grown in popularity and relevance over the last three decades or more, owing mostly to its proved applications in a wide range of seemingly disparate and diverse fields of science and engineering including fluid flow, economics, electrical networks, and etc. (see [25])..

It does, in fact, give some potentially valuable methods for solving differential and integral equations, as well as their usefulness in the modeling of a wide range of physical events involving very rapid and very small changes.

Furthermore, the fractional integral and fractional derivatives appear in the theory of control of dynamic systems, when the controlled system and -or- the controller is described by fractional differential equation.

While, the fractional Brownian motion was first introduced within a Hilbert space framework by Kolmogorov in 1940 in [73], where it was called Wiener Helix. It was further studied by Yaglom in [131]. The name fractional Brownian motion is due to Mandelbrot and Van Ness, who in 1968 provided in [85] a stochastic integral representation of this process in terms of a standard Brownian motion.

On the other hand, It is well known that the Gronwall-Bellman inequality [1, 10] and their generalizations can provide explicit bounds for solutions to differential and integral equations as well as difference equations.Many authors have researched various inequalities and investigated the boundedness, global existence, uniqueness, stability, and continuous dependence on the initial value and parameters of solutions to differential equations, integral equations see [2, 6, 20]. However, we notice that the existing results in the literature are inadequate for researching the qualitative and quantitative properties of solutions to some fractional integral equations see [11, 14, 22, 33, 34] . As far as the existence of such a theory is concerned, the foundations of the subject were laid by Liouville in a paper from 1832. The autodidact Oliver Heaviside introduce the practical use of fractional differential operators in electrical transmission line analysis circa 1890. Many authors have established a variety of inequalities for those fractional integral and derivative operators,for some of which have turned out to be useful in analyzing solutions of certain fractional integral and differential equations, for example, we refer the reader to [11, 20, 33, 34] and the references therein.

The main objective of the present thesis is to, gives in the first part a new Bihari's inequality with singular kernel and give a simple proof of the fractional Gronwall lemma. And in the second part,studied the Existence and uniqueness solutions for nonlinear fractional stochastic differential systems with nonlocal conditions of functional type.

So, for our purpose this thesis consist of five chapters.

In chapter 1 We present some definitions and property about the Fractional integrals and fractional derivatives,an introduction to the theory of specials functions as the Gamma function, Beta function and the Mittag-Leffler function are given.These function play a most important role in the study of fractional derivatives and fractional differential equations.

In Chapter 2 some results about the stochastic calculus and stochastic system theory are presented.

Chapter 3 we introduced the principal results about the fixed point theory, the theorem of non linear alternative of Leray-Shauder and others theorems are given. In addition the theory of C_0semi group and HIlle-Yosida theorem are given.

The chapters 4 and 5 are consecrated to presents our results

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Chapitre 1

Fractional Integrals and Fractional Derivatives

This chapter contains definitions and properties from such topics of Analysis as functional spaces, special functions, and some properties of fractional integrals and fractional derivatives of different types.

1.1. Banach Spaces of Continuous Functions

A topological space X is locally compact if, for every $x \in X$, there is an open set $U \subset X$ containing x such that $\bar{U}$ is compact. Assuming X is locally compact, let $C_b(X)$ denote the set of all functions $f : X \to \mathbb{C}$ that are continuous and bounded. Thus, for every $f \in C_b(X)$ means that there is an $R > 0$ such that $|f(x)| < R$ for all $x \in X$

Theorem 1.1.1. *If X is a locally compact space, then $C_b(X)$ is a Banach space, where the vector space operations are given by the usual pointwise operations, and where the norm of $f \in C_b(X)$ is defined by*

$$\|f\| = \sup_{x \in X} |f(x)|. \tag{1.1.1}$$

Proof. *It is elementary that $C_b(X)$ is a vector space and that* (1.1.1) *defines a norm on $C_b(X)$. Thus, it remains only to show that every Cauchy sequence in $C_b(X)$ is convergent in $C_b(X)$ Let $\{f_k\}_{k \in \mathbb{N}} \subset C_b(X)$ denote a Cauchy sequence. For each $x \in X$,*

$$\left|f_n(x) - f_m(x)\right| \leq \sup_{y \in X} \left|f_n(y) - f_m(y)\right| = \left\|f_n - f_m\right\|.$$

Since $\{f_k\}_{k \in \mathbb{N}}$ is a Cauchy sequence in $C_b(X)$, $\{f_k(x)\}_{k \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$ for each $x \in X$. Because $\mathbb{C}$ is complete, $\lim_k f_k(x)$ exists for every $x \in X$. Therefore,
define $f : X \to \mathbb{C}$ by $f(x) = \lim_k f_k(x)$, for each $x \in X$. We aim to show.

(i) that f is continuous and bounded, and
(ii) that $\{f_k\}_{k \in \mathbb{N}}$ converges to f in $C_b(X)$.

Let $\varepsilon > 0$. Because $\{f_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence, there exists $N_\varepsilon \in \mathbb{N}$ such that $\left\|f_n - f_m\right\| < \varepsilon$ for all $n, m \geq N_\varepsilon$. Assume that $n \geq N_\varepsilon$. Choose any $x \in X$; thus,

$$\begin{aligned}\left|f(x) - f_n(x)\right| &\leq \left|f(x) - f_m(x)\right| + \left|f_m(x) - f_n(x)\right| \\ &\leq \left|f(x) - f_m(x)\right| + \left\|f_m - f_n\right\|.\end{aligned}$$

As the inequalities above are true for all $m \in \mathbb{N}$

$$\begin{aligned}\left|f(x) - f_n(x)\right| &\leq \inf_{m\in\mathbb{N}}\left(\left|f(x) - f_m(x)\right| + \left\|f_m - f_n\right\|\right) \\ &\leq 0 + \varepsilon.\end{aligned}$$

This right-hand side of the inequality above is independent of the choice of $x \in X$. Hence, if $n \geq N_\varepsilon$ is fixed, then $f - f_n$ is a bounded function $X \to \mathbb{C}$ and

$$\sup_{x\in X}\left|f(x) - f_n(x)\right| \leq \varepsilon.$$

Since f is uniformly within ε of a continuous function, f is continuous at each $x \in X$. Furthermore, since the sum of bounded functions is bounded, $f_n + (f - f_n) = f$ is bounded. This proves that $f \in C_b(X)$. Finally, since $f \in C_b(X)$ satisfies $\left\|f - f_n\right\| \leq \varepsilon$ for all $n \geq N_\varepsilon$, the Cauchy sequence $\{f_k\}_{k\in\mathbb{N}}$ converges in $C_b(X)$ to $f \in C_b(X)$.

1.2. Banach Spaces of p-Integrable Functions

Proposition 1.2.1. *Suppose that (X, Σ, μ) is a measure space, and that $p \geq 1$. If*

$$\mathcal{L}^p(X, \Sigma, \mu) = \{f : X \to \mathbb{C} \mid f \text{ is } p\text{ -integrable }\}. \tag{1.2.1}$$

then $\mathcal{L}^p(X, \Sigma, \mu)$ is a complex vector space. Furthermore, if $\rho : \mathcal{L}^p(X, \Sigma, \mu) \to \mathbb{R}$ is given by

$$\rho(f) = \left(\int_X |f|^p d\mu\right)^{1/p}. \tag{1.2.2}$$

For all $f \in \mathcal{L}^p(X, \Sigma, \mu)$, then ρ is a semi-norm on $\mathcal{L}^p(X, \Sigma, \mu)$.

Proof. *It is clear that $\alpha f \in \mathcal{L}^p(X, \Sigma, \mu)$, for every $\alpha \in \mathbb{C}$ and $f \in \mathcal{L}^p(X, \Sigma, \mu)$. If $f, g \in \mathcal{L}^p(X, \Sigma, \mu)$, then $f + g \in \mathcal{L}^p(X, \Sigma, \mu)$, by Minkowski's inequality [1]. Hence, $\mathcal{L}^p(X, \Sigma, \mu)$ is a vector space To verify that ρ is a semi-norm, the only nontrivial fact to confirm is the triangle inequality holds. To this end, Minkowski's inequality yields :*

$$\begin{aligned}\rho(f+g) &= \left(\int_X |f+g|^p d\mu\right)^{1/p} \\ &\leq \left(\int_X |f|^p d\mu\right)^{1/p} + \left(\int_X |g|^p d\mu\right)^{1/p} \\ &= \rho(f) + \rho(g).\end{aligned} \tag{1.2.3}$$

Hence, ρ is a semi-norm.

1. If $1 \leq p < \infty$, then

$$\|u + v\|_p \leq \|u\|_p + \|v\|_p$$

Let $\Omega = [a, b](-\infty \leq a < b \leq \infty)$ be a finite or infinite interval of the real axis $\mathbb{R} = (-\infty, \infty)$. We denote by $L_p(a, b)(1 \leq p \leq \infty)$ the set of those Lebesgue complex-valued measurable functions f on Ω for which $\|f\|_p < \infty$, where

$$\|f\|_p = \left(\int_a^b |f(t)|^p dt\right)^{1/p} \quad (1 \leq p < \infty). \tag{1.2.4}$$

And

$$\|f\|_\infty = \operatorname*{ess\,sup}_{a \leq x \leq b} |f(x)|. \tag{1.2.5}$$

Here ess sup $|f(x)|$ is the essential maximum of the function $|f(x)|$ [see, for example, Nikol'skii [99], pp. 12 − 13)].

We also need the weighted L^p -space with the power weight. Such a space, which we denote by $X_c^p(a, b)(c \in \mathbb{R}; 1 \leq p \leq \infty)$, consists of those complex valued Lebesgue measurable functions f on $(a, \bar{b})$ for which $\|f\|_{X_c^p} < \infty$, with

$$\|f\|_{X_e^p} = \left(\int_a^b \left|t^c f(t)\right|^p \frac{dt}{t}\right)^{1/p} \quad (1 \leq p < \infty). \tag{1.2.6}$$

And

$$\|f\|_{X_c^\infty} = \operatorname*{ess\,sup}_{a \leq x \leq b} \left[x^c \mid f(x)\right]. \tag{1.2.7}$$

In particular, when $c = 1/p$, the space $X_c^p(a, b)$ coincides with the $L_p(a, b)$ -space :
$X_{1/p}^p(a, b) = L_p(a, b)$ Let now $[a, b](-\infty < a < b < \infty)$ be a finite interval and let $AC[a, b]$ be the space of functions f which are absolutely continuous on $[a, b]$. It is known that $AC[a, b]$ coincides with the space of primitives of Lebesgue summable functions :

$$f(x) \in AC[a, b] \Leftrightarrow f(x) = c^{ste} + \int_a^x \varphi(t)dt \quad (\varphi(t) \in L(a, b)). \tag{1.2.8}$$

And therefore an absolutely continuous function $f(x)$ and has a derivative $f'(x) = \varphi(x)$ almost everywhere on $[a, b]$. Thus (1.2.7) yields

$$\varphi(t) = f'(t) \text{ and } c = f(a). \tag{1.2.9}$$

For $n \in \mathbb{N} := \{1, 2, 3, \cdots\}$ we denote by $AC^n[a, b]$ the space of complex-valued functions $f(x)$ which have continuous derivatives up to order $n - 1$ on $[a, b]$ such that $f^{(n-1)}(x) \in AC[a, b]$:

$$AC^n[a, b] = \left\{f : [a, b] \to \mathbb{C} \text{ and } \left(D^{n-1} f\right)(x)\right\} \in AC[a, b] \quad \left(D = \frac{d}{dx}\right). \tag{1.2.10}$$

$\mathbb{C}$ being the set of complex numbers. In particular, $AC^1[a, b] = AC[a, b]$. This space is characterized by the following assertion [see Samko et al. ([120, Lemma 2.4]).

1.3. Special Functions

We presents in this section some definition about special function like Gamma function, Beta function . . . ,ect.

1.3.1. Gamma Function

Undoubtedly, the Euler's gamma function $\Gamma(z)$, is one of the basic functions of the fractional calculus is which generalizes the factorial $n!$ and allows $n!$ to take also non-integer and even complex values.

We will recall in this section some results on the gamma function which are important for other parts of this work.

Definition 1.3.1. *The gamma function $\Gamma(z)$ is defined by the integral*

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt. \tag{1.3.1}$$

Theorem 1.3.2. *Function $\Gamma(p)$ is convergent for $p > 0$.*

Proof. *The integral can be written as :*

$$\Gamma(p) = \int_0^1 e^{-x} x^{p-1} dx + \int_1^{\infty} e^{-x} x^{p-1} dx = I_1 + I_2. \tag{1.3.2}$$

Where $I_1 = \int_0^1 e^{-x} x^{p-1} dx$ is convergent. Since e^{-x} is decreasing on the interval $[0, 1]$, from $x = 0$, we have :

$$\int_0^1 e^{-x} x^{p-1} dx < \int_0^1 x^{p-1} dx = \frac{1}{p}. \tag{1.3.3}$$

Moreover, $I_2 = \int_1^{\infty} e^{-x} x^{p-1} dx$ is also convergent. We obtain :

$$1 \leq x \Rightarrow x^{p-1} e^{-x} \leq e^{-x/2} \Leftrightarrow x^{p-1} \leq e^{x/2} \Leftrightarrow \frac{x^{p-1}}{e^{x/2}} \leq 1. \tag{1.3.4}$$

Because $\lim_{x\to\infty} \frac{x^{p-1}}{e^{x/2}} = 0$, we have :

$$\int_1^{\infty} e^{-x} x^{p-1} dx \leq \int_1^{\infty} e^{-x/2} dx = 2e^{-1/2}.$$

The integral (1.3.1) *is convergent for $p > 0$ and divergent for $p \leq 0$.*

1.3.2. Some Properties of the Gamma Function

One of the basic properties of the gamma function is that it satisfies the following functional equation :

$$\Gamma(z + 1) = z\Gamma(z). \tag{1.3.5}$$

Which can be easily proved by integrating by parts :

$$\Gamma(z + 1) = \int_0^{\infty} e^{-t} t^{z} dt = \left[-e^{-t} t^{z}\right]_{t=0}^{t=\infty} + z \int_0^{\infty} e^{-t} t^{z-1} dt = z\Gamma(z). \tag{1.3.6}$$

Obviously, $\Gamma(1) = 1$, and using (1.3.5) we obtain for $z = 1, 2, 3, \ldots$:

$$\begin{aligned}
\Gamma(2) &= 1 \cdot \Gamma(1) = 1 = 1! \\
\Gamma(3) &= 2 \cdot \Gamma(2) = 2 \cdot 1! = 2! \\
\Gamma(4) &= 3 \cdot \Gamma(3) = 3 \cdot 2! = 3! \\
\cdots\cdots & \quad \cdots \quad \cdots \\
\Gamma(n + 1) &= n \cdot \Gamma(n) = n \cdot (n - 1)! = n!.
\end{aligned}$$

The second integral defines an entire function of the complex variable z. Indeed, let us write

$$\varphi(z) = \int_1^\infty e^{-t} t^{z-1} dt = \int_1^\infty e^{(z-1)\log(t)-t} dt. \tag{1.3.7}$$

The function $e^{(z-1)\log(t)-t}$ is a continuous function of z and t for arbitrary z and $t \geq 1$. Moreover, if $t \geq 1$ (and therefore $\log(t) \geq 0$), then it is an entire function of z. Let us consider an arbitrary bounded closed domain D in the complex plane ($z = x + iy$) and denote $x_0 = \max_{z \in D} \operatorname{Re}(z)$. Then we have :

$$\begin{aligned} \left|e^{-t} t^{z-1}\right| &= \left|e^{(z-1)\log(t)-t}\right| = \left|e^{(x-1)\log(t)-t}\right| \left|e^{iy\log(t)}\right| \\ &= \left|e^{(x-1)\log(t)-t|}\right| \leq e^{(x_0-1)\log(t)-t} = e^{-t} t^{x_0-1}. \end{aligned}$$

This means that the integral (1.3.7) converges uniformly in D and, therefore, the function $\varphi(z)$ is regular in D and differentiation under the integral in (1.3.7) is allowed. Because the domain D has been chosen arbitrarily, we conclude that the function $\varphi(z)$ has the above properties in the whole complex plane. Therefore, $\varphi(z)$ is an entire function allowing differentiation under the integral. Bringing together the above considerations, we see that.

$$\begin{aligned} \Gamma(z) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{k+z} + \int_1^\infty e^{-t} t^{z-1} dt \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{k+z} + \text{ entire function, .} \end{aligned}$$

and, indeed, $\Gamma(z)$ has only simple poles at the points $z = -n, n = 0, 1, 2, \ldots$

1.3.3. Limit Representation of the Gamma Function

The gamna function can be represented also by the limit

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{z(z+1)\ldots(z+n)}. \tag{1.3.8}$$

Where we initially suppose $\operatorname{Re}(z) > 0$ To prove (1.3.8), let us introduce an auxiliary function

$$f_n(z) = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt.$$

Performing the substitution $\tau = \frac{t}{n}$ and then repeating integration by parts we obtain ;

$$\begin{aligned} f_n(z) &= n^z \int_0^1 (1-\tau)^n \tau^{z-1} d\tau \\ &= \frac{n^z}{z} n \int_0^1 (1-\tau)^{n-1} \tau^z d\tau \\ &= \frac{n^z n!}{z(z+1)\ldots(z+n-1)} \int_0^1 \tau^{z+n-1} d\tau \\ &= \frac{n^z n!}{z(z+1)\ldots(z+n-1)(z+n)}. \end{aligned}$$

Taking into account the well-known limit

$$\lim_{n\to\infty}\left(1-\frac{t}{n}\right)^n = e^{-t}.$$

We may expect that

$$\lim_{n\to\infty} f_n(z) = \lim_{n\to\infty}\int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}dt = \int_0^\infty e^{-t}t^{z-1}dt. \tag{1.3.9}$$

Which ends the proof of the limit representation (1.3.8) of the gamma function, if the interchange of the limit and the integral in (1.3.9) is justified. To do this, let us estimate the difference

$$\begin{aligned}\Delta &= \int_0^\infty e^{-t}t^{z-1}dt - f_n(z)\\ &= \int_0^n \left[e^{-t}-\left(1-\frac{t}{n}\right)^n\right]t^{z-1}dt + \int_n^\infty e^{-t}t^{z-1}dt.\end{aligned}$$

Let us take an arbitrary $\varepsilon > 0$. Because of the convergence of the integral (1.3.9) there exists an N such that for $n \geq N$ we have

$$\left|\int_n^\infty c^{-t}t^{z-1}dt\right| \leq \int_n^\infty e^{-t}t^{x-1}dt < \frac{\varepsilon}{3}, \quad (x = \mathrm{Re}(z)).$$

Fixing now N and considering $n > N$ we can write Δ as a sum of three integrals :

$$\Delta = \left(\int_0^N + \int_N^n\right)\left[e^{-t}-\left(1-\frac{t}{n}\right)^n\right]t^{z-1}dt + \int_n^\infty e^{-t}t^{z-1}dt. \tag{1.3.10}$$

$$\Delta = \left(\int_0^N + \int_N^n\right)\left[e^{-t}-\left(1-\frac{t}{n}\right)^n\right]t^{z-1}dt + \int_n^\infty e^{-t}t^{z-1}dt. \tag{1.3.11}$$

The last term is less then $\frac{\varepsilon}{3}$. For the second integral we have :

$$\begin{aligned}\left|\int_N^n \left[e^{-t}-\left(1-\frac{t}{n}\right)^n\right]t^{z-1}dt\right| &\leq \int_N^n \left[e^{-t}-\left(1-\frac{t}{n}\right)^n\right]t^{x-1}dt\\ &< \int_N^\infty e^{-t}t^{x-1}dt < \frac{\varepsilon}{3}.\end{aligned}$$

Where, as above, $x = \mathrm{Re}(z)$. For the estimation of the first integral in (1.3.11) we need the following auxiliary inequality :

$$0 < e^{-t}-\left(1-\frac{t}{n}\right)^n < \frac{t^2}{2n}, \quad (0 < t < n). \tag{1.3.12}$$

Which follows from the relationships

$$1 - c^t\left(1-\frac{t}{n}\right)^n = \int_0^t e^\tau\left(1-\frac{\tau}{n}\right)^n\frac{\tau}{n}d\tau. \tag{1.3.13}$$

And

$$0 < \int_0^t e^\tau\left(1-\frac{\tau}{n}\right)^n\frac{\tau}{n}d\tau < \int_0^t e^\top\frac{\tau}{n}d\tau = e^t\frac{t^2}{2n}. \tag{1.3.14}$$

(Relationship (1.3.14) can be verified by differentiating both sides.) Using the auxiliary inequality (1.3.12) we obtain for large n and fixed N :

$$\left| \int_0^N \left[e^{-t} - \left(1 - \frac{t}{n}\right)^n \right] t^{z-1} dt \right| < \frac{1}{2n} \int_0^N t^{x+1} dt < \frac{\varepsilon}{3}. \tag{1.3.15}$$

Taking into account inequalities (1.3.10),(1.3.12) and (1.3.15) and the arbitrariness of ε we conclude that the interchange of the limit and the integral in (1.3.9) is justified.

This definitely completes the proof of the formula (1.3.8) for the limit representation of the gamma function for $\mathrm{Re}(z) > 0$.

With the help of (1.3.11) the condition $\mathrm{Re}(z) > 0$ can be weakened to $z \neq 0, -1, -2, \ldots$ in the following manner. If $-m < \mathrm{Re}(z) \leq -m + 1$, where m is a positive integer, then,

$$\begin{aligned}
\Gamma(z) &= \frac{\Gamma(z+m)}{z(z+1)\ldots(z+m-1)} \\
&= \frac{1}{z(z+1)\ldots(z+m-1)} \lim_{n\to\infty} \frac{n^{z+m} n!}{(z+m)\ldots(z+m+n)} \\
&= \frac{1}{z(z+1)\ldots(z+m-1)} \lim_{n\to\infty} \frac{(n-m)^{2+m}(n-m)!}{(z+m)(z+m+1)\ldots(z+n)} \\
&= \lim_{n\to\infty} \frac{n^z n!}{z(z+1)\ldots(z+n)}.
\end{aligned}$$

Therefore, the limit representation (1.3.1) holds for all z excluding $z \neq 0, -1, -2, \ldots$

1.3.4. Mittag-Leffler Functions

The exponentials function, e^Z. plays a crucial role in the theory of integer-order. differential equations . Its one-parameter generalizations, the function which is denoted by

$$E_{\alpha,\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}. \tag{1.3.16}$$

Was introduced by G.M.Mittag-Leffler[89], and studied also by A.Wiman [**?**] We present some properties as ; $\mathbb{E}_1(z) = e^z$ and $E_2(z) = \cosh(\sqrt{z})$

The two parameter function of the Mittag-Leffler type, which plays a very important role in the fractional calculus. was in fact introduced by Agarwal [4].

Definition 1.3.3. *A two-parameter function of the Mittag-Leffler type is defined by the series expansion [42]*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad (\alpha > 0, \beta > 0). \tag{1.3.17}$$

It follows from the definition 1.3.3 that

$$E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(z+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z. \tag{1.3.18}$$

And

$$E_{2,2}(z) = \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(2z+2)} = \frac{1}{z} \sum \frac{z^{2k+1}}{(2k+1)!} = \frac{\sinh z}{z}. \tag{1.3.19}$$

The hyperbolic functions of order n , which are generalizations of the hyperbolic sine and cosine, can also be expressed in terms of the Mittag-Leffler function[42] :

$$h_r(z,n) = \sum_{k=0}^{\infty} \frac{z^{nk+r-1}}{(nk+r-1)!} = z^{r-1} E_{n,r}(z^n) \quad (r = 1, 2, \ldots, n). \tag{1.3.20}$$

$$E_{1,3}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+3)} = \sum_{k=0}^{\infty} \frac{z^k}{(k+2)!} = \frac{1}{z^2} \sum_{k=0}^{\infty} \frac{z^{k+2}}{(k+2)} = \frac{e^z - 1 - z}{z^2}. \tag{1.3.21}$$

$$E_{1,m} = \frac{1}{z^{m-1}} \left\{ e^z - \sum_{k=0}^{m-2} \frac{z^k}{k!} \right\} \tag{1.3.22}$$

$$E_{1,2} = \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(2k+1)} = \cosh(z). \tag{1.3.23}$$

The trigonometric function of order n denoted by $k_r(z,n)$, which are the generalization of the sine and the cosine function

$$k_r(z,n) = \sum_{j=0}^{\infty} \frac{z^{nj+z-1}}{(nj+r-1)!} = z^{r-1} E_{n,r}(-z^n). \tag{1.3.24}$$

1.4. Riemann-Liouville Fractional Integrals and Fractional Derivatives.

Lemma 1.4.1. *Let I be the operator integral defined by following formula*

$$If(x) = \int_0^s f(s)ds \tag{1.4.1}$$

then the The following formula is true for any $n \in \mathbb{N}$

$$I^n f(x) = \int_0^x \frac{(x-s)^{n-1}}{(n-1)!} f(s)ds. \tag{1.4.2}$$

Proof. *By recurrence ;*

Assume that (1.4.2) *is true for $n = k$ and proof it for $n = k+1$ for $n = k$*

$$I^k f(x) = \int_0^x \frac{(x-s)^{k-1}}{(k-1)!} f(s)ds.$$

so

$$\begin{aligned} I^{k+1}f(x) &= I\left(I^k f(x)\right) \\ &= \int_0^x \frac{(x-s)^{k-1}}{(k-1)!} f(s)ds \\ &= \int_0^x \int_0^y \frac{(y-s)^{k-1}}{(k-1)!} f(s)dsdy. \end{aligned}$$

Changing the order of integration and using $0 \le y \le x, 0 \le s \le y$ we get

$$\begin{aligned} I^{k+1}f(x) &= \int_0^x \int_s^x \frac{(y-s)^{k-1}}{(k-1)!} f(s)dyds = \int_0^x f(s) \int_s^x \frac{(y-s)^{k-1}}{(k-1)!} dyds \\ &= \int_0^x f(s)\frac{(x-s)^k}{k(k-1)!} ds = \int_0^x \frac{(x-s)^k}{k!} f(s)ds. \end{aligned}$$

thus (1.4.2) *is trus for $n = k+1$ so it is trus for $n \in \mathbb{N}$*

Let us consider some of the starting points for a discussion of classical fractional calculus. One development begins with a generalization of repeated integration. In the same manner.

Lemma 1.4.2. *Consider a locally integrable*[2] *Real valued function $f : J \to \mathbb{R}$ whose domain of definition $J = [a, b] \subset \mathbb{R}$ is an interval with $-\infty \le a < b \le \infty$. Integrating n times gives the fundamental formula*

$$\begin{aligned} \left(I_{a+}^n f\right)(x) &= \int_a^x \int_a^{x_1} \cdots \int_a^{x_{n-1}} f\left(x_n\right) dx_n \ldots dx_2 dx_1 \\ &= \frac{1}{(n-1)!} \int_a^x (x-y)^{n-1} f(y)dy. \end{aligned} \tag{1.4.3}$$

Where $a < x < b$ and $n \in \mathbb{N}$. This formula may be proved by induction. It reduces $n-$fold integration to a single convolution integral. The subscript $a+$ indicates that the integration has a as its lower limit. An analogous formula holds with lower limit x and upper limit a. In that case the subscript a - will be used.

Definition 1.4.3 (Riemann-Liouville fractional integrals)**.** *Let $-\infty \le a < x < b \le \infty$. The Riemann-Liouville fractional integrals $I_{a+}^\alpha f, I_{b-}^\alpha$ of order $\alpha \in \mathbb{C}, \mathcal{R}(\alpha) > 0$ is defined for functions $f : [a, b] \to \mathbb{C}$ by*

$$\left(I_{a+}^\alpha f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt. \tag{1.4.4}$$

$$\left(I_{b-}^\alpha f\right)(x) = \left({}_x I_b^\alpha f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt. \tag{1.4.5}$$

Respectively. Here $\Gamma(\alpha)$ is the Gamma function . These integrals are called the left-sided and the right-sided fractional integrals. When $\alpha = n \in \mathbb{N}$, the definitions (1.4.4) and (1.4.5) coincide with the nth integrals defined in (1.4.3).

2.

[1] A function $f : J = [a, b] \subset \mathbb{R}$ is called locally integrable if it is integrable on all compact subsets $K \subset J$.

Definition 1.4.4 (Riemann-Liouville fractional Derivatives)**.** *The Riemann-Liouville fractional derivatives $D^{\alpha}_{a+}y$ and $D^{\alpha}_{b-}y$ of order $\alpha \in \mathbb{C}(\mathcal{R}(\alpha) > 0)$ are defined by*

$$\begin{aligned} D^{\alpha}_{a+}y &= \left(\frac{d}{dx}\right)^{n} (I^{n-\alpha}_{a+}y)(x) \\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^{n}\int_a^x \frac{y(t)dt}{(x-t)^{\alpha-n+1}} (n = [\mathcal{R}(\alpha)] + 1, x > a). \end{aligned} \tag{1.4.6}$$

And

$$\begin{aligned} D^{\alpha}_{b-}y &= \left(-\frac{d}{dx}\right)^{n} \left(I^{n-\alpha}_{b-}y\right)(x) \\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^{n}\int_x^b \frac{y(t)dt}{(t-x)^{\alpha-n+1}} (n = [\mathcal{R}(\alpha)] + 1, x < b). \end{aligned} \tag{1.4.7}$$

Respectively, where $[\mathcal{R}(\alpha)]$ means the integral part of α.

Property 1.4.1. *(a) when $\alpha = n \in \mathbb{N}$,*
then

$$\left(D^{0}_{a+}y\right)(x) = \left(D^{0}_{b-}y\right)(x) = y(x); (D^{n}_{a+}y)(x) = y^{(n)}(x).$$

and

$$\left(D^{n}_{b-}\right)(x) = (-1)^{n} y^{(n)} (n \in \mathbb{N})$$

where $(y^{(n)}x)$ is the usual derivative of $y(x)$ of order n.

(b) When $0 < \mathcal{R}(\alpha) < 1$, then

$$(D^{\alpha}_{a+}y)(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{y(t)dt}{(x-t)^{\alpha-[\mathcal{R}(\alpha)]}} (0 < \mathcal{R}(\alpha) < 1, x > a). \tag{1.4.8}$$

$$\left(D^{\alpha}_{b-}y\right)(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^b \frac{y(t)dt}{(t-x)^{\alpha-[\mathcal{R}(\alpha)]}} (0 < \mathcal{R}(\alpha) < 1, x < b). \tag{1.4.9}$$

(c) When $\alpha \in \mathbb{R}^{+}$,
then (1.4.6) *and* (1.4.7) *take the following forms,*

$$(D^{\alpha}_{a+}y)(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^{n}\int_a^x \frac{y(t)dt}{(x-t)^{\alpha-n+1}} (n = [\alpha] + 1, x > a). \tag{1.4.10}$$

And

$$\left(D^{\alpha}_{b}y\right)(x) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^{n}\int_x^b \frac{y(t)dt}{(t-x)^{\alpha-n+1}} (n = [\alpha] + 1, x < b). \tag{1.4.11}$$

While (1.4.8) *and* (1.4.9) *are given by*

$$(D^{x}_{a+}y)(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{y(t)dt}{(x-t)^{\alpha}} (0 < \alpha < 1, x > a). \tag{1.4.12}$$

And

$$\left(D^{x}_{b-}y\right)(x) = \frac{-1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^b \frac{y(t)dt}{(t-x)^{\alpha}} (0 < \alpha < 1, x < b). \tag{1.4.13}$$

(d) If $\mathcal{R}(\alpha) = 0, (\alpha \neq 0)$ then (1.4.6) *and* (1.4.7) *yield the Fractional derivatives of purely imaginary order*

$$\left(D^{i\theta}_{a+}y\right)(x) = \frac{1}{\Gamma(1-i\theta)}\frac{d}{dx}\int_a^x \frac{y(t)dt}{(x-t)^{i\theta}}(\theta \in \mathbb{R}^*, x > a). \tag{1.4.14}$$

$$\left(D^{i\theta}_{b-}y\right)(x) = \frac{-1}{\Gamma(1-i\theta)}\frac{d}{dx}\int_x^b \frac{y(t)dt}{(t-x)^{i\theta}}(\theta \in \mathbb{R}^*, x < b). \tag{1.4.15}$$

1.4.1. Basic Properties of RL Fractional Integrals

Property 1.4.2. *The Riemann-Liouville integral operator of order α is a linear operator. That means ;*

$$I^\alpha[af(x) + bg(x)] = aI^\alpha f(x) + bI^\alpha g(x)$$

Proof. *Using the definition ofI^α , we get*

$$\begin{aligned} I^\alpha[af(x) + bg(x)] &= \frac{1}{\Gamma(\alpha)}\int_0^x \frac{af(s) + bg(s)}{(x-s)^{1-\alpha}}ds \\ &= \frac{1}{\Gamma(\alpha)}\left[\int_0^x \frac{af(s)}{(x-s)^{1-\alpha}}ds + \int_0^x \frac{bg(s)}{(x-s)^{1-\alpha}}ds\right] \\ &= aI^\alpha(f(x)) + bI^\alpha(g(x)). \end{aligned}$$

Property 1.4.3. *If $\mathcal{R}(\alpha) \geq 0$ and $\beta \in \mathbb{C}(\mathcal{R}(\beta) > 0)$, then*

$$\left(I^\alpha_{a+}(t-a)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta+\alpha-1}(\mathcal{R}(\alpha) > 0). \tag{1.4.16}$$

$$\left(D^\alpha_{a+}(t-a)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta-\alpha-1}(\mathcal{R}(\alpha) \geq 0). \tag{1.4.17}$$

and

$$\left(I^\alpha_{b-}(b-t)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(b-x)^{\beta+\alpha-1}(\mathcal{R}(\alpha) > 0). \tag{1.4.18}$$

$$\left(D^\alpha_{b-}(t-a)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(b-x)^{\beta-\alpha-1}(\mathcal{R}(\alpha) \geq 0). \tag{1.4.19}$$

In particular, when $\beta = 1$ and $\mathcal{R}(\alpha) \geq 0$ we have,

$$\left(D^\alpha_{a+}1\right)(x) = \frac{(x-a)^{-a}}{\Gamma(1-\alpha)}, \text{ and } \left(D^\alpha_{b-}1\right)(x) = \frac{(b-x)^{-\alpha}}{\Gamma(1-\alpha)}(0 < \mathcal{R}(\alpha) < 1). \tag{1.4.20}$$

What mean that, the RL-Fractional derivatives of a constant, are in general not equal to zero. In the other hand, and for $j = 1, 2, ..., [\mathcal{R}(\alpha)] + 1$

$$\left(D^\alpha_{a+}(t-x)^{\alpha-j}\right)(x) = 0, \left(D^\alpha_{b-}(b-t)^{\alpha-j}\right)(x) = 0. \tag{1.4.21}$$

Property 1.4.4 (Integration by parts). *[69] The following results*

$$\int_a^b f(x)\left(I_{a+}^{\alpha} g(x)\right) dx = \int_a^b g(x)\left(I_{b-}^{\alpha} f(x)\right) dx, \quad f, g \in L^1(a,b).$$

is called the property of "integration by parts" for fractional integrals.

