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Equations et inclusions différentielles d'évolution à retard dépendant de l'état.

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Dedication

This thesis is specially dedicated to

My parents, whose sincerely raised me with their caring and offered me unconditional love, my special thanks go to them for all their support that makes me persevere actively throughout my life, to achieve my desired goals within my means.

My brothers, who have supported me all the time since the beginning of my studies.

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Publications

1. **Z. Hamlat**, **J.R. Graef** and **A. Ouahab**, Generalizations of the Nonlinear Henry Inequality and the Leary-schauder Fixed Point Theorem with Applications to Fractional Differential Inclusions, *Fractal Fract*, **2021**, 5, 173.

2. **Z. Hamlat** and **A. Ouahab**, Existence and properties of solutions for fractional neutral lattice system with delay, in preparation.

3. **Z. Hamlat** and **A. Ouahab**, Neutral Differential Inclusions with State-Dependent Delays, in preparation.

4. **Z. Hamlat** and **A. Ouahab**, a functional differential inclusions system of type neutral with state-dependent delay, in preparation.

5. **Z. Hamlat** and A. Ouahab, Existence of set solutions for functional differential inclusions system with state-dependent delays, in preparation.

Abstract

In this thesis, we aim to contribute to the study of various classes of differential inclusions and equations of integer or fractional order derivatives, with delay in a Banach space. We also generalize some fractional Gronwall-Bihari-Henry inequalities and we establish a multi-valued version of the Leray-Schauder alternative in strictly star-shaped sets. To obtain the existence of the solutions for the above problems, we propose sufficient and suitable conditions on the second member which are compatible with the means used in this study. Some topological and geometrical qualitative properties of the solution set have been also studies, including topological and geometric structures. The methods are based on fixed point theory combined with C_0-semigroup theory and the measure of noncompactness.

Key words and phrases:

Fractional differential inclusions, non-linear alternative, fractional inequalities, semi-group theory, qualitative properties topological and geometric, measure of non-compactness.

Résumé

Le but de cette thèse est de présenter quelques résultats d'existence des solutions de certaines classes d'équations et inclusions différentielles d'ordre entier ou fractionnaire avec retard dans un espace de Banach. Nous généralisons certaines inégalités fractionnaires de type Gronwall-Bihari-Henry, et nous établissons aussi une nouvelle version multi-valeur de l'alternative de Leray-Schauder dans des ensembles strictement étoilés. Pour obtenir l'existence des solutions aux problèmes présidents, on propose des conditions suffisantes et appropriées sur le second membre qui sont compatibles avec les moyens utilisés dans cette étude. Quelques propriétés qualitatives concernant la structure topologique et géométrique de l'ensemble des solutions a été présenté aussi. Les méthodes sont basé sur les théorèmes des points fixes combine avec la théorie de C_0-semi-group et la mesure de non-compacité.

Mots et Phrases Clefs :

Inclusions différentielles fractionnaires, alternative non linéaire, inégalité fractionnaire, semi-groupes, mesure de non-compacités, structure topologique et géométrique.

ملخص

نستهدف في هذه الأطروحة المساهمة في البحث عن وجود حلول لبعض مختلف الانتماءات و المعادلات التفاضلية ذات الدرجة الطبيعية أو الكسرية مع تأخر محدود مرتبط بالحل في فضاء عام لـ بناخ كما أننا تمكنا من تعميم بعض المتباينات الكسرية لـ ''غرونوال–بهري– هنري'' وأنجزنا صيغة جديدة لنظرية التبادل الغير خطي لـ ''لوراي–شودار'' في المجموعات ذات الشكل ألنجمي تماما. الخواص النوعية المتعلقة بالبنى الطبولوجية و الهندسية لمجموعات الحلول درست أيضا.

لدراسة وجود الحلول والصيغة الطبولوجية و الهندسية لمجموعة الحلول للمسائل المقترحة في هذه الأطروحة اعتمدنا على نظرية النقط الثابتة مشتركة مع نظريات الشبه زمر و مقياس عدم الاكتناز.

الكلمات المفتاحية

الانتماءات التفاضلية الكسرية، لبنى الطبولوجية و الهندسية، الخواص النوعية،المتباينات الكسرية، نظرية التبادل الغير خطي، الشبه زمر، مقياس عدم الاكتناز.

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Introduction

Differential equations and inclusions are one of the oldest and most important disciplines of mathematics. The study of differential equations and inclusions has become a very important area of research due to its applicability to many problems such as optimal control theory, as well as the modeling of many scientific phenomena: in physics, chemistry, medicine, etc. This extensive theory has been developed for several different types of differential equations and inclusions, such that the most important work of mathematicians since the discovery of equations and inclusions differential is the study of existence of solutions, in addition to their nature such as: uniqueness, signal, strength and weakness, they also became increasingly interested in the topological and geometrical structures of solutions set as: the compactness, the closure, the bounded, etc, in the topological sense and contractibility, $R_\delta - set$, acyclicity, etc, in the geometrical sense. All this is based on regularity conditions concerning the right-hand side. We refer the reader to [10, 24, 38, 60, 93].

It is well-known that inequalities, such as the Gronwall-Bellman-Bihari-Henry inequality, play an important role in the study of existence, uniqueness, boundedness, stability, and other qualitative properties of solutions of differential equations, integral equations, and differential inclusions (see, for example,[14, 80]). For nonlinear integral inequalities, perhaps one of the most important contributions was made by Bihari [20] for

$$v(t) \leq k + \int_0^t f(s)\psi(v(s))ds,\ k \in \mathbb{R}^+.$$

This integral inequality was generalized by many authors. For example, Agrawal *et al.* [5] replaced k, t, f, and ψ with the functions $a(t), (b_i(t))_{i=1}^p, (f_i)_{i=1}^p$, and $(\psi_i)_{i=1}^p$, respectively, and investigated the retarded Gronwall type inequality

$$v(t) \leq a(t) + \sum_{i=1}^{p} \int_{b_i(0)}^{b_i(t)} f_i(s)\psi_i(v(s))ds,\ b_i(t) \leq t,\ 0 \leq t \leq T.$$

In recent years, ordinary and partial differential equations and inclusions of fractional order have been investigated more in the literature due to their applicability to many problems in engineering and other scientific disciplines; see [15, 71] and the references therein for recent work. The question of the existence of solutions and other mathematical aspects of fractional differential equations and inclusions have been extensively studied and have attracted much attention; many important contributions

have been obtained so far (see the monographs [58, 79, 82] as well as papers listed in the references below). For recent results on the existence of solutions to fractional differential equations and inclusions with various types of delays we refer the reader to [2, 3, 18, 19, 50, 61, 81, 86, 92].

The concept of fractional substantial derivative was introduced in 2006 by Friedrich et al. [39] to consider retardation effects in Kramers-Fokker-Planck type equations, where these are expressed mathematically by a fractional substantial derivative, which is more suitable for characterize the global coupling in time and space then the classical fractional derivative. The fractional substantial derivative is widely applied in physics, [31, 91]. Moreover, the fractional substantial diffusion model is a tool in geophysics and financial applications [29, 67]. A main objective is to study the existence of solutions of fractional differential equations with the Caputo fractional substantial derivative,such as fractional lattice systems with delay. For more information about the application of fractional substantial derivative see [27, 28, 58].

Those who are interested in research in this area with all types of differential equations and inclusions which mentioned above, for example: Zarempa in (1936) [89], Marchaud (1938) [66] in a form different of the one presented here: at the left-hand side multi-valued derivative appears [90], it was Wazewski (1961) [87] who showed that these inclusion could be equivalently presented as inclusions with a regular single valued left derivative. In 1971 an article by A.F. Filippov [37] appeared, presenting the proof of existence of solutions to the Cauchy problem for a differential inclusion whose right-hand side was continuous (in the Hausdorff metric) with compact (but not necessarily convex) values. Through a very clever and simple construction, Filippov was able to build a sequence of approximate solution whose derivatives were piecewise constant maps. To the opposite of what happens in the single-valued case, however, in the case of a differential inclusion it was not possible arbitrarily decrease the jumps of the derivatives at the points of the partition of the time interval, by decreasing the mesh of the partition. To overcome this difficulty, Filippov was able to build a sequence of approximate solution whose derivatives yield a (strongly) (pre) compact sequence in L^∞. The compactness in L^p with this construction; is relatively easy to obtain, since it is possible to approximate in L^p integrable functions by smooth (continuous) maps, and apply the Ascoli-Arzela construction to the approximations; however, this process cannot be applied in L^∞. Alternative methods of proof were later provided to prove results in the spirit of this theorem of Filippov.

To simplify the construction, L^1 was used, as in the selections approach of Antosiewicz-Cellina in 1975 see [8] here the continuous selections were not taken for the set valued map F itself (since in general, continuous selections do not exist) but rather for the map $x(\cdot) \to \int_0^t F(s, x(s))ds$, seen as a map from the space of absolutely continuous maps (with the norm of derivative in L^1) to (the subset of) itself, the, so called, Nemytski operator.

We have mentioned these with a few details for clarity only. There are many

mathematicians and researchers in it, we mention them for instance
[9, 17, 23, 34, 36, 57, 64, 72, 75, 83, 84, 85], who are all interested in the study of the existence of solutions and its nature either individually or as a set of solutions, they have been able to summarize a lot of important works in this domain, all this has been based on very effective means, especially the fixed point theory combining with the semi-group theory and the measure of non-compactness.