Proof. *Putting for $f, g \in L^1(a,b)$ and using the Dirichlet's Formula [3] we get*

$$\begin{aligned} I_1 = \int_a^b \left(I_{a+}^{\alpha} g\right)(x) dx &= \frac{1}{\Gamma(\alpha)} \int_a^b f(x) \int_a^x g(t)(x-t)^{\alpha-1} dt dx \\ &= \frac{1}{\Gamma(\alpha)} \int_a^b \left\{ \int_t^b f(x)(x-t)^{\alpha-1} dx \right\} g(t) dt \\ &= \int_a^b \left\{ \frac{1}{\Gamma(\alpha)} \int_t^b f(x)(x-t)^{\alpha-1} dx \right\} g(t) dt. \end{aligned}$$

Changing t by x we obtain, $I_1 = \displaystyle\int_a^b g(x)\left(I_{b-}^{\alpha} f\right)(x) dx$

Lemma 1.4.5 (Semigroup property). *Let ϕ be integrable real valued function $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ then the Fractional integrals obey the following semigroup property :*

$$\begin{aligned} I_{a+}^{\alpha} I_{a+}^{\beta} \phi = I_{a+}^{\alpha+\beta} \phi = I_{a+}^{\beta} I_{a+}^{\alpha} \phi \\ I_{b-}^{\alpha} I_{b-}^{\beta} \phi = I_{b-}^{\alpha+\beta} \phi = I_{b-}^{\beta} I_{b-}^{\alpha} \phi. \end{aligned}$$

Proof.

$$\begin{aligned} I^{\alpha}\left[I^{\beta} f(x)\right] I &= I^{\alpha}\left[\frac{1}{\Gamma(\beta)} \int_0^x \frac{f(s)}{x-s}^{1-\beta} ds\right] \\ &= \frac{1}{\Gamma(\alpha)} \int_0^x \frac{1}{(x-y)^{1-\alpha}} \frac{1}{\Gamma(\beta)} \int_0^y \frac{f(s) ds}{(y-s)^{1-\alpha}} dy \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x \int_0^y \frac{1}{(x-y)^{1-\alpha}(y-s)^{1-\beta}} f(s) ds dy \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x \left[\int_s^x \frac{1}{(x-y)^{1-\alpha}} \frac{dy}{(y-s)^{1-\beta}}\right] f(s) ds. \end{aligned}$$

Denote now

$$A(x,s) = \int_0^x \frac{1}{(x-y)^{1-\alpha}} \frac{dy}{(y-s)1-\beta} \tag{1.4.22}$$

and putting $y - s = t$ then $dy = dt$,we gate

$$A(x,s) = \int_0^{x-s} \frac{dt}{(x-s-t)^{1-\alpha} t^{1-\beta}}$$

3. Let g continuous function, and let $\mu, \nu > 0$ then

$$\int_0^t f(x)(t-\xi)^{\mu-1} d\xi \int_0^{\xi} (\xi-x)^{\nu-1} g(\xi,x) dx = \int_0^t dx \int_x^t (t-\xi)^{\mu-1}(\xi-x)^{\nu-1} g(\xi,x) d\xi.$$

and by a change of variable, $t = (x-s)u$ and $dt = (x-s)du$, we have

$$\begin{aligned} A(x,s) &= \int_0^1 \frac{(x-s)du}{(x-s)^{1-(\alpha)}(1-u)^{1-\alpha}(x-s)^{1-\beta}} \\ &= \frac{1}{(x-s)^{1-(\alpha+\beta)}} \int_0^1 \frac{du}{(1-u)^{1-\alpha}u^{1-\beta}} \\ &= \frac{1}{(x-s)^{1-(\alpha+\beta)}} \beta(\alpha,\beta) \\ &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \end{aligned}$$

Now we can write

$$\int_s^x \frac{du}{(x-s)^{1-\alpha}(y-s)^{1-\beta}} = \frac{1}{(x-s)^{1-(\alpha+\beta)}} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Then

$$\begin{aligned} I^\alpha\left[I^\beta f(x)\right] &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x \frac{1}{(x-s)^{1-(\alpha+\beta)}} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} f(s)ds \\ &= \frac{1}{\Gamma(\alpha+\beta)} \int_0^x \frac{f(s)ds}{(x-s)^{1-(\alpha+\beta)}} = I^{\alpha+\beta} f(x). \end{aligned}$$

Lemma 1.4.6. *[120]*

(a) The fractional integral operator I_{a+}^α and I_{b-}^α, whith $\mathcal{R}(\alpha) > 0$ are bounded in $L^p(a,b)(1 \leq p \leq \infty)$

$$\|I_{a+}^\alpha f\|_p \leq K\|f\|_p, \text{ and } \|I_{b-}^\alpha f\|_p \leq K\|f\|_p. \tag{1.4.23}$$

Where $K = \frac{(b-a)^{\mathcal{R}(\alpha)}}{\mathcal{R}(\alpha)|\Gamma(\alpha)|}$

(b) If $0 < \alpha < 1$ and $0 < p < \frac{1}{\alpha}$, then the operator I_{a+}^α and I_{b-}^α are bounded from $L^p(a,b)$ into $L^q(a,b)$, where $q = p(1-\alpha p)$.

(c) If $\mathcal{R}(\alpha) > 0$ and $f(x) \in L^p(a,b)$ $1 < p \leq \infty$, then

$$\left(D_{a+}^\alpha I_{a+}^\alpha f\right)(x) = f(x), \text{ and } \left(D_{b-}^\alpha I_{b-}^\alpha f\right)(x) = f(x) \text{ where } (\mathcal{R}(\alpha) > 0). \tag{1.4.24}$$

Hold almost everywhere.

1.5. Caputo Fractional Derivatives

In this section the definitions and some properties of the Caputo fractional derivatives are presented[69]. Let $[a,b]$ be a finite interval of the real line $\mathbb{R}$, and let

$$(D_{a+}^\alpha[y(t)])(x) = (D_{a+}^\alpha y)(x) \text{ and } (D_{b-}^\alpha[y(t)]) = (D_{b-}^\alpha(y))(x)$$

be the Riemann-Liouville fractional derivatives of order $\alpha \in \mathbb{C}(\mathcal{R}(\alpha) > 0)$ defined by (1.4.6) and (1.4.7), respectively. The fractional derivatives $({}^cD_{a+}^\alpha y)(x)$ and $({}^cD_{b-}^\alpha y)(x)$ of order $\alpha \in \mathbb{C}\mathcal{R}(\alpha) \geq 0$ on $[a,b]$ are defined by ;

Definition 1.5.1. *Let $\alpha > 0, \quad n = \lceil \alpha \rceil$. The Caputo derivative operator of order α is defined as*

$$ {}_a^cD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-u)^{n-\alpha-1} \left(\frac{d}{du}\right)^n f(u)du. $$

For $a = 0$, we introduce the notation :

$$ {}^cD_t^\alpha f(t) = D^\alpha f(t). $$

Theorem 1.5.2. *For $t > 0, \alpha \in \mathbb{R}, n-1 < \alpha < n, n \in \mathbb{N}$, and a function $f(t)$ which obey the conditions of Taylor*[4] *theorem, the following representation is valid :*

$$ {}_aD_t^\alpha f(t) = {}_a^cD_t^\alpha f(t) + \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(1+k-\alpha)}(t-a)^{k-\alpha}. $$

Proof. *Proof In order to simplify our presentation, we consider $a = 0$. Because $f(t)$ can be expanded in Taylor series we can write*

$$ f(t) = \sum_{k=0}^{n-1} \frac{t^k}{\Gamma(k+1)} f^{(k)}(0) + R_{n-1}, $$

where :

$$ R_{n-1} = \int_0^t \frac{f^{(n)}(y)(t-y)^{n-1}}{(n-1)!}dy = \frac{1}{\Gamma(n)} \int_0^t f^{(n)}(y)(t-y)^{n-1}dy = I^n f^{(n)}. $$

If we apply the operator D^α we obtain successively :

$$ \begin{aligned} D^\alpha f(t) &= D^\alpha \left[\sum_{k=0}^{n-1} \frac{t^k}{\Gamma(k+1)} f^{(k)}(0) + R_{n-1} \right] \\ &= \sum_{k=0}^{n-1} \frac{D^\alpha t^k}{\Gamma(k+1)} f^{(k)}(0) + D^\alpha R_{n-1} \\ &= \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(0) + I^{n-\alpha} f^n(t) \\ &= \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(0) + D^\alpha f(t). \end{aligned} $$

Definition 1.5.3 (The Caputo Fractional Derivative in the Origin)**.** *For a function $f(t)$, for which $f(t) = 0$, if $t < 0$, it can be defined :*

$$ {}_0^cD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-u)^{n-\alpha-1} f^{(n)}(u)du, $$

where $\mathcal{R}(\alpha) = n$.

Property 1.5.1. *If C is a constant, then :*

$$ {}_0^cD_t^\alpha C = 0, $$

and the Riemann-Liouville Fractional Derivatives of C is :

$$ {}_0D_t^\alpha C = \frac{Cx^{-\alpha}}{\Gamma(1-\alpha)}, \alpha = 1, 2, \ldots $$

4. B.Taylor (1685-1731).

In what follows we note the Caputo derivative in the origin, simply, using the notation $D^\alpha f(x)$.

Theorem 1.5.4. *If $n-1<\alpha<n$, where $n\in\mathbb{N}$, and $\alpha\in\mathbb{R}$, then :*

1. $\lim_{\alpha\to n} D^\alpha f(t) = f^{(n)}(t)$,
2. $\lim_{\alpha\to n-1} D^\alpha f(t) = f^{(n-1)}(t) - f^{(n-1)}(0)$.

Proof. *In the formula*

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(y)dy}{(t-y)^{\alpha+1-n}},$$

we will use the integration by parts, obtaining :

$$\int_0^t u(y)v'(y)dy = u(y)v(y)\Big|_0^t - \int_0^t u'(y)v(y)dy,$$

$$u(y) = f^{(n)}(y), \quad v'(y) = (t-y)^{n-\alpha-1}$$
$$u'(y) = f^{(n+1)}(y), \quad v(y) = -(t-y)^{n-\alpha}.$$

It results :

$$\begin{aligned} D^\alpha f(t) =& \frac{1}{\Gamma(n-\alpha)}\Bigg[- f^{(n)}(y)\frac{(t-y)^{n-\alpha}}{n-\alpha}\Bigg|_0^t \\ &+\frac{1}{n-\alpha}\int_0^t (t-y)^{n-\alpha} f^{(n+1)}(y)dy\Bigg]. \end{aligned}$$

Using the property of Γ function

$$\Gamma(n-\alpha+1) = (n-\alpha)\Gamma(n-\alpha),$$

it results :

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha+1)}\left[f^{(n)}(0) + \int_0^t f^{(n+1)}(y)(t-y)^{n-\alpha}dy\right],$$

$$\lim_{\alpha\to n} D^\alpha f(t) = \left[f^{(n)}(0) + \int_0^t f^{(n+1)}(y)dy\right] = f^{(n)}(0) + f^{(n)}(y)\Big|_0^t = f^{(n)}(t).$$

$$\begin{aligned} \lim_{\alpha\to n-1} D^\alpha f(t) &= \left[f^{(n)}(0) + \int_0^t f^{(n+1)}(y)(t-y)dy\right] \\ &= f^{(n)}(0)t + (t-y)f^{(n)}(y)\Big|_0^t = f^{(n-1)}(t) - f^{(n-1)}(0). \end{aligned}$$

Example 1.5.5. *Let us calculate the FD for $\alpha>0, n-1<\alpha<n, \beta>n-1$ of the function $f(t)=t^\beta$ using the definitions, for the case : Riemann-Liouville, and Caputo in the origin,using the definition.*

Solution. 1. *For the Riemann-Liouville derivative, we can write :*

$$I = D^\alpha t^\beta = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t u^\beta(t-u)^{n-\alpha-1}du.$$

and we take :

$$u = vt, \quad du = tdv.$$

It follows :

$$\begin{aligned} I &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (vt)^\beta[(1-v)t]^{n-\alpha-1}tdv \\ &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (1-v)^{n-\alpha-1}v^\beta t^{n-\alpha+\beta}dv \\ I &= \frac{1}{\Gamma(n-\alpha)}\int_0^t (1-v)^{n-\alpha-1}v^\beta \frac{d^n}{dt^n}t^{n-\alpha+\beta}dv. \end{aligned}$$

but

$$\frac{d^n}{dt^n}t^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-n+1)}t^{\lambda-n},$$

$$B(p,q) = \int_0^1 v^{p-1}(1-v)^{q-1}dv,$$

so that it results :

$$I = \frac{1}{\Gamma(n-\alpha)}\frac{\Gamma(n-\alpha+\beta+1)}{\Gamma(-\alpha+\beta+1)}t^{-\alpha+\beta}\int_0^1 (1-v)^{n-\alpha-1}v^\beta dv$$

$$\int_0^1 (1-v)^{n-\alpha-1}v^\beta dv = B(n-\alpha,\beta+1) = \frac{\Gamma(n-\alpha)\Gamma(\beta+1)}{\Gamma(n-\alpha+\beta+1)}$$

$$D^\alpha t^\beta = I = \frac{\Gamma(\beta+1)}{\Gamma(-\alpha+\beta+1)}t^{\beta-\alpha}.$$

2. *In this case we apply the definition of the Caputo derivative of t^β :*

$$I = D^\alpha t^\beta = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{\left(u^\beta\right)^{(n)}}{(t-u)^{\alpha+1-\beta}}du$$

$$I = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{\Gamma(\beta+1)}{\Gamma(\beta-n+1)}u^{\beta-n}(t-u)^{n-\alpha-1}du.$$

We use the change of variable $u = vt$, resulting after calculations :

$$du = tdv,$$

$$I = \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha)\Gamma(\beta-n+1)}\int_0^1 (uv)^{\beta-n}\left[(t-v)^{n-\alpha-1}\right]tdv.$$

Finally, we obtain :

$$I = \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha)\Gamma(\beta-n+1)}B(\beta-n+1,n-\alpha) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}.$$

Example 1.5.6. *Find the Riemann-Liouville Fractional Integral and Fractional Derivative of*

$$f(t) = (t-a)^\beta.$$

Solution. *For the Fractional Integral we apply the Riemann-Liouville definition :*

$$I = {}_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-u)^{\alpha-1}(u-a)^\beta du.$$

The following change of variable

$$\frac{u-a}{t-a} = v,$$
$$du = (t-a)dv,$$

allows to calculate :

$$I = \frac{(t-a)^{\alpha+\beta}}{\Gamma(\alpha)} \int_0^1 (1-v)^{\alpha-1} v^{\beta} dv = \frac{(t-a)^{\alpha+\beta}}{\Gamma(\alpha)} B(\alpha, \beta+1),$$
$$I = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} (t-a)^{\alpha+\beta}.$$

For the Fractional Derivatives we apply the Riemann-Liouville definition :

$$Df = {}_aD_t^{\alpha}(t-a)^{\beta} = \frac{d^n}{dt^n} aI^{n-\alpha}(t-a)^{\beta},$$

and finally :

$$Df = \frac{\Gamma(\beta+1)}{\Gamma(\beta+n-\alpha+1)} \frac{d^n}{dt^n} (t-a)^{\beta+n-\alpha} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (t-a)^{\beta-\alpha}.$$

Chapitre 2

Stochastic Calculus

This chapter are focused on the Itô lemma for that we presente some definitions about Probability theory and others proprieties are given

2.1. Functions Calculus

2.1.1. Continuous and Differentiable Functions

Definition 2.1.1. *A function g is called continuous at the point $t = t_0$ if the increment of g over small intervals is small,*

$$\Delta g(t) = g(t) - g(t_0) \longrightarrow 0 \text{ as } \Delta t = t - t_0 \longrightarrow 0.$$

If g is continuous at every point of its domain of definition, it is simply called continuous. g is called differentiable at the point $t = t_0$ if at that point

$$\Delta g \sim C\Delta t \text{ or } \lim_{\Delta t \longrightarrow 0} \frac{\Delta g(t)}{\Delta t} = C,$$

this constant C is denoted by $g^{'}(t_0)$. If g is differentiable at every point of its domain, it is called differentiable. An important application of the derivative is a theorem on finite increments.

2.1.2. Right and Left-Continuous Functions

Definition 2.1.2. *A function g is called continuous at the point $t = t_0$ if*

$$\lim_{t \longrightarrow t_0} g(t) = g(t_0),$$

it is called right-continuous (left-continuous) at t_0 if the values of the function $g(t)$ approach $g(t_0)$ when t approaches t_0 from the right (left)

$$\lim_{t \uparrow t_0} g(t) = g(t_0), (\lim_{t \downarrow t_0} g(t) = g(t_0).)$$

If g is continuous it is, clearly, both right and left-continuous. The left-continuous version of g, denoted by $g(t-)$, is defined by taking left limit at each point,

$$g(t-) = \lim_{s \uparrow t} g(s).$$

From the definitions we have, g is left-continuous if $g(t) = g(t-)$. The concept of $g(t+)$ is defined similarly,

$$g(t+) = \lim_{s \downarrow t} g(s).$$

If g is a right-continuous function then $g(t+) = g(t)$ for any t, so that $g+ = g$.

2.1.3. Variation of a Function

If g is a function of real variable, its variation over the interval $[a, b]$ is defined as ;

$$V_g([a,b]) = \sup \sum_{i=1}^{n} |g(t_i^n) - g(t_{i-1}^n)|, \tag{2.1.1}$$

Clearly, (by the triangle inequality) the sums in (2.1.1) increase as new points are added to the partitions. Therefore variation of g is

$$V_g([a,b]) = \lim_{\delta_n \longrightarrow 0} \sum_{i=1}^{n} |g(t_i^n) - g(t_{i-1}^n)|, \tag{2.1.2}$$

where $\delta_n = \max_{1 \leq i \leq n}(t_i - t_{i-1})$. If $V_g([a,b])$ is finite then g is said to be a function of finite variation on $[a, b]$. If g is a function of $t \geq 0$, then the variation function of g as a function of t is defined by

$$V_g(t) = V_g([0,t]).$$

Clearly, $V_g(t)$ is a non-decreasing function of t.

Proof.

$$\begin{aligned}
[g](t) &= \lim_{\delta_n \longrightarrow 0} \sum_{i=0}^{n-1} (g(t_{i+1}^n) - g(t_i^n))^2 && (2.1.3)\\
&\leq \lim_{\delta_n \longrightarrow 0} \max_i |g(t_{i+1}^n) - g(t_i^n)| \sum_{i=0}^{n-1} |g(t_{i+1}^n) - g(t_i^n)| && (2.1.4)\\
&\leq \lim_{\delta_n \longrightarrow 0} \max_i |g(t_{i+1}^n) - g(t_i^n)| V_g(t). && (2.1.5)
\end{aligned}$$

Since g is continuous, it is uniformly continuous on $[0, t]$, hence $\lim_{\delta_n \longrightarrow 0} \max_i |g(t_{i+1}^n) - g(t_i^n)| = 0$, and the result follows.

2.1.4. Riemann Integral

Definition 2.1.3 (Riemann Integral)**.** *The Riemann Integral of f continue over interval $[a, b]$ is defined as the limit of Riemann sums ;*

$$\int_a^b f(t)dt = \lim_{\delta \longrightarrow 0} \sum_{i=1}^{n} f(\xi_i^n)(t_i^n - t_{i-1}^n), \tag{2.1.6}$$

where the t_i^n represent partitions of the interval,

$$a = t_0^n < t_1^n < \cdots < t_n^n = b, \delta = \max_{1\le i\le n}(t_i^n - t_{i-1}^n), \text{ and } t_{i-1}^n \le \xi_i^n \le t_i^n.$$

It is feasible to demonstrate that the Riemann Integral is well defined for continuous functions and that it may be extended to functions that are discontinuous at a finite number of points by breaking up the interval.

Theorem 2.1.4 (The fundamental theorem of calculus)). *If f is differentiable on $[a, b]$ and $f^{'}$ is Riemann integrable on $[a, b]$ then*

$$f(b) - f(a) = \int_a^b f^{'}(s)ds.$$

2.1.5. Stieltjes Integral

The Stieltjes Integral is an integral of the form $\int_a^b f(t)dg(t)$, where g is a finite variation function. Because a function of finite variation is the difference of two rising functions, defining the integral with regard to monotone functions suffices.

Stieltjes Integral with respect to Monotone Functions

Definition 2.1.5. *The Stieltjes Integral of f with respect to a monotone function g over an interval $(a, b]$ is defined as*

$$\int_a^b f dg = \int_a^b f(t)dg(t) = \lim_{\delta\longrightarrow 0} \sum_{i=1}^n f(\xi_i^n)(g(t_i^n) - g(t_{i-1}^n)), \tag{2.1.7}$$

with the quantities in the formulation being the same as for the Riemann Integral above. This integral is a generalization of the Riemann Integral, which may be recovered by taking $g(t) = t$. The Riemann-Stieltjes integral is another name for this integral.

Definition 2.1.6 (Change of Variables). *Let f have a continuous derivative ($f \in C^1$) and g be of finite variation and continuous, then*

$$f(g(t)) - f(g(0)) = \int_0^t f^{'}(g(s))dg(s) = \int_{g(0)}^{g(t)} f^{'}(u)du.$$

If g is of finite variation has jumps, and is right-continuous then

$$\begin{aligned} f(g(t)) - f(g(0)) \quad &= \quad \int_0^t f^{'}(g(s-))dg(s) &(2.1.8)\\ &+ \quad \sum_{0<s\le t}\Big(f(g(s)) - f(g(s-)) - f^{'}(g(s-))\Delta g(s)\Big), &(2.1.9)\end{aligned}$$

where $\delta g(s) = g(s) - g(s-)$ denotes the jump of g at s. This is known in stochastic calculus as Itô's formula.

Example 2.1.7. *Take $f(x) = x^2$, then we obtain*

$$g^2(t) - g^2(0) = 2\int_0^t g(s-)dg(s) + \sum_{s\leq t}(\Delta g(s))^2.$$

Remark 2.1.1. *Note that for a continuous f and finite variation g on $[0,t]$ the approximating sums converge as $\delta = \max_i \left(t_{i+1}^n - t_i^n\right) \to 0$,*

$$\sum_i f\left(g\left(t_i^n\right)\right)\left(g\left(t_{i+1}^n\right) - g\left(t_i^n\right)\right) \to \int_0^t f(g(s-))dg(s).$$

Theorem 2.1.8 (Lebesgue)**.** *A finite variation function g on $[a,b]$ is differentiable almost everywhere on $[a,b]$.*

2.2. Measure Theory

In this section, we will recall some definitions and results from measure theory. Our purpose here is to provide an introduction for readers who have not seen these concepts before and to review that material for those who have.

2.2.1. Probability Spaces

Definition 2.2.1 ($\sigma-$algebra)**.** *Consider a set Ω. a $\sigma-$algebra $\mathcal{A}$ of subsets of Ω is a collection $\mathcal{A}$ of subsets of Ω satisfying the following conditions :*
- *(a) $\emptyset \in \mathcal{A}$*
- *(b) If $B \in \mathcal{A}$ then its complement B^c is also in $\mathcal{A}$*
- *(c) If $B_1, B_2, \ldots$ is a countable collection of sets in $\mathcal{A}$ then their union $\cup_{n=1}^\infty B_n$ is in $\mathcal{A}$*

There are two extreme examples of sigma-algebras :
- — The collection $\{\emptyset, \Omega\}$ is a sigma-algebra of subsets of Ω
- — the set $\mathcal{P}(\Omega)$ of all subsets of Ω is a sigma-algebra

Any $\sigma-$algebra $\mathcal{A}$ of subsets of Ω lies between these two extremes :

$$\{\emptyset, \Omega\} \subset \mathcal{A} \subset \mathcal{P}(X).$$

Definition 2.2.2 (Mesure)**.** *A measure is a nonnegative countably additive set function ; that is, a function $\mu : \mathcal{A} \to \mathbb{R}$ with*
- *(i) $\mu(A) \geq \mu(\emptyset) = 0$ for all $A \in \mathcal{A}$, and*
- *(ii) if $(A_i)_{i\in I} \in \mathcal{A}$ is a countable sequence of disjoint sets, then*

$$\mu(\cup_{i\in I} A_i) = \sum_{i\in I} \mu(A_i)$$

If $\mu(\Omega) = 1$, we call μ a probability measure. In this thesis, probability measures are usually denoted by $\mathbb{P}$. The next result gives some consequences of the definition of a measure.

Theorem 2.2.3. *Let μ be a measure on $(\Omega, \mathcal{A})$*
- *(i) Monotonicity. If $A \subseteq B$ then $\mu(A) \leq \mu(B)$.*
- *(ii) Subadditivity. If $A \subseteq \cup_m A_m$ then $\mu(A) \leq \sum_m \mu(A_m)$.*

(iii) *Continuity from below. If $A_i \uparrow A$ (i.e., $A_1 \subset A_2 \subset \cdots$ and $\cup_i A_i = A$) then $\mu(A_i) \uparrow \mu(A)$.*
(iv) *Continuity from above. If $A_i \downarrow A$ (i.e., $A_1 \supset A_2 \supset \cdots$ and $\cap_i A_i = A$), with $\mu(A_1) < \infty$ then $\mu(A_i) \downarrow \mu(A)$.*

Property 2.2.1. *If $\mathcal{A}$ is a $\sigma-$algebra, and A_n is a sequence in $\mathcal{A}$, then the fol lowing properties follow immediately by checking the axioms :*

1. $\cap_n A_n \in \mathcal{A}$
2. $\limsup_n A_n := \cap_{n=1}^{+\infty} \cup_{k=n}^{+\infty} A_k \in \mathcal{A}$
3. $\liminf_n A_n := \cup_{n=1}^{+\infty} \cap_{k=n}^{+\infty} A_k \in \mathcal{A}$
4. *if $\mathcal{A}, \mathcal{B}$ are algebras, then $\mathcal{A} \cap \mathcal{B}$ is an algebra.*

Definition 2.2.4. *For any set C of subsets of Ω, we can define $\sigma(C)$, the smallest $\sigma-$algebra $\mathcal{A}$ which contains C. The $\sigma-$algebra $\mathcal{A}$ is the intersection of all $\sigma-$algebras which contain C. It is again a $\sigma-$algebra.*

Definition 2.2.5. (E, O) *is a topological space, where O is the set of open sets in E. then $\sigma(O)$ is called the Borel $\sigma-$algebra of the topological space. If $\mathcal{A} \subset \mathcal{B}$, then $\mathcal{A}$ is called a sub-algebra of $\mathcal{B}$. a set B in $\mathcal{B}$ is also called a Borel set.*

Remark 2.2.1. *One sometimes defines the Borel $\sigma-$algebra as the $\sigma-$algebra generated by the set of compact sets C of a topological space. Compact sets in a topological space are sets for which every open cover has a finite subcover. In Euclidean spaces $\mathbb{R}^n$, where compact sets coincide with the sets which are both bounded and closed, the Borel $\sigma-$algebra generated by the compact sets is the same as the one generated by open sets. The two definitions agree for a large class of topological spaces like "locally compact separable metric spaces".*

Remark 2.2.2. *Often, the Borel $\sigma-$algebra is enlarged to the $\sigma-$algebra of all Lebesgue measurable sets, which includes all sets B which are a subset of a Borel set A of measure 0. The smallest $\sigma-$algebra $\overline{\mathcal{B}}$ which contains all these sets is called the completion of $\mathcal{B}$. The completion of the Borel $\sigma-$algebra is the $\sigma-$algebra of all Lebesgue measurable sets. It is in general strictly larger than the Borel $\sigma-$algebra.*

Definition 2.2.6. *A probability space is a triple $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is a set of "outcomes," F is a set of "events," and $\mathbb{P} : \mathcal{F} \to [0, 1]$ is a function that assigns probabilities to events. We assume that $\mathcal{F}$ is a $\sigma-$field (or $\sigma-$algebra), i.e., a (nonempty) collection of subsets of Ω that satisfy*
(i) *if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$, and*
(ii) *if $A_i \in \mathcal{F}$ is a countable sequence of sets then $\cup_i A_i \in \mathcal{F}$.*

Definition 2.2.7. *A map X from a measure space $(\Omega, \mathcal{A})$ to an other measure space $(\Delta, \mathcal{B})$ is called measurable, if $X^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$. The set $X^{-1}(B)$ consists of all points $x \in \mathcal{A}$ for which $X(x) \in \mathcal{B}$. This pull back set $X^{-1}(B)$ is defined even if X is non-invertible. For example, for $X(x) = x^2$ on $(\mathbb{R}, \mathcal{B})$ one has $X^{-1}([1, 4]) = [1, 2] \cup [-2, -1]$.*

Definition 2.2.8. *function $X : \Omega \to \mathbb{R}$ is called a random variable, if it is a measurable map from $(\Omega, \mathcal{A})$ to $(\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ is the Borel $\sigma-$algebra of $\mathbb{R}$. Denote by C the set of all real random variables. The set C is an algebra under addition and multiplication : one can add and multiply random variables and gets new random variables. More generally, one can consider random variables taking values in a second measurable space $(E, \mathcal{B})$. If $E = \mathbb{R}^d$, then the random variable X is called a random vector. For a ran dom vector $X = (X_1, ..., X_d)$, each component X_i is a random variable.*

2.2.2. Convergence of Random Variables

In order to formulate the strong law of large numbers, we need some other notions of convergence.

Definition 2.2.9. *sequence of random variables X_n converges in probability to a random variable X, if*

$$\mathbb{P}\left[|X_n - X| \geq \varepsilon\right] \longrightarrow 0 \text{ for all } \varepsilon > 0.$$

Definition 2.2.10. *sequence of random variables X_n converges almost every where or almost surely to a random variable X, if $\mathbb{P}\left[X_n \longrightarrow X\right] = 1$.*

Definition 2.2.11. *sequence of $\mathcal{L}^p$ random variables X_n converges in $\mathcal{L}^p$ to a random variable X, if*

$$\|X_n - X\|_p \longrightarrow 0 \text{ for } n \longrightarrow \infty.$$

Definition 2.2.12. *sequence of random variables X_n converges fast in probability, or completely to a random variable X, if*

$$\sum_n \mathbb{P}\left[\|X_n - X\| \geq \varepsilon\right] < \infty \text{ for all } \varepsilon > 0.$$

We have so four notions of convergence of random variables $X_n \longrightarrow X$, if the random variables are defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. we will gives now the two equivalent but weaker notions convergence in distribution and weak convergence, which not necessarily assume X_n and X to be defined on the same probability space.

Definition 2.2.13. *sequence of random variables X_n converges in distribution to a random variable X,, if $F_{X_n}(x) \longrightarrow F_X(x)$ for all points x, where F_X is continuous.*

Definition 2.2.14. *sequence of random variables X_n converges in law to a random variable X, if the laws μ_n of X_n converge weakly to the law μ of X.*

Remark 2.2.3. *[72] In other words, X_n converges weakly to X if for every continuous function f on $\mathbb{R}$ of compact support, one has*

$$\int f(x) d\mu_n(x) \longrightarrow \int f(x) d\mu(x).$$

Property 2.2.2. *Given a sequence random variables $X_n \in \mathcal{L}^1$. The following is equivalent :*

(a) X_n converges in probability to X and $\{X_n\}_{n \in \mathbb{N}}$ is uniformly integrable.

(b) X_n converges in $\mathcal{L}^1$ to X.

Theorem 2.2.15 (Dominated convergence theorem). *Suppose $\{f_n\}_n$ is a sequence of measurable functions on a measure space $(\Omega, \mathcal{F}, \mu)$ such that*

$$f(x) = \lim_{n \to \infty} f_n(x), \quad \mu \text{ a.e.} \quad x \in \Omega.$$

If there is an integrable function g, i.e. $\int_\Omega g \, \mathrm{d}\mu < \infty$, such that

$$\left|f_n(x)\right| \leq g(x), \quad \text{for } \mu \text{ a.e. } x \in \Omega \text{ and for all } n \in \mathbb{N}.$$

then f is integrable and

$$\lim_{n \to \infty} \int_\Omega f_n \, \mathrm{d}\mu = \int_\Omega f \, \mathrm{d}\mu.$$

Theorem 2.2.16 (Bounded convergence theorem). *Let $\{f_n\}_n$ be a sequence of uniformly bounded and measurable functions on a bounded measure space $(\Omega, \mathcal{F}, \mu)$ such that*

$$f(x) = \lim_{n\to\infty} f_n(x), \quad \mu \text{ a.e. } x \in \Omega.$$

Then, f is integrable and

$$\lim_{n\to\infty} \int_\Omega f_n \, \mathrm{d}\mu = \int_\Omega f \, \mathrm{d}\mu.$$

2.3. Stochastic Processes

In order to study stochastic calculus, we must first understand stochastic processes. In this section, we define stochastic processes in general and give definitions of some of the most important processes.

Definition 2.3.1 (A stochastic process). *X is a family $\{X_t : t \in T\}$ of random variables which map the sample space Ω into the state space $S \subseteq \mathbb{R}$.*

We can observe stochastic processes in two manners : by studying fixed realizations of the process, or by studying the distributional properties of the process.

Definition 2.3.2. *The realization (or sample path) of X at ω for a fixed $\omega \in \Omega$ is the collection $\{X_t(\omega) : t \in T\}$ of members of S.*

Definition 2.3.3. *Let $t = (t_1, t_2, \ldots, t_n)$ be a vector with each $t_i \in T$. Then the vector $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$ has the joint distribution function $F_{\mathrm{t}} : \mathbb{R}^n \to [0, 1]$ defined by*

$$F_t(\mathbf{x}) = \mathbb{P}\left(X_{t_1} \leq x_1, X_{t_2} \leq x_2, \ldots, X_{t_n} \leq x_n\right).$$

Letting t range over all finite-length vectors of members of T, the collection $\{F_{\mathrm{t}}\}$ is called the collection of finite-dimensional distributions (fdds) of X.

A specific class of stochastic processes,witch have a particular interest, is Gaussian processes.

Definition 2.3.4. *A real-valued, meaning that $S = \mathbb{R}$, continuous-time, meaning that $T = [0, \infty)$, stochastic process X is called a Gaussian process if each finite-dimensional vector $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$ has the multivariate normal distribution $\mathcal{N}(\boldsymbol{\mu}(\mathbf{t}), \boldsymbol{\Sigma}(\mathbf{t}))$ for mean vector $\boldsymbol{\mu}(\mathbf{t})$ and covariance matrix $\boldsymbol{\Sigma}(\mathbf{t})$ which may depend on $\mathbf{t}$.*

Definition 2.3.5 (Filtrations and adapted processes). *A filtration $\mathcal{F} = (\mathcal{F}_t)_t \geq 0$ of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$is a family of $\sigma-$algebras $\mathcal{F}_t$ indexed by $t \in [0, \infty]$, all contained in $\mathcal{F}$, satisfying*

1. *if $s \leq t$ then $\mathcal{F}_s \subset \mathcal{F}_t$, and*
2. *$\mathcal{F}_\infty = \sigma(\cup_{t\geq 0} \mathcal{F}_t)$.*

A stochastic process $(X_t)_{t\geq 0}$ defined on $(\Omega; \mathcal{F}; \mathbb{P})$ is said to be adapted to the filtration $\mathcal{F}$ if for each $t \geq 0$ the random variable (or random vector, if the stochastic process is vector-valued) X_t is measurable relative to $\mathcal{F}_t$.

2.3.1. The Normal Distribution

The normal distribution is the most important one in all of probability and statistics. Many numerical populations have distributions that can be fit very closely by an appropriate normal curve.

Definition 2.3.6. *A continuous random variable X is said to have a normal distribution with parameters μ and σ (or μ and σ^2), where $-\infty < \mu < +\infty$ and $\sigma > 0$ if the pdf*[1] *of X is*

$$f(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \quad -\infty < x < +\infty, \tag{2.3.1}$$

Definition 2.3.7 (The Standard Normal Distribution)**.** *The normal distribution with parameter values $\mu = 0$ and $\sigma = 1$ is called the standard normal distribution. A random variable having a standard normal distribution is called a standard normal random variable and will be denoted by Z. The pdf of Z is*

$$f(z;0,1) = \frac{1}{\sqrt{2\pi}} e^{\frac{z^2}{2}} \quad -\infty < z < +\infty. \tag{2.3.2}$$

Definition 2.3.8 (The Multivariate Gaussian Distribution)**.** *A vector-valued random variable*

$$X = [X_1, X_2, X_3, \dots, X_n]^T$$

is said to have a multivariate normal (or Gaussian) distribution with mean $\mu \in \mathbb{R}^n$ and covariance matrix $\Sigma \in S^n_{++}$ [2] *if its probability density function is given by*

$$p(x;\mu,\Sigma) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left(-1/2(x-\mu)^T\Sigma^{-1}(x-\mu)\right).$$

We write this as $X \sim \mathcal{N}(\mu, \Sigma)$.

Definition 2.3.9. *A random vector $(X_1, \dots, X_n)$ is called a Gaussian random vector if there exists an $n \times m$ matrix A, and an n-dimensional vector b such that $X^{\mathrm{T}} = AY + b$, where Y is an m-dimensional vector with independent standard normal entries.*

2.4. Brownian Motion

The Brownian Motion (called Wiener process) is a stochastic process that will be important for our study of stochastic calculus and the applications of stochastic calculus.

Definition 2.4.1. *A Brownian motion (Wiener process) $B = \{B_t : t \geq 0\}$ is a real-valued Gaussian process such that :*

1. *For any $n, X_j = B_{t_j} - B_{s_j}$ where $1 \leq j \leq n$ are independent variables whenever the intervals $\left(s_j, t_j\right]$ are disjoint (B has independent increments).*

1. pdf : Probability distribution function
2. Recall from the section notes on linear algebra that S^n_{++} is the space of symmetric positive definite $n \times n$ matrices, defined as

$$S^n_{++} = \left\{A \in \mathbb{R}^{n\times n} : A = A^T \text{ and } x^TAx > 0 \text{ for all } x \in \mathbb{R}^n \text{ such that } x \neq 0\right\}.$$

2. *$B_{s+t} - B_s \sim \mathcal{N}\left(0, \sigma^2 t\right) \forall s, t \geq 0$, where σ^2 is a positive constant.*
3. *The sample paths of B are continuous. A Wiener process B is called standard if $B_0 = 0$ and $\sigma^2 = 1$.*

A standard d-dimensional Wiener process is a vector-valued stochastic process

$$B_t = (B_t^{(1)}, B_t^{(2)}, \dots, B_t^{(d)})$$

whose components $B_t^{(i)}$ are independent, standard one-dimensional Wiener processes.

2.4.1. Properties

In this section, we begin to prove some basic properties of Brownian motions that will become invaluable as we start delving into more complex proofs.

Lemma 2.4.2 (Scaling invariance). *Suppose $(B_t)_{t\geq 0}$ is a standard Brownian motion and let $a > 0$. Then, the process $X_t = \frac{1}{a} B_{a^2 t}$ is also a standard Brownian motion.*

Proof. *Continuity of the paths, independence and stationarity of the increments remain unchanged under the rescaling. Considering $B_{t+h} - B_t$, we have*

$$\begin{aligned} \mathbb{E}\left(X(t+h) - X(t)\right) &= \mathbb{E}\left(\frac{1}{a}B(a^2(t+h)) - \frac{1}{a}B(a^2 t)\right) \\ &= \frac{1}{a}\mathbb{E}\left(B(a^2 t + a^2 h) - B(a^2 t)\right), \end{aligned}$$

and we can deduce that $\mathbb{E}\left(X(t+h) - X(t)\right) = 0$ as the same way we proof that $\mathrm{Var}(X(t+h) - X(t)) = h$

$$\begin{aligned} \mathrm{Var}\left(X(t+h) - X(t)\right) &= \mathrm{Var}\left(\frac{1}{a}B(a^2(t+h)) - \frac{1}{a}B(a^2 t)\right) \\ &= \frac{1}{a^2}(a^2 h) = h \end{aligned}$$

Because the function $t \longrightarrow B(t)$ is almost surely continuous, the function

$$t \longrightarrow X(t) = \frac{1}{a}B(a^2 t).$$

Is the composition of (almost surely) continuous functions and is therefore almost surely continuous. thus $X(t)$ is also a standard Brownian motion.

Theorem 2.4.3 (Time inversion). *Suppose B_t is a standard Brownian motion. Then, the process defined by X_t,*

$$X_t = \begin{cases} 0 & \text{if } t = 0 \\ tB_{1/t} & \text{if } t > 0 \end{cases}.$$

is a standard Brownian motion.

Proof. *The increments of this process having an expected value of zero is immediate. To see that the other properties are satisfied, note that for Brownian motions, we have*

$$\begin{aligned}\operatorname{Cov}[B_t, B_{t+s}] = \mathbb{E}[B_t B_{t+s}] &= \mathbb{E}[B_t(B_{t+s} - B_t + B_t)] \\ &= \mathbb{E}\left[B_t^2\right] + \mathbb{E}[B_{t+s}]\,\mathbb{E}[B_t - B_s] = t.\end{aligned}$$

Then, for X_t we get

$$\begin{aligned}\operatorname{Cov}[X_t, X_{t+s}] &= \operatorname{Cov}\left[tB_{1/t}, (t+s)B_{1/(t+s)}\right] \\ &= t(t+s)\operatorname{Cov}\left[B_{1/t}, B_{1/(t+s}\right] = (t+s)\frac{t}{t+s} = t.\end{aligned}$$

Thus,

$$\begin{aligned}\operatorname{Cov}[X_t, X_{t+s} - X_t] &= \operatorname{Cov}[X_t, X_{t+s}] - \operatorname{Var}[X_t] = t - t = 0 \\ \operatorname{Var}[X_{t+s} - X_t] &= \operatorname{Var}[X_{t+s}] + \operatorname{Var}[X_t] - 2\operatorname{Cov}[X_{t+s}, X_t] \\ &= (t+s) + t - 2t = s.\end{aligned}$$

This shows that the increments have the right variance. Independence of increments holds due to X_t and X_{t+s} having zero covariance and being normal variables.

Lastly, we need to demonstrate continuity. When $t > 0$, this is clear. Now, recall that the distribution of X_t over the rationals is the same as the distribution for a Brownian Motion. This implies that for $t \in \mathbb{Q}$,

$$\lim_{t \to 0} X_t = 0.$$

Thus, completing the proof.

Corollary 2.4.4. *Let $\{B_t\}_{t\geq 0}$ be a Brownian motion with admissible filtration $\{\mathcal{F}_t\}_{t\geq 0}$, and let τ be a stopping time for this filtration. Let $\{B_s^*\}_{s\geq 0}$ be a second Brownian motion on the same probability space that is independent of the stopping field $\{\mathcal{F}_\tau\}_\tau$. Then the spliced process.*

$$\tilde{B}_t = \begin{cases} B_t & \text{for} \quad t \leq \tau \\ B_\tau + B^*_{t-\tau} & \text{for} \quad t \geq \tau. \end{cases}.$$

is also a Brownian motion.

Corollary 2.4.5 (Law of large numbers)**.** *Almost surely,*

$$\lim_{t \to \infty} \frac{B_t}{t} = 0.$$

Proof. *Let $\{X(t) : t \geq 0\}$ be as defined in Theorem 2.4.3 . Using this theorem, we see that*

$$\lim_{t \longrightarrow \infty} B(t)/t = \lim_{t \longrightarrow \infty} X(1/t) = X(0) = 0.$$

Lemma 2.4.6. *Let $\{B_t\}_{t\geq 0}$ be a Wiener process and $\{\mathcal{G}_t\}_{t\geq 0}$ a filtration of the probability space on which the Wiener process is defined. The filtration is said to be admissible for the Wiener process if*

(a) The Wiener process is adapted to the filtration, and

(b) for every $t \geq 0$, the process $\{B_{t+s} - B_t\}_{s\geq 0}$ is independent of the $\sigma-$ algebra $\mathcal{G}_t$.

2.4.2. Markov Process

Definition 2.4.7. *Given a measurable space $(S, \mathcal{B})$ called state space, where S is a set and $\mathcal{B}$ is a $\sigma-$algebra on S. A function $P : S \times \mathcal{B} \longrightarrow \mathbb{R}$ is called a transition probability function if $P(x, .)$ is a probability measure on $(S, \mathcal{B})$ for all $x \in S$ and if for every $B \in \mathcal{B}$,*
the map $s \longrightarrow P(s, \mathcal{B})$ is $\mathcal{B}-$measurable. Define $P^1(x, B) = P(x, B)$ and inductively the measures

$$P^{n+1}(x, B) = \int_s P^n(y, B)P(x, dy),$$

where we write $\int P(x, dy)$ for the integration on S with respect to the measure $P(x, .)$.

Definition 2.4.8. *Given a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with a filtration $\mathcal{A}_n$ of $\sigma-$algebras. An $\mathcal{A}_n-$adapted process X_n with values in S is called a discrete time Markov process if there exists a transition probability function P such that*

$$P\left(X_n \in B \backslash A_k\right)(\omega) = P^{n-k}\left(X_k \in B\right).$$

Definition 2.4.9. *If the state space S is a discrete space, a finite or countable set, then the Markov process is called a Markov chain, A Markov chain is called a denumerable Markov chain, if the state space S is countable, a finite Markov chain, if the state space is finite.*

Remark 2.4.1. *It follows from the definition of a Markov process that Xn satisfies the elementary Markov property : for $n > k$:,*

$$\mathbb{P}\left[X_n \in B \backslash X_1, \ldots, X_k\right] = \mathbb{P}\left[X_n \in B \backslash X_k\right].$$

This means that the probability distribution of X_n is determined by know ing the probability distribution of X_{n-i}. The future depends only on the present and not on the past.

Theorem 2.4.10 (Markov processes exist)**.** *For any state space $(S, \mathcal{B})$ and any transition probability function P, there exists a corresponding Markov process X.*

2.4.3. The Strong Markov Property

In this section, we begin to discuss multi-dimensional Brownian motion. Let us start with this definition.

Definition 2.4.11. *If $B_1, \ldots, B_d$ are all independent Brownian motions started in $x_1, \ldots, x_d$, then the random process B_t given by*

$$B_t = \left(B_1, \ldots, B_d\right).$$

is called a d-dimensional Brownian motion started in $(x_1, \ldots, x_d)$. If B_t starts at the origin it is termed a standard d-dimensional Brownian motion.

Theorem 2.4.12 (Markov Property)**.** *Let $\{B_t : t \geq 0\}$ is a Brownian motion started in $x \in \mathbb{R}^d$. Then the process $\{B_{t+s} - B_s : t, s > 0\}$ is a Brownian motion started at the origin and is independent of $\{B_t : 0 \leq t \leq s\}$.*

Proof. *Properties 1 and 3 follow from the cancellation of the $B(s)$ terms and the fact that $\{B(t)_{t\geq 0}\}$ is a Brownian motion. Because the map $t \longrightarrow (B(t + s) - B(s))$ is the composition of (almost surely) continuous functions, the map $t \longrightarrow (B(t + s)B(s))$ is continuous.*

Finally, $\{B(t+s)-B(s)\}_{t\geq 0}$ is a standard Brownian motion since $B(0+s)-B(s)=0$. Recall that two stochastic processes $\{X(t)_{t\geq 0}\}$ and $\{Y(t)_{t\geq 0}\}$ are said to be independent if for any sets of times $t_1, t_2, \dots, t_n \geq 0$ and $s_1, s_2, \dots, s_m \geq 0$ the vectors $(X(t_1), \dots, X(t_n))$ and $(Y(s_1), \dots, Y(s_m))$ are independent. Let $t_1, \dots, t_n \geq 0$ and $s \geq s_1, \dots, s_m \geq 0$. Because Brownian motion has independent increments, it follows that $(B(t_1+s)-B(s), \dots, B(t_n+s)-B(s))$ and $(B(s_1), \dots, B(s_m))$ are independent random vectors.

Definition 2.4.13. *The germ σ-algebra is defined as $\mathcal{F}^+(0)$, where*

$$\mathcal{F}^+(t) = \bigcap_{s>t} \mathcal{F}^0(s),$$

and $\left\{\mathcal{F}^0 : t \geq 0\right\}$ is the σ-algebra generated by $\{B_t : 0 \leq s \leq t\}$.

Definition 2.4.14. *The tail σ-algebra, $\mathcal{T}$ of a Brownian motion is defined as*

$$\mathcal{T} = \bigcap_{t\geq 0} \mathcal{G}(t),$$

where $\mathcal{G}(t)$ is the σ algebra generated by $\{B_s : s \geq t\}$.

Theorem 2.4.15. *For all $s \geq 0$, the random process $\{B_{t+s} - B_s : t \geq 0\}$ is independent of $\mathcal{F}^+(s)$.*

Proof. *By continuity $B(t+s) - B(s) = \lim_{n\longrightarrow\infty} B(s_n + t) - B(s_n)$ for a strictly decreasing sequence $\{s_n : n \in \mathbb{N}\}$ converging to s. By Theorem 2.4.12, for any $t_1, \dots, t_m = 0$, the vector*

$$(B(t_1+s)-B(s), \dots, B(t_m+s)-B(s)) = \lim_{j\uparrow\infty}(B(t_1+s_j)-B(s_j), \dots, B(t_m+s_j)-B(s_j)). \quad (2.4.1)$$

is independent of $\mathcal{F}^+(s)$, and so is the process $B(t+s)-B(s) : t \geq 0$.

Definition 2.4.16. *A nonnegative random variable τ (possibly taking the value $+\infty$) is a stopping time with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ if for every $t \geq 0$ the event $\{\tau \leq t\} \in \mathcal{F}_t$. The stopping time τ is proper if $\tau < \infty$ on Ω. The stopping field $\mathcal{F}_\tau$ associated with a stopping time τ is the σ-algebra consisting of all events $B \subset \mathcal{F}_\infty$ such that $B \cap \{\tau \leq t\} \in \mathcal{F}_t$ for every $t \geq 0$.*

Theorem 2.4.17 (Strong Markov property)**.** *For every almost surely finite stopping time T, the process $\{B_{T+t} - B_T : t \geq 0\}$ is a standard Brownian motion independent of $\mathcal{F}^+(T)$.*

Proof. *Let T be a stopping time. We can then define*

$$T_n = (m+1)2^{-n}, \text{ where } m/2^n \leq T < (m+1)/2^n.$$

This can be thought of as a discrete approximation which stops at the first dyadic rational next to the original. Keeping in mind that this definition implies that T_n is also as stopping time, we then define the following :

$$B_k(t) = B_{t+k/2^n} - B_{k/2^n} \text{ and } B_k = \{B_k(t) : t \geq 0\}$$
$$B_*(t) = B_{t+T_n} - B_{T_n} \text{ and } B_* = \{B_*(t) : t \geq 0\}.$$

Now, take $E \in \mathcal{F}^+(T_n)$ and the event $\{B_* \in A\}$. We have

$$\mathbb{P}\left(\{B_* \in A\} \bigcap E\right) = \sum_{k=1}^{\infty} \mathbb{P}\left(\{B_k \in A\} \bigcap E \bigcap \{T_n = k/2^n\}\right).$$

Note, however, that $E \bigcap \{T_n = k/2^n\} \in \mathcal{F}^+(k/2^n)$, *which by Theorem 2.4.15 is independent of* $\{B_k \in A\}$. *Thus, we have*

$$\mathbb{P}\left(\{B_* \in A\} \bigcap E\right) = \sum_{k=0}^{\infty} \mathbb{P}\{B_k \in A\} \mathbb{P}\left(E \bigcap \{T_n = k/2^n\}\right).$$

Now, using the Markov property 2.4.17we see that for all $k \in \mathbb{N}, \mathbb{P}\{B \in A\} = \mathbb{P}\{B_k \in A\}$. *This yields*

$$\begin{aligned}\sum_{k=0}^{\infty} \mathbb{P}\{B_k \in A\} \mathbb{P}\left(E \bigcap \{T_n = k/2^n\}\right) &= \mathbb{P}\{B \in A\} \sum_{k=0}^{\infty} \mathbb{P}\left(E \bigcap \{T_n = k/2^n\}\right) \\ &= \mathbb{P}\{B \in A\}\mathbb{P}(E).\end{aligned}$$

Thus, B_* *is independent of every E and hence independent of* $\mathcal{F}^+(T_n)$. *Now, recall that the sequence* T_n *is a uniformly decreasing sequence that converges to* T, *hence* $\mathcal{F}^+(T_n) \subset \mathcal{F}^+(T)$ *is independent of the Brownian motion* $(B_{s+T_n} - B_{T_n})$. *Then, the random process* $(B_{r+T} - B_T)$, *defined by the increments*

$$B_{s+t+T} - B_{t+T} = \lim_{n\to\infty} (B_{s+t+T_n} - B_{t+T_n}),$$

is independent, $\mathcal{N}(0, s)$, *and almost surely continuous. Thus, it is a Brownian motion and independent of* $\mathcal{F}^+(T)$.

2.5. Itô Integral

we define the stochastic integral $\int_0^{+\infty} \psi(s)dBs$, as the mean square limit of Riemann-Stieltjes sums.

We first define the integral for a simpler set of stochastic processes, random step functions, as a Riemann-Stieltjes sum. We then show that, for stochastic processes $\psi(s)$ satisfying certain conditions, the process can be expressed as the limit of a sequence of random step functions. We define the Itô integral to be the mean square limit of the sequence of integrals of the random step functions.

Definition 2.5.1. *Denote by* $\{\mathcal{F}_t\}_t$ *the filtration generated by the one-dimensional Brownian motion* B_t *and by* $\mathcal{B}$ *the Borel* σ*-algebra on* $[0, \infty)$. *Let* $\mathcal{V} = \mathcal{V}(v, w)$ *be the class of functions*

$$f : [0, \infty) \times \Omega \to \mathbb{R}.$$

Such that

(i) $(t, \omega) \to f(t, \omega)$ *is* $\mathcal{B} \times \mathcal{F}$ *measurable.*

(ii) $f(t, \omega)$ *is* $\mathcal{F}_t$ *adapted.*

(iii) $E\left[\int_v^w (f(t, \omega))^2 \, dt\right] < \infty$.

Definition 2.5.2 (Elementary function)**.** *A function* $\phi \in \mathcal{V}$ *is called elementary if it has the form*

$$\phi(t, \omega) = \sum_j e_j(\omega) I_{[t_j, t_{j+1})}(t).$$

- Note that since $\phi \in \mathcal{V}$ each function e_j must be $\mathcal{F}_{t_j}$-measurable - For elementary functions $\phi(t, \omega)$ we define the integral as :

$$\int_v^w \phi(t, \omega) dB_t(\omega) = \sum_{j\geq 0} e_j(\omega) \left[B_{t_{j+1}} - B_{t_j}\right](\omega).$$

Lemma 2.5.3 (The Itô isometry). *If $\phi(t,\omega)$ is bounded and elementary then*

$$\mathbb{E}\left[\left(\int_v^w \phi(t,\omega)\mathrm{d}B_t(\omega)\right)^2\right] = \mathbb{E}\left[\int_v^w (\phi(t,\omega))^2 \,\mathrm{d}t\right].$$

Proof. *Let $\Delta B_j = B_{t_{j+1}} - B_{t_j}$. Then*

$$\mathbb{E}\left[e_i e_j \Delta B_i \Delta B_j\right] = \begin{cases} 0 & \text{if } i \neq j \\ \mathbb{E}\left[e_j^2\right]\left(t_{j+1} - t_j\right) & \text{if } i = j \end{cases}.$$

using independence of $e_i e_j \Delta B_i$ and ΔB_j if $i < j$. Thus

$$\begin{aligned} \mathbb{E}\left[\left(\int_v^w \phi \mathrm{d}B\right)^2\right] &= \sum_{i,j} \mathbb{E}\left[e_i e_j \Delta B_i \Delta B_j\right] \\ &= \sum_j \mathbb{E}\left[e_j^2\right]\left(t_{j+1} - t_j\right) \\ &= \mathbb{E}\left[\int_v^w \phi^2 \,\mathrm{d}t\right]. \end{aligned}$$

We now use the Itô isometry to extend the definition from elementary functions to functions on $\mathcal{V}$.

Lemma 2.5.4. *Let $g \in \mathcal{V}$ be bounded and $g(\cdot,\omega)$ continuous for each ω. Then there exist elementary functions $\phi_n \in \mathcal{V}$ such that*

$$\mathbb{E}\left[\int_v^w \left(g - \phi_n\right)^2 \,\mathrm{d}t\right] \to 0 \quad \text{as } n \to \infty.$$

Proof. *Define $\phi_n(t,\omega) = \sum_j g\left(t_j,\omega\right) I_{\left[t_j,t_{j+1}\right)}(t)$. Then ϕ_n is elementary since $g \in \mathcal{V}$, and, for each ω*

$$\int_v^w \left(g - \phi_n\right)^2 \,\mathrm{d}t \to 0 \quad \text{as } n \to \infty,$$

since $g(\cdot,\omega)$ is continuous for each ω. Hence $\mathbb{E}\left[\int_V^W \left(g - \phi_n\right)^2 \,\mathrm{d}t\right] \to 0$ as $n \to \infty$ by bounded convergence 2.2.16.

Lemma 2.5.5. *Let $h \in \mathcal{V}$ be bounded. Then there exist bounded functions $g_n \in \mathcal{V}$ such that $g_n(\cdot,\omega)$ is continuous for all ω and n, and*

$$\mathbb{E}\left[\int_v^w \left(h - g_n\right)^2 \,\mathrm{d}t\right] \to 0 \quad \text{as } n \to \infty.$$

Proof. *Suppose $|h(t,\omega)| \leq M$ for all (t,ω). For each n let ψ_n be a non-negative, continuous function on $\mathbb{R}$ such that;*

(i) $\psi_n(x) = 0$ *for* $x \leq -1/n$ *and* $x \geq 0$

(ii) $\int_{-\infty}^{+\infty} \psi_n(x)\mathrm{d}x = 1$.

Let us define

$$g_n(t,\omega) = \int_0^t \psi_n(s-t)h(s,\omega)\mathrm{d}s.$$

Clearly, $g_n(\cdot,\omega)$ is continuous for each ω and $\left|g_n(t,\omega)\right| \leq M$. Since $h \in \mathcal{V}$, $g_n(t,\cdot)$ is $\mathcal{F}_t$-measurable for all t. Moreover,

$$\int_v^w (h(s,\omega) - g_n(s,\omega))^2 \, \mathrm{d}s \to 0 \text{ as } n \to \infty, \text{ for each } \omega,$$

By bounded convergence we get

$$\mathbb{E}\left[\int_v^w (h(s,\omega) - g_n(s,\omega))^2 \, \mathrm{d}s\right] \to 0, \quad \text{as } n \to \infty.$$

Lemma 2.5.6. *Let $f \in \mathcal{V}$. There exists a sequence of bounded functions $\{h_n\} \subset \mathcal{V}$ such that*

$$\mathbb{E}\left[\int_v^w (f - h_n)^2 \, \mathrm{d}t\right] \to 0 \quad \text{as } n \to \infty.$$

Proof. *Let us define*

$$h_n(t,\omega) = \begin{cases} -n & \text{if } f(t,\omega) < -n \\ f(t,\omega) & \text{if } -n < f(t,\omega) < n. \\ n & \text{if } f(t,\omega) > n \end{cases}.$$

Then h_n is bounded for each $n \in \mathbb{N}$ and

$$\int_v^w (f(s,\omega) - h_n(s,\omega))^2 \, \mathrm{d}s \to 0 \text{ as } n \to \infty, \text{ for each } \omega.$$

The result then follows by dominated convergence2.2.15.

- We can now complete the definition of the Itô integral

$$\int_v^w f(t,\omega)\mathrm{d}B_t(\omega), \quad \text{for } f \in \mathcal{V}.$$

- If $f \in \mathcal{V}$ by Lemmas 2.5.4 to 2.5.6 we can choose elementary functions $\phi_n \in \mathcal{V}$ such that

$$\mathbb{E}\left[\int_v^w \left(f - \phi_n\right)^2 \, \mathrm{d}t\right] \to 0 \quad \text{as } n \to \infty$$

- We can then define

$$\mathcal{I}[f](\omega) = \int_v^w f(t,\omega)\mathrm{d}B_t(\omega) = \lim_{n\to\infty} \int_S^T \phi_n(t,\omega)\mathrm{d}B_t(\omega).$$

- The limit above exists as an element of $L^2(P)$ since

$$\left\{\int_v^w \phi_n(t,\omega)\mathrm{d}B_t(\omega)\right\}.$$

is a Cauchy sequence in $L^2(P)$ by the Itô Isometry.

2.5.1. The Itô Integral

Definition 2.5.7 (The Itô integral)**.** *Let $f \in \mathcal{V}(v,w)$. Then the Itô integral of f (from v to w) is defined by*

$$\int_v^w f(t,\omega)\mathrm{d}B_t(\omega) = \lim_{n\to\infty}\int_v^w \phi_n(t,\omega)\mathrm{d}B_t(\omega) \quad \textit{limit in } L^2(P), \tag{2.5.1}$$

where $\{\phi_n\}$ is a sequence of elementary functions such that

$$E\left[\int_v^w \left(f(t,\omega) - \phi_n(t,\omega)\right)^2 \mathrm{d}t\right] \to 0 \quad \textit{as } n \to \infty. \tag{2.5.2}$$

Note that :
- such a sequence $\{\phi_n\}$ exists by Lemmas 2.5.4 to 2.5.6 .
- the limit in 2.5.1 exists and does not depend on the choice of $\{\phi_n\}$, as long as 2.5.2 holds.

Corollary 2.5.8 (The Itô isometry)**.**

$$\mathbb{E}\left[\left(\int_v^w f(t,\omega)\mathrm{d}B_t(\omega)\right)^2\right] = \mathbb{E}\left[\int_v^w (f(t,\omega))^2 \,\mathrm{d}t\right] \quad \textit{for all } f \in \mathcal{V}(v,w) \tag{2.5.3}$$

Corollary 2.5.9. *If $f(t,\omega) \in \mathcal{V}(v,w)$, $f_n(t,\omega) \in \mathcal{V}(t,\omega)$ for $n = 1,2,\dots$ and*

$$\mathbb{E}\left[\int_v^w (f_n(t,\omega) - f(t,\omega))^2 \,\mathrm{d}t\right] \to 0 \quad \textit{as } n \to \infty,$$

then

$$\int_v^w f_n(t,\omega)\mathrm{d}B_t(\omega) \to \int_v^w f(t,\omega)\mathrm{d}B_t(\omega) \quad \textit{in } L^2(P) \textit{ as } n \to \infty.$$

Example 2.5.10. *Assume $B_0 = 0$. Then*

$$\int_0^t B_s \,\mathrm{d}B_s = \frac{1}{2}B_t^2 - \frac{1}{2}t.$$

To prove this we consider the sequence of elementary functions

$$\phi_n(t,\omega) = \sum_j B_j(\omega) I_{[t_j,t_{j+1})}(t),$$

where $B_j = B_{t_j}$. Then

$$\begin{aligned}
\mathbb{E}\left[\int_0^t \left(\phi_n - B_s\right)^2 \,\mathrm{d}s\right] &= \mathbb{E}\left[\sum_j \int_{t_j}^{t_{j+1}} \left(B_j - B_s\right)^2 \,\mathrm{d}s\right] \\
&= \sum_j \int_{t_j}^{t_{j+1}} \left(s - t_j\right)\mathrm{d}s \\
&= \sum_i \frac{1}{2}\left(t_{j+1} - t_j\right)^2 \to 0 \textit{ as } \Delta t_j \to 0.
\end{aligned}$$

By the previous corollary, we get that

$$\int_0^t B_s \, \mathrm{d}B_s = \lim_{\Delta t_j \to 0} \int_0^t \phi_n \, \mathrm{d}B_s = \lim_{\Delta t_j \to 0} \sum_j B_j \Delta B_j.$$

We now note that

$$\Delta\left(B_j^2\right) = B_{j+1}^2 - B_j^2 = \left(B_{j+1} - B_j\right)^2 + 2B_j\left(B_{j+1} - B_j\right) = \left(\Delta B_j\right)^2 + 2B_j \Delta B_j,$$

and therefore

$$B_t^2 = \sum_j \Delta\left(B_j^2\right) = \sum_j \left(\Delta B_j\right)^2 + 2\sum_j B_j \Delta B_j.$$

that is

$$\sum_j B_j \Delta B_j = \frac{1}{2}B_t^2 - \frac{1}{2}\sum_j \left(\Delta B_j\right)^2.$$

Noting that $\sum_j \left(\Delta B_j\right)^2 \to t$ *in* $L^2(P)$ *as* $\Delta t_j \to 0$, *we obtain the result.*

2.5.2. The Itô Lemma

Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function. If we consider a sample path of a Wiener process $B_t(\omega)$, we have the following Taylor expansion, writing $dB_t = B_{t+dt} - B_t$:

$$f(B_t + dB_t) - f(B_t) = f'(B_t)dB_t + \frac{1}{2}f''(B_t)(dB_t)^2 + \dots.$$

Then, because $(dB_t)^2 \overset{\text{m.s.}}{\to} dt$ as $dt \to 0$, third and higher order terms have negligible contribution to the Taylor expansion, so we have :

$$f(B_t) - f(B_s) = \int_v^w f'(B_x)\, dB_x + \frac{1}{2}\int_v^w f''(B_x)\, dx.$$

where $\int_v^w f'(B_x)\, dB_x$ is an Itô integral, and $\int_v^w f''(B_z)\, dx$ is a Riemann integral, and equality is in the mean square sense.
If we let $f : \mathbb{R}^2 \to \mathbb{R}$ have infinite partial derivatives, the Taylor expansion yields :

$$\begin{aligned} f(t + dt, B_{t+dt}) - f(t, B_t) =& f_1(t, B_t)\, dt + f_2(t, B_t)\, dB_t \\ &+ \frac{1}{2}\left[f_{11}(t, B_t)(dt)^2 + 2f_{12}(t, B_t)\, dt dB_t + f_{22}(t, B_t)(dB_t)^2\right. \end{aligned}$$

Because $(dB_t)^2 \overset{\text{m.s.}}{\to} dt$ as $dt \to 0$, we have the following :

1. The contribution of third and higher order terms to the Taylor expansion are negligible.
2. The contribution of the $dtdB_t$ term is negligible. Therefore, we have :

$$f(t, B_t) - f(s, B_s) = \int_v^w \left[f_1(x, B_\alpha) + \frac{1}{2}f_{22}(x, B_x)\right] dx + \int_v^w f_2(x, B_x)\, dB_x.$$

   where again $\int_v^w f_2(x, B_x)\, dB_x$ is an Itô integral, and $\int_v^w \left[f_1(x, B_x) + \frac{1}{2}f_2(x, B_x)\right] dx$ is a Riemann integral, and equality is in the mean square sense.

2.5.3. Itô Processes

There are a subset of stochastic processes, called Itô processes, which can be represented as the solution to a stochastic differential equation (SDE).

Definition 2.5.11. *A stochastic process X_t is an Itô process if it is a solution to the stochastic differential equation :*

$$X_t = X_0 + \int_0^t \mu(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dB_s.$$

where μ and σ are stochastic processes, $\int_0^t \mu(s, X_s)\, ds$ is a Riemann integral, and $\int_0^t \sigma(s, X_s)\, dB_s$ is an Itô integral. We abbreviate this SDE *to :*

$$dX = \mu(t, X)dt + \sigma(t, X)dW.$$

We state without proof the following facts :

1. So long as weak conditions on μ, σ, and X_0 are satisfied, the SDE

$$dX = \mu(t, X)dt + \sigma(t, X)dB$$

   has a unique solution which has continuous sample paths.
2. If X is an Itô process, the processes μ and σ are uniquely determined :

$$\mu_1(t, X)dt + \sigma_1(t, X)dW = dX = \mu_2(t, X)dt + \sigma_2(t, X)dW \Rightarrow \mu_1 = \mu_2 \text{ and } \sigma_1 = \sigma_2.$$

We will now extend the Itô Lemma to stochastic processes that are functions of Itô processes. a justification for the formula rather than a rigorous proof. Suppose X_t is an Itô process with

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dB_t.$$

Let $f : \mathbb{R}^2 \to \mathbb{R}$ have infinite partial derivatives. Then a Taylor expansion for

$$f(t + dt, X_{t+dt}) - f(t, X_t)$$

yields :

$$\begin{aligned} f(t + dt, X_{t+dt}) - f(t, X_t) =& f_1(t, X_t)\, dt + f_2(t, X_t)\, dX_t \\ &+ \frac{1}{2}\left[f_{11}(t, X_t)(dt)^2 + 2f_{12}(t, X_t)\, dtdX_t + f_{22}(t, X_t)(dX_t)^2\right] \\ &\dots \\ =& f_1(t, X_t)\, dt + \mu f_2(t, X_t)\, dt + \sigma f_2(t, X_t)\, dB_t \\ &+ \frac{1}{2}\left[\sigma^2 f_{22}(t, X_t)(dB_t)^2\right]. \end{aligned}$$

Then we have the following extension of the Itö Lemma :

$$\begin{aligned} f(t, X_t) - f(s, X_s) =& \int_s^t \left[f_1\left(y, X_y\right) + \mu f_2\left(y, X_y\right) + \frac{\sigma^2}{2} f_{22}\left(y, X_y\right)\right] dy \\ &+ \int_s^t \sigma f_2\left(y, X_y\right) dB_y. \end{aligned}$$

2.5.4. Geometric Brownian motion

Let $\mu, \sigma \in \mathbb{R}$ with $\sigma > 0$. Consider the SDE :

$$dX_t = X_t\left(\mu dt + \sigma dB_t\right).$$

where W_t is a standard Wiener process. The unique solution to this SDE is given by :

$$X_t = f\left(t, B_t\right) = X_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t}.$$

and X_t is called a geometric Brownian motion. The geometric Brownian motion of this form is clearly an Itö process with

$$\mu\left(s, X_s\right) = \mu X_s \text{ and } \sigma\left(s, X_s\right) = \sigma X_s.$$

where $\mu\left(s, X_s\right)$ and $\sigma\left(s, X_s\right)$ are stochastic processes determining the Itô process, and μ and σ are constant parameters of the geometric Brownian motion.

Definition 2.5.12 (Stochastic variables)**.** *A random (or stochastic) variable* $\mathrm{X}(\omega), \omega \in \Omega$ *is a real valued function defined on the sample space* Ω*. In the following we omit the parameter* ω *whenever no confusion is possible.*

Definition 2.5.13 (Probability of an event)**.** *The probability of an event equals the number of elementary outcomes divided by the total number of all elementary outcomes, provided that all cases are equally likely.*

Definition 2.5.14 (Probability distribution function and probability density)**.** *In the continuous case, the probability distribution function* (PDF)$\mathrm{F_X}(x)$ *of a vectorial stochastic variable* $\mathrm{X} = (\mathrm{X}_1, \ldots, \mathrm{X}_n)$ *is defined by the monotonically increasing real function*

$$\mathrm{F_X}\left(x_1, \ldots, x_n\right) = \mathbb{P}\left(\mathrm{X}_1 \leq x_1, \ldots, \mathrm{X}_n \leq x_n\right). \tag{2.5.4}$$

where we used the convention that the variable itself is written in upper case letters, whereas the actual values that this variable assumes are denoted by lower case letters.

The probability density $\mathrm{p_X}\left(x_1, \ldots, x_n\right)$ (PD) *of the random variable is then defined by*

$$\mathrm{F_X}\left(x_1, \ldots, x_n\right) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \mathrm{p_X}\left(u_1, \ldots, u_n\right) \mathrm{d}u_1 \cdots \mathrm{d}u_n. \tag{2.5.5}$$

and this leads to

$$\frac{\partial^n \mathrm{F_X}}{\partial x_1 \cdots \partial x_n} = \mathrm{p_X}\left(x_1, \ldots, x_n\right).$$

Note that we can express (2.5.4) *and* (2.5.5) *alternatively if we put*

$$\begin{aligned} &\mathbb{P}\left(x_{11} \leq \mathrm{X}_1 \leq x_{12}, \ldots, x_{n1} \leq \mathrm{X}_n \leq x_{n2}\right) \\ &= \int_{x_{11}}^{x_{12}} \cdots \int_{x_{n1}}^{x_{n2}} \cdots \mathrm{p_X}\left(x_1, \ldots, x_n\right) \mathrm{d}x_1 \cdots \mathrm{d}x_n. \end{aligned}$$

The conditions to be imposed on the PD *are given by the positiveness and the normalization condition*

$$\mathbb{P}_\mathrm{X}\left(x_1, \ldots, x_n\right) \geq 0; \qquad \int \cdots \int \mathrm{p_X}\left(x_1, \ldots, x_n\right) \mathrm{d}x_1 \cdots \mathrm{d}x_n = 1.$$

In the latter equation we used the convention that integrals without explicitly given limits refer to integrals extending from the lower boundary $-\infty$ *to the upper boundary* ∞*. In a continuous phase space the PD may contain Dirac delta functions*

$$\mathbb{P}(x) = \sum_k \mathrm{q}(k)\delta(x-k) + \hat{\mathbb{P}}(x); \quad \mathrm{q}(k) = \mathbb{P}(x = k).$$

where q(k) represents the probability that the variable x of the discrete set equals the integer value k. We also dropped the index X in the latter formula. We can interpret it to correspond to a PD of a set of discrete states of probabilities q(k) that are embedded in a continuous phase space S. The normalization condition (1.4) yields now

$$\sum_k \mathrm{q}_k + \int_s \hat{\mathrm{p}}(x)\mathrm{d}x = 1.$$

Chapitre 3

Fixed Point Theorems & C_0−semigroup

In this chapter we present some definitions and theorem of fixed point and others properties of the semi-group theory are presented.