This thesis is structured in six chapters and each chapter contains several sections. It is arranged as follows:

In chapter 1, we introduce notations, definitions, lemmas and fixed point theorems which are used throughout this thesis.

In chapter 2 we are interested in the existence of the mild solution when the delay is finite and dependents on the state for the first order class of differential inclusion of neutral type of the following form

$$\begin{cases} \frac{d}{dt}(x(t) - g(t, x(t - \eta(t)))) \in F(t, x_t, x(t - \tau(t, x_t, \sigma))) \ , & \text{t} \in J := [0, T], \\ x(t) = \varphi(t) \ , & \text{t} \in I := [-r, 0], \end{cases}$$

where: $(E, \|\cdot\|)$. a Banach space, $F : J \times C(I, E) \times E \to \mathcal{P}(E)$, a multi-function, $g : J \times E \to E$ a given continuous function in its two variables, $\varphi : I \to E$, $\tau : J \times C(I, E) \times \Sigma \to [0, r]$ and $\eta : J \to [0, r]$ are given continuous functions where Σ is a parameter space menu of the norm $|\cdot|_\Sigma$. For any function x defined on $[-r, T]$ and any $t \in J$ we denote by x_t the element of $C(I, E)$ defined by

$$x_t(\theta) = x(t + \theta), \ \theta \in I.$$

Here $x_t(\cdot)$ represents the history of the state from time $t - r$, up to the present time t.

In chapter 3, we consider the following problem of first-order semi-linear differential inclusions system with delays:

$$\begin{cases} \frac{d}{dt}(x(t) - g_1(t, x_t, y_t)) - A_1 x(t) \in F(t, x_t, y_t), & \text{t} \in J := [0, b] \\ \frac{d}{dt}(y(t) - g_2(t, x_t, y_t)) - A_2 y(t) \in G(t, x_t, y_t), & \text{t} \in J \\ x(t) = \varphi(t) \ and \ y(t) = \psi(t), & \text{t} \in I := [-r, 0], \end{cases}$$

where A_1 and A_2 are infinitesimal generators of two strongly continuous semigroups $\{T_1(t), T_2(t)\}_{t \geq 0}$ on a Banach space E, with the operators A_i not necessarily bounded $F, G : J \times C(I, E) \times C(I, E) \to \mathcal{P}(E)$, two multi-valued maps, $g_i : J \times C(I, E) \times C(I, E) \to E$, with $i = 1.2$ and $\varphi, \psi : I \to E$, are given continuous functions. And we will interested in the study of the existence of mild solutions, using either fixed point theory. We will also study the qualitative properties of the solution set. For any function (x, y) defined on $[-r, b]^2$ and any $t \in J$ we denote by (x_t, y_t) the element of $(C(I, E))^2$ defined by

$$(x_t(\theta), y_t(\theta)) = (x(t + \theta), y(t + \theta)), \ \theta \in [-r, 0].$$

In chapter 4, we recall some basic concepts which used in fractional calculus, such as: the Euler Gamma function and Bêta function with their properties, some definitions

and results of fractional integral calculus within the meaning of Riemann-Liouville, the Caputo fractional derivative and fractional substantial derivative, and finally we wish to establish some nonlinear integral inequalities that can be used as the most important elements in the study of fractional differential equations and inclusions.

In chapter 5, in the convex case we use some of the inequalities obtained in section 4.3 of chapter 4 combining with the new Leray-Schauder type fixed point theorem obtained in section 1.4 of chapter 1, and the Filippov theorem in the non-convex case for proving the existence of solutions for the Cauchy problem of fractional differential inclusions with delay

$$\begin{cases} {}^cD^{\beta}x(t) \in F(t, x_t) \ , & \text{t} \in J, \\ x(t) = \varphi(t) \ , & \text{t} \in J_0, \end{cases}$$

where: $0 < \beta \leq 1$, $J := [0, T]$, $J_0 := [-r, 0]$, and $F : J \times C(J_0, E) \to \mathcal{P}_{cp,cv}(E)$ is a multivalued function. For any function x defined on $[-r, T]$ and any $t \in J$ we denote by x_t the element of $C(J_0, E)$ defined by

$$x_t(\theta) = x(t + \theta), \ \theta \in J_0.$$

In chapter 6, we study the existence of global solution for fractional neutral lattice system with delay:

$$\begin{cases} D_s^{\alpha}[u_i(t) - g_i(t, u_i(t - \eta(t)))] - u_{i-1}(t) - 2u_i(t) + \lambda u_i(t) = F_i(t, u_{it}, u_{it,\tau}) \ , & \text{t} \in J, \\ u_i(t) = \varphi_i(t) \ , & \text{t} \in J_0, \end{cases}$$

where $F_i(t, u_{it}, u_{it,\tau}) = -f_i(t, u_{it}, u_i(t - \tau(t, u_{it}, \sigma)))$ $\lambda \in \mathbb{R}$, $0 < \alpha < 1$, $i \in \mathbb{Z}$ and D_s^{α} the Caputo fractional substantial, Γ is the Gamma function, $f_i : J \times C(I, E) \times C(I, E) \to E$, are given functions, $g_i : J \times E \to E$ are given continuous functions in its two variables, $\varphi : I \to E$, $\tau : J \times C(I, E) \times \Sigma \to [0, h]$ and $\eta : J \to [0, h]$ are given continuous functions. For any function u_i defined on $[-h, T]$ and any $t \in J$ we denote by u_{it} the element of $C(I, E)$ defined by

$$u_{it}(\theta) = u_i(t + \theta), \ \theta \in I.$$

We are interested in the study of the existence of solutions for this last problem, we use as a tool in this study: the fixed point theory (Leray-Schauder's fixed point theorem).

Finally, we give a conclusion with some perspectives. In addition to bibliography.

Chapter 1

Preliminaries

In this chapter, we introduce notations, definitions, lemmas and fixed point theorems which are used throughout this thesis.

Let $(E, \|\cdot\|)$ be a Banach space. $C([-r, T], E)$ is the Banach space of all continuous functions from $[-r, T]$ into E with the norm

$$\|x\|_\infty = \sup_{t\in[-r,T]} \|x(t)\|.$$

$L^1([0, T], E)$ denotes the Banach space of measurable functions $x : [0, T] \to E$ which are Bochner integrable and is normed by

$$\|x\|_{L^1} = \int_0^T \|x(t)\|dt,$$

where r, T are strictly positive reals. In the normed space $(X, \|\cdot\|_X)$, the open ball around a point x_0 with radius R is denoted by $B_X(x_0, R)$, i.e., $B_X(x_0, R) := \{x \in X : \|x - x_0\|_X < R\}$, and the corresponding closed ball is denoted by $\bar{B}_E(x_0, R)$, i.e,

$$\bar{B}_E(x_0, R) := \{x \in E : \|x - x_0\|_E \leq R\}.$$

Let $B(E)$ be the Banach space of bounded linear operators from E into E.

Definition 1.0.1. *A linear map $T : E \to X$ is said to be compact if for any bounded sequence (x_n) in E, $(T(x_n))$ has a convergent subsequence.*

Definition 1.0.2. *Let E be a real normed space. A mapping $T : D(T) \subset E \to E$ is called compact if T maps every bounded subset of $D(T)$ to a relatively compact subset in E. T is said to be completely continuous if T is continuous and compact.*

In the rest of this chapter we denote by J the interval $[0, T]$.

1.1 Multi-Valued Analysis

Let (X, d) be a metric space and A be a subset of X. Denote by

- $P(X) = \{A \subset X : A \neq \emptyset\}$;
- $P_{cl}(X) = \{A \in P(X) : A \text{ closed}\}$;
- $P_b(X) = \{A \in P(X) : A \text{ bounded}\}$;
- $P_{cv}(X) = \{A \in P(X) : A \text{ convex}\}$;
- $P_{cp}(X) = \{A \in P(X) : A \text{ compact}\}$;
- $P_{wcp}(X) = \{A \in P(X) : A \text{ weakly compact}\}$;
- $P_{cv,cp}(X) = \{A \in P(X) : A \text{ convex}, \text{compact}\}$.

We will denote by $H_{d_\star}$ the Hausdorff pseudo-metric distance on $P(X)$ defined by:

$$H_{d_\star} : P(X) \times P(X) \to \mathbb{R}^+ \cup \{\infty\}, H_{d_\star}(A, B) = \max\{\sup_{a \in A} d_\star(a, B), \sup_{b \in B} d_\star(A, b)\},$$

where

$$d_\star(a, B) = \inf_{b \in B} d(a, b), \text{and } d_\star(A, b) = \inf_{a \in A} d(a, b).$$

Definition 1.1.1. *A multi-valued map F on a space X to a space Y is a relation which associates with any element x of X a subset $F(x)$ of Y. We will note $F : X \to P(Y)$.*

Definition 1.1.2. *We call the graph of the multi-function F the set noted*

$$Gr(F) := \{(x, y) \in X \times Y : y \in F(x)\}.$$

F is has closed graph if $Gr(F)$ is closed on $X \times Y$. We will also say that F is closed.

Definition 1.1.3. *(i) We call image of F the set $ImF := \{y \in Y, \exists x \in X : y \in F(x)\}$. (ii) We call image of $A \in P(X)$ by F the set $F(A) := \cup_{x \in A} F(x)$, and domain of F the set $Dom(F) := \{x \in X : F(x) \neq \emptyset\}$.*

Definition 1.1.4. *(a) A multi-function $F : X \to P(Y)$ has closed (respec, convex) values if $F(x)$ is closed (respec, convex) for each $x \in X$.*
(b) F is could bounded on the bounded subsets if the set $F(A) = \cup_{x \in A} F(x)$, is bounded in Y, for every subset $A \in P_b(X)$, i.e., $\sup_{x \in A}\{\sup\{|y| : y \in F(x)\}\} < \infty$.

The continuity of a multi-function is characterized by the following notion

Definition 1.1.5. *We say that a multi-function $F : X \to P(Y)$ is upper semi-continuous (u.s.c) at a point $x^\star \in Dom(F)$ if for any neighborhood V of $F(x^\star)$ there exists $r > 0$ such that $F(x) \subset V$ for all $x \in B_X(x^\star, r)$. F will be called upper semi-continuous (u.s.c) if it is east at each point of its domain.*

Definition 1.1.6. *A multi-function $F : X \to P(Y)$ is said lower semi-continuous (l.s.c) in a point $x^\star \in Dom(F)$ if for all $y^\star \in F(x^\star)$ and for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset Dom(F)$ convergent to $x^\star$, there exists a sequence of element $(y_n) \in F(x_n)$ which converge to $y^\star$.*

Definition 1.1.7. *A multi-function F is said continuous in a point $x^\star \in Dom(F)$ if it is at once upper semi-continuous and lower semi-continuous at this point. It is said to be continuous if and only if it is at each point of its domain.*

Definition 1.1.8. *A multi-function $F : X \to P(Y)$ is said convex if for all $x, x' \in X$ and $\lambda \in [0, 1], \lambda F(x) + (1 - \lambda)F(x') \subset F(\lambda x + (1 - \lambda)x')$.*

Definition 1.1.9. *We say that a multi-function $F : X \to P(Y)$ is compact at a point $x^\star \in Dom(F)$ if any sequence $(x_n, y_n) \in Gr(F)$ such that, (x_n) converges to $x^\star$, has a subsequence (x_{n_k}, y_{n_k}) that converges in $Gr(F)$. If this is true at any point in $A \subset Dom(F)$ we say that F is compact on A.*

Theorem 1.1.1. *[57] The following assertions are equivalent:*

(a) The multi-function F is closed.

(b) For any $(x, y) \in X \times Y$ such that $y \notin F(x)$, there exists are neighborhoods $V(x)$ of x and $W(y)$ of y, such that $F(V(x)) \cap W(y) = \emptyset$.

(c) For generalized sequences $\{x_n\}_{n\in\mathbb{N}} \subset X, \{y_n\}_{n\in\mathbb{N}} \subset Y$, if $x_n \to x$, and $y_n \in F(x_n), y_n \to y$ then $y \in F(x)$.

Definition 1.1.10. *A multi-function $F : X \to P(Y)$ is called:*

(a) Compact if the image $F(X)$ is relatively compact in Y, i.e $\overline{F(X)}$ is compact in Y.

(b) Locally compact if every point $x \in X$ has a neighborhood $V(x)$ such that the restriction of F a $V(x)$ is compact (i.e. $F(V(x))$ is locally compact).

(c) Quasi-compact if the restriction of F on any compact subset $A \subseteq X$ is compact, (i.e.$F(A)$ is compact).

(d) Completely continuous if it is u.s.c. and for every bounded set $A \subseteq X, F(A)$ is relatively compact.

Lemma 1.1.2. *([34, Proposition 1.2]) Let $F : X \to P_{cl}(Y)$ a multi-function, if F is u.s.c., then $Gr(F)$ is a closed subset of $X \times Y$, i.e for every sequences $(x_n)_{n\in\mathbb{N}} \subset X$ and $(y_n)_{n\in\mathbb{N}} \subset Y$, if when $n \to \infty, x_n \to x^\star, y_n \to y^\star$ and $y_n \in F(x_n)$, then $y^\star \in F(x^\star)$. Conversely, if F has nonempty compact values, is locally compact and has a closed graph, then it is u.s.c.*

Definition 1.1.11. *A multi-function $F : X \to P(Y)$ is said measurable if for every open $U \in Y$, the set $F_+^{-1}(U) := \{x \in X : F(x) \in U\}$ is Lebesgue measurable.*

Definition 1.1.12. *Let (Ω, Σ) a measurable space, we call that $f : \Omega \to X$ is a measurable selection of F if $f(\omega) \in F(\omega), \forall \omega \in \Omega$.*

The following two lemmas are concerned the measurability of multi-functions. The first one is the celebrated Kuratowski-Ryll-Nardzewski selection theorem.

Lemma 1.1.3. *([42] Kuratowski-Ryll-Nardzewski selection theorem 19.7). Let E be a separable metric space and $F : \Omega \to P(E)$ a measurable multi-valued map with nonempty closed values. Then F has a measurable selection.*

Lemma 1.1.4. *([93], Lemma 3.2) Let $F : J \to P(E)$ be a multi-function and $u : J \to E$ a measurable function. Then for any measurable function $v : J \to \mathbb{R}^+$ there exists a measurable selection f of F such that for every $t \in J$,*

$$| u(t) - f(t) | \leq d(u(t), F(t)) + v(t).$$

Proposition 1.1.5. *(Lemma 1.5 in [38]). The multi-function $F : \Omega \to P_{cl}(X)$ is measurable if and only if for each $x \in X, \omega \mapsto d(x, F(\omega)) := \inf\{d(x, y) : y \in F(x)\}$ is measurable.*

Definition 1.1.13. *A multivalued map $F : [a, b] \to P_{cp}(X)$ is strongly measurable if there exists a sequence $\{F_n : n \in \mathbb{N}\}$ of step multifunctions such that*

$$H_d(F_n(t), F(t)) \to 0 \text{ as } n \to \infty \text{ for a.e } t \in [a, b],$$

where H_d is the Hausdorff metric on $P_{cp}(X)$.

Lemma 1.1.6. *(Lemma 1.4 in [38]) Let F a multi-valued map measurable in the first variable and $p(t) - Lipschitzian$ on $y(t) + \beta B_X(0, 1)$ in the second variable, with $\beta > 0$, $p \in L^1(J, X)$, $y \in C(J, X)$, the map $t \mapsto d(g(t), F(t, y(t)))$ is measurable and $g(t) \in L^1(J, X)$. Taking $x \in C(J, X)$ such that $\|x - y\|_C \leq \beta$. Then the map $t \mapsto F(t, x(t))$ is measurable.*

Definition 1.1.14. *Let F a multi-values map we define the set*

$$S_{F,x} := \{v \in L^1(J, \mathbb{R}^+) : v(t) \in F(t, x(t)), p.p.t \in J\}.$$

The set $S_{F,x}$ is could the selection set. Is closed. Is convex if and only if F is convex $p.p.t \in J$. When F is a multi-function L^1-Carathéodory, we have a Lasota and Opial [63] result which ensures that for each $x \in C(J, \mathbb{R}^+)$, the set $S_{F,x}$ is nonempty. This allows us to define the multi-operator

$$S_F : C(J, \mathbb{R}^+) \to P(C(J, \mathbb{R}^+)),\ x \mapsto S_F(x) = S_{F,x}.$$

We define the norm of a multi-function $F : J \times E \times E \to P(E)$ by

$$\|F(t, x)\|_P := \sup\{|v| : v \in F(t, x)\}.$$

Definition 1.1.15. *F is called a multi-valued Carathéodory function if,*

(a) the function $t \mapsto F(t,x)$ is strongly measurable for each $x \in E$,

(b) for each $t \in J$, the map $x \mapsto F(t,x)$ is upper semi-continuous. Furthermore, F is L^1-Carathéodory if it is locally integrable bounded, i.e., for each positive r, there exists $h_r \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t,x)\|_P \leq h_r(t), \ \textit{for a.e. } t \in J \textit{ and all } x \in X : \|x\| \leq r.$$

Lemma 1.1.7. *([63]). Let J be a compact interval and E be a Banach space. Let F be an L^1-Carathéodory multi-valued map with $S_{F,x} \neq \emptyset$, and let Γ be a linear continuous mapping from $L^1(J,E)$ into $C(J,E)$. Then the operator,*

$$\Gamma o S_F : C(J,E) \to P_{cp,cv}(C(J,E)), \ x \mapsto \Gamma o S_F(x) = \Gamma(S_{F,x}),$$

has a closed graph in $C(J,E) \times C(J,E)$.

Theorem 1.1.8. *(Theorem III.41. in [30]) Let $(X, \mathcal{A})$ be a measurable space and E a separable Banach space. Let $f : X \to E$ be a measurable map and $\gamma : X \to [0, +\infty]$ a measurable function. Then*
1) $t \mapsto \overline{B}_X(f(t), \gamma(t))$ is a measurable multifunction,
2) moreover if $\Gamma : X \to P_{cl}(E)$ is measurable, then

$$t \mapsto U(t) := \{x \in \Gamma(t) : \|f(t) - x\| = d(f(t), \Gamma(t))\}$$

is measurable (and eventually nonempty value)

Definition 1.1.16. *([11], Definition 2.6.2) Let X be a metric space and E be a Banach space. A multivalued map $F : X \to P(X)$ is upper hemicontinuous if for each $x^\star \in E^\star$, the function $\sigma(x^\star, F(\cdot)) : X \to \mathbb{R} \cup \{+\infty\}$ given by*

$$\sigma(x^\star, F(x)) = \sup_{y \in F(x)} \langle x^\star, y \rangle$$

is upper semicontinuous.