3.1. Generalized Metric and Banach Spaces

In this section we define vector metric spaces and generalized Banach spaces and prove some properties. If, $x, y \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, by $x \le y$ we mean $x_i \le y_i$ for all $i = 1, \ldots, n$. Also $|x| = (|x_1|, \ldots, |x_n|)$ and $\max(x, y) = \max(\max(x_1, y_1), \ldots, \max(x_n, y_n))$. If $c \in \mathbb{R}$, then $x \le c$ means $x_i \le c$ for each $i = 1, \ldots, n$. For $x \in \mathbb{R}^n$, $(x)_i = x_i$, $i = 1, \ldots, n$.

Definition 3.1.1. *Let X be a nonempty set. By a vector-valued metric on X we mean a map $d : X \times X \to \mathbb{R}^n$ with he following properties :*
(i) $d(u, v) \ge 0$ for all $u, v \in X$; if $d(u, v) = 0$ then $u = v$
(ii) $d(u, v) = d(v, u)$ for all $u, v \in X$
(iii) $d(u, v) \le d(u, w) + d(w, v)$ for all $u, v, w \in X$.
Note that for any $i \in \{1, \ldots, n\}$ $(d(u, v))_i = d_i(u, v)$ is a metric space in X.

We call the pair (X, d) generalized metric space .For $r = (r_1, r_2, \ldots, r_n) \in \mathbb{R}^n_+$,we will denote by

$$B(x_0, r) = \{x \in X : d(x_0, x) < r\}.$$

the open ball centrad in x_0 with radius r and

$$\overline{B(x_0, r)} = \{x \in X : d(x_0, x) \le r\}.$$

the closed ball centered in x_0 with radius r.

Definition 3.1.2. *Let E be a vector space on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. By a vector-valued norm on E we mean a map $\| \cdot \| : E \to \mathbb{R}^n_+$ with the following properties :*
(i) $\|x\| \ge 0$ for all $x \in E$; if $\|x\| = 0$ then $x = 0$.

(ii) $\|\lambda x\| = |\lambda|\|x\|$ *for all* $x \in E$ *and* $\lambda \in \mathbb{K}$.
(iii) $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in E$.

The pair $(E, \|\cdot\|)$ is called a generalized normed space. If the generalized metric generated by $\|\cdot\|$ (i.e $d(x, y) = \|x - y\|$) is complete then the space $(E, \|\cdot\|)$ is called a generalized Banach space, where

$$\|x - y\| = \begin{pmatrix} \|x - y\|_1 \\ \dots \\ \|x - y\|_n \end{pmatrix}.$$

Notice that $\|\cdot\|$ is a generalized Banach space on E if and only if $\|\cdot\|_i$, $i = 1, \dots, n$ are norms on E. In the following, we are interested by giving some fixed point theorems with related notions :

Definition 3.1.3. *Let E and F two Banach spaces and A be an application defined on E in F. We say that A is completely continuous if it is continuous and transforms any bounded of E into a relatively compact set in F.A is called compact if* $A(E)$ *is relatively compact in F.*

3.2. Fixed Point Theorems

Theorem 3.2.1 ([50])**.** *[Banach contraction principle]Let E be a Banach space. If* $A : E \to E$ *is a contraction, then A has a unique fixed point in E.*

Theorem 3.2.2 (Schauder's fixed point theorem)**.** *Let* $\mathcal{M}$ *be a closed convex subset of a Banach space E. If* $A : \mathcal{M} \to \mathcal{M}$ *is continuous and the set* $\overline{A(\mathcal{M})}$ *is compact, then A has a fixed point in* $\mathcal{M}$.

Theorem 3.2.3. *[50][Non linear Alternative of Leray-Shauder type for contractive maps] Suppose U is an open subset of Banach space X,* $0 \in U$ *and* $N : \overline{U} \to X$ *a contraction with* $N(\overline{U})$ *bounded.Then either*

1. *N has a fixed point in* $\overline{U}$
   Or
2. *There exist* $\lambda \in (0; 1)$ *and* $u \in \partial U$ *with* $u = \lambda N(u)$

Theorem 3.2.4 (Perov)**.** *[112] Let* $(X; d)$ *be a complete generalized metric space and* $T : X \to X$ *a generalized contraction with Lipschitz matrix M : Then T has a unique .fixed point* x^* *and for each* $x \in X$ *we have*

$$d(T^k(x); x^*) \leq M^k(I - M)^{-1} d(x; T(x)); k \in \mathbb{N}$$

Theorem 3.2.5. *[112]Let* (X, d) *be a complete generalized metric space with* $d : X \times X \longrightarrow \mathbb{R}^n$ *and let* $N : X \longrightarrow X$ *be such that*

$$d(N(x), N(y)) \leq Md(x, y).$$

For all $x, y \in X$ *and some square matrix M of nonnegative numbers. If the matrix M is convergent to zero, that is* $M^k \longrightarrow 0$ *as* $k \longrightarrow \infty$*, then N has a unique fixed point* $x_* \in X$

$$d(N^k(x_0), x_*) \leq M^k(I - M)^{-1} d(N(x_0), x_0).$$

For every $x_0 \in X$ *and* $k \geq 1$.

Lemma 3.2.6. *If* $A \in M_{n\times n}(\mathbb{R}^+)$ *is a matrix with* $\rho(A) < 1$*; then* $\rho(A + B) < 1$ *for every matrix* $B \in M_{n\times n}(\mathbb{R}^+)$ *whose elements are small enough.*

The role of matrices with spectral radius less than one in the study of semilinear operator systems was pointed out in [113], also in connection with other abstract principles from nonlinear functional analysis.

Theorem 3.2.7 (Schauder). *Let X be a Banach space, $D \subset X$ a nonempty closed bounded convex set and $T : D \to D$a completely continuous operator (i.e., T is continuous and $T(D)$ is relatively compact). Then T has at least one fixed point.*

Theorem 3.2.8 (Leray-Schauder). *Let $(X; |.|_X)$ be a Banach space, $R > 0$ and*

$$T : \overline{B}_X(0; R) \to X$$

a completely continuous operator. If $|u|_X < R$ for every solution u of the equation $u = \lambda T(u)$ and any $\lambda \in (0; 1)$; then T has at least one fixed point.

Theorem 3.2.9 (Krasnosel'skii, M.A. (1955)). *[75] Let M be a closed, convex, bounded and nonempty subset of a Banach space X. Let A_1 and A_2 be two operators such that*

a) $A_1x + A_2y \in M$ for all $x, y \in M$;

b) A_1 is a completely continuous operator (continuous, and compact, that is, it maps bounded sets into relatively compact sets);

c) A_2 is a contraction mapping.

Then there exists $z \in M$ such that $z = A_1z + A_2z$.

3.3. Semigroup Theory

Definition 3.3.1. *Let X be a (real or complex) Banach space. A one-parameter semigroup on X is a function $T : [0, 1) \to L(X)$ (where $L(X)$ denotes the space of bounded linear operators in X, with domain all of X), satisfying,*

(i) $T(t + s) = T(t)T(s)$, for all $t, s > 0$.
If additionally

(ii) $\lim_{t\to 0+} T(t)x = x$ for all $x \in X$,

then T is called a C_0-semigroup (on X) (also a strongly continuous semigroup). If T is defined on $\mathbb{R}$ instead of $[0, 1)$, and (i) holds for all $t, s \in \mathbb{R}$, then T is called a one-parameter group, and if additionally (ii) holds, then T is called a C_0-group.

Remark 3.3.1. *(a) Property (i) implies that for $t, s \geq 0$ the operators $T(t), T(s)$ commute also, if $t_1, t_2, \cdots, t_n \geq 0$, then $T\left(\sum_{j=1}^{n} t_j\right) = \prod_{j=1}^{n} T(t_j)$.*

(b) Preperty (i) implies that $T(0) = T(0)^2$ is a projection.

(c) If T is a C_0-semigroup, then $T(0)x = \lim_{t\to 0_+} T(t)T(0)x = \lim_{t\to 0_+} T(t)x = x$ for all $x \in X$, ie., $T(0) = I$

Lemma 3.3.2. *[54] Let $A \in \mathcal{L}(X)$, and define*

$$T(t) = e^{tA} = \sum_{j=0}^{\infty} \frac{(tA)^j}{j!} \text{ for } t \in \mathbb{R}. \tag{3.3.1}$$

Then T is C_0-semigroup.

Lemma 3.3.3 ([54]). *Let T be a one-parameter semigroup on X, and assume that there exists $\delta > 0$ such that $M := \sup_{0\leq t<\delta} \|T(t)\| < \infty$. Then there exists $\omega \in \mathbb{R}$ such that*

$$\|T(t)\| \leq Me^{\omega t} \text{ for all } t \geq 0$$

Proposition 3.3.4 ([54]). *Let T be a semi-group on X*

(a) Then there exist $M \geq 0$ and $\omega \in \mathbb{R}$ such that

$$\|T(t)\| \leq Me^{\omega t} \text{ for all } t \geq 0.$$

(b) For all $x \in X$ the function $t \in [0,\infty) \mapsto T(t)x$ is continuous. In other words, the function T is strongly continuous.

(c) If T is C_0−group on X, then there exist $M \geq 0$ and $\omega \in \mathbb{R}$ such that

$$\|T(t)\| \leq Me^{\omega|t|} \text{ for all } t \in \mathbb{R}.$$

For all $x \in X$ the function $\mathbb{R} \ni t \mapsto T(t)x \in X$ is continuous.

Lemma 3.3.5 ([54]). *Let T be a one-parameter semigroup on X. Assume that*

$$\sup_{0\leq t<1} \|T(t)\| < \infty$$

and that there exists a dense subset D of X such that $\lim_{t\to 0-+} T(t)x = x$. Then T is a C_0−semigroup.

This lemma is an immediate consequence of the following fundamental fact of functional analysis, which we insert here, also for further reference.

Theorem 3.3.6 ([54]). *Let X, Y be Banach spaces over the same field, and let $a, b \in \mathbb{R}, a < b$.*

(a) Let $f : [a,b] \to X$ be continuous, and let $A \in \mathcal{L}(X,Y)$. Then

$$A\int_a^b f(t)dt = \int_a^b Af(t)dt.$$

(b) (Hille's theorem) Let A be a closed operator from X to Y. Let $f : [a,b] \to X$ be continuous, $f(t) \in \mathrm{dom}(A)$ for all $t \in [a,b]$, and $t \mapsto Af(t) \in Y$ continuous. Then $\int_a^b f(t)dt \in \mathrm{dom}(A)$, and

$$A\int_a^b f(t)dt = \int_a^b Af(t)dt.$$

3.3.1. The Generator of a C_0−semigroup

Let X, Y be two vector spaces over the same field $K \in \mathbb{R}, \mathbb{C}$. For a linear relation in $X \times Y$, i.e., a subspace $A \subseteq X \times Y$, we define the domain of A,

$$\mathrm{dom}(A) := \{x \in X;\ \text{there exists } y \in Y \text{ such that } (x,y) \in A\}.$$

the range of A,

$$\mathrm{ran}(A) := \{y \in Y;\ \text{there exists } x \in X \text{ such that } (x,y) \in A\}.$$

and the kernel (or null space) of A,

$$\ker(A) := \{x \in X; (x,0) \in A\}.$$

Definition 3.3.7 ([54]). *Let X be a Banach space. For a C_{0-}semigroup T we define the generator (also called the infinitesimal generator) A, an operator in X, by*

$$A := \left\{(x, y) \in X \times X; y = \lim_{h\to 0+} h^{-1}(T(h)x - x) \text{ exists.}\right\}.$$

Theorem 3.3.8 ([54]). *Let T be a C_{0-}semigroup on X, with generator A. Then :*

(a) For $x \in \mathrm{dom}(A)$ one has $T(t)x \in \mathrm{dom}(A)$ for all $t > 0$, the function $t \to T(t)x$ is continuously differentiable on $[0, 1)$, *and*

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax(t \leq 0).$$

(b) For all $x \in X, t > 0$ one has $\int_0^t T(s)xds \in \mathrm{dom}(A)$,

$$A\int_0^t T(s)xds = T(t)x - x.$$

(c) $\mathrm{dom}(A)$ *is dense in X, and A is a closed operator.*

3.4. Hille-Yosida Theorem

Let E be a Banach space and let $A : \mathrm{dom}(A) \subseteq E \to E$ be an unbounded linear operator. One says that A is m-accretive if $\overline{\mathrm{dom}(A)} = E$ and for every $\lambda > 0$, $I + \lambda A$ is bijective from $D(A)$ into E with $\|(I + \lambda A)^{-1}\|_{\mathcal{L}(E)} \leq 1$.

Theorem 3.4.1. *Hille-yosida[23] Let A be m-accretive. Then given any $u_0 \in \mathrm{dom}(A)$ there exists a unique function*

$$u \in C^1([0, +\infty); E) \cap C([0, +\infty); \mathrm{dom}(A)).$$

Such that

$$\begin{cases} \frac{du}{dt}Au = 0 & \text{on } [0; +\infty) \\ u(0) = u_0 \end{cases} \tag{3.4.1}$$

Moreover

$$\|u(t)\| \leq \|u0\| \text{ and } \left\|\frac{du}{dt}\right\|(t) = \|Au(t)\| \leq \|Au_0\| \forall t \geq 0.$$

The map $u_0 \to u(t)$ extended by continuity to all of E is denoted by $S_A(t)$. It is a continuous semigroup of contractions on E. Conversely, given any continuous semigroup of contractions $S(t)$, there exists a unique m-accretive operator A such that $S(t) = S_A(t) \forall t \geq 0$.

3.4.1. The Exponential Formula

There are numerous iteration techniques for solving (3.4.1). Let us mention a basic method

Theorem 3.4.2. *[54] Assume that A is m-accretive. Then for every $u_0 \in D(A)$ the solution u of* (3.4.1) *is given by the "exponential formula"*

$$u(t) = \lim_{n\to 0} \left[\left(I + \frac{t}{n}A\right)^{-1}\right]^n u_0. \tag{3.4.2}$$

Consider, in a Banach space E, the problem

$$\begin{cases} \frac{du}{dt}t + Au(t) = f(t) & \text{on } [0, T] \\ u(0) = u_0 \end{cases} \tag{3.4.3}$$

The following holds.

Theorem 3.4.3 ([54]). *Assume that A is m-accretive. Then for every $u_0 \in \mathrm{dom}(A)$ and every $f \in C^1([0, T]; E)$ there exists a unique solution u of* (3.4.3) *with*

$$u \in C^1([0, T]; E) \cap C([0, T]; \mathrm{dom}(A)).$$

Moreover, u is given by the formula

$$u(t) = S_A(t)u_0 + \int_0^t S_A(t-s)f(s)ds, \tag{3.4.4}$$

where $S_A(t)$ is the semigroup.

Chapitre 4

Fractional Bihari Inequalities and Applications

The Gronwall-Bellman and the Bihari inequalities provide excellent tools in the qualitative theory of differential equations (see [14, 104, 114, 115]). There are many generalizations in the literature both linear and nonlinear cases [5, 6, 105, 106, 117]. Bihari's inequality [22] is perhaps the most important generalization of the Gronwall-Bellman inequality.

Fractional differential equations have now proved to be valuable tools modeling many real world phenomena (like physics and chemistry [45, 82, 84, 96]). Moreover, there has also been a major theoretical development in fractional differential equations ; see the monographs of Abbas *et al.* [1], Kilbas *et al.* [68], Podlubny [109] and Samko *et al.* [119].

One main advantage of the fractional Gronwall lemma is in the study of qualitative properties of solutions of fractional differential equations, integral equations with singular kernels and impulsive fractional differential equations.

In 1981, Henry [53] established the following *Gronwall-like* nonlinear integral inequality :

If

$$v(t) \leq w(t) + a\int_0^t \frac{v(s)}{(t-s)^\gamma}ds, \quad \text{for every} \quad t \in [0,b],$$

for some function w and constants $a > 0$ and $0 < \gamma < 1$, then there exists a constant $K = K(\gamma)$ such that

$$v(t) \leq w(t) + Ka\int_0^t \frac{w(s)}{(t-s)^\gamma}ds, \quad \text{for every} \quad t \in [0,b].$$

Ye *et al.* [136] studied the following fractional integral inequality :

Let $v : [0,b) \to [0,\infty)$ be a real function,supposed nonnegative and locally integrable on $0 \leq t < b$. and $w(\cdot)$ be a nonnegative, locally integrable function on $[0,b)$ (some $b \leq +\infty$), and let $a(t)$ be a nonnegative, nondecreasing continuous function defined on $0 \leq t < b$, $a(t) \leq M$ (constant).

Assume $\gamma > 0$ such that

$$v(t) \leq w(t) + a(t) \int_0^t \frac{v(s)}{(t-s)^{1-\gamma}} ds.$$

Then

$$v(t) \leq w(t) + \int_0^t \phi(s) w(s) ds, \quad \text{for every } t \in [0, b),$$

where

$$\phi(s) = \sum_{n=1}^{\infty} \frac{(a(t)\Gamma(\gamma))^n}{\Gamma(n\gamma)} (t-s)^{n\gamma-1}.$$

Researchers have developed many useful and recent integral inequalities based on the above inequalities, primarily motivated by their uses in various branches of fractional differential equations (see [2, 3, 36, 43, 53, 83, 126, 136] and the references therein).

However, in certain situations, such as some classes of fractional differential equations or fractional integral equations, where the right sides have nonlinear growth, it will be desirable to explore some new Bihari inequalities in order to obtain some estimates. In this paper, we discuss a class of integral inequalities with singular kernels. Mathematical analysis techniques, combined with Young's and Hölder's inequalities, are used to obtain explicit estimates. Finally, to illustrate the applications of our results some examples are given.

4.1. The Bihari Fractional Inequality

First, we present a nonlinear version of Gronwall's lemma for singular kernels.

Theorem 4.1.1. *Let $0 < \alpha < 1$ and $k, \bar{k} > 0$. Suppose f is a nonnegative function, which is integrable (or locally integrable) on $I = [0, b]$, and $\psi : [0, \infty) \to [0, \infty)$ is a nondecreasing continuous function. If $u(t)$ is a continuous function on I satisfying*

$$u(t) \leq k + \int_0^t f(s)\psi(u(s)) ds + \bar{k} \int_0^t (t-s)^{\alpha-1} u(s) ds,$$

then

$$u(t) \leq \Psi^{-1}\left(\Psi(k_*) + \left[1 + \sum_{n=1}^{\infty} \frac{(\bar{k}\Gamma(\alpha) b^{\alpha})^n}{\Gamma(n\alpha+1)}\right] \int_0^t f(s) ds\right),$$

where

$$\Psi(z) = \int_1^z \frac{dy}{\psi(y)}, \; k_* = 1 + \sum_{n=1}^{\infty} \frac{(\bar{k}\Gamma(\alpha) b^{\alpha})^n}{\Gamma(n\alpha+1)},$$

and Ψ^{-1} is the inverse function of Ψ, and for every $t \in [0, b]$,

$$\Psi(k_*) + \left[1 + \sum_{n=1}^{\infty} \frac{(\bar{k}\Gamma(\alpha) b^{\alpha})^n}{\Gamma(n\alpha+1)}\right] \int_0^t f(s) ds \in Dom(\Psi^{-1}), \quad t \in [0, b],$$

with $Dom(\Psi^{-1})$ denoting the domain of Ψ^{-1}.

Proof. *Let us consider*

$$v(t) = k + \int_0^t f(s)\psi(u(s))ds, \quad t \in [0, b].$$

Then

$$v'(t) = f(t)\psi(u(t)), \quad v(0) = k.$$

and

$$u(t) \leq v(t) + \int_0^t (t-s)^{\alpha-1} u(s) ds.$$

Hence

$$\begin{aligned} u(t) &\leq v(t) + \int_0^t \sum_{n=1}^{\infty} \frac{(\bar{k}\Gamma(\alpha))^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} v(s) ds \\ &\leq v(t) + \int_0^t \sum_{n=1}^{\infty} \frac{(\bar{k}\Gamma(\alpha))^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} v(t) ds. \end{aligned}$$

Therefore

$$u(t) \leq \left[1 + \sum_{n=1}^{\infty} \frac{(\bar{k}\Gamma(\alpha) b^{\alpha})^n}{\Gamma(n\alpha+1)} \right] v(t).$$

We set

$$w(t) = \left[1 + \sum_{n=1}^{\infty} \frac{(\bar{k}\Gamma(\alpha) b^{\alpha})^n}{\Gamma(n\alpha+1)} \right] v(t).$$

Since ψ is a nondecreasing function, we obtain

$$\frac{v'(t)}{\psi(w(t))} \leq f(t), \quad t \in [0, b].$$

By integrating both sides of the above inequality from 0 *to* t*, we get*

$$\int_0^t \frac{v'(s)}{\psi\left(\left(1 + \sum_{n=1}^{\infty} \frac{(\bar{k}\Gamma(\alpha) b^{\alpha})^n}{\Gamma(n\alpha+1)}\right) v(s)\right)} ds \leq \int_0^t f(s) ds.$$

Then, we obtain

$$\int_{w(0)}^{w(t)} \frac{dz}{\psi(z)} \leq \left[1 + \sum_{n=1}^{\infty} \frac{(\bar{k}\Gamma(\alpha) b^{\alpha})^n}{\Gamma(n\alpha+1)} \right] \int_0^t f(s) ds.$$

Thus, it follows that

$$u(t) \leq \Psi^{-1}\left(\Psi(k_*) + \left[1 + \sum_{n=1}^{\infty} \frac{(\bar{k}\Gamma(\alpha) b^{\alpha})^n}{\Gamma(n\alpha+1)} \right] \int_0^t f(s) ds \right).$$

Remark 4.1.1.

- *For $\alpha \to 1$ and $f = 0$, we obtain the classical result that*

$$u(t) \leq k + \bar{k} \int_0^t u(s)ds$$

implies that

$$u(t) \leq ke^{\bar{k}t}.$$

- *If $f = 0$ and $\alpha \in (0, 1)$ we obtain the Henry [53] inequality.*
- *If $\bar{k} = 0$, we obtain the classical Bihari [22] inequality.*

Next, we present a version of the fractional Bihari inequality on a bounded interval.

Theorem 4.1.2. *Let $u : [0, b] \to [0, \infty)$ be a continuous function and $\psi : [0, \infty) \to [0, \infty)$ be a non-negative non-decreasing continuous function such that $\psi(0) = 0$. Assume that there are constants $k > 0$, $\widetilde{k} \geq 0$, $0 < \alpha < 1$ and $p\left(1 - \frac{1}{q}\right) = 1$, $q > \frac{1}{\alpha}$ such that*

$$u(t) \leq k + \widetilde{k} \int_0^t (t - s)^{\alpha - 1} \psi(u(s))ds.$$

Then, for every $t \in [0, b]$ we have

$$u(t) \leq \Phi^{-1}\left(\Phi(\bar{k}) + \frac{t}{q}\right), \tag{4.1.1}$$

where

$$\Phi(z) = \int_k^z \frac{du}{(\psi(u))^q} du, \quad \bar{k} = k + \frac{\widetilde{k} b^{p(\alpha-1)+1}}{p(p(\alpha - 1) + 1)}$$

and

$$\Phi(\bar{k}) + \frac{t}{q} \in Dom(\Phi^{-1}).$$

Proof. *Using Young's inequality, we have*

$$(t - s)^{\alpha - 1} \psi(u(s)) \leq \frac{1}{p}(t - s)^{p(\alpha - 1)} + \frac{1}{q}(\psi(u(s)))^q, \quad s \in [0, t).$$

This implies that

$$u(t) \quad \leq \quad k + \frac{1}{p} \int_0^t (t - s)^{p(\alpha - 1)} ds + \frac{1}{q} \int_0^t (\psi(u(s)))^q ds.$$

Since $q > \frac{1}{\alpha}$ and $p\left(1 - \frac{1}{q}\right) = 1$, hence $p < \frac{1}{1 - \alpha}$, so $p(\alpha - 1) + 1 > 0$. We obtain immediately

$$u(t) \quad \leq \quad k + \frac{\widetilde{k} t^{p(\alpha-1)+1}}{p(p(\alpha - 1) + 1)} + \frac{1}{q} \int_0^t (\psi(u(s)))^q ds$$

and

$$u(t) \quad \leq \quad k + \frac{\widetilde{k} b^{p(\alpha-1)+1}}{p(p(\alpha - 1) + 1)} + \frac{1}{q} \int_0^t (\psi(u(s)))^q ds.$$

Putting

$$v(t) = k + \frac{\widetilde{k}b^{p(\alpha-1)+1}}{p(p(\alpha-1)+1)} + \frac{1}{q}\int_0^t (\psi(u(s)))^q ds,$$

it is obvious that

$$v'(t) = \frac{1}{q}(\psi(u(t)))^q, \quad v(0) = \bar{k}.$$

Since $u(t) \leq v(t)$ and ψ is a nondecreasing function, we get that

$$v'(t) \leq \frac{1}{q}\left(\psi(v(t))\right)^q, \quad t \in [0, b].$$

By integrating the above inequality over 0 to t, we write

$$\int_0^t \frac{v'(s)}{(\psi(v(s)))^q} ds \leq \frac{t}{q}.$$

We then have

$$\int_{\bar{k}}^{v(t)} \frac{dz}{(\psi(z))^q} \leq \frac{t}{q}.$$

This means that

$$\Phi(v(t)) = \int_k^{v(t)} \frac{dz}{(\psi(z))^q} \leq \Phi(k) + \frac{t}{q}.$$

Therefore

$$v(t) \leq \Phi^{-1}\left(\Phi(\bar{k}) + \frac{t}{q}\right), \quad t \in [0, b],$$

which gives us the desired estimate (4.1.2).

By the same argument of the above theorem, we can easily prove the following corollary.

Corollary 4.1.3. *Let $u : \mathbb{R}_+ \to [0, \infty)$ be a continuous function and $\psi : [0, \infty) \to [0, \infty)$ be a non-negative non-decreasing continuous function such that $\psi(0) = 0$. Assume there are constants $k > 0$, $\widetilde{k} \geq 0$, $0 < \alpha < 1$ and $p\left(1 - \frac{1}{q}\right) = 1$, $q > \frac{1}{\alpha}$ such that*

$$u(t) \leq k + \widetilde{k}\int_0^t (t-s)^{\alpha-1} \sqrt[q]{\psi(u(s))} ds.$$

Then, for every $t \in [0, b]$ we have

$$u(t) \leq \Phi^{-1}\left(\Phi(\bar{k}) + \frac{t}{q}\right), \tag{4.1.2}$$

where

$$\Phi(z) = \int_k^z \frac{du}{\psi(u)} du, \quad \bar{k} = k + \frac{\widetilde{k}b^{p(\alpha-1)+1}}{p(p(\alpha-1)+1)}$$

and

$$\Phi(\bar{k}) + \frac{t}{q} \in Dom(\Phi^{-1}).$$

By using Hölder's inequality, we give a natural generalization of the above inequality.

Theorem 4.1.4. *Let $k > 0$ and $u, f : [0,\infty) \to (0,\infty)$ be continuous functions and $\psi : [0,\infty) \to [0,\infty)$, be continuous, nondecreasing and $\psi(0) = 0$. If*

$$u(t) \leq k + \int_0^t (t-s)^{\alpha-1} f(s)\psi(u(s))ds$$

then

$$u(t) \leq \left[\Psi^{-1}\left(\frac{2^q b^{q(p(\alpha-1)+1)}}{(p(\alpha-1)+1)^q}\int_0^t f^q(s)ds\right)\right]^{\frac{1}{q}} \quad t \in [0,b], \tag{4.1.3}$$

where

$$\Psi(z) = \int_{2^q k^q}^{z} \frac{dx}{(\psi(\sqrt[q]{x}))^q}, \quad z \geq 2^q k^q,\ p\left(1-\frac{1}{q}\right) = 1, \quad q > \frac{1}{\alpha}.$$

For $c > 2^q k^q$,

$$u(t) \leq \left[\Psi^{-1}\left(\Psi(c) + \frac{2^q b^{q(p(\alpha-1)+1)}}{(p(\alpha-1)+1)^q}\int_0^t f^q(s)ds\right)\right]^{\frac{1}{q}} \quad t \in [0,b]. \tag{4.1.4}$$

Proof. *Applying Hölder's inequality, we obtain*

$$u(t) \leq k + \left(\int_0^t (t-s)^{p(\alpha-1)}ds\right)^{\frac{1}{p}}\left(\int_0^t f^q(s)\psi^q(u(s))ds\right)^{\frac{1}{q}}.$$

For every $t \in [0,b]$, we obtain

$$u(t) \leq k + \frac{b^{p(\alpha-1)+1}}{p(\alpha-1)+1}\left(\int_0^t f^q(s)\psi^q(u(s))ds\right)^{\frac{1}{q}}.$$

Then

$$u^q(t) \leq 2^q k^q + \frac{b^{qp(\alpha-1)+q}}{(p(\alpha-1)+1)^q}\int_0^t f^q(s)\psi^q(u(s))ds.$$

Hence

$$u(t) \leq 2\left(k^q + 2^q \frac{b^{qp(\alpha-1)+q}}{(p(\alpha-1)+1)^q}\int_0^t f^q(s)\psi^q(u(s))ds\right)^{\frac{1}{q}}.$$

Define the function

$$v(t) = 2^q k^q + \frac{2^q b^{qp(\alpha-1)+q}}{(p(\alpha-1)+1)^q}\int_0^t f^q(s)\psi^q(u(s))ds, \quad t \in [0,b].$$

Then

$$v'(t) = \frac{2^q b^{qp(\alpha-1)+q}}{(p(\alpha-1)+1)^q} f^q(t)\psi^q(u(t)).$$

From the definitions of Ψ and v, it follows that

$$\begin{aligned}\frac{d\Psi(v(t))}{dt} &= \Psi'(v(t))v'(t) = \frac{v'(t)}{\psi^q\left(\sqrt[q]{v(t)}\right)} \\ &= \frac{b^{qp(\alpha-1)+q} f^q(t)\psi^q(u(t))}{(p(\alpha-1)+1)^q\psi^q\left(\sqrt[q]{v(t)}\right)}.\end{aligned}$$

Because ψ is a nondecreasing function, we can get

$$\frac{d\Psi(v(t))}{dt} \le \frac{2^q b^{qp(\alpha-1)+q} f^q(t)}{(p(\alpha-1)+1)^q}.$$

Integrating this from 0 *to* t *and using* $\Psi(v(0)) = \Psi(2^q k^q) = 0$,

$$\Psi(v(t)) \le \frac{2^q b^{qp(\alpha-1)+q}}{(p(\alpha-1)+1)^q} \int_0^t f^q(s)ds.$$

As a result, since Ψ *is strictly decreasing,we have*

$$v(t) \le \Psi^{-1}\left(\frac{2^q b^{qp(\alpha-1)+q}}{(p(\alpha-1)+1)^q} \int_0^t f^q(s)ds\right), \quad t \in [0, b],$$

which concludes the proof of theorem.

We now consider more classes of functions in developing a new Gronwall-Bellman-Bihari type fractional integral inequality on bounded or unbounded intervals.

Definition 4.1.5. *A function* $\psi : [0, \infty) \to [0, \infty)$ *is said to belong to the class* H *if*
H_1) $\psi(z)$ *is continuous, nondecreasing for* $z \ge 0$ *and positive for every* $z > 0$.
H_2) *There exists a continuous function* $\phi : [0, \infty) \to [0, \infty)$ *(called a "multiplier function") such that*

$$\psi(\gamma z) \le \phi(\gamma)\psi(z), \quad \text{for every } z \ge 0,\ \gamma > 0.$$

For examples about this class of functions we suggest [34].

Now we give the first of our main results of this part.

Theorem 4.1.6. *Let* $u, f : [0, \infty) \to [0, \infty)$ *be two continuous functions and* $\psi \in H$, *with corresponding multiplier function* ψ *on* $[0, \infty)$, *and* $h(t) > 0$ *be a monotonic, nondecreasing and continuous function on* $[0, \infty)$. *If*

$$u(t) \le h(t) + \int_0^t (t-s)^{\alpha-1} f(s)\psi(u(s))ds,\ t \in [0, b],$$

then

$$u(t) \le h(t)\Psi^{-1}\left(\Psi(k') + \frac{1}{q}\int_0^t \left(\frac{\phi(h(s))f(s)}{h(s)}\right)^q ds\right), \quad t \in [0, b],$$

where

$$\Psi(z) = \int_1^z \frac{du}{(\psi(u))^q}, \quad p\left(1 - \frac{1}{q}\right) = 1, q \ge \frac{1}{\alpha},\ k' = 1 + \frac{b^{p(\alpha-1)+1}}{p(p(\alpha-1)+1)}$$

and

$$\Psi(k') + \frac{1}{q}\int_0^t \left(\frac{\phi(h(s))f(s)}{h(s)}\right)^q ds \in Dom(\Psi^{-1}), \quad t \in [0, b].$$

Proof. *According to the hypothesis, we have*

$$u(t) \le h(t) + \int_0^t (t-s)^{\alpha-1} f(s)\psi(u(s))ds,\ t \in [0, b].$$

Then

$$\begin{aligned}\frac{u(t)}{h(t)} &\leq 1+\int_0^t (t-s)^{\alpha-1}\frac{f(s)}{h(s)}\psi(u(s))ds\\ &\leq 1+\int_0^t (t-s)^{\alpha-1}\frac{\phi(h(s))f(s)}{h(s)}\psi\left(\frac{u(s)}{h(s)}\right)ds.\end{aligned}$$

From Young's inequality, we get

$$\begin{aligned}\frac{u(t)}{h(t)} &\leq 1+\int_0^t (t-s)^{\alpha-1}\frac{f(s)}{h(s)}\psi(u(s))ds\\ &\leq 1+\frac{1}{p}\int_0^t (t-s)^{p(\alpha-1)+1}ds+\frac{1}{q}\int_0^t\left(\frac{\phi(h(s))f(s)}{h(s)}\right)^q\left(\psi\left(\frac{u(s)}{h(s)}\right)\right)^q ds.\end{aligned}$$

This implies that

$$\frac{u(t)}{h(t)} \leq 1+\frac{b^{p(\alpha-1)+1}}{p(p(\alpha-1)+1)}+\frac{1}{q}\int_0^t\left(\frac{\phi(h(s))f(s)}{h(s)}\right)^q\left(\psi\left(\frac{u(s)}{h(s)}\right)\right)^q ds.$$

Define

$$v(t)=1+\frac{b^{p(\alpha-1)+1}}{p(p(\alpha-1)+1)}+\frac{1}{q}\int_0^t\left(\frac{\phi(h(s))f(s)}{h(s)}\right)^q\left(\psi\left(\frac{u(s)}{h(s)}\right)\right)^q ds.$$

It follows that

$$v'(t)=\left(\frac{\phi(h(t))f(t)}{h(t)}\right)^q\left(\psi\left(\frac{u(t)}{h(t)}\right)\right)^q,\quad v(0)=k'.$$

Since ψ is a nondecreasing function, then

$$v'(t)\leq\left(\frac{\phi(h(t))f(t)}{h(t)}\right)^q(\psi(v(t)))^q.$$

By integration from 0 *to* t, *we obtain*

$$\int_{v(0)}^{v(t)}\frac{dz}{\psi(z)}\leq\frac{1}{q}\int_0^t\left(\frac{\phi(h(s))f(s)}{h(s)}\right)^q ds.$$

Consequently, it follows that

$$v(t)\leq\Psi^{-1}\left(\Psi(k')+\frac{1}{q}\int_0^t\left(\frac{\phi(h(s))f(s)}{h(s)}\right)^q ds\right),\quad t\in[0,b].$$

Hence

$$u(t)\leq h(t)\Psi^{-1}\left(\Psi(k')+\frac{1}{q}\int_0^t\left(\frac{\phi(h(s))f(s)}{h(s)}\right)^q ds\right),\quad t\in[0,b].$$

Corollary 4.1.7. *Let $k>0$ and $f,g:[0,\infty)\to(0,\infty)$ be continuous functions and $\psi\in H$, with corresponding multiplier function ψ on $[0,\infty)$. If*

$$u(t)\leq k+\int_0^t f(s)\psi(u(s))ds+\int_0^t (t-s)^{\alpha-1}g(s)\psi(u(s))ds,$$

then

$$u(t) \le \left[\Psi^{-1}\left(\frac{2^q b^{q(p(\alpha-1)+1)}}{(p(\alpha-1)+1)^q}\int_0^t f^q(s)ds\right)\right]^{\frac{1}{q}} \quad t \in [0,b], \tag{4.1.5}$$

where

$$\Psi(z) = \int_{2^q k^q}^{z} \frac{dx}{(\psi(\sqrt[q]{x}))^q}, \quad z \ge 2^q k^q, \ q\left(1-\frac{1}{p}\right) = 1, \quad q > \frac{1}{\alpha}.$$

For $c > 2^q k^q$,

$$u(t) \le \left[\Psi^{-1}\left(\Psi(c) + \frac{2^q b^{q(p(\alpha-1)+1)}}{(p(\alpha-1)+1)^q}\int_0^t f^q(s)ds\right)\right]^{\frac{1}{q}} \quad t \in [0,b]. \tag{4.1.6}$$

Proof. *For every* $t \in [0,b]$ *we have*

$$\begin{aligned} u(t) &\le k + \int_0^t f(s)\psi(u(s))ds + \int_0^t (t-s)^{\alpha-1} g(s)\psi(u(s))ds \\ &= k + \int_0^t (t-s)^{1-\alpha}(t-s)^{\alpha-1} f(s)\psi(u(s))ds + \int_0^t (t-s)^{\alpha-1} g(s)\psi(u(s))ds \\ &\le k + b^{1-\alpha}\int_0^t (t-s)^{\alpha-1} f(s)\psi(u(s))ds + \int_0^t (t-s)^{\alpha-1} g(s)\psi(u(s))ds. \end{aligned}$$

Then

$$u(t) \le k + \int_0^t (t-s)^{\alpha-1} L(s)\psi(u(s))ds, \quad t \in [0,b],$$

where

$$L(t) = b^{1-\alpha} f(t) + g(t), \quad t \in [0,b].$$

From Theorem 4.1.6, we obtain

$$u(t) \le k\Psi^{-1}\left(\Psi(k') + \frac{\phi^q(k)}{k^q q}\int_0^t L^q(s)ds\right), \quad t \in [0,b],$$

and this concludes the proof of corollary.