Lemma 1.1.9. *([21]) If $F : X \to P_{wcp,cv}(E)$ is upper hemicontinuous, then it is weakly u.s.c.*

Lemma 1.1.10. *([21]) $F : X \to P_{wcp,cv}(E)$ is weakly u.s.c. if and only if for sequences $(x_n)_{n\in\mathbb{N}} \subset X$ and $(y_n)_{n\in\mathbb{N}} \subset E$, if $x_n \rightharpoonup x^\star$, as $n \to \infty$ with $y_n \in F(x_n)$, then there is a subsequence $y_{n_k} \rightharpoonup y^\star \in F(x^\star)$.*

Theorem 1.1.11. *([77]) Let X be a reflexive Banach space and $F : X \to P_{cl,cv}(X)$ be a upper hemicontinuous multivalued map. Let J be a finite interval of $\mathbb{R}$ and the sequences $(x_n : J \to X)_{n\in\mathbb{N}}$ and $(y_n : J \to X)_{n\in\mathbb{N}}$ satisfy the following conditions:*
1) $(x_n)_{n\in\mathbb{N}}$ converges to a function $x : J \to X$;
2) $(y_n)_{n\in\mathbb{N}}$ converges weakly to $y \in L^p(J,X), 1 \leq p < \infty$;
3) $y_n(t) \in \overline{co}B(F(B(x_n(t), \varepsilon_n), \varepsilon_n))$ for a.e. all $t \in J$, where $\varepsilon \to 0$ as $n \to \infty$.

Then $y \in F(x(t))$ for a.e. all $t \in J$.

Lemma 1.1.12. *([7], Theorem 1.4.13). If $G : X \to \mathcal{P}_{cp}(X)$ is u.s.c, then for any $x_0 \in X$*

$$\limsup_{x \to x_0} G(x) = G(x_0).$$

Lemma 1.1.13. *([7], Lemma 1.1.9). Let $(K_n)_{n\in\mathbb{N}} \subset K \subset X$ be a sequence of subset where K is compact in the separable Banach space X. Then*

$$\overline{co}(\limsup_{n \to +\infty} K_n) = \cap_{m>0} \overline{co}(\cup_{n \geq m} K_n),$$

where $\overline{co}B$ refers to the closure of the convex hull of B.

For further readings and details on multi-valued analysis, we refer to,[11, 10, 33, 34, 42, 55, 57, 85].

1.2 Topological structure of solutions sets

In what follows (X, d) and (Y, d') stand for two metric spaces.

Definition 1.2.1. . *A subset $B \subset X$ is called a retract of X if there exists a continuous mapping $\pi : X \to B$ such that $\pi(x) = x, \forall x \in B$.*

Definition 1.2.2. . *A subset $B \subset X$ is called a contractible space provided there exists a continuous homotopy $h : B \times [0, 1] \to B$ and $x_0 \in B$ such that*

(a) $h(x, 0) = x, \forall x \in B$,

(b) $h(x, 1) = x_0, \forall x \in B$, B is homotopically equivalent to a one point space.

Note that any closed convex subset of X is contractible. Which means the class of contractible sets is much larger then the class of closed convex sets.

Definition 1.2.3. *A compact nonempty space X is called an $R_\delta -$ set provided there exists a decreasing sequence of compact nonempty contractible space $(X_n)_{n\in\mathbb{N}}$ such that $X = \cap_{n=1}^{\infty} X_n$.*

The following notion, strictly connected with extendability, was introduced by K. Borsuk [22].

Definition 1.2.4. . *A space X is called an absolute retract (in short $X \in AR$) provided that for every space Y, every closed subset $C \subseteq Y$ and any continuous map $f : C \to X$, there exists a continuous extension $\widetilde{f} : Y \to X$ of f over Y, i.e. $\widetilde{f}(x) = f(x)$ for every $x \in C$. If the set $f(X)$ is a retract of U for every open neighborhood U of B in Y we say that $X \in ANR$ and call X to be an absolute neighborhood retract.*

Lemma 1.2.1. . *Let $X \in AR$ and $\widetilde{X}$ be such that $X \simeq \widetilde{X}$. Then $\widetilde{X} \in AR$.*

Definition 1.2.5. *A space B is closed acyclic if:*

(a) $H_0(B) = \mathbb{Q}$,

(b) $H_n(B) = 0$, *for every* $n > 0$, *where* $H_\star = \{H_n\}_{n>0}$ *is the* $\check{C}ech - homology$ *functor with compact carriers and coefficients in the field of rational* $\mathbb{Q}$. *In other words, a space B is acyclic if the map* $i : \{p\} \to X$, $i(p) = x_0 \in B$, *induces an isomorphism* $i_\star : H_\star(\{p\}) \to H_\star(B)$.

Remark 1.2.1. *An* $R_\delta -$ *space is a compact, connected space acyclic with respect to the* $\check{C}ech - homology$ *functor; it has the same homology as a one-point space.*

Definition 1.2.6. *A u.s.c map* $F : X \to \mathcal{P}(Y)$ *is called acyclic if for each* $x \in X$, $F(x)$ *is compact acyclic.*

From the continuity of $\check{C}ech - homology$ functor, we have:

Lemma 1.2.2. *[1]. Let X be a compact metric space. Then X is an acyclic space and its structure corresponds to one of the following type:*

1) X is convex,

2) X is contractible,

3) X is AR,

4) X is an $R_\delta -$ *set.*

The next definitions were introduced in [45].

Definition 1.2.7. *A metric space X is called acyclically contractible if there exists an acyclic homotopy* $\Pi : X \times [0,1] \to \mathcal{P}(X)$ *such that*
(a) $x_0 \in \Pi(x, 1)$, *for every* $x \in X$ *and for some* $x_0 \in X$,
(b) $x \in \Pi(x, 0)$, *for every* $x \in X$.

Remark 1.2.2.

(a) Clearly, any contractible space is acyclically contractible. In addition, acyclic, compact metric space is acyclically contractible. Indeed, it is enough any acyclically contractible space is acyclic.

(b) From the continuity of the $\check{C}$ech-homology functors, if X is a contractible compact space, then it is acyclic.

(c) An R_δ-contractible space has the same homology as one-point space $\{p\}$; so it is acyclic. Hence connected.

Definition 1.2.8. *A metric space X is called R_δ-contractible if there exists a multivalued homotopy* $\Pi : X \times [0,1] \to \mathcal{P}(X)$ *which is u.s.c. and satisfies*

(a) $x \in \Pi(x, 1)$, *for every* $x \in X$,

(b) $\Pi(x,0) = B$, *for every* $x \in X$ *and for some* $B \subset X$,

(c) $\Pi(x,\alpha)$ *is an* $R_\delta - set$, *for every* $\alpha \in [0,1]$ *and* $x \in X$.

We conclude that, for any compact space, we have

$$convex \subset AR \subset contractible \subset R_\delta \subset acyclic.$$

Definition 1.2.9. *([45], Definition 9.15) Let* X *be a metric space and* E *be a Banach space. A map* $N : X \to E$ *is proper if it is continuous and for every compact* $K \in E$, *the set* $N^{-1}(K)$ *is compact.*

Definition 1.2.10. *([45], Definition 2.29) A map* $L : X \to Y$ *is said to be a Vietoris map if the following conditions are satisfied:*

(i) $L : X \to Y$ *is proper;*

(ii) L *is surjective;*

(iii) the set $L^{-1}(y)$ *is acyclic for every* $y \in Y$.

It is clear that any multivalued operator $F : X \to P(Y)$ admits the standard factorization through the graph $Gr(F)$, i.e., there exists, $p_F : Gr(F) \to X$ and $q_F : Gr(F) \to Y$ are the projections such that

$$F(x) = q_F(p_F^{-1}(x)) \ for \ all \ x \in X.$$

Also we present a result about the topological structure of the set of solutions of some nonlinear fractional inclusions due to N. Aronszajn and developed by F. Browder and Ch.P. Gupta in [26] (see also [7] Theorem 1.2).

Theorem 1.2.3. *Let* X *be a metric space,* E *a Banach space and* $f : X \to E$ *a proper map. Assume further that for each* $\varepsilon > 0$, *a proper map* $f_\varepsilon : X \to E$ *is given and the following two conditions are satisfied:*

(a) $\|f_\varepsilon(x) - f(x)\| < \varepsilon$, *for every* $x \in X$,

(b) for every $\varepsilon > 0$ *and* $u \in E$ *in a neighborhood of the origin such that* $\|u\| \leq \varepsilon$, *the equation* $f_\varepsilon(x) = u$ *has exactly one solution* x_k. *Then the set* $f^{-1}(0)$ *is an* R_δ*-set.*

Lemma 1.2.4. *(Lasota-Yorke Approximation theorem [44]) Let* E *be a normed space,* X *be a metric space and* $f : X \to E$ *be a continuous map. Then, for each* $\varepsilon > 0$, *there is a locally Lipschitz map* $f_\varepsilon : X \to E$ *such that*

$$\|f(x) - f_\varepsilon(x)\| < \varepsilon, \ for \ every \ x \in X.$$

This suggests the following definition and properties.