Using Hölder's inequality, we establish a simple proof of the fractional Gronwall inequality.

Theorem 4.1.8. *Let* $u, g : [0,\infty) \to [0,\infty)$ *be two continuous functions and* $h(t) > 0$ *be a monotonic, nondecreasing and continuous function on* $[0,\infty)$. *If*

$$u(t) \le h(t) + g(t)\int_0^t (t-s)^{\alpha-1} u(s)ds, \quad t \in [0,b],$$

then

$$u(t) \le h(t)\left(1 + g_*(t)\frac{\left(\int_0^t m(s)ds\right)^{\frac{1}{q}}}{1-(1-m(s))^{\frac{1}{q}}}\right), \quad t \in [0,\infty), \tag{4.1.7}$$

where

$$m(t) = \exp\left(-\int_0^t g_*^q(s)ds\right), \quad g_*(t) = g(t)t^{\frac{p(\alpha-1)+1}{p}}, \ p\left(1-\frac{1}{q}\right) = 1, \quad q > \frac{1}{\alpha}.$$

Proof. *By Hölder's inequality, we obtain*

$$u(t) \leq h(t) + g(t)\left(\int_0^t (t-s)^{p(\alpha-1)}ds\right)^{\frac{1}{p}}\left(\int_0^t u^q(s)ds\right)^{\frac{1}{q}}.$$

Since h is a monotonic, nondecreasing and positive function, we have

$$\frac{u(t)}{h(t)} \leq 1 + g(t)\left(\int_0^t (t-s)^{p(\alpha-1)}ds\right)^{\frac{1}{p}}\left(\int_0^t \left(\frac{u(s)}{h(s)}\right)^q ds\right)^{\frac{1}{q}}.$$

Using a Pachpatte inequality [104], the result follows from (4.1.7).

Now, we give a nonlinear version of the above inequality on an unbounded interval.

Theorem 4.1.9. *Let $u, f_1, f_2, g : [0,\infty) \to [0,\infty)$ be continuous functions and $\psi \in H$, with corresponding multiplier function ϕ on $[0,\infty)$, and $h(t) > 0$ be a monotonic, nondecreasing and continuous function on $J = [0,\infty)$ and $\frac{f_1}{h} \in L^p(J, \mathbb{R}_+)$. If*

$$u(t) \leq h(t) + g(t)\int_0^t (t-s)^{\alpha-1}u(s)ds + \int_0^t f_1(s)f_2(s)\psi(u(s))ds,$$

then, for $t \in J$,

$$u(t) \leq h_*(t)\Psi^{-1}\left[\Psi(1+k) + \frac{1}{q}\int_0^t \left(f_2(s)\phi(h(s))\right)^q\left(\phi\left(1 + g_*(s)\frac{\left(\int_0^s m(r)dr\right)^{\frac{1}{q}}}{1-(1-m(s))^{\frac{1}{q}}}\right)\right)^{q'} ds\right],$$

where

$$h_*(t) = h(t)\left(1 + g_*(t)\frac{\left(\int_0^t m(s)ds\right)^{\frac{1}{q}}}{1-(1-m(t))^{\frac{1}{q}}}\right),\ t \in J,$$

$$\Psi(z) = \int_1^z \frac{dx}{(\psi(x))^q}, m(t) = \exp\left(-\int_0^t g_*^q(s)ds\right), \quad g_*(t) = g(t)t^{\frac{p(\alpha-1)+1}{p}},\ t \in J,$$

and

$$k = \frac{1}{p}\left\|\frac{f_1}{h}\right\|_{L^p}^p,\ p\left(1-\frac{1}{q}\right) = 1, \quad q \geq \frac{1}{\alpha}.$$

Proof. *From Young's inequality, we obtain that,*

$$\begin{aligned}\frac{u(t)}{h(t)} \quad &\leq \quad 1 + g(t)\int_0^t (t-s)^{\alpha-1}\frac{u(s)}{h(s)}ds + \frac{1}{p}\int_0^t \left(\frac{f_1(s)}{h(s)}\right)^p ds \\ &\quad + \frac{1}{q}\int_0^t f_2^q(s)\left(\psi(u(s))\right)^q ds.\end{aligned}$$

This implies

$$\begin{aligned}\frac{u(t)}{h(t)} \quad &\leq \quad 1 + g(t)\int_0^t (t-s)^{\alpha-1}\frac{u(s)}{h(s)}ds + \frac{1}{p}\left\|\frac{f_1}{h}\right\|_{L^p}^p \\ &\quad + \frac{1}{q'}\int_0^t \left(f_2(s)\phi(h(s))\right)^q\left(\psi\left(\frac{u(s)}{h(s)}\right)\right)^q ds.\end{aligned}$$

Let

$$v(t) = 1 + \frac{1}{p}\left\|\frac{f_1}{h}\right\|_{L^p}^p + \frac{1}{q}\int_0^t \left(f_2(s)\phi(h(s))\right)^q \left(\psi\left(\frac{u(s)}{h(s)}\right)\right)^q ds.$$

Hence

$$v'(t) = \frac{1}{q}\left(f_2(t)\phi(h(t))\right)^q \left(\psi\left(\frac{u(t)}{h(t)}\right)\right)^q,\ v(0) = 1 + k,\ k := \frac{1}{p}\left\|\frac{f_1}{h}\right\|_{L^p}^p,$$

and

$$\frac{u(t)}{h(t)} \leq v(t) + g(t)\int_0^t (t-s)^{\alpha-1}\frac{u(s)}{h(s)}ds.$$

By Theorem 4.1.8,

$$\frac{u(t)}{h(t)} \leq v(t)\left(1 + g_*(t)\frac{\left(\int_0^t m(s)ds\right)^{\frac{1}{q}}}{1-(1-m(t))^{\frac{1}{q}}}\right).$$

Since ψ is nondecreasing and has corresponding multiplier function ϕ,

$$\left(\psi\left(\frac{u(t)}{h(t)}\right)\right)^q \leq \left(\psi\left(v(t)\left(1 + g_*(t)\frac{\left(\int_0^t m(s)ds\right)^{\frac{1}{q}}}{1-(1-m(t))^{\frac{1}{q}}}\right)\right)\right)^q.$$

Then

$$\frac{v'(t)}{(\psi(v(t)))^q} \leq \frac{1}{q}\left(f_2(t)\phi(h(t))\right)^q \left(\phi\left(1 + g_*(t)\frac{\left(\int_0^t m(s)ds\right)^{\frac{1}{q}}}{1-(1-m(t))^{\frac{1}{q}}}\right)\right)^q.$$

Integration from 0 *to* t *yields*

$$\int_{1+k}^{v(t)} \frac{dz}{(\psi(z))^q} \leq \frac{1}{q}\int_0^t \left(f_2(s)\phi(h(s))\right)^q \left(\phi\left(1 + g_*(s)\frac{\left(\int_0^s m(r)dr\right)^{\frac{1}{q}}}{1-(1-m(s))^{\frac{1}{q}}}\right)\right)^q ds.$$

Finally, we have

$$v(t) \leq \Psi^{-1}\left[\Psi(1+k) + \frac{1}{q}\int_0^t \left(f_2(s)\phi(h(s))\right)^q \left(\phi\left(1 + g_*(s)\frac{\left(\int_0^s m(r)dr\right)^{\frac{1}{q}}}{1-(1-m(s))^{\frac{1}{q}}}\right)\right)^q ds\right].$$

4.1.1. Fractional Cauchy Problems

In this section, we assume the usual definitions of $I^\alpha h(t)$, $D^\alpha h(t)$ and ${}^cD^\alpha h(t)$ for, respectively, the Riemann-Liouville fractional integral, Riemann-Liouville fractional derivative and Caputo fractional derivative.

For $b > 0$ and $\alpha \in (0,1]$, in this part, we consider the following Caputo fractional differential equation :

$$\begin{cases} {}^cD^\alpha y(t) &= f(t, y(t)), \quad t \in [0,b], \\ y(0) &= a. \end{cases} \tag{4.1.8}$$

where $f\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function and $a \in \mathbb{R}^n$. We will now establish the first result of this section.

Theorem 4.1.10. *Let $q > \frac{1}{\alpha}$ and $p(1-\frac{1}{q}) = 1$.Assume there exists $\psi : [0,\infty) \to [0,\infty)$ a continuous, nondecreasing function and $\psi(0) = 0$ such that*

$$\|f(t,x) - f(t,y)\| \le \psi(\|x - y\|), \quad \text{for all } x, y \in \mathbb{R}^n. \tag{4.1.9}$$

If

$$\int_0^\infty \frac{dr}{(\psi(\sqrt[q]{r}))^q} = \infty, \quad \text{for every } x > 0, \tag{4.1.10}$$

then the problem (4.1.8) *has unique solution on* $[0,\infty)$.
In addition, if

$$b_\infty = \int_0^\infty \frac{dr}{(\psi(\sqrt[q]{r}))^q} < \infty,$$

then for each $b < b_\infty$, the problem (4.1.8) *has unique solution on* $[0,b]$.

Proof. *For every $b > 0$, we consider the following Cauchy problem*

$$\begin{cases} {}^cD^\alpha y(t) &=& f(t,y(t)), \quad t \in [0,b], \\ y(0) &=& a. \end{cases} \tag{4.1.11}$$

Step 1 : *Uniqueness of a solution. For this, suppose there exist two solutions x and y of* (4.1.11). *Then*

$$x(t) = a + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x(s))ds, \quad t \in [0,b]$$

and

$$y(t) = a + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,y(s))ds, \quad t \in [0,b].$$

It follows from (4.1.9) *that*

$$\begin{aligned} \|x(t) - y(t)\| &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s,y(s)) - f(s,x(s))\| ds \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \psi(\|x(s) - y(s)\|) ds \end{aligned}$$

Thus, for each $\varepsilon > 0$, we obtain

$$\|x(t) - y(t)\| \le \varepsilon + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \psi(\|x(s) - y(s)\|) ds.$$

Theorem 4.1.4, implies that

$$\|x(t) - y(t)\| \le \Psi^{-1}\left(\frac{b^{p(\alpha-1)+1}}{p(\alpha-1)+1} \int_0^t f^q(s) ds \right),$$

and from the condition (4.1.10) *we have that*

$$x(t) = y(t), \quad \text{for all } t \in [0,b].$$

Step 2 : *Existence of the solution. Indeed, since f is a continuous function, then it is easy to prove that the operator $N : C([0,b],\mathbb{R}^n) \to C([0,b],\mathbb{R}^n)$ defined by*

$$N(y)(t) = a + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,y(s))ds, \quad t \in [0,b]$$

is completely continuous.

- A priori bounds on solutions.

Let $y = \gamma N(y)$ for some $0 < \gamma < 1$. This implies by (4.1.9),

$$\begin{aligned} \|y(t)\| &\leq \|a\| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,y(s))\|ds \\ &\leq \|a\| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\psi(\|y(s)\|)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,0)\|ds. \end{aligned}$$

Hence

$$\|y(t)\| \quad \leq \quad \|a\| + 1 + \frac{b^{\alpha}\|f(\cdot,0)\|_\infty}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\psi(\|y(s)\|)ds.$$

From Theorem 4.1.4, we have

$$\|y(t)\| \leq \left[\Psi^{-1}\left(\Psi(\bar{k}) + \frac{2^q t^{qp(\alpha-1)+2q}}{(\Gamma(\alpha)(p(\alpha-1)+1))^q}\right)\right]^{\frac{1}{q}},$$

where

$$\Psi(z) = \int_1^z \frac{dx}{\psi^q(\sqrt[q]{x})}.$$

Then

$$\|y\|_\infty \leq \left[\Psi^{-1}\left(\Psi(\bar{k}) + \frac{2^q b^{qp(\alpha-1)+2q}}{(\Gamma(\alpha)(p(\alpha-1)+1))^q}\right)\right]^{\frac{1}{q}} =: M.$$

Set

$$U := \{y \in C([0,b],\mathbb{R}^n) : \|y\|_\infty < M+1\},$$

and consider the operator $N : \overline{U} \to C([0,b],\mathbb{R}^n)$. From the choice of U, there is no $y \in \partial U$ such that $y = \gamma N(y)$ for some $\gamma \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [50, 37], we deduce that N has a fixed point y in U which is a solution of the problem (4.1.11). We can conclude that for every $b > 0$ the Problem (4.1.8) has unique solution on $[0,b)$.

Now, we show that the problem has unique solution defined on $[0,\infty)$.

Let

$$b_\infty = \sup\{b \in \mathbb{R}_+ : \text{the problem (4.1.8) has unique solution on } [0,b)\}.$$

If $b_\infty < \infty$, then for $b_* = b_\infty + 1$, we introduce similarly that the fractional Cauchy problem

$$\left\{ \begin{array}{rcl} {}^cD^\alpha y(t) &=& f(t,y(t)), \quad t \in t \in [0,b_*], \\ y(0) &=& a. \end{array} \right. \tag{4.1.12}$$

has unique solution defined on $[0,b_*]$, therefore $b_* = b_\infty + 1 \leq b_\infty$ which is contradiction. This concludes the proof of the existence of a global solution of the problem (4.1.8) on $\mathbb{R}_+$.

As a consequence of above theorem we have :

Corollary 4.1.11. *Let* $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ *be a continuous function and* $f(\cdot, 0) = 0$. *Assume that there exists* $K > 0$ *such that*

$$\|f(t,x) - f(t,y)\| \leq K\|x - y\|, \quad \text{for all } x, y \in \mathbb{R}^n, \quad t \in \mathbb{R}_+. \tag{4.1.13}$$

Then the problem (4.1.8) *has a unique solution on* $\mathbb{R}_+$.

Proof. *From the condition* (4.1.13), *we have*

$$\|f(t,x)\| \leq K\|x\|, \quad \text{for all } x \in \mathbb{R}^n,\ t \in \mathbb{R}_+.$$

Let $\psi : \mathbb{R}_+ \to [0, \infty)$ *be defined by*

$$\psi(x) = Kx, \quad x \in \mathbb{R}_+.$$

It is clear that $\psi(x)$ *is a continuous, nondecreasing function,* $\psi(0) = 0$, *and*

$$\int_0^\infty \frac{dxr}{\psi(\sqrt[q]{r})} = \int_0^x \frac{dr}{\sqrt[q]{Kr}} = \infty, \quad q > \frac{1}{\alpha}.$$

From Theorem 4.1.10 the problem (4.1.8) *has unique solution defined on* $\mathbb{R}_+$.

4.1.2. Stochastic Fractional Differential Equations

Stochastic differential equations play a retrograde role in various applied fields, including physics, biology and engineering problems ; see for instance the monographs of Arnold [10], Han and Kloeden [52], Øksendal, [103], Pardoux and Rascanu [107], Tsokos and Padgett [125] and Sobczyk [123]. However few publications treat stochastic differential differential equations involving fractional derivatives. The most of these papers have attempted to prove the nature and uniqueness of solutions under Lipschitz and linear growth conditions. The existence and uniqueness of solutions, for some classes of stochastic differential equations with integer and fractional order derivative, by employing the fixed point theory have been discussed in [20, 30, 33, 36, 38, 118, 124, 127, 132, 135, 134] and the references therein.

In this subsection, we relax the Lipschitz and linear growth conditions for the existence and uniqueness of solutions for fractional stochastic differential equations of the type,

$$\begin{cases} {}^cD^\alpha y(t) = f(t, y(t)) + g(t, y(t))\dfrac{dw(t)}{dt}, & t \in \mathbb{R}_+, \\ y(0) = y_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}), \end{cases} \tag{4.1.14}$$

where $\frac{1}{2} < \alpha < 1$, $f, g : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^m$ are continuous and $W(t) = (W_1(t), W_2(t), \ldots, W_m(t))^T$ is an m-dimensional Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., right continuous and $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets).

For each $t \in \mathbb{R}_+, L^2(\Omega, \mathcal{F}_t, \mathbb{P})$ denotes the space of all $\mathcal{F}_t$-measurable, mean square integrable functions $x = (x_1, \ldots, x_m) : \Omega \to \mathbb{R}^m$ with

$$\|x\|_{ms} = \sqrt{\mathbb{E}\|x\|_2^2}, \ \ \|x\|_2^2 = \sum_{i=1}^m |x_i|^2.$$

Definition 4.1.12. *A process $y : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^m$ is said to be $\mathbb{F}-$adpted if for every $t \in \mathbb{R}_+$, we have $y(t) \in L^2(\Omega, \mathcal{F}_t, \mathbb{P})$.*

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a complete probability space furnished with a complete family of right continuous increasing σ-algebras $\{\mathcal{F}_t, t \in [0, b]\}$ satisfying $\mathcal{F}_t \subset \mathcal{F}$. Let $L^2(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a space of all square random variables with values in $\mathbb{R}^m$, that are measurable with respect to $\{\mathcal{F}_t, t \in [0, b]\}$. Let $\widehat{M}_2([0, b], \mathbb{R}^m)$ denote the class of $\mathbb{R}^m$-valued stochastic processes $\{\xi(t) : t \in [0, b]\}$ which are $\mathcal{F}_t$-adapted and have finite second moments, that is,

$$\|\xi\|_{\widehat{M}_2} = \sup_{t\in[0,b]} (E|\xi(t)|^2)^{\frac{1}{2}} < \infty.$$

It is easy to verify that $\widehat{M}_2$ furnished with the norm topology as defined above, is a Banach space. White noise is usually regarded as informal time derivative $W'(t)$ of Brownian motion or Wiener process $W(t)$. In Itô's theory of stochastic integration an integral with respect to $W'(t)$ is rewritten as one with respect to $dW(t)$, that is,

$$\int_a^b \psi(t)dW(t) = \int_a^b \psi(t)W'(t)dt.$$

The Itô integral $\int_a^b \psi(t)dW(t)$ is defined for any process $\psi(t)$ which satisfies the conditions,

(1) ψ is nonanticipating,

(2) Almost all sample paths of ψ belong to $L^2([a, b])$. Moreover, $\int_a^b \psi(t)dW(t) \in L^2(\Omega)$ if and only if $\psi \in L^2([a, b] \times \Omega)$. In fact the following equality holds

$$\mathbb{E}\left|\int_a^b \psi(t)dW(t)\right|^2 = \mathbb{E}\int_a^b |\psi(t)|^2 dt.$$

Definition 4.1.13. *An $F-$adpted process $y : \mathbb{R}_+ \times \Omega \to \mathbb{R}^m$ is called a solution of* (4.1.14) *with initial condition $y(0) = y_0$ if the following integral stochastic equation holds for all $t \in [0, \infty)$,*

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y(s))ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s, y(s))dW(s).$$

Theorem 4.1.14. *Let $f, g : [0, b] \times \mathbb{R}^m \to \mathbb{R}^m$ be continuous functions. Assume that there exist a continuous nondecreasing function $\psi : \mathbb{R}_+ \to (0, \infty)$ and $K > 0$ such that*

$$\mathbb{E}\|f(t,x) - f(t,y)\|^2 \le K\psi(\mathbb{E}\|x-y\|^2), \quad \forall x, y \in \mathbb{R}^m, \quad \int_0^b \|f(s,0)\|^2 ds < \infty, \tag{4.1.15}$$

$$\mathbb{E}\|g(t,x) - g(t,y)\|^2 \le K\psi(\mathbb{E}\|x-y\|^2), \quad \forall\, x, y \in \mathbb{R}^m, \tag{4.1.16}$$

and

$$\int_0^x \frac{dz}{\psi(z)} = \infty, \quad \textit{for all } x > 0. \tag{4.1.17}$$

Consequently, the following fractional stochastic differential equation,

$$\begin{cases} {}^cD^\alpha y(t) = f(t, y(t)) + g(t, y(t))\dfrac{dW(t)}{dt}, & t \in [0, b], \\ y(0) = y_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}), \end{cases} \tag{4.1.18}$$

has a unique solution on $[0, b]$.

Proof. *We define the operator* $S : \widehat{M}_2([0,b],\mathbb{R}^m) \to \widehat{M}_2([0,b],\mathbb{R}^m)$ *by*

$$Sy(t) = y_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,y(s))ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s,y(s))dW(s), t \in [0,b].$$

Step 1 : *First, we show that the operator* S *is well-defined.*
Let $y \in \widehat{M}_2([0,b],\mathbb{R}^m)$*, then*

$$\begin{aligned}
\|Sy(t)\|^2 &\leq 3\|y_0\|^2 + \frac{3}{\Gamma^2(\alpha)}\left\|\int_0^t (t-s)^{\alpha-1} f(s,y(s))ds\right\|^2 \\
&+ \frac{3}{\Gamma^2(\alpha)}\left\|\int_0^t (t-s)^{\alpha-1} g(s,y(s))dW(s)\right\|^2 \\
&\leq 3\|y_0\|^2 + \frac{6}{\Gamma^2(\alpha)}\left\|\int_0^t (t-s)^{\alpha-1}(f(s,y(s)) - f(s,0))ds\right\|^2 \\
&+ \frac{6}{\Gamma^2(\alpha)}\left\|\int_0^t (t-s)^{\alpha-1} f(s,0)ds\right\|^2 \\
&+ \frac{6}{\Gamma^2(\alpha)}\left\|\int_0^t (t-s)^{\alpha-1}(g(s,y(s)) - g(s,0))dW(s)\right\|^2 \\
&+ \frac{6}{\Gamma^2(\alpha)}\left\|\int_0^t (t-s)^{\alpha-1} g(s,0)dW(s)\right\|^2.
\end{aligned}$$

Then

$$\begin{aligned}
\mathbb{E}\|Sy(t)\|^2 &\leq 3\mathbb{E}\|y_0\|^2 + \frac{6}{\Gamma^2(\alpha)}\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1}(f(s,y(s)) - f(s,0))ds\right\|^2 \\
&+ \frac{6}{\Gamma^2(\alpha)}\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1} f(s,0)ds\right\|^2 \\
&+ \frac{6}{\Gamma^2(\alpha)}\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1}(g(s,y(s)) - g(s,0))dW(s)\right\|^2 \\
&+ \frac{6}{\Gamma^2(\alpha)}\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1} g(s,0)dW(s)\right\|^2.
\end{aligned}$$

Combining the Hölder inequality and Itô isometry, we obtain

$$
\begin{aligned}
\mathbb{E}\|Sy(t)\|^2 &\leq 3\mathbb{E}\|y_0\|^2 + \frac{6b^{2\alpha-1}}{\alpha\Gamma^2(\alpha)}\int_0^t \mathbb{E}(\|f(s,y(s)) - f(s,0)\|^2)ds \\
&+ \frac{6b^{2\alpha-1}}{\alpha\Gamma^2(\alpha)}\int_0^b \|f(s,0)\|^2 ds + \frac{6}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2\alpha-2}\mathbb{E}\|g(s,y(s)) - g(s,0)\|^2 ds \\
&+ \frac{6}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2\alpha-2}\|g(s,0)\|^2 ds \\
&\leq 3\mathbb{E}\|y_0\|^2 + \frac{6b^{2\alpha}}{\alpha\Gamma^2(\alpha)}\int_0^t K\psi(\mathbb{E}(\|y(s)\|^2)ds \\
&+ \frac{6b^{2\alpha-1}}{\alpha\Gamma^2(\alpha)}\int_0^b \|f(s,0)\|^2 ds + \frac{6}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2\alpha-2}K\psi(\mathbb{E}\|y(s)\|^2)ds \\
&+ \frac{6}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2\alpha-2}\|g(s,0)\|^2 ds.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\|Sy)\|^2_{\widehat{M}_2} &\leq 3\mathbb{E}\|y_0\|^2 + \frac{6b^{2\alpha}K}{\alpha\Gamma^2(\alpha)}\psi(\|y\|^2_{\widehat{M}_2}) + \frac{6b^{2\alpha-1}}{\alpha\Gamma^2(\alpha)}\|f(\cdot,0)\|^2_{L^2} \\
&+ \frac{6b^{2\alpha}K}{(2\alpha-1)\Gamma^2(\alpha)}\psi(\|y\|^2_{\widehat{M}_2}) + \frac{6b^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)}\|g(\cdot,0)\|^2_{\infty}.
\end{aligned}
$$

This implies that, the operator S is well-defined.
Clearly, the fixed points of operator S are solutions of problem (4.1.18).

Step 2 : *S is continuous.*

Let $(y_n)_{n\in\mathbb{N}}$ be a sequence such that $y_n \to y$ in $\widehat{M}_2$. Then, for $t \in [0,b]$, by the dominated convergence theorem, we have

$$
\begin{aligned}
\mathbb{E}\|Sy_n(t) - Sy(t)\|^2 &\leq \frac{2}{\Gamma(\alpha)}\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1}[f(s,y_n(s)) - f(s,y(s))]ds\right\|^2 \\
&+ \frac{2}{\Gamma(\alpha)}\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1}[g(s,y_n(s)) - g(s,y(s))]dW(s)\right\|^2 \\
&\leq \frac{2b^{2\alpha-1}}{(2\alpha-1)\Gamma(\alpha)}\int_0^t \mathbb{E}\|f(s,y_n(s)) - f(s,y(s))\|^2 ds \\
&+ \frac{2}{\Gamma(\alpha)}\int_0^t (t-s)^{2\alpha-2}\mathbb{E}\|g(s,y_n(s)) - g(s,y(s))\|^2 ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
\|Sy_n - Sy\|^2_{\widehat{M}_2} &\leq \frac{2b^{2\alpha}K}{(2\alpha-1)\Gamma(\alpha)}\psi(\|y_n - y\|^2_{\widehat{M}_2}) \\
&+ \frac{2b^{2\alpha-1}K}{(2\alpha-1)\Gamma(\alpha)}\psi(\|y_n - y\|^2_{\widehat{M}_2}) \to 0, \text{ as } n \to \infty.
\end{aligned}
$$

Thus S is continuous on $\widehat{M}_2([0,b],\mathbb{R}^m)$.

Step 3 : *S maps bounded sets into bounded sets in* $\widehat{M}_2([0,b],\mathbb{R}^m)$.

Indeed, it is enough to show that for any $q>0$*, there exists* $l>0$ *such that for each*

$$y\in\mathcal{B}_r=\{y\in\widehat{M}_2([0,b],\mathbb{R}^m):\|y\|^2_{\widehat{M}_2}\le r\},$$

one has $\|Sy\|^2_{\widehat{M}_2}\le l$.
Let $y\in\mathcal{B}_r$*, then for each* $t\in[0,b]$*, we have*

$$\begin{aligned}
\mathbb{E}\left\|Sy(t)\right\|^2 &= \mathbb{E}\left\|y_0+\tfrac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,y(s))ds+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}g(s,y(s))dW(s)\right\|^2\\
&\le \mathbb{E}\left\|y_0\right\|^2+\tfrac{3}{\Gamma^2(\alpha)}\mathbb{E}\left\|\int_0^t(t-s)^{\alpha-1}f(s,y(s))ds\right\|^2\\
&+ \tfrac{3}{\Gamma^2(\alpha)}\mathbb{E}\left\|\int_0^t(t-s)^{\alpha-1}g(s,y(s))dW(s)\right\|^2\\
&\le \mathbb{E}\left\|y_0\right\|^2+\tfrac{3Kb^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)}\psi(r)+\tfrac{3b^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)}\left\|f(\cdot,0)\right\|^2_{L^2}\\
&+ \tfrac{3Kb^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\psi(r)+\tfrac{3b^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\left\|g(\cdot,0)\right\|_\infty:=l.
\end{aligned}$$

Therefore, we obtain

$$\|Sy\|^2_{\widehat{M}_2}\le l.$$

Step 4 : *The map S is equicontinuous.*
Let $\tau_1,\tau_2\in[0,b]$, $\tau_1<\tau_2$ *and* $y\in\mathcal{B}_r$*, we have*

$$\begin{aligned}
\mathbb{E}\|Sy(\tau_2)-Sy(\tau_1)\|^2 &\le \frac{4}{\Gamma^2(\alpha)}\mathbb{E}\left\|\int_0^{\tau_1}[(\tau_1-s)^{\alpha-1}-(\tau_2-s)^{\alpha-1}]\|f(s,y(s))\|ds\right\|^2\\
&+ \frac{8}{\Gamma^2(\alpha)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{2\alpha-2}ds\int_0^b\mathbb{E}\|f(s,y(s))\|^2ds\\
&+ \frac{8}{\Gamma^2(\alpha)}\int_0^{\tau_1}[(\tau_1-s)^{\alpha-1}-(\tau_2-s)^{\alpha-1}]^2\mathbb{E}\|g(s,y(s))\|^2ds\\
&+ \frac{8}{\Gamma^2(\alpha)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{2\alpha-2}\mathbb{E}\|g(s,y(s))\|^2ds.
\end{aligned}$$

Combining Young's inequality, Hölder's inequality and Itô's isometry, we obtain

$$\begin{aligned}
\mathbb{E}\|Sy(\tau_2)-Sy(\tau_1)\|^2 &\le \frac{8(bK\psi(r)+\|f(\cdot,0)\|^2_{L^2})}{\Gamma^2(\alpha)}\left(\int_0^{\tau_1}((\tau_1-s)^{\alpha-1}-(\tau_2-s)^{\alpha-1})ds\right)^2\\
&+ \frac{8(bK\psi(r)+\|f(\cdot,0)\|^2_{L^2})}{\Gamma^2(\alpha)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{2\alpha-2}ds\\
&+ \frac{8(K\psi(r)+\|g(\cdot,0)\|^2_\infty}{\Gamma^2(\alpha)}\int_0^{\tau_1}((\tau_1-s)^{\alpha-1}-(\tau_2-s)^{\alpha-1})^2ds\\
&+ \frac{8(K\psi(r)+\|g(\cdot,0)\|^2_\infty)}{\Gamma^2(\alpha)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{2\alpha-2}ds.
\end{aligned}$$

Since for every $a \geq \bar{a} \geq 0$, we have $(a-\bar{a})^2 \leq a^2 - \bar{a}^2$, then we obtain

$$\begin{aligned}
\mathbb{E}\|Sy(\tau_2) - Sy(\tau_1)\|^2 &\leq \frac{8(bK\psi(r) + \|f(\cdot,0)\|^2_{L^2})}{\Gamma^2(\alpha)}\left(\frac{\tau_1^\alpha}{\alpha} - \frac{\tau_2^\alpha}{\alpha} + \frac{(\tau_2-\tau_1)^\alpha}{\alpha}\right)^2 \\
&+ \frac{8(bK\psi(r) + \|f(\cdot,0)\|^2_{L^2})}{(2\alpha-1)\Gamma^2(\alpha)}(\tau_2-\tau_1)^{2\alpha-1} \\
&+ \frac{8(K\psi(r) + \|g(\cdot,0)\|^2_{\infty})}{\Gamma^2(\alpha)}\int_0^{\tau_1}((\tau_1-s)^{2\alpha-2} - (\tau_2-s)^{2\alpha-2})ds \\
&+ \frac{8(K\psi(r) + \|g(\cdot,0)\|^2_{\infty})}{\Gamma^2(\alpha)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{2\alpha-2}ds.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\leq \frac{8(bK\psi(r) + \|f(\cdot,0)\|^2_{L^2})}{\Gamma^2(\alpha)}\left(\frac{\tau_1^\alpha}{\alpha} - \frac{\tau_2^\alpha}{\alpha} + \frac{(\tau_2-\tau_1)^\alpha}{\alpha}\right)^2 \\
&+ \frac{8(bK\psi(r) + \|f(\cdot,0)\|^2_{L^2})}{(2\alpha-1)\Gamma^2(\alpha)}(\tau_2-\tau_1)^{2\alpha-1} \\
&+ \frac{8(K\psi(r) + \|g(\cdot,0)\|^2_{\infty})}{(2\alpha-1)\Gamma^2(\alpha)}(\tau_1^{2\alpha-1} + (\tau_2-\tau_1)^{2\alpha-1} - \tau_2^\alpha) \\
&+ \frac{8(K\psi(r) + \|g(\cdot,0)\|^2_{\infty})}{(2\alpha-1)\Gamma^2(\alpha)}(\tau_2-\tau_1)^{2\alpha-1}.
\end{aligned}$$

The right-hand side tends to zero as $\tau_2 - \tau_1 \to 0$. Then $S(B_r)$ is equicontinuous. As a consequence of Steps 1 to 3 together with the Arzelà-Ascoli we can conclude that

$$S : \widehat{M}_2([0,b],\mathbb{R}^m) \to \widehat{M}_2([0,b],\mathbb{R}^m),$$

is completely continuous.

Step 5 : A priori *bounds on solutions.*

Let $y = \gamma S(y)$ for some $0 < \gamma < 1$. Then,

$$y(t) = \gamma\left(y_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,y(s))ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s,y(s))dW(s)\right).$$

This implies, for each $t \in J$, that

$$\begin{aligned}
\mathbb{E}\|y(t)\|^2 &\leq 3\mathbb{E}\|y_0\|^2 + \frac{3}{\Gamma(\alpha)}\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1}f(s,y(s))ds\right\|^2 \\
&+ \frac{3}{\Gamma^2(\alpha)}\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1}g(s,y(s))dW(s)\right\|^2.
\end{aligned}$$

Using the Hölder inequality and Itô isometry together, we obtain

$$\begin{aligned}
\mathbb{E}\|y(t)\|^2 &\leq 3\mathbb{E}\|y_0\|^2 + \frac{3b^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)}\int_0^t \mathbb{E}\|f(s,y(s))\|^2 ds \\
&+ \frac{3}{\Gamma(\alpha)}\int_0^t (t-s)^{2\alpha-2}\mathbb{E}\|g(s,y(s))\|^2 ds \\
&\leq 3\mathbb{E}\|y_0\|^2 + \frac{6b^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)}\int_0^t K\psi(\mathbb{E}\|y(s)\|^2) ds \\
&+ \frac{6b^{2\alpha}}{(2\alpha-1)\Gamma(\alpha)}\int_0^t K\|f(s,0)\|^2 ds \\
&+ \frac{6}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2\alpha-2}K\psi(\mathbb{E}\|y(s)\|^2) ds + \frac{6}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2\alpha-2}\|g(s,0)\|^2) ds.
\end{aligned}$$

Thus, we obtain

$$\mathbb{E}\|y(t)\|^2 \leq K_0 + K_1\int_0^t \psi(\mathbb{E}\|y(s)\|^2) ds + K_2\int_0^t (t-s)^{\gamma-1}\psi(\mathbb{E}\|y(s)\|^2)) ds. \tag{4.1.19}$$

From corollary 4.1.7, there exists a constant M such that

$$\mathbb{E}\|y(t)\|^2 \leq M,\ t \in [0,b].$$

Thus

$$\|y\|_{\hat{M}_2} \leq \sqrt{M}.$$

Let

$$U = \left\{y \in \hat{M}_2([0,b],\mathbb{R}^n); \|y\|_{\hat{M}_2} \leq \sqrt{M}+1\right\}$$

The operator $S : \bar{U} \longrightarrow \hat{M}_2([0,b],\mathbb{R}^n)$ is completely continuous.Then by the choice of U there no $y \in \partial U$ such $U = S(U)$for some $\gamma \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type5.2.6, we deduce that S has a fixed point y in U which is a solution of the problem4.1.11 .And we can conclude that for every $b > 0$ the problem 4.1.14 has a unique solution.

Step 6 : *Now we show that the problem* (4.1.18), *has a unique solution.*
Let x and y be two solutions of (4.1.18), *then*

$$x(t) = y_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,x(s))ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s,x(s))dW(s), \quad t \in [0,b],$$

and

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,y(s))ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s,x(s))ds, \quad t \in [0,b].$$

Then

$$\begin{aligned}
\mathbb{E}\|x(t)-y(t)\|^2 &\leq \frac{2}{\Gamma(\alpha)}\mathbb{E}\Big\|\int_0^t (t-s)^{\alpha-1}[f(s,x(s))-f(s,y(s))]ds\Big\|^2\\
&+ \frac{2}{\Gamma(\alpha)}\mathbb{E}\Big\|\int_0^t (t-s)^{\alpha-1}[g(s,x(s))-g(s,y(s))]dW(s)\Big\|^2\\
&\leq \frac{2b^{2\alpha-1}}{(2\alpha-1)\Gamma(\alpha)}\int_0^t \mathbb{E}\|f(s,x(s))-f(s,y(s))\|^2 ds\\
&+ \frac{2}{\Gamma(\alpha)}\int_0^t (t-s)^{2\alpha-2}\mathbb{E}\|g(s,x(s))-g(s,y(s))\|^2 ds\\
&\leq \frac{2b^{2\alpha-1}}{(2\alpha-1)\Gamma(\alpha)}\int_0^t (t-s)^{2-2\alpha}(t-s)^{(2\alpha-1)-}\psi(\mathbb{E}\|x(s)-y(s)\|^2)ds\\
&+ \frac{2}{\Gamma(\alpha)}\int_0^t (t-s)^{(2\alpha-1)-2}\psi(\mathbb{E}\|x(s)-y(s)\|^2)ds.
\end{aligned}$$

Therefore

$$\mathbb{E}\|x(t)-y(t)\|^2 \leq K\int_0^t (t-s)^{\beta-1}\psi(\mathbb{E}\|x(s)-y(s)\|^2)ds$$

where

$$K=\frac{2}{\Gamma(\alpha)}+\frac{2b}{(2\alpha-1)\Gamma(\alpha)},\ \beta=2\alpha-1.$$

For each $\varepsilon>0$ *we have*

$$\mathbb{E}\|x(t)-y(t)\|^2 \leq \varepsilon + K\int_0^t (t-s)^{\beta-1}\psi(\mathbb{E}\|x(s)-y(s)\|^2)ds.$$

Theorem 4.1.4 implies that

$$\mathbb{E}\|x(t)-y(t)\|^2 \leq \Psi^{-1}\left(\frac{b^{p(\alpha-1)+1}}{p(\alpha-1)+1}\int_0^t f^q(s)ds\right) := x,$$

and from the condition (4.1.17) *we conclude that*

$$\mathbb{E}\|x(t)-y(t)\|^2 = 0, \quad \textit{for all } t\in[0,b].$$

The uniqueness is proved.

With a very similar proof as in Theorem 3.1 we can establish the next result.

Theorem 4.1.15. *Assume there exists* $\psi:[0,\infty)\to[0,\infty)$ *be a continuous function, nondecreasing and* $\psi(0)=0$ *such that*

$$\mathbb{E}\|f(t,x)-f(t,y)\|^2 \leq \psi(\mathbb{E}\|x-y\|^2), \quad \forall x,y\in\mathbb{R}^n \tag{4.1.20}$$

and

$$\mathbb{E}\|g(t,x)-g(t,y)\|^2 \leq \psi(\mathbb{E}\|x-y\|^2), \quad \forall x,y\in\mathbb{R}^n. \tag{4.1.21}$$

If

$$\int_0^x \frac{dx}{(\psi(\sqrt{x}))^2} = \infty, \quad \textit{for every } x>0, \tag{4.1.22}$$

then the problem (4.1.8) *has unique solution on* $[0, \infty)$.
If

$$b_\infty = \int_0^\infty \frac{dx}{(\psi(\sqrt{x}))^2} < \infty$$

then the for each $b < b_\infty$ *the problem* (4.1.8) *has unique solution on* $[0, b]$.

4.1.3. Random fractional problem

In this part, we prove the existence of solutions to the following fractional differential equations with random effect :

$$\begin{cases} {}^cD^\alpha x(t,\omega) = f(t, x(t,\omega), \omega) + g(t, x(t,\omega), \omega),\ 0 < \alpha < 1,\ t \in [0, b], \\ x(0,\omega) = x_0(\omega),\ \omega \in \Omega, \end{cases} \tag{4.1.23}$$

where $f, g : [0, b] \times \mathbb{R}^m \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m$, $(\Omega, \mathcal{A})$ is a measurable space and $x_0 : \Omega \to \mathbb{R}^m$ is a random variable. Random fractional differential equations, seem to be a natural extensions of deterministic ones. For quantitative and qualitative results, we cite [15, 65, 80, 81, 122, 129] and the references therein.

Definition 4.1.16. *The random variable* $x : \Omega \to C([0, b], \mathbb{R})$ *is said to be a solution of problem (4.1.23) if*

$$\begin{aligned} x(t,\omega) &= x_0(\omega) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s,\omega), \omega) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, x(s,\omega), \omega) ds, \quad t \in [0, b]. \end{aligned}$$

For every $\tau > 0$, we define

$$\|x\|_\tau := \sup_{t\in[0,b]} \frac{\|x(t)\|}{E_\alpha(\tau t^\alpha)} \quad \text{for all } x \in C([0, b], \mathbb{R}^m),$$

where $E_\alpha(\cdot)$ is the Mittag-Leffler function such that

$$E_\alpha(t) := \sum_{k=1}^\infty \frac{t^k}{\Gamma(\alpha k + 1)} \text{ for all } t \in \mathbb{R}.$$

For more details about Mittag-Leffler functions, see [35]. We observe that for every $x \in (C([0, b], \mathbb{R}^m)$ we have

$$\|x\|_\infty \leq M\|x\|_\tau, \quad M := \sup_{t\in[0,b]} E(\tau t^\alpha)$$

and

$$\|x\|_\tau \leq \|x\|_\infty.$$

Thus, the norms $\|\cdot\|_\infty$ and $\|\cdot\|_\tau$ are equivalent. Hence $(C([0, b], \mathbb{R}^m), \|\cdot\|_\tau)$ is a Banach space.

Proposition 4.1.17. *For any* $\alpha \in (0, 1)$ *and* $\tau > 0$, *the following inequality holds :*

$$\frac{\tau}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} E_\alpha(\tau s^\alpha) ds \leq E_\alpha(\tau t^\alpha).$$

Proof. *Let $0 < \tau \leq 1$. We consider first the linear problem*

$$^{c}D^{\alpha}x(t) = \tau x(t), \quad t \in \mathbb{R}_{+}. \tag{4.1.24}$$

From [35, Theorem 7.2 and Remark 7.1], the function $x(t) = E(\tau t^{\alpha})$ is solution of (4.1.24) and for any $t \in \mathbb{R}_{+}$ we have

$$E(\tau t^{\alpha}) = 1 + \frac{\tau}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} E_{\alpha}(\tau s^{\alpha}) ds.$$

This yields the proof of proposition.

We introduce the following hypotheses.

($\mathcal{H}_1$) For every $\omega \in \Omega$, the functions $f(\cdot,\cdot,\omega)$ and $g(\cdot,\cdot,\omega)$ are continuous and $\omega \to f(\cdot,\cdot,\omega)$, $\omega \to g(\cdot,\cdot,\omega)$ are measurable.

($\mathcal{H}_2$) There exists a measurable function $\gamma : \Omega \to \mathbb{R}_{+}$ and $\psi \in H$ such that

$$\|f(t,x,y,\omega)\| \leq \gamma(\omega)\psi(\|x\|),$$

for all $t \in [0,b]$, $\omega \in \Omega$ and $x, y \in \mathbb{R}^{m}$.

($\mathcal{H}_3$) There exists random variable $p_1 : \Omega \to \mathbb{R}_{+}$ such that

$$\|g(t,x,\omega) - g(t,\widetilde{x},\omega)\| \leq p_1(\omega)\|x - \widetilde{x}\| \quad x, \widetilde{x} \in \mathbb{R}^{m}, \ \omega \in \Omega.$$

Now, we establish the existence of a solution of problem (4.1.23) by using the Krasnosel'skii random fixed point theorem type in a Banach space.

Theorem 4.1.18. *[49, 59] Let E be a separable Banach space. Suppose that T and B are two random operators from $\Omega \times E$ into E such that*

($\mathcal{A}_1$) *T be a completely continuous random operator.*

($\mathcal{A}_2$) *B be a continuous random operator.*

If

$$\mathcal{M} = \left\{ x : \Omega \to E \text{ is measurable} \mid \lambda(\omega)T(\omega,x) + \lambda(\omega)B\left(\frac{x}{\lambda(\omega)}, \omega\right) = x \right\},$$

is bounded for all measurable $\lambda : \Omega \to \mathbb{R}$ with $0 < \lambda(\omega) < 1$ on Ω. Then the random equation

$$x = T(\omega,x) + B(\omega,x), \quad x \in E,$$

has at least one random solution.

Let $\mathcal{N}_{\omega} : C([0,b],\mathbb{R}^{m}) \to C([0,b],\mathbb{R}^{m})$ be an operator defined by

$$\mathcal{N}_{\omega}(x,y) = G_{\omega}(x) + K_{\omega}(x), \quad x, \in C([0,b],\mathbb{R}^{m}),$$

where

$$G_{\omega}(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} f(s, x(s,\omega), \omega) ds + x_0(\omega), \quad t \in [0,b]$$

and

$$K_{\omega}(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} g(s, x(s,\omega), \omega) ds, \quad t \in [0,b].$$

Lemma 4.1.19. *Under assumption ($\mathcal{H}_3$). The operator K_{ω} is contraction on $C([0,b],\mathbb{R}^{m})$.*

Proof. *Indeed let* $(x(\cdot,\omega),\widetilde{x}(\cdot,\omega)) \in C([0,b],\mathbb{R}^m)\times C([0,b],\mathbb{R}^m)$. *Then*

$$\begin{aligned}\|K_\omega(x(t)) - K_\omega(\widetilde{x}(t))\| &= \left\|\int_0^t \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(g(s,x(s,\omega),\omega) - g(s,\widetilde{x}(s,\omega),\omega)\right)ds\right\| \\ &\le \frac{1}{\Gamma(\alpha)}\int_0^t p_1(\omega)(t-s)^{\alpha-1}\|x(s,\omega)-\widetilde{x}(s,\omega)\|ds.\end{aligned}$$

Let $\tau > 1$. *Then,*

$$\|K_\omega(x(t)) - K_\omega(\widetilde{x}(t))\| \le \frac{\tau p_1(\omega)}{\tau\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}E_\alpha(\tau p_1(\omega)s^\alpha)ds\|x(\cdot,\omega)-\widetilde{x}(\cdot,\omega)\|_{\tau p_1(\omega)}.$$

By Proposition 4.1.17, we obtain

$$\|K_\omega(x(t)) - K_\omega(\widetilde{x}(t))\| \le \frac{E_\alpha(\tau p_1(\omega)t^\alpha)}{\tau}\|x(\cdot,\omega)-\widetilde{x}(\cdot,\omega)\|_{\tau p_1(\omega)}, \quad t\in[0,b].$$

Together, we have

$$\|K_\omega(x) - K_\omega(\widetilde{x})\|_{\tau p_1(\omega)} \le \frac{1}{\tau}\|x-\widetilde{x}\|_{\tau p_1(\omega)}, \quad \textit{for all } x,\widetilde{x}\in C([0,b],\mathbb{R}^m).$$

We conclude that K_ω *is a contraction.*

Lemma 4.1.20. *Under assumption* $(\mathcal{H}_1)$. *The operator* $G_\omega : C([0,b],\mathbb{R}^m) \to C([0,b],\mathbb{R}^m)$ *is completely continuous.*

Proof. *The proof will be given in several steps.*

— **Step 1.** G_ω *is continuous.*
Let $x_n(\cdot,\omega)$ *be a sequence such that* $x_n(\cdot,\omega)\to x(\cdot,\omega)\in C([0,b],\mathbb{R}^m)$ *as* $n\to\infty$. *Then*

$$\|G_\omega(x_n(\cdot,\omega)) - G_\omega(x(\cdot,\omega))\|_\infty \le \frac{b^\alpha}{\Gamma(\alpha+1)}\|f(\cdot,x_n(\cdot,\omega)) - f(\cdot,x(\cdot,\omega))\|_\infty.$$

Since f *is a continuous function, thus*

$$\|G_\omega(x_n(\cdot,\omega)) - G_\omega(x(\cdot,\omega))\|_\infty \to 0 \textit{ as } n\to\infty.$$

— **Step 2.** G_ω *maps bounded sets into bounded sets in* $C([0,b],\mathbb{R}^m)$.
Indeed, it suffices to show that for any $q>0$ *there exists a positive constant* l *such that for each* $x(\cdot,\omega)\in B_q = \{x\in C([0,b],\mathbb{R}) : \|x(\cdot,\omega)\|_\infty \le q\}$, *we have*

$$\|G_\omega(x(\cdot,\omega))\|_\infty \le l.$$

In that direction, for each $t\in[0,b]$, *we get*

$$\begin{aligned}\|G_\omega(x(t,\omega))\| &= \left\|x_0(\omega) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,x(s,\omega))ds\right\| \\ &\le \|x_0(\omega)\| + \frac{\gamma_1(\omega)}{\Gamma(\alpha)}\int_0^b \|f(s,x(s,\omega))\|ds.\end{aligned}$$

So, from $(\mathcal{H}_2)$,

$$\|G_\omega(x(\cdot,\omega))\|_\infty \le \|x_0(\omega)\| + \frac{2b^\alpha\psi(q)}{\Gamma(\alpha+1)}\gamma_1(\omega) := l.$$

— ***Step 3.*** *G_ω maps bounded sets into equicontinuous sets of $C([0,b],\mathbb{R}^m)$.*
Let B_q be a bounded set in $C([0,b],\mathbb{R}^m)\times C([0,b],\mathbb{R}^m)$ as in Step 2. Let $r_1,r_2\in J, r_1<r_2$ and $u\in B_q$. Thus we have

$$\begin{aligned}\|G_\omega(x(r_2,\omega))-G_\omega(x(r_1,\omega))\| &\leq \frac{2q\gamma_1(\omega)}{\Gamma(\alpha)}\left[\int_{r_1}^{r_2}(r_2-s)^{\alpha-1}ds\right.\\ &\left.+\int_0^{r_1}(r_1-s)^{\alpha-1}-(r_2-s)^{\alpha-1}ds\right].\end{aligned}$$

Hence

$$\|G_\omega(x(r_2,\omega))-G_\omega(x(r_1,\omega))\| \leq \frac{4\psi(q)\gamma_1(\omega)}{\Gamma(\alpha+1)}(r_2-r_1)^\alpha.$$

The right-hand term tends to zero as $|r_2-r_1|\to 0$. As a consequence of Steps 1 to 3 together with the Arzelà-Ascoli, we conclude that G_ω maps B_q into a precompact set in $C([0,b],\mathbb{R})\times C([0,b],\mathbb{R})$.

Lemma 4.1.21. *Assume that $(\mathcal{H}_1)-(\mathcal{H}_3)$ hold. Then the set*

$$\mathcal{A}(\omega)=\left\{x(\cdot,\omega)\in C([0,b],\mathbb{R}^m):\ x(\cdot,\omega)=\lambda(\omega)G_\omega(x(\cdot,\omega))+\lambda(\omega)K_\omega\left(\frac{x(\cdot,\omega)}{\lambda(\omega)}\right),\lambda(\omega)\in(0,1)\right\}$$

is bounded.

Proof. *Let $x\in\mathcal{A}(\omega)$. Then $x(\cdot,\omega)=\lambda(\omega)G_\omega(x(\cdot,\omega))+\lambda(\omega)K_\omega\left(\frac{x(\cdot,\omega)}{\lambda(\omega)}\right)$. Thus, for $t\in[0,b]$, we have*

$$\begin{aligned}\|x(t,\omega)\| &\leq \|x_0(\omega)\|+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\|f(s,x(s,\omega),\omega)\|ds\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\|g(s,x(s,\omega),\omega)\|ds\\ &\leq \|x_0(\omega)\|+\frac{1}{\Gamma(\alpha)}\int_0^t\gamma_1(\omega)(t-s)^{\alpha-1}\psi(\|x(s,\omega)\|)ds\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}p_1(\omega)\|x(s,\omega)\|ds+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\|g(s,0,\omega)\|ds.\end{aligned}$$

Hence

$$\begin{aligned}\|x(t,\omega)\| &\leq c(\omega)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}p_1(\omega)\|x(s,\omega)\|ds\\ &+\frac{\gamma_1(\omega)}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\psi(\|x(s,\omega)\|)ds,\end{aligned}$$

where

$$c(\omega)=\|x_0(\omega)\|+\frac{b^\alpha\|g(\cdot,0,\omega)\|_\infty}{\Gamma(\alpha+1)}+1.$$

By Theorem 4.1.9, there exists $K(\alpha,\omega)>0$ such that

$$\|x(t,\omega)\|\leq K(\alpha,\omega),\ \textit{for each}\ t\in[0,b].$$

Consequently

$$\|x\|_\infty \leq K(\alpha, \omega).$$

This shows that $\mathcal{A}(\omega)$ is bounded.

We are now in the position to prove our main existence result for (4.1.23).

Theorem 4.1.22. *Assume that the following conditions $(\mathcal{H}_1)-(\mathcal{H}_3)$ hold. Then the problem (4.1.23) has at least on random solution.*

Proof. *Let $N : C([0,b],\mathbb{R}^m)\times\Omega \to C([0,b],\mathbb{R}^m)$,*

$$x \mapsto G(x,\omega) + K(x,\omega)$$

where

$$G(x,\omega) \quad = \quad \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, x(s,\omega))ds + x_0(\omega)$$

and

$$K(x,\omega) \quad = \quad \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{1-\alpha} g(s, x(s,\omega),\omega)ds.$$

First we show that N is a random operator on $C([0,b],\mathbb{R}^m)$. Since f and g are Carathédory functions, then $\omega \to f(t,x,\omega)$ and $\omega \to g(t,x,\omega)$ are measurable maps. Further, the integral is a limit of a finite sum of measurable functions, and therefore, the maps

$$\omega \to G(x(t,\omega),\omega), \ \omega \to K(x(t,\omega),\omega)$$

are measurable. As a result, N is a random operator on $C([0,b],\mathbb{R}^m)\times\Omega$ into $C([0,b],\mathbb{R}^m)$.

Now we show that all the conditions of Theorem 4.1.18 are satisfied.

We observe that from Lemmas 4.1.19 and 4.1.20, the operator N is a contraction and K is completely continuous. It is clear by Lemma 4.1.21 that the set

$$\mathcal{A}(\omega) = \left\{x(\cdot,\omega) \in C([0,b],\mathbb{R}^m) : \ x(\cdot,\omega) = \lambda(\omega)G_\omega(x(\cdot,\omega)) + \lambda(\omega)K_\omega\left(\frac{x(\cdot,\omega)}{\lambda(\omega)}\right), \lambda(\omega) \in (0,1)\right\}$$

is bounded. As a consequence of Theorem 4.1.18, we deduce that N has at least one random fixed point, which is a solution to the problem (4.1.23).

Remark 4.1.2. *Many researchers have established existence of a unique solution for fractional differential equations with, or without, impulses for Cauchy-Lipschitz problems with some restrictive conditions on the Lipschitz constant. But via application of Proposition 4.1.17, we can establish those results without the restrictive conditions. For example, in the problem when $f = 0$, Lemma 4.1.19 yields a unique solution of (4.1.23).*

4.1.4. R_δ Solutions Sets

In this part, we recall some elementary concepts and definitions from geometric topology. For more information about this section, we recommend [37, 48, 78]. In what follows (X,d) and (Y,d') stand for two metric spaces. Denote by $\mathcal{P}(X) = \{Y \subset E : \ Y \neq \emptyset\}$. Let E be a Banach space and $\mathcal{P}_{cv,cl}(E) = \{Y \in \mathcal{P}(E) : \ Y \text{ convex, closed}\}$.

Definition 4.1.23. *Let $A \in \mathcal{P}(X)$. The set A is said a contractible space if there exists a continuous homotopy $H : A \times [0,1] \to A$ and $x_0 \in A$ such that*
(a) $H(x,0) = x$, for every $x \in A$,
(b) $H(x,1) = x_0$, for every $x \in A$,
i.e. if the identity map is homotopic to a constant map (A is homotopically equivalent to a point).

Note that if $A \in \mathcal{P}_{cv,cl}(X)$, then A is contractible, but the class of contractible sets is much larger than the class of closed convex sets.

Definition 4.1.24. *A compact nonempty space X is called an $R_\delta-$set if there exists a decreasing sequence of compact nonempty contractible spaces $\{X_n\}_{n=1}^{\infty}$ such that*

$$X = \bigcap_{n=1}^{\infty} X_n.$$

The next result deals with the topological structure of the solution set of some nonlinear functional equations is due to Aronszajn and developed by Browder and Gupta in [24] (see also [9, Th. 1.2]).

Theorem 4.1.25. *Let (X,d) be a metric space, $(E, \|\cdot\|)$ a Banach space and $F : X \to E$ a proper map, i.e., F is continuous and for every compact $K \subset E$, the set $F^{-1}(K)$ is compact. Assume further that for each $\varepsilon > 0$, a proper map $F_\varepsilon : X \to E$ is given, and the following two conditions are satisfied :*
(a) $\|F_\varepsilon(x) - F(x)\| < \varepsilon$, for every $x \in X$,
(b) for every $\varepsilon > 0$ and $u \in E$ in a neighborhood of the origin such that $\|u\| \leq \varepsilon$, the equation $F_\varepsilon(x) = u$ has exactly one solution x_ε.
Then the set $S = F^{-1}(0)$ is an $R_\delta-$set.

Lemma 4.1.26. *Let E be a Banach space, $C \subset E$ be a nonempty closed bounded subset of E and $F : C \to E$ be a completely continuous map, then $G = Id - F$ is a proper map.*

Under classical Lipschitz and linear growth conditions the solutions set of ordinary and fractional differential equations has been studied by many authors; see, for example, the monographs and papers [31, 37, 40, 51, 55] and the references therein.

In the following, we will study the existence, compactness and R_δ properties of solutions sets of the fractional problem (4.1.8) on a compact interval $[0,b]$. Denote the solution sets of the problem (4.1.8) by

$$S(f,a) = \{y \in C([0,b], \mathbb{R}^m) : y \text{ is a solution of the problem (4.1.8)}\}.$$

Theorem 4.1.27. *Assume that f is Carathédory function and satisfies the following condition :*
$(\mathcal{H}_1)$ There exists $g \in L^\infty([0,b], \mathbb{R}_+)$ such that

$$\|f(t,x)\| \leq g(t)\psi(\|x\|), \quad \text{for all } x \in \mathbb{R}^n,\ t \in [0,b]$$

where $\psi : [0,\infty) \to (0,\infty)$ is a continuous, increasing function, $\psi(0) = 0$ and $q > \frac{1}{\alpha}$ with $p\left(1 - \frac{1}{q}\right) = 1$.
Then the solution set $S(f,a)$ is an $R_\delta-$set.

Proof. *Let $N : C([0,b],\mathbb{R}^n) \to C([0,b],\mathbb{R}^n)$ be defined by*

$$N(y)(t) = a + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,y(s))ds, \quad t \in [0,b].$$

Thus Fix $N = S(f,a)$ and we show that $S(f,a) \neq \emptyset$. In particlar, we shall show that N satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof will be given in several steps.

Step 1 *: N is continuous.*

Let $\{y_m\}$ be a sequence such that $y_m \to y$ in $C([0,b],\mathbb{R}^n)$. Then

$$\begin{aligned} \|N(y_n)(t) - N(y)(t)\| &\leq \int_0^t (t-s)^{\alpha-1}\|f(s,y_m(s)) - f(s,y(s))\|ds \\ &\leq \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{p(\alpha-1)}ds\right)^{\frac{1}{p}}\left(\int_0^b \|f(s,y_m(s)) - f(s,y(s))\|^q ds\right)^{\frac{1}{q}}. \end{aligned}$$

This implies

$$\|N(y_m) - N(y)\|_\infty \leq \frac{b^{p(\alpha-1)+1}}{(p(\alpha-1)+1)\Gamma(\alpha)}\|f(\cdot,y_m(\cdot)) - f(\cdot,y(\cdot))\|_{L^q}.$$

By $(\mathcal{H}_1)$ and since f is a Carathéodory function, the Lebesgue dominated convergence theorem implies

$$\|N(y_m) - N(y)\|_\infty \to 0 \text{ as } m \to \infty.$$

Step 2 *: N maps bounded sets into bounded sets in $C([0,b],\mathbb{R}^n)$.*

Indeed, it suffices to show that there exists a positive constant ℓ such that for each

$$y \in B_r = \{y \in C([0,b],\mathbb{R}^n) : \|y\|_\infty \leq r\},$$

one has $\|N(y)\|_\infty \leq \ell$.
Let $y \in B_r$. Then for each $t \in [0,b]$, we have

$$N(y)(t) = a + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,y(s))ds.$$

By $(\mathcal{H}_1)$ we have for each $t \in [0,b]$

$$\begin{aligned} \|N(y)(t)\| &\leq \|a\| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{1-\alpha}\|f(s,y(s))\|ds \\ &\leq \|a\| + \frac{\psi(r)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s)ds. \end{aligned}$$

From Hölder's inequality, we get

$$\|N(y)(t)\| \leq \|a\| + \left(\frac{\psi^p(r)t^{p(\alpha-1)+1}}{(p(\alpha-1)+1)\Gamma(\alpha)}\right)^{\frac{1}{p}}\left(\int_0^b g^q(s)ds\right)^{\frac{1}{q}}.$$

Then for each $y \in B_r$ we have

$$\|N(y)\|_\infty \leq \|a\| + \left(\frac{\psi^p(r)b^{p(\alpha-1)+1}}{(p(\alpha-1)+1)\Gamma(\alpha)}\right)^{\frac{1}{p}} \|g\|_{L^q} := \ell.$$

Step 3 *: N maps bounded set into equicontinuous sets of* $C([0,b],\mathbb{R}^n)$.

Let $\tau_1, \tau_2 \in [0,b]$, $\tau_1 < \tau_2$ and B_r be a bounded set of $C([0,b],\mathbb{R}^n)$ as in Step 2. Let $y \in B_r$ and $t \in [0,b]$. We have

$$\begin{aligned}
\|N(y)(\tau_2) - N(y)(\tau_1)\| &\leq \frac{1}{\Gamma(\alpha)}\int_0^{\tau_1} |(\tau_2-s)^{\alpha-1} - (\tau_1-s)^{\alpha-1}|\|f(s,y(s))\|ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2} (\tau_2-s)^{\alpha-1}\|f(s,y(s))\|ds \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{\tau_1} |(\tau_2-s)^{\alpha-1} - (\tau_1-s)^{\alpha-1}|g(s)\psi(\|y(s)\|)ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2} (\tau_2-s)^{\alpha-1}g(s)\psi(\|y(s)\|)ds \\
&\leq \frac{\psi(r)}{\Gamma(\alpha)}\int_0^{\tau_1} |(\tau_2-s)^{\alpha-1} - (\tau_1-s)^{\alpha-1}|g(s)ds \\
&\quad + \frac{\psi(r)}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2} (\tau_2-s)^{\alpha-1}g(s)ds.
\end{aligned}$$

Since $g \in L^\infty([0,b],\mathbb{R}_+)$, then $g \in L^q$. Using Hölder's inequality, we obtain

$$\begin{aligned}
\|N(y)(\tau_2) - N(y)(\tau_1)\| &\leq \frac{\psi(r)}{\Gamma(\alpha)}\int_0^{\tau_1} |(\tau_2-s)^{\alpha-1} - (\tau_1-s)^{\alpha-1}|^p ds\|g\|_{L^\infty} \\
&\quad + \frac{\psi(r)}{\Gamma(\alpha)}\left(\int_{\tau_1}^{\tau_2} (\tau_2-s)^{p(\alpha-1)}ds\right)^{\frac{1}{p}} \|g\|_{L^q}.
\end{aligned}$$

Then

$$\begin{aligned}
\|N(y)(\tau_2) - N(y)(\tau_1)\| &\leq \frac{\psi(r)}{\Gamma(\alpha+1)}\left((\tau-\tau_1)^\alpha - \tau_2^\alpha + \tau_1^\alpha\right)\|g\|_{L^\infty} \\
&\quad + \frac{\psi(r)}{(\alpha-1+\frac{1}{p})\Gamma(\alpha)}\|g\|_{L^q}(\tau_2-\tau_1)^{\alpha-1+\frac{1}{p}}.
\end{aligned}$$

As $\tau_2 \longrightarrow \tau_1$ the right-hand side of the above inequality tends to zero. Then $N(B_r)$ is equicontinuous. As a consequence of Steps 1 to 3 together with the Arzelà-Ascoli theorem we can conclude that $N : C([0,b],\mathbb{R}^n) \to C([0,b],\mathbb{R}^n)$ is completely continuous.

Step 4 : A priori *bounds on solutions.*

Let $y = \gamma N(y)$ for some $0 < \gamma < 1$. This implies

$$\begin{aligned}
\|y(t)\| &\leq \|a\| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,y(s))\|ds \\
&\leq \|a\| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)\psi(\|y(s)\|)ds.
\end{aligned}$$

Therefore

$$\|y(t)\| \quad \leq \quad \|a\| + 1 + \frac{\|g\|_{L^\infty}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\psi(\|y(s)\|)ds.$$

From Theorem 4.1.2, we have

$$\|y(t)\| \leq \Phi^{-1}\left(\Phi(\bar{k}) + \frac{t}{q}\right), \quad t \in [0,b],$$

where

$$\Phi(z) = \int_k^z \frac{dx}{(\psi(x))^q}, \; k = \|a\| + 1, \; \bar{k} = k + \frac{\|g\|_{L^\infty} b^{p(\alpha-1)+1}}{p(p(\alpha-1)+1)}.$$

Hence

$$\|y\|_* \leq \Phi^{-1}\left(\Phi(\bar{k}) + \frac{b}{q}\right) := \overline{M}.$$

Set

$$U := \{y \in C([0,b], \mathbb{R}^n) : \|y\|_\infty < M + 1\},$$

and consider the operator $N : \overline{U} \to C([0,b], \mathbb{R}^n)$. *From the choice of* U, *there is no* $y \in \partial U$ *such that* $y = \gamma N(y)$ *for some* $\gamma \in (0,1)$. *As a consequence of the nonlinear alternative of Leray-Schauder type [50, 37], we deduce that* N *has a fixed point* y *in* U *which is a solution of the problem (4.1.8).*

Now, we prove that $S(f,a)$ *is compact. Let* $\{y_m\}_{m\geq 1}$ *be a sequence in* $S(f,a)$, *then*

$$y_m(t) = a + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_m(s))ds, \; m \geq 1, \; t \in [0,b].$$

As in Steps 3 and 4 we can easily prove that there exists $M > 0$ *such that*

$$\|y_m\|_\infty < M, \textit{ for all } m \geq 1,$$

and the set $\{y_m : m \geq 1\}$ *is equicontinuous in* $C([0,b], \mathbb{R}^n)$. *Hence by the Arzelà-Ascoli Theorem we conclude that there exists a subsequence of* $\{y_m : m \geq 1\}$ *converging to* y *in* $C([0,b], \mathbb{R}^n)$. *Using that fact that* f *is Carathédory we can prove that*

$$y(t) = a + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s))ds, \; t \in [0,b].$$

Thus $S(f,c)$ *is compact. Define*

$$\widetilde{f}(t,x) = \begin{cases} f(t,x), & \textit{if} \quad \|x\| \leq \overline{M}, \\ f\left(t, \frac{\overline{M}x}{\|x\|}\right), & \textit{if} \quad \|x\| \geq \overline{M}. \end{cases}$$

Since f *is Carathédory, the function* $\widetilde{f}$ *is Carathédory and is also bounded. So there exists* $M_* > 0$ *such that*

$$\|\widetilde{f}(t,x)\| \leq M_*, \quad \textit{for all } x \in \mathbb{R}^n, \textit{ a.e. } t \in [0,b]. \tag{4.1.25}$$

Consider the following modified problem :

$$\begin{cases} {}^cD^\alpha y(t) \;=\; \widetilde{f}(t, y(t)), \quad \textit{a.e.} \quad t \in [0,b], \\ y(0) = a. \end{cases}$$

We can easily prove that $S(f,a) = S(\widetilde{f},a) = Fix\widetilde{N}$, *where* $\widetilde{N} : C([0,b],\mathbb{R}^n) \to C([0,b],\mathbb{R}^n)$ *is as defined by*

$$\widetilde{N}(y)(t) = a + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\widetilde{f}(s,y(s))ds, \quad t \in [0,b].$$

By the inequality (4.1.25), we deduce that there exists $R > 0$ *such that*

$$\|\widetilde{N}(y)\|_\infty \quad \leq \quad R.$$

Then $\widetilde{N}$ *is uniformly bounded. As in Steps 2 and 3, we can prove that*

$$\widetilde{N} : C([0,b],\mathbb{R}^n) \to C([0,b],\mathbb{R}^n),$$

is compact which allows us to define the compact perturbation of the identity $\widetilde{G}(y) = y - \widetilde{N}(y)$ *which is a proper map. From the compactness of* $\widetilde{N}$, *we can easily prove that all conditions of Theorem 4.1.25 are met. Therefore the solution set* $S(\widetilde{f},a) = \widetilde{G}^{-1}(0)$ *is an* R_δ *set, hence an acyclic space.*

Theorem 4.1.28. *Assume that* f *is a Carathédory function and satisfies the following condition :*

($\mathcal{H}_1$) *There exists* $g \in L^\infty_{loc}(\mathbb{R}_+,\mathbb{R}_+)$ *such that*

$$\|f(t,x)\| \leq g(t)\psi(\|x\|), \quad \textit{for all } x \in \mathbb{R}^n,\ t \in [0,b]$$

where $\psi : [0,\infty) \to (0,\infty)$ *is a continuous, increasing function,* $\psi(0) = 0$ *and* $q > \frac{1}{\alpha}$ *with* $p\left(1 - \frac{1}{q}\right) = 1$.

Then the problem (4.1.8) *has at least one solution.*

4.1.5. Fractional Differential Inclusions on Banach lattices

Multivalued analysis and differential inclusions have been investigated by many authors from different points of view. A comprehensive overview of this theory can be found in [11, 49, 56, 57, 74, 67] among others.

In 2009, Michta and J. Motyl [86], introduced a new class of multivalued maps in Banach lattices, a class they called "upper separated." The notion of an upper separated multifunction function F is necessary and sufficient for proving the existence of a convex selection of F. The deterministic and stochastic differential inclusions have been considered by Michta and Motyl in [87, 91, 92, 93, 94, 95].

The aim of this section is to give the existence of some classes of fractional differential inclusions in Euclidean spaces satisfying the property of an order complete Banach lattice. More precisely, we will consider the following problem,

$$\begin{cases} {}^cD^\alpha x(t) & \in & F(t,x(t)), \quad \text{a.e. } t \in [0,b], \\ x(0) & = & x \in \mathbb{R}^m, \end{cases} \tag{4.1.26}$$

where $F\colon [0,b]\times\mathbb{R}^m \to \mathcal{P}_{cp,cv}(\mathbb{R}^m)$ is a multifunction.

First, we recall some notations, and basic definitions from multivalued analysis and Banach lattices which will be used in the sequel.