Definition 1.2.11. *([44], Definition 40.1) A multivalued operator* $F : X \to P(Y)$ *is called admissible if there exists a metric space* $\widetilde{X}$ *and two continuous maps* $\widetilde{p} : \widetilde{X} \to X$ *and* $\widetilde{q} : \widetilde{X} \to Y$ *such that*

(i) $\widetilde{p}$ *is a Vietoris map;*

(ii) $F(x) = \widetilde{q}(\widetilde{p}^{-1}(x))$ *for any* $x \in X$.

Proposition 1.2.5. *([44]) If* $F : X \to P(Y)$ *and* $G : Y \to P(Z)$ *are two admissible maps, then the composition* $G \circ F$ *is admissible.*

1.3 Measure of noncompactness

Next, we define in this Section the measures of noncompactness (**MNC** for short) and give their basic properties.

Definition 1.3.1. *([45], Definition 2.91) Let* E *be a Banach space and* $(\mathcal{A}, \geq)$ *be a partially ordered set. A map* $\alpha : P(X) \to \mathcal{A}$ *is called a measure of noncompactness on* X (MNC)*, if*

$$\alpha(\overline{co}\Omega) = \alpha(\Omega)$$

for every $\Omega \in P(X)$.

Notice that if D is dense in Ω, then $\overline{co}\Omega = \overline{co}D$ and hence

$$\alpha(\Omega) = \alpha(D).$$

The measure of noncompactness satisfies the following properties:

Definition 1.3.2. *([45] Definition 2.29) Let* $A, B, D \in P(X)$. *A measure of noncompactness* α *is:*

(a) Monotone if $A \subset B$, *implies* $\alpha(A) \leq \alpha(B)$.

(b) Nonsingular if $\alpha(\{a\} \cup D) = \alpha(D)$ *for every* $a \in X$.

(c) Invariant with respect to the union with compact sets if $\alpha(K \cup D) = \alpha(D)$ *for every relatively compact set* $K \subset X$.

(d) Real if $\mathcal{A} = \overline{\mathbb{R}}_+ = [0, \infty]$ *and* $\alpha(D) < \infty$ *for every bounded* D.

(e) Semi-additive if $\alpha(A \cup B) = \max(\alpha(A), \alpha(B))$.

(f) Regular if the condition $\alpha(D) = 0$, *is equivalent to the relative compactness of* D.

As an example of a measure of noncompactness, consider the Hausdorff measure of noncompactness

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \; has \; a \; finite \; \varepsilon - net\}.$$

Recall that a bounded set $A \subset X$ has a finite ε-net if there exists a finite subset $S \subset X$ such that $A \subset S + \varepsilon\overline{B}$ where $\overline{B}$ is a closed ball in X. Other examples are given by the following measures of noncompactness defined on the space of continuous functions $C([0, b], X)$ with values in a Banach space X:

(i) The modulus of fiber noncompactness

$$\varphi(\Omega) = \sup_{t\in[0.b]} \chi_X(\Omega(t)),$$

where χ_X is the Hausdorff measure of noncompactness in X and $\Omega(t) = \{y(t) : y \in \Omega\}$.

(ii) The modulus of equicontinuity

$$mod_C(\Omega) = \lim_{\delta\to 0} \sup_{|\tau_1-\tau_2|\leq\delta} \|y(\tau_1) - y(\tau_2)\|.$$

It should be mentioned that these measures of noncompactness satisfy all above-mentioned properties except regularity.

Definition 1.3.3. *([59]) Let (X, d) be a complete metric space and $\mathcal{B}$ the family of bounded subsets of X. For every $B \in \mathcal{B}$ we define the Kuratowski (1896-1980) measure of noncompactness $\alpha(B)$ of the set B as the infimum of the numbers d such that B admits a finite covering by sets of diameter smaller than d.*

Remark 1.3.1. *The diameter of a set B is the number* $\sup\{d(x, y) : x \in B, y \in B\}$ *denoted by $diam(B)$, with $diam(\emptyset) = 0$.*
It is clear that $0 \leq \alpha(B) \leq diam(B) < \infty$ for each nonempty bounded subset B of X and that $diam(B) = 0$ if and only if B is an empty set or consists of exactly one point.

Definition 1.3.4. *Let E be a Banach space and Ω_E the family of bounded subsets of E. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_E \to [0, \infty)$ defined by*

$$\alpha(B) = \inf\{\varepsilon > 0 \ : \ B \subseteq \cup_{i=1}^{n} B_i \text{ and } diam(B_i) \leq \varepsilon\},$$

where

$$diam(B_i) = sup\{||x - y|| : x, y \in B_i\}.$$

Remark 1.3.2. *The α-measure of noncompactness was introduced by Kuratowski in order to generalize the Cantor intersection theorem*

Theorem 1.3.1. *([59]) Let (X, d) be a complete metric space and $\{B_n\}$ be a decreasing sequence of nonempty, closed and bounded subsets of X and $\lim_{n\to\infty} \alpha(B_n) = 0$. Then the intersection B_∞ of all B_n is nonempty and compact.*

In [54], Horvath has proved the following generalization of the Kuratowski theorem:

Theorem 1.3.2. *([59]) Let (X, d) be a complete metric space and $\{B_i\}_{i\in I}$ be a family of nonempty of closed and bounded subsets of X having the finite intersection property. If $\inf_{i\in I} \alpha(B_i) = 0$ then the intersection B_∞ of all B_i is nonempty and compact.*

Lemma 1.3.3. *([48]) If $V \subset C(J,E)$ is a bounded and equicontinuous set, then*

(i) the function $t \to \alpha(V(t))$ is continuous on J, and

$$\alpha_c(V) = \sup_{0\le t\le T} \alpha(V(t)),$$

(ii) for $V(s) = \{x(s) : x \in V\},\ s \in J$ we have

$$\alpha\left(\int_0^T x(s)ds : x \in V\right) \le \int_0^T \alpha(V(s))ds.$$

In the definition of the Kuratowski measure we can consider balls instead of arbitrary sets. In this way we get the definition of the Hausdorff measure:

Definition 1.3.5. *([59]) The Hausdorff measure of noncompactness $\chi(B)$ of the set B is the infimum of the numbers r such that B admits a finite covering by balls of radius smaller than r.*

Theorem 1.3.4. *([59]) Let $B(0,1)$ be the unit ball in a Banach space X. Then*

$$\alpha(B(0,1)) = \chi(B(0,1)) = 0$$

if X is finite dimensional, and $\alpha(B(0,1)) = 2$, $\chi(B(0,1)) = 1$ otherwise.

Theorem 1.3.5. *([59]) Let $S(0,1)$ be the unit sphere in a Banach space X. Then $\alpha(S(0,1)) = \chi(S(0,1)) = 0$ if X is finite dimensional, and $\alpha(S(0,1)) = 2$, $\chi(S(0,1)) = 1$ otherwise.*

Theorem 1.3.6. *([59]) The Kuratowski and Hausdorff MNCs are related by the inequalities*

$$\chi(B) \le \alpha(B) \le 2\chi(B).$$

In the class of all infinite dimensional Banach spaces these inequalities are the best possible.

Example 1.3.1. *Let l^∞ be the space of all real bounded sequences with the supremum norm, and let A be a bounded set in l^∞. Then $\alpha(A) = 2\chi(A)$.*

Definition 1.3.6. *([45]) Let $\mathcal{M}$ be a closed subset of a Banach space X and $\alpha : P(X) \to (\mathcal{A}, \ge)$ be a MNC on E. A multivalued map $F : \mathcal{M} \to P_{cp}(X)$ is said to be α-condensing if for every $\Omega \subset \mathcal{M}$, the relation*

$$\alpha(F(\Omega)) \ge \alpha(\Omega)$$

implies the relative compactness of Ω.

We end this section by the following compactness criterion in L^1.

Lemma 1.3.7. *([57], Theorem 5.1.1) Let $N : L^1([a,b],X) \to C([a,b],X)$ be an operator satisfying the following conditions:*
(S_1) N is L-Lipschitz: there exists $L > 0$ such that for every $f, g \in L^1([a,b],X)$,

$$\|Nf(t) - Ng(t)\| \leq L \int_a^b \|f(s) - g(s)\| ds, \ \ for\ all\ t \in [a,b].$$

(S_2) N is weakly-strongly sequentially continuous on compact subsets: for any compact $K \subset X$ and any sequence $\{f_n\}_{n=1}^{\infty} \subset L^1([a,b],X)$ such that $\{f_n\}_{n=1}^{\infty} \subset K$ for a.e. $t \in [a,b]$, the weak convergence of $f_n \rightharpoonup f_0$ implies the strong convergence of $N(f_n) \to N(f_0)$ as $n \to \infty$. Then for every semi-compact sequence $\{f_n\}_{n=1}^{\infty} \subset L^1([a,b],X)$, the image sequence $(N\{f_n\}_{n=1}^{\infty})$ is relatively compact in $C([a,b],X)$.

Lemma 1.3.8. *([57], Theorem 5.2.2) Let an operator $N : L^1([a,b],X) \to C([a,b],X)$ satisfy conditions $(S_1) - (S_2)$ together with*
(S_3) there exists $\eta \in L^1([a,b],X)$ such that for every integrably bounded sequence $\{f_n\}_{n=1}^{\infty}$, we have

$$\chi(\{f_n\}_{n=1}^{\infty}) \leq \eta(t), \ \ for\ a.e.\ t \in [a,b],$$

where χ is the Hausdorff MNC. Then

$$\chi(\{N(f_n)(t)\}_{n=1}^{\infty}) \leq 2L \int_a^b \eta(s) ds, \ \ for\ all\ t \in [a,b],$$

where L is the constant in S_1.