Let $\mathcal{X}$ be a Banach space and $(\mathcal{Y},\leq)$ be a Banach lattice space generated by a positive cone $\mathcal{K}^+$. We use the notation $x \leq y$ if $x - y \in \mathcal{K}^+$. We denote the space of linear bounded operators by $B(\mathcal{X},\mathcal{Y})$.

Definition 4.1.29. *The Banach lattice space $(\mathcal{Y}, \preceq)$ is called complete, if every nonempty majorized set of $\mathcal{Y}$ has a supremum in $\mathcal{Y}$.*

- *A set $A \subset \mathcal{Y}$ is called order bounded, if there exist $a, b \in \mathcal{Y}$ such that*

$$A \subset [a, b] = \{y \in Y : a \preceq t \preceq b\}.$$

- *A set A of $\mathcal{Y}$ is called order convex (or full) if for every $x, y \in A$ we have $[x, y] \subset A$.*

We adjoin to $\mathcal{Y}$ the greatest element $+\infty$ and the lowest element $-\infty$. We extend the space $\mathcal{Y}$ in a natural way $\bar{\mathcal{Y}} = \mathcal{Y} \cup \{+\infty, -\infty\}$. Now we define an extended function $g : \mathcal{X} \to \bar{\mathcal{Y}}$. Let $Dom(g) = \{x \in \mathcal{X} : g(x) \neq \pm\infty\}$ and we define the epigraph of g by

$$Epi(g) = \{(x, y) \in X \times Y : \ g(x) \preceq y\}.$$

Definition 4.1.30. *A function $g : \mathcal{X} \to \mathcal{Y}$ is called order convex if for every $x, \bar{x} \in \mathcal{X}$ and $\lambda \in [0, 1]$ we have*

$$g(\lambda x + (1 - \lambda)\bar{x}) \preceq \lambda g(x) + (1 - \lambda)g(\bar{x}).$$

The function g is locally order Lipshitz if for all $x_0 \in \mathcal{X}$ there exist an open neighbourhood $\mathcal{U}_{x_0}$ and $a \in \mathcal{K}^+$ such hat

$$|g(x) - g(\bar{x})| \preceq a\|x - \bar{x}\| \quad \text{for all } x, \bar{x} \in \mathcal{U}_{x_0}.$$

— A multifunction $G : \mathcal{X} \to \mathcal{P}(\mathcal{Y})$ is called *upper semi-continuous* (*u.s.c.* for short) if the set

$$G_-^{-1}(V) = \{x \in \mathcal{X} : \ G(x) \subset V\}$$

is open for any open set V in $\mathcal{Y}$.

— G is called *lower semi-continuous* (*l.s.c.* for short) on $\mathcal{X}$ if the set

$$G_+^{-1}(V) = \{x \in \mathcal{X}, \ G(x) \cap V \neq \emptyset\}$$

is open for any open set V in $\mathcal{Y}$.

Definition 4.1.31. *We say that the multifunction $F : \mathcal{X} \to \mathcal{P}(\mathcal{Y})$ is majorized in neighborhood of x_0 if there exist an open $\mathcal{U}_{x_0}$ and $y \in \mathcal{Y}$ such that for every $x \in \mathcal{U}_{x_0}$ we have*

$$a \preceq y \quad \text{for any } a \in F(x).$$

Let $\mathcal{V}, \mathcal{W} : \mathcal{X} \to \mathcal{Y}$ be two functions defined by

$$\mathcal{V}(x) = \sup\{a : a \in F(x)\}$$

and

$$\mathcal{W}(x) = \inf\{b : b \in F(x)\}.$$

We denote by $\prod_{F(x)}(c)$ the metric projection of $c \in \mathcal{Y}$ onto the set $F(x)$ and we define

$$\mathcal{V}(x) = \begin{cases} \prod_{F(x)}(\mathcal{V}(x)) & \text{if } x \in Dom(\mathcal{V}), \\ +\infty & \text{if } x \notin Dom(\mathcal{V}), \end{cases}$$

and

$$\mathcal{W}(x) = \begin{cases} \prod_{F(x)}(\mathcal{W}(x)) & \text{if } x \in Dom(\mathcal{W}), \\ -\infty & \text{if } x \notin Dom(\mathcal{W}). \end{cases}$$

Definition 4.1.32. *Let $F : \mathcal{X} \to \mathcal{P}(\bar{\mathcal{Y}})$ be a multivalued map.*

We say that F is upper separated if for all $x \in \mathcal{X}$ and $\varepsilon > 0$ there exists a hyperplane $H(x, \varepsilon)$ strongly separating a point $(x, \overline{\mathcal{W}}(x) - x)$ from the set $Epi(\overline{\mathcal{V}})$.

The term "separated" is in the following sense : for every $x \in \mathcal{X}$ and all $\varepsilon \in \mathcal{K}^+\backslash\{0\}$ there exist $A \in B(\mathcal{X}, \mathcal{Y})$, $a \in \mathbb{R}$ and $\eta \in \mathcal{K}^+\backslash\{0\}$ such that for any $y \in Dom\overline{\mathcal{V}}$ and each $b \in \mathcal{K}^+$ we have

$$A(x) - A(y) + a(\overline{\mathcal{W}}(x) - \overline{\mathcal{V}}(y) - \varepsilon - b) - \eta \in \mathcal{K}^+.$$

For more information about Banach lattice we refer the reader to [8, 108, **?**, 121].

Let $(\Omega, \mathcal{F}, \mu)$ be a complete $\sigma-$finite measurable space, $\mathcal{B}(\mathcal{X})$ be the Borel $\sigma-$algebra of $\mathcal{X}$ and $\mathcal{F} \otimes \mathcal{B}(\mathcal{X})$ be a product $\sigma-$algebra of $\Omega \times \mathcal{X}$.

Definition 4.1.33. *A multi-valued map $F : \Omega \to \mathcal{P}(\mathcal{Y})$ is said to be measurable provided for every open $\mathcal{U} \subset \mathcal{Y}$, the set $F_+^{-1}(\mathcal{U}) \in \mathcal{F}$.*

Definition 4.1.34. *A multifunction F is called a Carathéodory function if*
(a) the multifunction $t \mapsto F(t, x)$ is measurable for each $x \in \mathcal{X}$;
(b) for a.e. $t \in \Omega$, the map $x \mapsto F(t, x)$ is continuous.

The space $\mathbb{R}^m$, is equipped with the Euclidean norm and the following canonical order :
— If, $x, y \in \mathbb{R}^m$, $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m)$, by $x \leq y$ we mean $x_i \leq y_i$ for all $i = 1, \ldots, m$. We define the positive cone by

$$\mathcal{K}^+ = \{x \in \mathbb{R}^m : x_i \geq 0,\ i = 1, \ldots, m\}.$$

Then $(\mathbb{R}^m, \leq)$ is a complete Banach lattice with order unit $e = (1, \ldots, 1)$.

Theorem 4.1.35. *[90, 91] Let $F : \Omega\times\mathcal{X} \to \mathcal{P}_{cp,cv}(\mathbb{R}^m)$ be a multivalued map. Suppose the following conditions hold :*
1) F is a Carathédory map.
2) For every $\omega \in \Omega$, the multifunction $F(\omega, \cdot)$ is upper separated.
Then there exists a single-valued function $f : \Omega \times \mathcal{X} \to \mathcal{Y}$ such that

a) f is $\mathcal{F} \otimes \mathcal{B}(\mathcal{X})-$measurable.

b) For any $(\omega, x) \in \Omega \times \mathcal{X}$ we have
$$f(t, x) \in F(t, x).$$

c) For all $\omega \in \Omega$, $f(\omega, \cdot)$ is order-convex.

Proposition 4.1.36. *[91] Let $(\Omega, \mathcal{F}, \mu)$ be a complete measurable space, $(\mathcal{X}, \|\cdot\|)$ be a separable Banach space and $F : \Omega \times \mathcal{X} \to \mathcal{P}(\mathbb{R}^m)$ be a multivalued map. Assume that there exist $h : \Omega \to \mathbb{R}_+$ a measurable function and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous nondecreasing function such that*

$$\|F(\omega, x)\|_{\mathcal{P}} = \sup\{\|v\| : v \in F(\omega, x)\} \leq h(t)\psi(\|x\|) \quad \textit{for all } \omega \in \Omega,\ x \in \mathbb{R}^m.$$

Then every Carathédory order convex selection of F is continuous.

We introduce the following conditions
$(\mathcal{H}_4)$ F is a Carathédory multifunction.

($\mathcal{H}_5$) For all $t \in [0, b]$, the multifunction $F(t, \cdot)$ is upper separated.
($\mathcal{H}_6$) There exist $\psi \in H$ and $\bar{f} \in L^q([0, b], \mathbb{R}_+)$ such that

$$\|F(t, x)\|_{\mathcal{P}} = \sup\{\|v\| : v \in F(t, x)\} \leq \bar{f}(t)\psi(\|x\|) \quad \text{for all } x \in \mathbb{R}^m.$$

Theorem 4.1.37. *Assume that the conditions* ($\mathcal{H}_4$) – ($\mathcal{H}_6$) *are satisfied. Then the problem* (4.1.26)*, has at least one solution.*

Proof. *Let $\Omega = [0, b]$, $\mathcal{F} = \mathcal{B}([0, b])$ be a Borel $\sigma-$algebra and $\mathcal{X} = \mathcal{Y} = \mathbb{R}^m$. The hypotheses* ($\mathcal{H}_4$) *and* ($\mathcal{H}_5$) *imply that F satisfies the conditions of Theorem 4.1.35 and Proposition 4.1.36. Then there exists a Carathédory function $f : [0, b] \times \mathbb{R}^m \to \mathbb{R}$ such that*

$$f(t, x) \in F(t, x) \quad \textit{for any } (t, x) \in [0, b] \times \mathbb{R}^m.$$

We consider the following problem

$$\begin{cases} {}^cD^\alpha x(t) &= f(t, x(t)), \quad a.e.\ t \in [0, b], \\ x(0) &= x \in \mathbb{R}^m. \end{cases} \tag{4.1.27}$$

We define the operator $L : C([0, b], \mathbb{R}^m) \to C([0, b], \mathbb{R}^m)$ by

$$L(x(t)) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s, x(s))ds, \quad t \in [0, b].$$

Step 1 : *L is continuous.*

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence such that $x_n \to x$ in $C([0, b], \mathbb{R}^m)$. Then, for $t \in [0, b]$, we have by the Lebesgue dominated convergence theorem

$$\begin{aligned} \|L(x_n(t)) - L(x(t))\| &= \frac{1}{\Gamma(\alpha)} \left\| \int_0^t (t - s)^{\alpha-1}[f(s, x_n(s)) - f(s, x(s))]ds \right\| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1}\|f(s, x_n(s)) - f(s, x(s))\|ds \\ &\leq \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t - s)^{p(\alpha-1)}ds \right)^{\frac{1}{p}} \left(\int_0^t \|f(s, x_n(s)) - f(s, x(s))\|^q ds \right)^{\frac{1}{q}}. \end{aligned}$$

Then

$$\|L(x_n) - L(x)\|_\infty \leq \frac{b^{p(\alpha-1)+1}}{(p(\alpha - 1) + 1)\Gamma(\alpha)} \|f(\cdot, x_n(\cdot)) - f(\cdot, x(\cdot))\|_{L^q} \to 0, \textit{ as } n \to \infty.$$

This implies that L is continuous on $C([0, b], \mathbb{R}^m)$.

Step 2 : *L maps bounded sets into bounded sets in $C([0, b], \mathbb{R}^m)$.*

Indeed, it is enough to show that for any $q > 0$, there exists $l > 0$ such that for each

$$y \in \mathcal{B}_r = \{x \in C([0, b], \mathbb{R}^m) : \|x\|_\infty \leq r\},$$

one has $\|L(x)\|_\infty \leq l$.
Let $x \in \mathcal{B}_r$, then for each $t \in [0, b]$, we have

$$\begin{aligned} \|L(x(t))\| &= \left\|y_0 + \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x(s))ds\right\| \\ &\leq \|x_0\| + \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s,x(s))\| ds \\ &\leq \tfrac{\psi(r)}{\Gamma(\alpha)} \left(\int_0^t \|\bar{f}(s)\|^q ds\right)^{\frac{1}{q}}. \end{aligned}$$

Therefore, we obtain

$$\|L(x)\|_\infty \leq \frac{b^{p(\alpha-1)+1}\psi(r)}{(p(\alpha-1)+1)\Gamma(\alpha)} \|\bar{f}\|_{L^q}.$$

Step 3 : *The map L is equicontinuous.*
Let $\tau_1, \tau_2 \in [0,b]$, $\tau_1 < \tau_2$ and $y \in \mathcal{B}_r$, we have

$$\begin{aligned} \|L(x(\tau_2)) - L(x(\tau_1))\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^{\tau_1} |(\tau_1 - s)^{\alpha-1} - (\tau_2 - s)^{\alpha-1}| \|f(s,x(s))\| ds \\ &+ \frac{\psi(r)}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{p(\alpha-1)+1} ds \|\bar{f}\|_{L^q}. \end{aligned}$$

Therefore

$$\begin{aligned} \|L(x(\tau_2)) - L(x(\tau_1))\| &\leq \frac{\psi(r)\|\bar{f}\|_{L^q}}{\Gamma(\alpha)} \left(\int_0^{\tau_1} ((\tau_1 - s)^{\alpha-1} - (\tau_2 - s)^{\alpha-1})^p ds\right)^{\frac{1}{p}} \\ &+ \frac{\psi(r)}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{p(\alpha-1)} ds \|\bar{f}\|_{L^q} \\ &\leq \frac{\psi(r)\|\bar{f}\|_{L^q}}{\Gamma(\alpha)} \left(\int_0^{\tau_1} ((\tau_1 - s)^{p(\alpha-1)} - (\tau_2 - s)^{p(\alpha-1)}) ds\right)^{\frac{1}{p}} \\ &+ \frac{\psi(r)}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{p(\alpha-1)} ds \|\bar{f}\|_{L^q} \\ &\leq \frac{\psi(r)\|\bar{f}\|_{L^q}}{\Gamma(\alpha)\sqrt[p]{p(\alpha-1)+1}} \left(\tau_2^{p(\alpha-1)+1} - \tau_1^{p(\alpha-1)+1} - (\tau_2 - \tau_1)^{p(\alpha-1)+1}\right)^{\frac{1}{p}} \\ &+ \frac{\psi(r)\|\bar{f}\|_{L^q}}{\Gamma(\alpha)\sqrt[p]{p(\alpha-1)+1}} (\tau_2 - \tau_1)^{p(\alpha-1)+1}. \end{aligned}$$

The right-hand side tends to zero as $\tau_2 - \tau_1 \to 0$. Then $L(\mathcal{B}_r)$ is equicontinuous. As a consequence of Steps 1 to 3 together with the Arzelà-Ascoli theorem, we can conclude that
$L : C([0,b], \mathbb{R}^m)) \to C([0,b], \mathbb{R}^m)$ is completely continuous.

Step 4 : A priori *bounds on solutions.*

Let $x = \gamma L(x)$ for some $0 < \gamma < 1$. Then,

$$x(t) = \gamma \left[x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x(s))ds\right] \quad t \in [0,b].$$

This implies, for each $t \in [0,b]$,

$$\|x(t)\| \leq \|x_0\| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s,x(s))\| ds.$$

Thus, we obtain

$$\|x(t)\| \leq \|x_0\| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \psi(\|x(s)\|) ds.$$

From Theorem 4.1.6, there exists a constant M such that

$$\|x\|_\infty \leq M.$$

Let

$$\mathcal{U} := \{x \in C([0,b], \mathbb{R}^m) : \|x\|_\infty < M+1\},$$

and consider the operator $L : \overline{\mathcal{U}} \to C([0,b], \mathbb{R}^m)$*. From the choice of* $\mathcal{U}$*, there is no* $y \in \partial\mathcal{U}$ *such that* $x = \gamma L(x)$ *for some* $\gamma \in (0,1)$*. As a consequence of the Leray-Schauder nonlinear alternative, we deduce that L has a fixed point y in* $\mathcal{U}$ *which is a solution of problem (4.1.27).*

Chapitre 5

Fractional Stochastic Differential Systems with Nonlocal Conditions

5.1. Introduction

The more realistic way to describe many scientific phenomena, such as in economics, finance, chemistry, physics, and biology, is to use differential equations involving fractional derivatives in time. In 1940, Kolmogorov [73] introduced fractional Brownian motion within a Hilbert space framework where it was called a Wiener Helix. In 1968, Mandelbrot and Van Ness used the term fractional Brownian motion after introducing a stochastic integral description of this process in terms of a standard Brownian motion.

The nature, uniqueness, and asymptotic behavior of mild solutions to stochastic delay evolution equations with fractional Brownian motion have only been studied in a few papers. Ciu and Yan [33] used Sadovskii's fixed point theorem to explore the presence of a mild solution to neutral stochastic integro-differential equations with infinite delay. Sakthivel et al. [116] proved the existence of a mild solution to a nonlocal fractional stochastic differential equation, and more recently Jingyun et al. [66] gave sufficient condition for the existence and uniqueness of mild solutions to a system with nonlocal fractional stochastic Brownian motion and Hurst index $H > 1/2$.

The purpose of this paper is examine the existence and uniqueness of a mild solution to the system of fractional differential equations driven by Brownian motion

$$\begin{cases} {}^cD^q x(t) = [A_1 x(t) + f^1(s, x(s), y(s))]ds + \sigma^1(t)dB_t^{H_1}, \quad 1/2 < q \leq 1, \ J = [0, b], \\ {}^cD^q y(t) = [A_2 y(t) + f^2(s, x(s), y(s))]ds + \sigma^2(t)dB_t^{H_2}, \\ x(0) = \alpha[x, y], \\ y(0) = \beta[x, y], \end{cases} \tag{5.1.1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order $q \in \left(\frac{1}{2}, 1\right]$ with the lower limit 0. We assume that a probability space $(\Omega, \mathcal{F}_b, \mathbb{P})$ together with a normal filtration $\{\mathcal{F}_t\}_{t\in[0,b]}$ are given. The stochastic process $\{X(t)\}_{t\in[0,b]}$ takes values in the real separable Hilbert space W. We take A to be the infinitesimal generator of a strongly continuous semigroup $\{S(t) : t \geq 0\}$

in W. Here, for $i = 1,2$, $B_t^{H_i} = \left\{B^{H_i}(t) : t \in J\right\}$ is a fractional Brownian motion (fBm) with Hurst index $H_i \in \left(\frac{1}{2}, 1\right)$ on a real separable Hilbert space V. We will need the following spaces

$$L(V,W) = \{g : V \to W \,|\, g \text{ is a bounded linear operator}\};$$

$$L^2(\Omega, \mathcal{F}_b; W) := \{f : \Omega \to W \,|\, f \text{ is a } \mathcal{F}_b\text{-measurable square integrable random variable}\};$$

$$C(J, L^2(\Omega, \mathcal{F}_b; W)) := \{X : J \to L^2(\Omega, \mathcal{F}_b, W) \mid \text{is a continuous mapping from } J \text{ into } L^2(\Omega, \mathcal{F}_b; W)\}$$

such that $\sup_{t\in J} \mathbb{E}\left[\|X(t)\|^2\right] < \infty$;

$$\mathcal{C} := \left\{X : J \times \Omega \to W \,|\, X \in C\left(J, L^2(\Omega, \mathcal{F}_b; W)\right) \text{ is an } \mathcal{F}_t\text{-adapted stochastic process}\right\}.$$

For $X \in \mathcal{C}$, define a norm by $\|X\|_{\mathcal{C}} = \left(\sup_{t\in J} \mathbb{E}\left[\|X(t)\|^2\right]\right)^{\frac{1}{2}}$. It is clear that $(\mathcal{C}, \|\cdot\|_c)$ is a Banach space.

Here, for each $i = 1,2$, the linear operator $-A_i : D(A_i) \subseteq C_i \to C_i$ generates a strongly continuous semigroup of contractions $\{S_q(t) : t \geq 0\}$ on the Banach space $(C_i, \|\cdot\|_{C_i})$. We let $f^i : [0,T] : X_1 \times X_2 \to X_i$ and $\sigma^i : [0,T] \to X_i$, $i = 1,2$, be given functions. It will be convenient to write the constraints in the equivalent form of nonlocal conditions, namely,

$$x(0) = \alpha[x,y], \quad y(0) = \beta[x,y].$$

As mentioned above, the main purpose of this paper is to study the existence of mild solutions to the above described system.

5.2. Some Mathematical Preliminaries

Definition 5.2.1. *Let X be a nonempty set. By a vector-valued metric on X we mean a map $d : X \times X \to \mathbb{R}^n$ satisfying :*

(i) $d(u,v) \geq 0$ for all $u, v \in X$, and if $d(u,v) = 0$, then $u = v$;
(ii) $d(u,v) = d(v,u)$ for all $u, v \in X$;
(iii) $d(u,v) \leq d(u,w) + d(w,v)$ for all $u, v, w \in X$.

Note that for any $i \in \{1, \cdots, n\}$, $(d(u,v))_i = d_i(u,v)$ is a metric space on X.

We call the pair (X,d) a generalized metric space. For $r = (r_1, r_2, \cdots, r_n) \in \mathbb{R}_+^n$,

$$B(x_0, r) = \{x \in X : d(x_0, x) < r\}$$

denotes the open ball centered at x_0 with radius r, and by $\overline{B(x_0, r)}$ its closure.

Definition 5.2.2. *A square matrix of real numbers is said to be convergent to zero if its spectral radius $\rho(M)$ is strictly less than 1. This means that all the eigenvalues of M are in the open unit disc $|\lambda| < 1$ for every $\lambda \in \mathbb{C}$ with $\det(M - \lambda I) = 0$, where I denotes the identity matrix in $\mathcal{M}_{n\times n}(\mathbb{R})$.*

Theorem 5.2.3. ([112]) *Let $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$. The following assertions are equivalent :*

(i) M is convergent to zero;
(ii) $M^k \to 0$ as $k \to \infty$;
(iii) The matrix $(I - M)$ is nonsingular and

$$(I - M)^{-1} = I + M + M^2 + \ldots + M^k + \ldots;$$

(iv) The matrix $(I - M)$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements.

Definition 5.2.4. *Let (X, d) be a generalized metric space. An operator $N : X \to X$ is said to be contractive if there exists a convergent to zero matrix M such that*

$$d(N(x), N(y)) \leq Md(x, y) \text{ for all } x,\ y \in X.$$

For $n = 1$, this reduces to the classical Banach contraction fixed point result.

5.2.1. Fixed point theorems

The following fixed point theorems are the tools to be used in our proofs.

Theorem 5.2.5. (Perov [112]) *Let (X, d) be a complete generalized metric space with*

$$d : X \times X \longrightarrow \mathbb{R}^n$$

and let $N : X \longrightarrow X$ satisfy

$$d(N(x), N(y)) \leq Md(x, y)$$

for all $x,\ y \in X$ and some square matrix M of nonnegative numbers. If the matrix M is convergent to zero, then N has a unique fixed point $x_* \in X$ and

$$d(N^k(x_0), x_*) \leq M^k(I - M)^{-1}d(N(x_0), x_0)$$

for every $x_0 \in X$ and $k \geq 1$.

The role of matrices with spectral radius less than one in the study of semilinear operator systems as well as their connection to other abstract principles from nonlinear functional analysis was pointed out in [113].

Theorem 5.2.6. (Leray-Schauder Theorem) *Let $(X, |\cdot|_X)$ be a Banach space, $R > 0$,*

$$\overline{B}_X(0, R) = \{x \in X : |x|_X \leq R\},$$

and

$$T : \overline{B}_X(0, R) \to X$$

be a completely continuous operator. If $|u|_X < R$ for every solution u of the equation $u = \lambda T(u)$ and any $\lambda \in (0, 1)$, then T has at least one fixed point.

5.2.2. Fractional Brownian motion

We first give the definition of a one-dimensional fBm.

Definition 5.2.7. *A one-dimensional fBm, $B_t^H = \{B^H(t) : t \in J\}$, of Hurst index $H \in (0,1)$ is a continuous and centered Gaussian process with covariance function*

$$R^H(t,s) = \mathbb{E}\Big[B^H(t)B^H(s)\Big] = \frac{1}{2}\Big(t^{2H} + s^{2H} - |t-s|^{2H}\Big), \quad t,\ s \in J. \tag{5.2.1}$$

Remark 5.2.1. (i) If $H = \frac{1}{2}$, then $B_t^{\frac{1}{2}}$ is a standard Brownian motion.

(ii) For $\frac{1}{2} < H < 1$, B_t^H can be represented over a finite interval as

$$B^H(t) = \int_0^t K^H(t,s)\, dW(s),$$

where $W = \{W(t) : t \in J\}$ is a Wiener process,

$$K^H(t,s) = c_H\left(H - \frac{1}{2}\right)s^{\frac{1}{2}-H}\int_s^t (u-s)^{H-\frac{3}{2}}u^{H-\frac{1}{2}}\, du,$$

and c_H is a constant depending on H.

Notice that if $H = 1$, the process B_t is a standard Brownian motion, but if $H \neq 1$, then it does not have independent increments. From (5.2.1), it follows that $\mathbb{E}[B_t \cdot B_s]^2 = |t-s|^{2H}$. As a consequence, the process B_t^H has λ-Hölder continuous paths for all $\lambda \in (0, H)$.

For what follows it will be convenient to have the following definition.

Definition 5.2.8. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(E, \mathcal{A})$ be a measurable space and $X : \Omega \to E$ be a random variable. The law of X is the probability measure $\mu_X : \mathcal{A} \to \mathcal{R}_+$ defined by*

$$\mu_X(A) = \mathbb{P}(X^{-1}(A)), \quad A \in \mathcal{A}.$$

From (5.2.1) we see that a standard fBm B_t^H has the following properties :

1. $B^H(0) = 0$ and $\mathbb{E}[B_t^H] = 0$ for all $t \geq 0$.
2. B_t^H has homogeneous increments, i.e., $B^H(t+s) - B^H(s)$ has the same law as $B^H(t)$ for $s, t \geq 0$.
3. B_t^H is a Gaussian process and $\mathbb{E}[(B^H(t))^2] = t^{2H}$, $t \geq 0$, for all $H \in (0,1)$.
4. B_t^H has continuous trajectories.

In the remainder of this paper we will assume that $H \in (\frac{1}{2}, 1)$.

Denote by ε the linear space of step functions on J of the form

$$\phi(t) = \sum_{i=1}^{n-1} a_i I_{(t_i, t_{i+1}]}(t),$$

where $0 = t_1 < t_2 < \cdots < t_n = b$, $n \in N$, $a_i \in R$. We let $\mathcal{H}$ be the closure of ε with respect to the scalar product $\langle I_{[0,t]}, I_{[0,s]}\rangle_{\mathcal{H}} = R^H(t,s)$. The Wiener integral of $\phi \in \varepsilon$ with respect to B^H is given by

$$\int_0^b \phi(s)\, dB^H(s) = \sum_{i=1}^{n-1} a_i\Big(B^H(t_{i+1}) - B^H(t_i)\Big).$$

Moreover, the mapping

$$\phi \to \int_0^b \phi(s)\, dB^H(s)$$

is an isometry between ε and the linear space $\operatorname{span}\{B^H(t) : t \in J\}$ viewed as a subspace of $L^2(\Omega)$. This mapping can be extended to an isometry between $\mathcal{H}$ and the first Wiener chaos of the fBm

$$\overline{\operatorname{span}}^{L^2(\Omega)}\{B^H(t) : t \in J\}.$$

The image of an element $h \in \mathcal{H}$ by this isometry is also called the Wiener integral of h with respect to B_t^H.

For any $\tau \in [0, b]$, consider the linear operator $K_\tau^* : \varepsilon \to L^2[0, b]$ given by

$$\left(K_\tau^* \phi\right)(s) = \int_s^\tau \phi(t) \frac{\partial K^H(t,s)}{\partial t} dt.$$

The operator K_b^* induces an isometry between ε and $L^2[0, b]$ that can be extended to $\mathcal{H}$.

We have the following relation between the Wiener integral with respect to the fBm B_t^H and the Itô integral with respect to the Wiener process :

$$\int_0^b h(s)\, dB^H(s) = \int_0^b \left(K_b^* h\right)(s)\, dW(s), \quad h \in \mathcal{H}, \text{ iff } K_b^* h \in L^2[0, b].$$

For $t \in [0, b]$, $\int_0^t h(s)\, dB^H(s)$ is defined by

$$\int_0^t h(s)\, dB^H(s) := \int_0^t h(s) I_{[0,t]}(s)\, dB^H(s).$$

Moreover, we have

$$\int_0^t h(s)\, dB^H(s) = \int_0^t \left(K_t^* h\right)(s)\, dW(s),\ t \in [0, b],\ hI_{[0,t]} \in \mathcal{H}, \text{ provided } K_t^* h \in L^2[0, b].$$

Define $L^2_{\mathcal{H}}[0, b]$ by

$$L^2_{\mathcal{H}}[0, b] = \left\{h \in \mathcal{H} : K_b^* h \in L^2[0, b]\right\}.$$

For $H > \frac{1}{2}$, we have that (see [10])

$$L^{\frac{1}{H}}[0, b] \subset L^2_{\mathcal{H}}[0, b]. \tag{5.2.2}$$

Next, we define the infinite dimensional fBm and give the definition of the corresponding stochastic integral.

Let $Q \in L(V, W)$ be a non-negative self-adjoint trace class operator defined by $Qe_n = \lambda_n e_n$ with

$$\operatorname{tr} Q = \sum_{n=1}^{\infty} \lambda_n < \infty,$$

where $\lambda_n \geq 0$, $n = 1, 2, \ldots$, are real numbers and $\{e_n\}$, $n = 1, 2, \ldots$ is a complete orthonormal basis for V. Define the V-valued Q-cylindrical fBm on $(\Omega, \mathcal{F}_b, P)$ with covariance operator Q by

$$B^H(t) = \sum_{n=1}^{\infty} Q^{\frac{1}{2}} e_n B_n^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n B_n^H(t),$$

where B_n^H are real, independent, one-dimensional fBm. Define the space $L_Q^0(V,Y)$ by

$$L_Q^0(V,W) = \{\xi : V \to W \,|\, \xi \text{ is a } Q\text{-Hilbert-Schmidt operator}\}.$$

Note that $\xi \in L(V,W)$ is called a Q-Hilbert-Schmidt operator if

$$\|\xi\|^2_{L_Q^0(V,W)} := \sum_{n=1}^{\infty} \| \sqrt{\lambda_n}\xi e_n\|^2 < \infty.$$

The space $L_Q^0(V,W)$ equipped with the inner product

$$\langle \xi, \zeta\rangle_{L_Q^0(V,W)} = \sum_{n=1}^{\infty}\langle \xi e_n, \zeta e_n\rangle$$

is a separable Hilbert space.

Definition 5.2.9 ([?],[?]). *Let $\Lambda : [0,b] \to L_Q^0(V,W)$ satisfy*

$$\sum_{n=1}^{\infty}\left\|K_b^*\left(\Lambda Q^{\frac{1}{2}}\right)e_n\right\|_{L^2([0,b],W)} < \infty. \tag{5.2.3}$$

Then its stochastic integral with respect to the fBm B^H is defined as

$$\begin{aligned}\int_0^t \Lambda(s)\, dB^H(s) &:= \sum_{n=1}^{\infty}\int_0^t \Lambda(s)Q^{\frac{1}{2}}e_n\, dB_n^H(s)\\ &= \sum_{n=1}^{\infty}\int_0^t \left(K_b^*\left(\Lambda Q^{\frac{1}{2}}e_n\right)\right)(s)\, dW(s), \quad t \in [0,b].\end{aligned}$$

Remark 5.2.2. Notice that if

$$\sum_{n=1}^{\infty}\left\|\Lambda Q^{\frac{1}{2}}e_n\right\|_{L^{\frac{1}{H}}([0,b],W)} < \infty, \tag{5.2.4}$$

then (5.2.3) follows immediately from (5.2.2).

Lemma 5.2.10. ([?], [?]) *If*

$$\Lambda : [0,b] \to L_Q^0(V,W)$$

satisfies (5.2.4), *then for any $0 \le s < t \le b$,*

$$\mathbb{E}\left[\left\|\int_s^t \Lambda(\tau)\, dB^H(\tau)\right\|^2_{L_Q^0(V,W)}\right] \le C_H(t-s)^{2H-1}\sum_{n=1}^{\infty}\int_s^t \left\|\Lambda(\tau)Q^{\frac{1}{2}}e_n\right\|^2_{L_Q^0(V,W)} d\tau,$$

where C_H is a constant depending on the Hurst index H. If, in addition, $\sum_{n=1}^{\infty}\left\|\Lambda(t)Q^{\frac{1}{2}}e_n\right\|_{L_Q^0(V,W)}$ is uniformly convergent for $t \in [0,b]$, then

$$\mathbb{E}\left[\left\|\int_s^t \Lambda(\tau)\, dB^H(\tau)\right\|^2_{L_Q^0(V,W)}\right] \le C_H(t-s)^{2H-1}\int_s^t \left\|\Lambda(\tau)\right\|^2_{L_Q^0(V,W)} d\tau.$$

We now give some basic definitions and properties from the fractional calculus. Here, $\Gamma(\cdot)$ is the Gamma function and $[q]$ is the integer part of q

Definition 5.2.11. ([12]) *The fractional integral of the function* $f : [0,\infty) \to \mathbb{R}$ *of order* q *with lower limit* 0 *is defined as*

$$I^{q}_{0^+} f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}}\, ds, \quad t > 0,\ q > 0,$$

provided that the right side is point-wise defined on $[0,\infty)$.

Definition 5.2.12. ([12]) *The Riemann-Liouville's derivative of the function* $f : [0,\infty) \to \mathbb{R}$ *of order* q *with lower limit* 0 *is given by*

$${}^{L}D^{q} f(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{q+1-n}}\, ds, \quad t > 0,\ n = [q]+1.$$

Definition 5.2.13. ([12]) *The Caputo derivative of the function* $f : [0,\infty) \to \mathbb{R}$ *of order* q *with lower limit* 0 *is defined as*

$${}^{c}D^{q} f(t) = {}^{L}D^{q}\left[f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right], \quad t > 0,\ n = [q]+1.$$

Moreover, if $f^{(n)} \in C[0,\infty)$, *then*

$${}^{c}D^{q} f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{q+1-n}}\, ds, \quad n = [q]+1.$$

Next, we define what is meant by a mild solution to system (5.1.1). To do this, we need the following concepts.

Definition 5.2.14. *A filtration* $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ *of a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*is a family of* $\sigma-$*algebras* $\mathcal{F}_t$, *indexed by* $t \in [0,\infty]$ *and all belonging to* $\mathcal{F}$, *satisfying*

1. *if* $s \leq t$ *then* $\mathcal{F}_s \subset \mathcal{F}_t$, *and*
2. $\mathcal{F}_\infty = \sigma(\cup_{t\geq 0}\mathcal{F}_t)$.

Definition 5.2.15. *A stochastic process* $(X_t)_{t\geq 0}$ *defined on* $(\Omega, \mathcal{F}, \mathbb{P})$ *is said to be adapted to the filtration* $\mathcal{F}$ *if for each* $t \geq 0$, *the random variable* X_t *is measurable relative to* $\mathcal{F}_t$.

Here is our definition of a mild solution.

Definition 5.2.16. *A real-valued stochastic process* $u = (x, y) \in C \times C$ *is said to be a solution of* (5.1.1) *with respect to the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *if :*

1) $x(0) = \alpha[x, y]$ *and* $y(0) = \beta[x, y]$;
2) $u(t)$ *is* $\mathcal{F}_t$*-adapted for all* $t \in J = (0, b]$;
3) $u(t)$ *is right continuous and has a limit from the left at all* $t \in J$;

4) $u(t)$ satisfies

$$\begin{cases} x(t) = S_q(t)\alpha[x,y] + \displaystyle\int_0^t (t-s)^{q-1} T_q(t-s) f^1(s, x(s), y(s))\, d(s) \\ \qquad + \displaystyle\int_0^t (t-s)^{q-1} T_q(t-s)\sigma^1(s) dB^{H_1}(s), \quad t \in J, \\ \\ y(t) = S_q(t)\beta[x,y] + \displaystyle\int_0^t (t-s)^{q-1} T_q(t-s) f^2(s, x(s), y(s)) d(s) \\ \qquad + \displaystyle\int_0^t (t - s^{q-1} T_q(t-s))\sigma^2(s) dB^{H_2}(s), \quad t \in J, \end{cases}$$

where

$$S_q(t) = \int_0^\infty \xi_q(\theta) S\left(t^q \theta\right)\, d\theta, \qquad T_q(t) = q\int_0^\infty \theta \xi_q(\theta) S\Big(t^q \theta\Big)\, d\theta,$$

$\{S(t) : t \geq 0\}$ *is our strongly continuous semigroup in* W*,*

$$\xi_q(\theta) = \frac{1}{q}\theta^{-(1+\frac{1}{q})}\overline{\omega}_q\Big(\theta^{-\frac{1}{q}}\Big) \geq 0,$$

$$\overline{\omega}_q(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-nq-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \quad \theta \in (0,\infty),$$

and ξ_q *is a probability density function defined on* $(0,\infty)$ *such that*

$$\xi_q(\theta) \geq 0 \quad \textit{for} \quad \theta \in (0,\infty), \quad \textit{and} \quad \int_0^\infty \xi_q(\theta)\, d\theta = 1.$$

Lemma 5.2.17. ([60]) *The following properties are satisfied :*

(i) $S_q(t)$ *and* $T_q(t)$ *are bounded linear operators for each fixed* $t \geq 0$*. In particular, there is a constant* $M > 0$ *such that*

$$\left\|S_q(t)x\right\| \leq M\|x\| \quad \textit{and} \quad \left\|T_q(t)x\right\| \leq \frac{qM}{\Gamma(q+1)}\|x\|, \quad \textit{for } x \in X;$$

(ii) $\{S_q(t) : t \geq 0\}$ *and* $\{T_q(t) : t \geq 0\}$ *are strongly continuous ;*

(iii) if for every $t > 0$*,* $S(t)$ *is compact, then* $S_q(t)$ *and* $T_q(t)$ *are also compact operators.*

This system can be viewed as a fixed point problem in $C([0,T], X_1) \times C([0,T], X_2)$ for the nonlinear operator

$$T = (T_1, T_2) : C \times C \to C \times C \tag{5.2.5}$$

defined by

$$\begin{cases} T_1(x,y) &= S_q(t)\alpha[x,y] + \displaystyle\int_0^t (t-s)^{q-1} T_q(t-s) f^1(s, x(s), y(s)) ds \\ & \quad + \displaystyle\int_0^t (t-s)^{q-1} T_q(t-s)\sigma^1(s) dB^{H_1}(s), \\ T_2(x,y) &= S_q(t)\beta[x,y] + \displaystyle\int_0^t (t-s)^{q-1} T_q(t-s) f^2(s, x(s), y(s)) ds \\ & \quad + \displaystyle\int_0^t (t-s)^{q-1} T_q(t-s)\sigma^2(s) dB^{H_2}(s). \end{cases}$$

5.3. Existence and Uniqueness Results

We begin by introducing the following conditions that will be use to obtain our first existence result.

(H_1) There exist nonnegative numbers a_i and b_i for $i \in \{1,2\}$ such that

$$\begin{cases} \|f^1(t,x(s),y(s)) - f^1(t,\overline{x}(s),\overline{y}(s))\|^2 \le a_1\mathbb{E}[(|x-\overline{x}|)^2] + b_1\mathbb{E}[(|y-\overline{y}|)^2], \\ \|f^2(t,x(s),y(s)) - f^2(t,\overline{x}(s),\overline{y}(s))\|^2 \le a_2\mathbb{E}[(|x-\overline{x}|)^2] + b_2\mathbb{E}[(|y-\overline{y}|)^2], \end{cases}$$

for all $x, y, \overline{x}, \overline{y} \in \mathbb{R}$ and $t \in J$.

(H_2) There exist positive constants A_i and B_i for $i = 1, 2$ such that

$$\begin{cases} \|\alpha[x,y] - \alpha[\overline{x},\overline{y}]\|^2 \le A_1\|x-\overline{x}\|^2 + B_1\|y-\overline{y}\|^2, \\ \|\beta[x,y] - \beta[\overline{x},\overline{y}]\|^2 \le A_2\|x-\overline{x}\|^2 + B_2\|y-\overline{y}\|^2, \end{cases}$$

for all $x, y, \overline{x}, \overline{y} \in \mathbb{R}$.

Here is our first existence and uniqueness result. Its proof is based on Perov's fixed point theorem.

Theorem 5.3.1. *Assume that conditions* (H_1)–(H_2) *are satisfied and the matrix*

$$\mathcal{M} = M\sqrt{2}\begin{pmatrix} \sqrt{A_1 + \frac{b^{2q-1}}{(\Gamma(q))^2(2q-1)}a_1} & \sqrt{B_1 + \frac{b^{2q-1}}{(\Gamma(q))^2(2q-1)}b_1} \\ \sqrt{A_2 + \frac{b^{2q-1}}{(\Gamma(q))^2(2q-1)}a_2} & \sqrt{B_2 + \frac{b^{2q-1}}{(\Gamma(q))^2(2q-1)}b_2} \end{pmatrix}$$

converges to zero. Then the problem (5.1.1) *has a unique solution.*

Démonstration. We will show that the hypotheses of Perov's fixed point theorem are satisfied. Now

$$\begin{aligned} \Xi_1 &= \mathbb{E}[\|T_1(x(t),y(t)) - T_1(\overline{x}(t),\overline{y}(t))\|^2] \\ &\le 2\mathbb{E}[\left\|S_q(t)\left(\alpha[x,y] - \alpha[\overline{x},\overline{y}]\right)\right\|^2] \\ &\quad + 2\mathbb{E}\left[\left\|\int_0^t (t-s)^{q-1}T_q(t-s)\left(f^1(s,x(s),y(s)) - f^1(s,\overline{x}(s),\overline{y}(s))\right)ds\right\|^2\right], \end{aligned}$$

and using Lemma 5.2.17, Fubini's stochastic theorem, Hölder's inequality, and conditions (H_1)–(H_2), we obtain

$$\begin{aligned} \Xi_1 &\le 2M^2\mathbb{E}[\|\alpha[x,y] - \alpha[\overline{x},\overline{y}]\|^2] \\ &\quad + 2\frac{(qM)^2}{(\Gamma(q+1))^2}\mathbb{E}\left[\int_0^t \left\|f^1(s,x(s),y(s)) - f^1(s,\overline{x}(s),\overline{y}(s))\right\|^2 ds\right] \\ &\le 2M^2\mathbb{E}[\|\alpha[x,y] - \alpha[\overline{x},\overline{y}]\|^2] \\ &\quad + 2\frac{(qM)^2}{(\Gamma(q+1))^2}\mathbb{E}\left[\int_0^t \left\|f^1(s,x(s),y(s)) - f^1(s,\overline{x}(s),\overline{y}(s))\right\|^2 ds\right] \\ &\le 2M^2\left(A_1\|x-\overline{x}\|_C^2 + B_1\|y-\overline{y}\|_C^2\right) + 2\frac{(qM)^2b^{2q-1}}{(\Gamma(q+1))^2(2q-1)}\left(a_1\|x-\overline{x}\|_C^2 + b_1\|y-\overline{y}\|_C^2\right). \end{aligned}$$

Hence,

$$\Xi_1 = \mathbb{E}[\|T_1(x(t), y(t)) - T_1(\overline{x}(t), \overline{y}(t))\|^2] \leq \left(2M^2 A_1 + \frac{2M^2 b^{2q-1}}{(\Gamma(q))^2(2q-1)} a_1\right) \|x - \overline{x}\|_C^2 + \left(2M^2 B_1 + \frac{2M^2 b^{2q-1}}{(\Gamma(q))^2(2q-1)} b_1\right) \|y - \overline{y}\|_C^2.$$

Similarly, we have

$$\Xi_2 = \mathbb{E}[\|T_2(x(t), y(t)) - T_2(\overline{x}(t), \overline{y}(t))\|^2] \leq \left(2M^2 A_2 + \frac{2M^2 b^{2q-1}}{(\Gamma(q))^2(2q-1)} a_2\right) \|x - \overline{x}\|_C^2 + \left(2M^2 B_2 + \frac{2M^2 b^{2q-1}}{(\Gamma(q))^2(2q-1)} b_2\right) \|y - \overline{y}\|_C^2.$$

Therefore,

$$\begin{aligned} \|T(x,y) - T(\overline{x},\overline{y})\|_C &= \begin{pmatrix} \|T_1((x,y) - T_1(\overline{x},\overline{y})\|_C \\ \|T_2(x,y) - T_2(\overline{x},\overline{y})\|_C \end{pmatrix} \\ &\leq 2M^2 \begin{pmatrix} A_1 + \frac{b^{2q-1}}{(\Gamma(q))^2(2q-1)} a_1 & B_1 + \frac{b^{2q-1}}{(\Gamma(q))^2(2q-1)} b_1 \\ A_2 + \frac{b^{2q-1}}{(\Gamma(q))^2(2q-1)} a_2 & B_2 + \frac{b^{2q-1}}{(\Gamma(q))^2(2q-1)} b_2 \end{pmatrix} \begin{pmatrix} \|x - \overline{x}\|_C \\ \|y - \overline{y}\|_C \end{pmatrix}, \end{aligned} \tag{5.3.1}$$

so

$$\|T(x,y) - T(\overline{x},\overline{y})\|_C \leq \mathcal{M} \begin{pmatrix} \|x - \overline{x}\|_C \\ \|y - \overline{y}\|_C \end{pmatrix}.$$

From Perov's fixed point theorem (Theorem 5.2.5 above) the mapping T has a unique fixed point $(x, y) \in C \times C$ that is a unique solution of problem (5.1.1). □

We will now give an existence result based on the nonlinear alternative of Leray-Schauder type. We need the following conditions to obtain our result.

(H_3) The functions f^1 and f^2 are L^1–Carathédory functions.

(H_4) There are constants $\tilde{A}_1, \tilde{B}_1, K_1, \tilde{A}_2, \tilde{B}_2$, and K_2 such that

$$\begin{cases} \|\alpha[x,y]\|^2 \leq \tilde{A}_1 \|x\|_C^2 + \tilde{B}_1 \|y\|_C^2 + K_1, \\ \|\beta[x,y]\|^2 \leq \tilde{A}_2 \|x\|_C^2 + \tilde{B}_2 \|y\|_C^2 + K_2, \end{cases}$$

for all $x, y \in C[0, b]$.

(H_5) There exist functions $p, q, h, \tilde{p}, \tilde{q}, \bar{h} \in L^1([0, b], \mathbb{R}^+)$ such that

$$\left\|f^1(t,x,y)\right\|^2 \leq p(t)\,\|x\| + q(t)\left\|y\right\|^2 + h(t)$$

and

$$\left\|f^2(t,x,y)\right\|^2 \leq \tilde{p}(t)\,\|x\|^2 + \tilde{q}(t)\left\|y\right\|^2 + \tilde{h}(t)$$

for each $t \in J$ and $x, y \in C[0, b]$.

(H_6) The functions $\sigma^i : J \to L_Q^0(V, Y)$, $i = 1, 2$, are measurable and there exist constants $\xi_i > 0$ for $i = 1, 2$ such that :

(i) $\sup\limits_{0 \leq s \leq b} \|\sigma^i(s)\|^2_{L_Q^0(V,W)} \leq \xi_i$;

(ii) $\sum_{n=1}^{\infty} \|\sigma^i Q^{\frac{1}{2}} \varepsilon_n\|_{L_Q^0(V,W)} < \infty$;

(iii) $\sum_{n=1}^{\infty} \|\sigma^i Q^{\frac{1}{2}} \varepsilon_n\|_W$ is uniformly convergent for $t \in [0, b]$.

Theorem 5.3.2. *Assume that* (H_3)–(H_6) *hold and*

$$M_* = \max(3M^2(\widetilde{A}_i + \widetilde{B}_i)) < 1, i = 1, 2. \tag{5.3.2}$$

Then problem (5.1.1) *has at least one solution on* J.

Démonstration. It is easy to see that the fixed points of the operator T given in (5.2.5) are solutions to (5.1.1). In order to apply Theorem 5.2.6, we first show that T is completely continuous. The proof will be given in several steps.

Step 1. *$T = (T_1, T_2)$ is continuous.* Let (x_n, y_n) be a sequence such that $(x_n, y_n) \to (\widetilde{x}, \widetilde{y}) \in C \times C$ as $n \to \infty$. Then,

$$\begin{aligned}
&\mathbb{E}[\|T_1(x_n(t), y_n(t)) - T_1(\widetilde{x}, \widetilde{y})\|^2] \\
&\quad \leq \mathbb{E}\Bigg[\Bigg\| S_q(t)\alpha[x_n, y_n] + \int_0^t (t-s)^{q-1} T_q(t-s) f^1(s, x_n(s), y_n(s)) d(s) \\
&\quad\quad + \int_0^t S_q(t)(t-s)^{q-1} T_q(t-s)\sigma^1(s) dB^H(s) - S_q(t)\alpha_1\left[\widetilde{x}, \widetilde{y}\right] \\
&\quad\quad - \int_0^t (t-s)^{q-1} T_q(t-s) f^1(s, \widetilde{x}(s), \widetilde{y}(s)) d(s) \\
&\quad\quad - \int_0^t S_q(t)(t-s)^{q-1} T_q(t-s)\sigma^1(s) dB^H(s) \Bigg\|^2\Bigg].
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathbb{E}[\|T_1(x_n(t), y_n(t)) - T_1(\widetilde{x}(t), \widetilde{y}(t))\|^2] \\
&\quad \leq 2\mathbb{E}\left[\left\| S_q(t)\left(\alpha[x_n, y_n] - \alpha[\widetilde{x} - \widetilde{y}]\right)\right\|^2\right] \\
&\quad\quad + 2\mathbb{E}\left[\left\| \int_0^t (t-s)^{q-1} T_q(t-s)\left(f^1(s, x_n(s), y_n(s)) - f^1(s, \widetilde{x}(s), \widetilde{y}(s))\right) ds \right\|^2\right] \\
&\quad =: I_1 + I_2.
\end{aligned}$$

Applying condition (H_2),

$$I_1 \leq \mathbb{E}[A_1\|x_n - \widetilde{x}\|^2 + B_1\|y_n - \widetilde{y}\|^2], \tag{5.3.3}$$

and by the Lebesgue dominated convergence theorem, $I_1 \to 0$ as $n \to \infty$ since $(x_n, y_n) \to (\widetilde{x}, \widetilde{y})$. Now using Lemma 5.2.17 and Hölder's inequality,

$$I_2 \leq \frac{2M^2 b^{2q-1}}{\Gamma(q)^2(2q-1)} \mathbb{E}\left[\int_0^t \left\| f^1(s, x_n(s), y_n(s)) - f^1(s, \widetilde{x}, \widetilde{y}) \right\| ds\right] \tag{5.3.4}$$

Since f^1 is an L_1-Carathédory function, by the Lebesgue dominated convergence theorem, $I_2 \to 0$ again since $(x_n, y_n) \to (\widetilde{x}, \widetilde{y})$. Similarly,

$$\begin{aligned}
\mathbb{E}[||T_2(x_n(t),y_n(t)) - T_2(\widetilde{x}(t), \widetilde{y}(t))||^2] &\\
\leq 2\mathbb{E}\left[\left\|S_q(t)\beta[x_n, y_n] - \beta[\widetilde{x}, \widetilde{y}]\right\|^2\right] &\\
+ 2\mathbb{E}\left[\left\|\int_0^t (t-s)^{q-1}T_q(t-s)\left(f^1(s, x_n(s), y_n(s)) - f^1(s, \widetilde{x}(s), \widetilde{y}(s))\right)\right\|^2\right] ds &\\
=: \widetilde{I_1} + \widetilde{I_2} &
\end{aligned}$$

and again $\widetilde{I_i} \to 0$ for $i = 1, 2$. Therefore, T is continuous.

Step 2. *T maps bounded sets into bounded sets in $C \times C$.* It suffices to show that for any $K > 0$, there exists a positive constant $l = (l_1, l_2)$ such that, for

$$(x, y) \in B_K = \{(x, y) \in C \times C : ||x||_C \leq K,\ ||y||_C \leq K\},$$

we have

$$||T(x, y)||_C \leq l. \tag{5.3.5}$$

Now for each $t \in J$,

$$\begin{aligned}
\mathbb{E}[\left\|T_1(x(t), y(t))\right\|^2] &= \mathbb{E}\left[\left\|S_q(t)\alpha[x, y] + \int_0^t (t-s)^{q-1}T_q(t-s)f^1(s, x(s), y(s))d(s)\right.\right.\\
&\quad \left.\left. + \int_0^t (t-s)^{q-1}T_q(t-s)\sigma^1(s)dB^H(s)\right\|^2\right]\\
&\leq 3\mathbb{E}[\left\|S_q(t)\alpha[x, y]\right\|^2]\\
&\quad + 3\mathbb{E}\left[\left\|\int_0^t (t-s)^{q-1}T_q(t-s)f^1(s, x(s), y(s))d(s)\right\|^2\right]\\
&\quad + 3\mathbb{E}\left[\left\|\int_0^t (t-s)^{q-1}T_q(t-s)\sigma^1(s)dB^H(s)\right\|^2\right]\\
&=: j_1 + j_2 + j_3,
\end{aligned}$$

where

$$\begin{aligned}
j_1 &= 3\mathbb{E}\left[||S_q(t)\alpha[x, y]||^2\right]\\
&\leq 3M^2(\widetilde{A}_1||x||_C^2 + \widetilde{B}_1||y||_C^2 + K_1) = \widetilde{l}_{11}
\end{aligned}$$

by (H_4). Using Hölder's inequality, Lemma 5.2.17, and condition (H_5), we see that

$$\begin{aligned}
j_2 &= 3\mathbb{E}\left[\left\|\int_0^t (t-s)^{q-1}T_q(t-s)f^1(s,x(s),y(s))d(s)\right\|^2\right] \\
&\leq \frac{3M^2}{(\Gamma(q))^2}\mathbb{E}\left[\left\|\int_0^t (t-s)^{q-1}f^1(s,x(s),y(s))ds\right\|^2\right] \\
&\leq \frac{3M^2}{(\Gamma(q))^2}\left(\int_0^t (t-s)^{q-1}ds\right)^2 \mathbb{E}\left[\int_0^t \|f^1(s,x(s),y(s))ds\|^2\right] \\
&\leq \frac{3M^2 b^{2q-1}}{(\Gamma(q))^2(2q-1)}\mathbb{E}\left[\int_0^t \left(p(s)\|x\|_C + q(s)\|y\|_C + h(s)\right)ds\right] \\
&\leq \frac{3M^2 b^{2q-1}\left(K\|p\|_{L^1} + K\|q\|_{L^1} + \|h\|_{L^1}\right)}{(\Gamma(q))^2(2q-1)} = \widetilde{l_{21}},
\end{aligned}$$

and for j_3, we use Lemma 5.2.17 and condition (H_6) to obtain

$$\begin{aligned}
j_3 &= 3\mathbb{E}\left[\left\|\int_0^t (t-s)^{q-1}T_q(t-s)\sigma^1(s)dB^H(s)\right\|^2\right] \\
&\leq 3C_H t^{2H-1}\int_0^t \left\|(t-s)^{q-1}T_q(t-s)\sigma(s)\right\|^2_{L_Q^0(V,W)} ds \\
&\leq \frac{3C_H M^2 \xi_2 b^{2H+2q+2}}{(\Gamma(q))^2(2q-1)} = \widetilde{l_{31}}.
\end{aligned}$$

Therefore,

$$\|T_1\|_C^2 = \mathbb{E}[\|T_1(x(t),y(t))\|^2] \leq \widetilde{l_{11}} + \widetilde{l_{21}} + \widetilde{l_{31}} = l_1.$$

Similarly, we have

$$\|T_2\|_C^2 = \mathbb{E}[\|T_2(x(t),y(t))\|^2] \leq \widetilde{l_{12}} + \widetilde{l_{22}} + \widetilde{l_{32}} = l_2,$$

where

$$\widetilde{l_{12}} = 3M^2(\widetilde{A_2}\|x\|_C^2 + \widetilde{B_2}\|y\|_C^2 + K_2),$$

$$\widetilde{l_{22}} = \frac{3M^2 b^{2q-1}\left(K\|\widetilde{p}\|_{L^1} + K\|\widetilde{q}\|_{L^1} + \|\widetilde{h}\|_{L^1}\right)}{(\Gamma(q))^2(2q-1)},$$

and

$$\widetilde{l_{32}} = \frac{3C_H M^2 c_2 b^{2H+2q+2}}{(\Gamma(q))^2(2q-1)}.$$

Hence, (5.3.5) holds.

Step 3. *T maps bounded sets into equicontinuous sets of* $C\times C$. Let B_K be a bounded set in $C\times C$

as in Step 2. Let $r_1, r_2 \in J$ with $r_1 < r_2$ and $u = (x, y) \in B_K$. Then,

$$\begin{aligned}
\Theta_1 &:= \mathbb{E}\left[\|T_1(x(r_2), y(r_2)) - T_1(x(r_1), y(r_1))\|^2\right] \\
&\leq 3\mathbb{E}\left[\left\|(S_q(r_2) - S_q(r_1))\alpha[x, y]\right\|^2\right] \\
&\quad + 3\mathbb{E}\left[\left\|\int_0^{r_2}(r_2 - s)^{q-1}T_q(r_2 - s)f^1(s, x(s), y(s))ds\right.\right. \\
&\quad - \left.\left.\int_0^{r_1}(r_1 - s)^{q-1}T_q(r_1 - s)(f^1(s, x(s), y(s))d(s)\right\|^2\right] \\
&\quad + 3\mathbb{E}\left[\left\|\int_0^{r_2}(r_2 - s)^{q-1}T_q(r_2 - s)\sigma^1(s)dB^H(s)\right.\right. \\
&\quad - \left.\left.\int_0^{r_1}(r_1 - s)^{q-1}T_q(r_1 - s)\sigma^1(s)dB^H(s)\right\|^2\right] \\
&=: k_1 + k_2 + k_3.
\end{aligned}$$

By the strong continuity of $S_q(t)$, we see that $\lim_{r_2 \to r_1}\left(S_q(r_2) - S_q(r_1)\right)(\alpha[x, y])$ =0. Also, using Lemma 5.2.17 and condition (H_4), we have

$$\mathbb{E}[\|(S_q(r_2) - S_q(r_1))\alpha[x, y]\|^2] \leq (S_q(r_2) - S_q(r_1))\left(\widetilde{A}_1\|x\|_C^2 + \widetilde{B}_1\|y\|_C^2 + K_1\right) \tag{5.3.6}$$

By the Lebesgue dominated convergence theorem,

$$\lim_{r_2 \to r_1} k_1 = 3\mathbb{E}[\left\|(S_q(r_2) - S_q(r_1))\alpha[x, y]\right\|^2] = 0.$$

Now

$$\begin{aligned}
k_2 &\leq 6\mathbb{E}\left[\left\|\int_0^{r_1}(r_2 - s)^{q-1}T_q(r_2 - s)f^1(s, x(s), y(s))ds\right.\right. \\
&\quad + \int_{r_1}^{r_2}(r_2 - s)^{q-1}T_q(r_2 - s)f^1(s, x(s), y(s))ds \\
&\quad - \left.\left.\int_0^{r_1}(r_1 - s)^{q-1}T_q(r_1 - s)f^1(s, x(s), y(s))ds\right\|^2\right] \\
&\leq 6\mathbb{E}\left[\left\|\int_0^{r_1}[(r_2 - s)^{q-1}T_q(r_2 - s) - (r_1 - s)^{q-1}T_q(r_1 - s)]f^1(s, x(s), y(s))ds\right\|^2\right] \\
&\quad + \mathbb{E}\left[\left\|\int_{r_1}^{r_2}(r_2 - s)^{q-1}T_q(r_2 - s)f^1(s, x(s), y(s))dB^H(s)\right\|^2\right] \\
&\leq k_{21} + k_{22}.
\end{aligned}$$

Using Lemma 5.2.17, Fubini's stochastic theorem, Hölder's inequality, and condition (H_5),

$$\begin{aligned}
k_{21} &\leq 6\mathbb{E}\left[\left(\int_0^{r_1}[(r_2-s)T_q(r_2-s)-(r_1-s)^{q-1}T_q(r_1-s)]ds\right)^2\right]\mathbb{E}\left[\int_0^{r_1}\|f^1(s,x(s),y(s))\|^2ds\right]\\
&\leq 6\left(\int_0^{r_1}[(r_2-s)^{q-1}T_q(r_2-s)-(r_1-s)^{q-1}T_q(r_1-s)]ds\right)^2\\
&\quad\times\mathbb{E}\left[\int_0^{r_1}(p(s)\|x\|_C+q(s)\|y\|_C+h(s))^2ds\right]\\
&\leq \frac{6M^2[K\|p\|_{L^1}+K\|q\|_{L^1}+\|h(s)\|_{L^1}]}{\Gamma^2(q)(q-2)^2}[(r_2-r_1)^{q-1}+r_2^{q-2}-r_1^{q-2}].
\end{aligned}$$

Clearly, the right hand side of the above inequality tends to zero as $r_2 \to r_1$. Hence, $\lim_{r_2\to r_1} k_{21} = 0$. Similarly,

$$\begin{aligned}
k_{22} &\leq 6\mathbb{E}\left[\left\|\int_{r_1}^{r_2}(r_2-s)^{q-1}T_q(r_2-s)f^1(s,x(s),y(s))ds\right\|^2\right]\\
&\leq 6\left(\int_{r_1}^{r_2}(r_2-s)^{q-1}T_q(r_2-s)ds\right)^2\mathbb{E}\left[\int_{r_1}^{r_2}\|f^1(s,x(s,y(s)))\|^2ds\right]
\end{aligned}$$

By Lemma 5.2.17 and condition (H_5),

$$k_{22} \leq \frac{6M^2b^{2q-1}\xi_2[K\|p\|_{L^1}+K\|q\|_{L^1}+\|h\|_{L^1}]}{\Gamma^2(q)(2q-1)}[(r_2-r_1)^{2q-1}],$$

so $\lim_{r_2\to r_1} k_{22} = 0$.

Finally,

$$\begin{aligned}
k_3 \leq 3\mathbb{E}\Bigg[\Bigg\|&\int_0^{r_1}(r_2-s)^{q-1}T_q(r_2-s)\sigma^1(s)dB^H(s)\\
&-\int_0^{r_1}(r_1-s)T_q(r_2-s)\sigma^1(s)dB^H(s)\\
&+\int_{r_1}^{r_2}(r_2-s)^{q-1}T_q(r_2-s)\sigma^1(s)dB^H(s)\Bigg\|^2\Bigg],\\
=:&\ k_{31}+k_{32}
\end{aligned}$$

and

$$\begin{aligned}
k_{31} &= 6\mathbb{E}\left[\left\|\int_0^{r_1}\left((r_2-s)^{q-1}T_q(r_2-s)-(r_1-s)^{q-1}T_q(r_1-s)\right)\sigma^1(s)dB^H(s)\right\|^2\right]\\
&\quad+6\mathbb{E}\left[\left\|\int_{r_1}^{r_2}(r_2-s)^{q-1}T_q(r_2-s)\sigma^1(s)dB^H(s)\right\|^2\right].
\end{aligned}$$

Using (H_5) and Lemma 5.2.17,

$$\begin{aligned}
k_{31} &\leq 6C_Ht^{2H-1}\int_0^{r_1}\|\left[(r_2-s)^{q-1}-(r_1-s)^{q-1}\right]\sigma^1(s)\|^2_{L_Q^0(V,W)}ds\\
&\leq \frac{6C_Ht^{2H-1}\xi_2M^2}{(\Gamma(q))^2(2q-1)}\left[r_1^{2q-1}+(r_2-r_1)^{2q-1}-r_2^{2q-1}\right].
\end{aligned}$$

Hence, $\lim_{r_2\to r_1} k_{31} = 0$. Similarly,

$$k_{32} \le 6\mathbb{E}\left[\left\|\int_{t_1}^{t_2}(r_2-s)^{q-1} - T_q(r_2-s)\sigma^1(s)dB^H(s)\right\|^2\right]$$
$$\le \frac{6C_H t^{2H-1}\zeta_2 M^2(r_2-r1)^{2q-1}}{\Gamma^2(q)(2q-1)} \to 0$$

as $r_2 \to r_1$, so $\lim_{r_2\to r_1} k_{32} = 0$. Thus,

$$\lim_{r_2\to r_1} \mathbb{E}\left[\|T(x(r_2), y(r_2)) - T(x(r_1, y(r_1))\|^2\right] = 0.$$

Therefore, the function $t \to T(x(t), y(t))$ is continuous on $[0, b]$, so by the Arzelà-Ascoli theorem, $T : B_K \to C \times C$ is completely continuous.

Step 4. *Solutions are a priori bounded.* For $t \in J$, we have

$$\begin{aligned}
\mathbb{E}[\|x(t)\|^2] &\le 3\mathbb{E}[\|S_q(t)\alpha[x,y]\|^2] + \mathbb{E}\left[\left\|\int_0^t (t-s)^{q-1}T_q(t-s)f^1(s,x(s),y(s))ds\right\|^2\right] \\
&+ 3\mathbb{E}\left[\left\|\int_0^t (t-s)^{q-1}T_q(t-s)\sigma^1(s)dB^H(s)\right\|^2\right] \\
&\le 3M^2[\widetilde{A}_1\mathbb{E}[\|x(t)\|^2] + \widetilde{B}_1\mathbb{E}[\|y(t)\|^2] + K_1] \\
&+ \frac{3M^2t^{2q-1}}{\Gamma^2(q)(2q-1)}\int_0^t (p(s)\mathbb{E}[\|x(s)\|^2] + q(s)\mathbb{E}[\|y(s)\|^2] + \|h(s)\|)ds \\
&+ \frac{3M^2C_H t^{2H+2b-1}\xi_1}{\Gamma^2(q)(2q-1)}.
\end{aligned}$$

by (H_4), (H_5), and (H_6)(i).

Similarly,

$$\begin{aligned}
\mathbb{E}[\|y(t)\|^2] &\le 3M^2[\widetilde{A}_2\mathbb{E}[\|x(t)\|^2] + \widetilde{B}_2\mathbb{E}[\|y(t)\|^2] + K_1] \\
&+ \frac{3M^2t^{2q-1}}{\Gamma^2(q)(2q-1)}\int_0^t (p(s)\mathbb{E}[\|x(s)\|^2] + q(s)\mathbb{E}[\|y(s)\|^2] + \|\widetilde{h}(s)\|)ds \\
&+ \frac{3M^2C_H t^{2H+2b-1}\xi_2}{\Gamma^2(q)(2q-1)}.
\end{aligned}$$

Therefore,

$$\mathbb{E}[\|x(t)\|^2] + \mathbb{E}[\|y(t)\|^2] \le \widetilde{M} + \int_0^t \gamma(t)(\mathbb{E}[\|x(s)\|^2] + \mathbb{E}[\|y(s)\|^2])ds,$$

where

$$\widetilde{M} = \frac{3M^2}{1-M_*}\left(K_1 + K_2 + \frac{C_H b^{2H+2b-1}(\xi_1+\xi_2)}{\Gamma^2(q)(2q-1)} + \frac{b^{2q-1}(\|h\|_{L^1} + \|\widetilde{h}\|_{L^1})}{\Gamma^2(q)(2q-1)}\right)$$

and

$$\gamma(t) = \frac{3M^2b^{2q-1}}{\Gamma^2(q)(2q-1)(1-M_*)}\left(p(t) + q(t) + \widetilde{p}(t) + \widetilde{q}(t)\right).$$

Hence, by Gronwall's inequality, there exists $M > 0$ such that

$$\mathbb{E}[\|x(t)\|^2] + \mathbb{E}[\|y(t)\|^2] \leq M, \quad \text{for all } t \in [0, b].$$

Set

$$U := \{(x, y) \in C \times C : \|x\|_C < \sqrt{M} + 1, \|y\|_C < \sqrt{M} + 1\},$$

and consider the operator $T : \overline{U} \to C \times C$. From the choice of U, there is no $(x, y) \in \partial U$ such that $(x, y) = \lambda T(x, y)$ for some $\lambda \in (0, 1)$. As a consequence of Theorem 5.2.6, T has a fixed point (x, y) in U that in turn is a solution of the problem (5.1.1). $\square$

We conclude this paper with an example of our results.

Example 5.3.3. Consider the partial neutral stochastic functional differential system

$$\begin{cases} du(t,\xi) + \frac{\partial^2}{\partial \xi^2} u(t,\xi) = F(t, u(t,\xi), v(t,\xi)) + \sigma(t)\frac{dB^H}{dt}, & t \in [0, b],\ 0 \leq \xi \leq \pi, \\ dv(t,\xi) + \frac{\partial^2}{\partial \xi^2} v(t,\xi) = G(t, u(t,\xi), v(t,\xi)) + \sigma(t)\frac{dB^H}{dt}, & t \in [0, b],\ 0 \leq \xi \leq \pi, \\ u(t, 0) = u(t, \pi) = 0, & t \in [0, b], \\ v(t, 0) = v(t, \pi) = 0, & t \in [0, b], \\ u(0, \xi) = \int_0^\pi k(\xi, y) x(t, y) dy, & 0 \leq \xi \leq \pi, \\ v(0, \xi) = \int_0 v^\pi k(\xi, x) y(t, x) dx, & 0 \leq \xi \leq \pi, \end{cases} \tag{5.3.7}$$

where B^H denotes a fractional Brownian motion, and G, $F : [0, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $k : [0, b] \times [0, \pi] \to \mathbb{R}$ are continuous functions. Let

$$\begin{aligned} x(t)(\xi) = u(t, \xi),\ y(t)(\xi) = v(t, \xi), &\quad \xi \in [0, \pi], \\ f(t, x(t), y(t))(\xi) = F(t, u(t, \xi), v(t, \xi)), &\quad \xi \in [0, \pi], \\ g(t, x(t), y(t))(\xi) = G(t, u(t, \xi), v(t, \xi)), &\quad \xi \in [0, \pi] \end{aligned}$$

for all $t \in J$. Take $\mathcal{K} = \mathcal{H} = L^2([0, \pi])$ and define the operator A by $Au = u''$ with domain

$$D(A) = \{u \in \mathcal{H} : u', u'' \in \mathcal{H} \text{ and } u(0) = u(\pi) = 0\}.$$

Then, it is well known that

$$Az = -\sum_{n=1}^{\infty} e^{-n^2 t} \langle z, e_n \rangle e_n, \quad z \in \mathcal{H},$$

and A is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t \geq 0}$ on $\mathcal{H}$ given by

$$S(t)u = \sum_{n=1}^{\infty} e^{-n^2 t} \langle u, e_n \rangle e_n,\ u \in \mathcal{H},$$

and $e_n(u) = (2/\pi)^{1/2} \sin(nu)$, $n = 1, 2, \cdots$, is the orthogonal set of eigenvectors of A. The analytic semigroup $\{S(t)\}_{t>0}$, $t \in J$, is compact, and there exists a constant $M \geq 1$ such that $\|S(t)\|^2 \leq M$. Thus, problem (5.3.7) can be written in the abstract form

$$\begin{cases} dx(t) = [A_1 x(t) + f(t, x, y)]dt + \sigma^1(t) dB^H(t), & t \in J := [0, T], \\ dy(t) = [A_2 y(t) + g(t, x, y)]dt + \sigma^2(t) dB^H(t), & t \in J := [0, T], \\ x(0) = \alpha[x, y], \\ y(0) = \beta[x, y]. \end{cases}$$

We now take

$$f(t,u,v) = \frac{tu}{1+u^2+v^2} \text{ and } g(t,u,v) = \frac{tv}{1+u^2+v^2},$$

which are clearly are continuous functions and note that

$$|f(t,u,v)|^2 \leq b|u|^2 \text{ and } |g(t,u,v)|^2 \leq b|v|^2.$$

Hence, conditions (H_3), (H_5), and (H_6) hold. If we assume that there exist

$$\alpha(u,v) = \int_0^{\pi} k(\xi,y)u(t,y)dy \text{ and } \beta(u,v) = \int_0^{\pi} k(\xi,x)v(t,x)dx$$

satisfying condition (H_4), then we can apply Theorem 5.3.2 to see that the problem (5.3.7) has a unique solution on $[0,b] \times [0,\pi]$.

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