For further facts concerning measures of noncompactness and their properties we refer to [6, 10, 12, 59] and the references therein.

1.4 Some fixed point theorems

In this section we present some old and new fixed point theorems.

Definition 1.4.1. *Let (X,τ) be a topological space $F : X \to P(X)$ a multi-function. We say that $x \in X$ is a fixed point of F if $x \in F(x)$.*

Theorem 1.4.1. *([44]) Let $D \subset X$ be a nonempty closed convex and bounded subset of X and $N : D \to P(D)$ an β-condensing admissible multivalued operator, where β is a nonsingular measure of noncompactness defined on subsets of X. Then $Fix(N)$ is nonempty and compact.*

The next result is concerned with nonlinear alternative for β-condensing u.s.c. multi-valued maps.

Lemma 1.4.2. *Let $V \subset X$ be a bounded open neighborhood of zero and $N : \overline{V} \to P(X)$ be a β-condensing admissible multi-valued map, where β is a nonsingular measure of noncompactness defined on subsets of X, that satisfies the Leray-Schauder boundary condition*

$$x \notin \lambda N(x)$$

for all $x \in \partial V$ and $0 < \lambda < 1$. Then $FixN$ is nonempty and compact.

Proof. Let C be the set defined by

$$C = \{x \in \overline{V} : x \in \lambda N(x) \ for \ some \ \lambda \in [0,1]\}.$$

It is clear that C is nonempty set since $0 \in C$. To show that C is closed, let $(x_n)_{n\in\mathbb{N}} \subset C$ be a sequence converging to x; then there exists $(\lambda_n)_{n\in\mathbb{N}} \subset [0,1]$ such that

$$x_n \in \lambda_n N(x_n) \ for \ n \in \mathbb{N}.$$

Since $[0,1]$ is compact, there exists a subsequence of $(\lambda_n)_{n\in\mathbb{N}}$ that converges to $\lambda \in [0,1]$. If $\lambda \in (0,1]$, then using the fact that N is admissible, we see that N has a closed graph, so $x \in \lambda N(x)$. If $\lambda = 0$, it is clear that $N\left(\overline{\{x_n : n \in \mathbb{N}\}}\right)$ is a compact set, so there exists $M > 0$ such that

$$\|y\| \leq M \ for \ any \ y \in N\left(\overline{\{x_n : n \in \mathbb{N}\}}\right).$$

This implies

$$\|x_n\| \leq \lambda_n M, \ n \in \mathbb{N},$$

that is, $x = 0 \in 0N(0)$, and we conclude that C is closed set in X. Since $C \cap X \setminus V = \emptyset$, Urysohn's lemma guarantees the existence of a continuous function $\mu : X \to [0,1]$ such that $\mu(x) = 1$ for $x \in C$ and $\mu(x) = 0$ for $x \in X \setminus V$. Let $r : X \to \overline{V}$ be a retraction of the space X onto $\overline{V}$.

We introduce the multivalued operator $\widetilde{N} : X \to P(X)$ defined by

$$\widetilde{N}(x) = \begin{cases} \mu(x)N(x), & \text{x} \in \overline{V}, \\ \{0\}, & \text{x} \in X \setminus \overline{V}. \end{cases}$$

Observe that

$$\widetilde{N}(x) = f \circ (\mu(x) \times N(r(x))), \ x \in X,$$

where $f : [0,1] \times X \to X$ is defined by

$$f(\lambda, x) = \lambda x, \ x \in X.$$

Let $C_\star = \overline{co}(N(\overline{V}) \cup \{0\})$; it is easy to show that

$$\widetilde{N}(C_\star) \subset C_\star.$$

It follows from the definition of $\widetilde{N}$ that it is an admissible multivalued map. Next, we show that $\widetilde{N}$ is β-condensing. Let $D \in P_b(C_\star)$ be such that

$$\beta(D) \leq \beta(\widetilde{N}(D)).$$

If $D \cap \overline{V} = \emptyset$, then from the definition of $\widetilde{N}$, we have

$$\beta(D) \leq \beta(\widetilde{N}(D)) \leq \beta(\{0\}) = 0,$$

so D is relatively compact. On the other hand, if $D \cap \overline{V} \neq \emptyset$, then

$$\beta(D \cap \overline{V}) \leq \beta(D) \leq \beta(\widetilde{N}(D)) \leq \beta(\widetilde{N}(D \cap \overline{V})).$$

From the definition of $\widetilde{N}$,

$$\beta(\widetilde{N}(D \cap \overline{V})) = \beta(\mu(D \cap \overline{V})N(D \cap \overline{V})) = \beta\left(\bigcup_{x \in D \cap \overline{V}} \mu(x)N(D \cap \overline{V})\right).$$

From Definition 1.3.2, for every $x \in X$, we have

$$\beta(\mu(x)N(D \cap \overline{V})) = |\mu(x)|\beta(N(D \cap \overline{V})) \leq \beta(N(D \cap \overline{V})).$$

Hence,

$$\beta(D \cap \overline{V}) \leq \beta(D) \leq \beta(\widetilde{N}(D)) \leq \beta(N(D \cap \overline{V})).$$

Since N is β-condensing, we obtain that $D \cap \overline{V}$ is relatively compact. Moreover, since N is u.s.c., $\beta(D) = 0$, so D is relatively compact. Therefor, $\widetilde{N}$ is β-admissible since it is admissible and β-condensing. We now use Theorem 1.4.1 to show that $\widetilde{N}$ has at least one fixed point $x \in C_\star$ such that $x \in \widetilde{N}(x)$. Since $0 \in V, x \in \overline{V}$ and $x \in \mu(x)N(x)$. Hence, $x \in C$. This implies that $x \in N(x)$ so $Fix(N) \neq \emptyset$. To see that $Fix(N)$ is compact, first note that

$$Fix(N) \subset N(Fix(N)).$$

Therefore,

$$\beta(Fix(N)) \leq \beta(N(Fix(N))).$$

Since N is β-condensing, we conclude that $Fix(N)$ is compact. This completes the proof of the lemma. □

An additional result on the set of fixed points of N is contained in the following proposition.

Proposition 1.4.3. *Let X be a Banach space and $N : X \to P(X)$ be a β-admissible multivalued map, where β is a nonsingular measure of noncompactness defined on subsets of X. If the set*

$$\Lambda = \{x \in X : x \in \lambda N(x), \ for \ some \ \lambda \in (0, 1)\}$$

is bounded, then $FixN \neq \emptyset$ and compact.

Proof. Since Λ is a bounded set, there exists $d > 0$ such that

$$\|x\|_\infty < d \ for \ each \ x \in \Lambda.$$

We can easily prove that $N : \overline{B(0,d)} \to P(X)$ is a β-admissible operator and

$$x \notin \lambda N(x) \ for \ all \ x \in \partial\overline{B(0,d)}.$$

Consequently, from Lemma 1.4.2, the set $Fix(N) \neq \emptyset$ and compact. □

Similarly, we have the following result.

Lemma 1.4.4. ([57]) *Let W be a closed subset of a Banach space E and $N : W \to P_{cp}(X)$ be a closed β-condensing multivalued map, where β is a nonsingular measure of noncompactness defined on subsets of X. If the fixed point set $Fix(N)$ is bounded, then it is compact.*

In what follows, we wish to replace the Leray-Schauder boundary condition with the following "star-shaped" condition.

Definition 1.4.2. *([11], Definition 4.2.7) Let X be a Banach space. An open bounded neighborhood of the origin, V is strictly star-shaped with respect to the origin if for any $x \in \partial V$, we have*

$$\{\lambda x : \lambda > 0\} \cap \partial V = \{x\}.$$

Remark 1.4.1. *If V is a bounded open neighborhood of the origin, the following strict inclusions hold (see ([56], Proposition 1)):*

$$Convex \subset Strictly \ Star - shaped \subset Star - shaped.$$

For any strictly star-shaped open and bounded neighborhood V of the origin, we can define the Minkowski function $\mu_V : X \to [0, \infty)$ by

$$\mu_V(x) = \inf\{\lambda > 0 : x \in \lambda V\}.$$

This function satisfies the properties:

- $\mu_V(tx) = t\mu_V(x)$, *for all* $x \in X$ *and* $t \geq 0$.
- $\mu_V(x) > 0$ *for any* $x \in X$.
- $V = \{x : \mu_V(x) < 1\}, \overline{V} = \{x : \mu_V(x) \leq 1\}$, *and* $\partial V = \{x : \mu_V(x) = 1\}$.

Remark 1.4.2. *For strictly star set, the Minkowski function can equivalently be defined as: $\mu_V(0) = 0$ and, for $x \neq 0$, $\mu_V(x)$ is the unique positive number such that $\frac{x}{\mu_V(x)} \in V$.*

Proposition 1.4.5. *([41]) Let X be a Banach space and V be an bounded open neighborhood of zero. If V is strictly star-shaped, then the Minkowski function μ_V is continuous and the mapping $R_V : X \to \overline{V}$ given by*

$$R_V(x) = \begin{cases} \frac{x}{\mu_V(x)}, & x \in X \setminus \overline{V}, \\ x\ , & x \in \overline{V}, \end{cases}$$

is a continuous retract of X into closure of V.

In 2005, Jimenez-Melado and Morales [56] introduced the so-called interior condition for single valued maps. In the following definition, we introduce a multivalued version of this condition.

Definition 1.4.3. *([56], Page 501) Let X be a real Banach space and V be open subset of X with $0 \in V$. We say that the multivalued map $N : \overline{V} \to P(X)$ satisfies the interior condition if there exists $\delta > 0$ such that*

$$\lambda x \notin N(x) \text{ for } x \in V_\delta,\ \lambda > 1, \text{ and } N(x) \cap X \setminus \overline{V} \neq \emptyset,$$

where

$$V_\delta = \{x \in V : d(x, \partial V) < \delta\}.$$

The following result is taken from Gonzalez, Jimenez-Melado and Liorens-Fuster ([41], Proposition 2).

Proposition 1.4.6. *Let X be a Banach space and V be a strictly star-shaped open bounded neighborhood of the origin. Let $0 \leq k \leq \widetilde{k}$, where $k = d(0, \partial V)$ and $\widetilde{k} = \sup\{\|x\| : x \in \partial V\}$, and choose $\delta \in (0, k], r = \frac{\delta}{k} \in [0, 1]$. Define $L_V^\delta : X \to X$ by*

$$L_V^\delta(x) = \begin{cases} \frac{r+(1-r)\mu_V(x)}{\mu_V^2(x)} x, & x \in X \setminus \overline{V}, \\ x\ , & x \in \overline{V}. \end{cases}$$

Then L_V^δ is continuous, $L_V^\delta(X \setminus \overline{V}) \subset V_\delta$, and $L_V^\delta(x) \in \overline{co}(\{0\} \cup \{x\})$ for all $x \in X$.

The following fixed point theorem is for multivalued maps satisfying the interior condition.

Theorem 1.4.7. *Let X be a real Banach space and let V be a open bounded and strictly star-shaped subset of X with $0 \in V$. If the multivalued map $N : \overline{V} \to P(X)$ is β-condensing, admissible, and satisfies the interior condition, then N has at least one fixed point.*

Proof. Let $R_V : X \to \overline{V}$ be a retraction of X into $\overline{V}$. Define the multivalued operator $N^\star : \overline{V} \to P(X)$ by

$$N^\star(x) = \begin{cases} \frac{r+(1-r)\mu_V(x)}{\mu_V^2(x)} N(R_V(x)), & \text{x} \in X \setminus \overline{V}, \\ N(x)\ , & \text{x} \in \overline{V}. \end{cases}$$

We can write $N^\star$ as

$$\widetilde{N}^\star(x) = g \circ (\widetilde{\mu}(x) \times N(R_V(x))),\ x \in X,$$

where $g : [0,1] \times X \to X$ is defined by

$$g(\lambda, x) = \lambda x,\ x \in X,$$

and

$$\widetilde{\mu}(x) = \begin{cases} \frac{r+(1-r)\mu_V(x)}{\mu_V^2(x)}, & \text{x} \in X \setminus \overline{V}, \\ 1\ , & \text{x} \in \overline{V}, \end{cases}$$

are continuous function. Then, by proposition 1.2.5, $N^\star$ is an admissible operator, and we see that $\overline{co}(N(\overline{V}) \cup \{0\})$ is invariant under $N^\star$. Let $D \in P_b(\overline{V})$; then from the definition of $N^\star$,

$$\beta(N^\star(D)) = \beta(N(D)).$$

Therefore,

$$\beta(N^\star(D)) \leq \beta(N(D))\ for\ all\ D \in P_b(\overline{V}).$$

From this it follows that $N^\star$ is a β-condensing admissible operator. We wish to show that the conditions of Monch's theorem [69] hold. Let $D \subset \overline{V}$ with $D \subset \overline{co}(\{0\} \cup N^\star(D))$; then

$$\beta(D) \leq \beta(\overline{co}(\{0\} \cup N^\star(D))) = \beta(\overline{co}(N^\star(D))).$$

Hence,

$$\beta(D) \leq \beta(N^\star(D)).$$

Using the fact that $N^\star$ is β-condensing, we see that D is relatively compact.

Next, we show that $N^\star$ satisfies the Leray-Schauder condition. Since N satisfies the interior condition, there exists $\delta \in I := (0, \widetilde{k})$ such that

$$\lambda x \notin N(x)\ for\ x \in V_\delta,\ \lambda > 1,\ and\ N(x) \cap X \setminus \overline{V} \neq \emptyset,$$

where

$$\widetilde{k} = \sup\{\|x\| : x \in \partial V\}.$$

As in the proof of ([56], Theorem 1), for any $t \in (1 - \frac{\delta}{\widetilde{k}}, 1)$, the set $V_t = \{tx : x \in V\}$ is an open subset of V, $\overline{tV} = t\overline{V}$, and $t\overline{V} \subset V$. Additionally, we can easily show that

$$\partial V \subset \bigcup_{t \in I} \partial \overline{V}_t \subset \bigcup_{t \in I} \partial V_t.$$

Now suppose there exist $\lambda_t > 1$ and $x_t \in \partial V_t$ such that $\lambda_t x_t \in N^\star(x_t)$. Then by the definition of $N^\star$,

$$\lambda_t x_t \in N(x_t),\ \lambda_t > 1,\ and\ x_t \in \partial V_t. \tag{1.4.1}$$

Since V is strictly star-shaped and $x_t \in \partial V_t \subset V$, there exists a unique $z_t \in \partial V$ such that $x_t = tz_t$ (see Remark 1.4.2). Hence, we obtain

$$d(x_t, \partial V) \leq \|x_t - z_t\| \leq (1-t)\widetilde{k} < \delta, \tag{1.4.2}$$

and consequently, by the interior condition, we have $\lambda_t x_t \in \overline{V}$. This implies that $\lambda_t x_t \in [x_t, z_t]$ since that is a strictly star-shaped set. Thus, we conclude that

$$\|x_t\| \leq \|\lambda_t x_t\| \leq \|z_t\| \leq \widetilde{k}.$$

Therefore, the set

$$D^\star = \{x_t : x_t \; satisfies \; the \; relation \; (1.4.1), \; t \in I\}$$

is bounded and $D^\star \subset \overline{co}\{N(D^\star) \cup \{0\}\}$. Since N is β-condensing, $D^\star$ is relatively compact, and hence there exists a sequence $(x_n)_{n\in\mathbb{N}}$ converging to x in $\overline{V}$ and a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that

$$\lambda_n x_n \in N(x_n), \; \lambda_n \to 1, \; t_n \to 1, \; and \; x_n \to x \; as \; n \to \infty.$$

It is easy to see that (1.4.2), implies $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x$ and there exists $n \in \mathbb{N}$ such that $x_n \in V$. We then have that

$$d(x_n, \partial V) < \delta \; implies \; x_n \in V_\delta.$$

The interior condition of N implies that

$$\lambda_n x_n \notin N(x_n) \; and \; N(x_n) \cap X \setminus \overline{V} = \emptyset,$$

which contradicts (1.4.1). Therefore, $N^\star$ satisfies the Leray-Schauder condition. Since $\overline{V}$ is a retract of X, $N^\star$ is an admissible map and satisfies Monch's theorem. Then, by ([78], Theorem 5), there exists $x \in \overline{V}$ such that $x \in N^\star(x)$. From the definition of $N^\star$, it follows that $x \in N(x)$, and this completes the proof of the theorem. □

Nonlinear Alternative of Leray-Schauder

Theorem 1.4.8. *Let $(X, \|\cdot\|)$ a normed space and $F : X \to P_{cl,cv}(X)$ a multi-valued operator upper semi-continuous and compact. Then at least one of the following assertions is verified:*
(i) F has at least one fixed point on X,
(ii) the set

$$B_\lambda := \{x \in X : x \in \lambda F(x), \lambda \in (0,1)\}$$

is unbounded.

Theorem 1.4.9. *(Leray-Schauder's fixed point theorem [47, 73] Let N be a continuous and compact mapping of a Banach space X, such that the set $\{x \in X : x = \lambda Nx, \; for \; some \; \lambda \in (0,1)\}$ is bounded. Then N possesses a fixed point.*

More details on fixed point theory with the proof be found on the books [4, 7, 43].

1.5 Semigroup of Linear Operator

Definition 1.5.1. *A one-parameter family $T(t)$ for of bounded linear operators on a Banach space X is a C_0-semigroup (or strongly continuous) on X if*

(i) $T(t) \circ T(s) = T(t+s)$, *for* $t, s \geq 0$, *(semigroup property),*

(ii) $T(0) = I$, *(the identity on X),*

(iii) the map $t \to T(t)x$ *is strongly continuous, for each* $x \in X$, *i.e,*

$$\lim_{t \to 0} T(t)(x) = x, \, \forall x \in X.$$

Remark 1.5.1. *A semigroup of bounded linear operators $(T(t))_{t \geq 0}$ is uniformly continuous if*

$$\lim_{t \to 0} \|T(t) - I\|_{B(E)} = 0.$$

Here I denotes the identity operator in E. A strongly continuous semigroup of bounded linear operators on X will be called a semigroup of class C_0 or simply a C_0 semigroup. If only (i) and (ii) are satisfied we say that the family $(T(t))_{t \geq 0}$ of bounded linear operators is a semigroup.

Definition 1.5.2. *Let $T(t)$ be a semigroup of class (C_0) defined on X. The infinitesimal generator A of $T(t)$ is the linear operator defined by*

$$A(x) = \lim_{h \to 0} = \frac{T(h)(x) - x}{h}, \text{ for } x \in D(A),$$

where $D(A) = \{x \in X \mid \lim_{h \to 0} = \dfrac{T(h)(x) - x}{h} \text{ exists in } X\}$.

Let us recall the following property:

Theorem 1.5.1. *[76] If $T(t)$ is a C_0-semigroup, then there exist $\omega \geq 0$ and $M \geq 1$ such that*

$$\|T(t)\|_{B(E)} \leq M \exp(\omega t), \text{ for } 0 \leq t < \infty.$$

Remark 1.5.2. *If, $M = 1$ and $\omega = 0$, i.e; $\|T(t)\|_{B(E)} \leq 1$, for $t \geq 0$, then the semigroup $T(t)$ is called a contraction semigroup (C_0).*

Theorem 1.5.2. *[76] Let $T(t)_{t \geq 0}$ be a C_0-semigroup and A be its infinitesimal generator. Then*

(a) For $x \in X$,

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} T(s)x ds = T(t)x.$$

(*b*) *For* $x \in X$, $\int_0^t T(s)xds \in D(A)$ *and*

$$A\left(\int_0^t T(s)xds\right) = T(t)x - x.$$

(*c*) *For* $x \in D(A)$, $T(t) \in D(A)$ *and*

$$\frac{d}{dt}T(t)(x) = A(T(t)(x)) = T(t)(A(x)). \tag{1.5.1}$$

(*d*) *For* $x \in D(A)$

$$T(t)x - T(s)x = \int_s^t T(\tau)Axd\tau = \int_s^t AT(\tau)xd\tau,$$

$$\lim_{t\to 0} T(t)(x) = x,\ \forall x \in X.$$

Corollary 1.5.3. *[76] If A is the infinitesimal generator of a C_0 semigroup $(T(t)_{t\geq 0})$ then $D(A)$ the domain of A, is dense in X and A is closed linear operator.*

Theorem 1.5.4. *A linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator.*

Theorem 1.5.5. *[76] Let $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ be two C_0 semigroup on X, generated respectively by A and B. If $A = B$ then $T(t) = S(t)$, $t \geq 0$.*

Proof. Assume $A = B$ and let $x \in D(A) = D(B)$.
Define $\alpha(s) := T(t-s)S(s)x$, $s \in [0,t]$.
From Theorem 1.5.2 part (c) it follows that α is differentiable and that
$\alpha'(s) = \frac{d}{ds}T(t-s)S(s)x = -T(t-s)AS(s)x + T(t-s)BS(s)x = 0$, since $A = B$.
It follows $\alpha(s) = constant$. In particular, its values at $s = 0$ and $s = t$ are the same that is $T(t)x = S(s)x\ \forall x \in D(A)$. By Corollary 1.5.3 $D(A)$ is dense in X and $T(t)$, $S(s)$ are closed;
therefore $T(t)x = S(s)x$; $\forall x \in X$. □

Let E an complex Banach space and

$$\Delta = \{z \in \mathbb{C} : \theta_1 < \arg(z) < \theta_2\},$$

with $\theta_1 < 0 < \theta_2$. for all $z \in \Delta$, we consider $T(z)$ an linear bounded operator.

Definition 1.5.3. *We say that the family $\{T(z) \mid z \in \Delta\}$ is an analytic semi-group in Δ if:*

(*A*) $T(z) \circ T(w) = T(z+w)$, *for* $z, w \in \Delta$,

(*B*) $T(0) = I$,

(*C*) *the map* $z \mapsto T(z)x$ *is strongly continuous, for each* $x \in E$, *i.e,*

$$\lim_{z \to 0} T(z)(x) = x, \ \forall x \in E,$$

(*D*) $\forall x \in E, z \mapsto T(z)x$ *is analytic.*

Therefore, according to this definition we note that the restriction of an analytic semi-group to the real half-axis positif is a strongly continuous semi-group and therefore it is concluded that any semi-group strongly continuous is analytic.

Let A be the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t \geq 0}$ of uniformly bounded linear operators on E. Let $0 \in \rho(A)$, with $\rho(A)$ is the resolvent set of A. Then for $0 < \beta < 1$, it is possible to define the fractional power A^β as a closed linear operator on its domain $D(A^\beta)$. For the analytic semigroup $\{T(t)\}_{t \geq 0}$, the following properties will be used.

Properties 1.5.6. *(i) There is a constant* $M \geq 1$ *such that*

$$M := \sup_{t \in [0, +\infty)} \|T(t)\|_{B(E)} < \infty,$$

(ii) for any $\beta \in (0, 1)$, *there exists a positive constant* C_β *such that*

$$\|A^\beta T(t)\|_{B(E)} \leq \frac{C_\beta}{t^\beta}, \ t \in (0, b].$$

For more details about the above preliminaries, we refer to
[4, 7, 49, 52, 59, 76, 87, 88].

Chapter 2

Neutral Differential Inclusions with State-Dependent Delays

In this chapter we give some existence results of solutions and Filippov's theorem of neutral differential equation with state-dependent delay. More, presley, we consider the following:

$$\begin{cases} \frac{d}{dt}(x(t) - g(t, x(t - \eta(t)))) \in F(t, x_t, x(t - \tau(t, x_t, \sigma))) \ , & \text{t} \in J, \\ x(t) = \varphi(t) \ , & \text{t} \in I, \end{cases} \tag{2.0.1}$$

where, $(E, \|\cdot\|)$ a Banach space, $J := [0, T]$, $I := [-r, 0]$, r, T are strictly positive constants, $F : J \times C(I, E) \times E \to \mathcal{P}(E)$, a multi-function, $g : J \times E \to E$ a given continuous function in its two variables, $\varphi : I \to E$, $\tau : J \times C(I, E) \times \Sigma \to [0, r]$ and $\eta : J \to [0, r]$ are given continuous functions, Σ is a parameter space menu of the norm $|\cdot|_\Sigma$. For any function x defined on $[-r, T]$ and any $t \in J$ we denote by x_t the element of $C(I, E)$ defined by

$$x_t(\theta) = x(t + \theta), \ \theta \in I.$$

2.1 Existence of solutions

In this section we give our main existence result for problem (2.0.1). Before starting and proving this result, we give the definition of its solution. Let's denote by Ω the set $C([-r, T], E)$.

Definition 2.1.1. *The function $x \in \Omega$ is solution of the problem* (2.0.1) *if there exists one function $v \in L^1(J, E)$, such that*
$v(t) \in F(t, x_t, x(t - \tau(t, x_t, \sigma)))$, for each $t \in J$, i.e. $v(t) \in S_{F,x}$, for each $t \in J$, with

$$x(t) = \begin{cases} \varphi(t), & t \in I \\ \varphi(0) - g(0, x(-\eta(0))) + g(t, x(t - \eta(t))) + \int_0^t v(s)ds \ , & t \in J. \end{cases}$$

2.1.1 Convex case

For the convex case we introduce the following hypotheses:

(H_1) The multifunction $F : J \times C(I,E) \times E \to \mathcal{P}(E)$ has convex values and there exist a continuous and nondecreasing function $\psi : [0,+\infty) \to (0,\infty)$ and there exist a function $p \in L^1(J,\mathbb{R}^+)$ such that

$$\|F(t,u,v)\|_E \leq p(t)\psi(\|u\|_\infty + \|v\|_E)$$

for every $t \in J$ and $u \in C(I,E)$, $v \in E$.

(H_2) (i) There exist constants $0 < d_1 < \frac{1}{2}$ and $d_2 \geq 0$ such that

$$\|g(t,u)\|_E \leq d_1\|u\|_E + d_2, \text{ for each } t \in J, u \in E,$$

(ii) the subset $\{t \mapsto g(t,u) : u \in B\}$ is equicontinuous for all subset bounded B of E.

(H_3) There exist an constant $M > 0$ such that

$$\alpha(g(t,B)) \leq M\alpha(B)$$

for each $t \in J$ and for all subset bounded $B \subset E$.

(H_4) There exist functions $Q_1, Q_2 \in L^1(J,\mathbb{R}^+)$ such that

$$\alpha(F(t,A,B)) \leq Q_1(t)\sup_{s\in I}\alpha(A(s)) + Q_2(t)\alpha(B)$$

for each $t \in J$ and for all subsets bounded $A \subset C(I,E), B \subset E$, with $M + \|Q\|_{L^1} < 1$.

Let $\nu = -\eta(t)$, and $\omega = -\tau(t,x_t,\sigma)$, hance $\nu,\omega \in [-r,0]$, then we obtain $x(t-\eta(t)) = x(t+\nu) = x_t(\nu)$, and $x(t-\tau(t,x_t,\sigma)) = x(t+\omega) = x_t(\omega)$ this implies that, $g(t,x(t-\eta(t))) = g(t,x_t(\nu))$, and $F(t,x_t,x(t-\tau(t,x_t,\sigma))) = F(t,x_t,x_t(\omega))$.

Our result is given by

Theorem 2.1.1. *Assume that* (H_1)*-* (H_4) *hold, and if further*

$$\frac{1}{1-2d_1}\int_0^T p(s) < \int_c^\infty \frac{d\gamma}{\psi(2\gamma)}, \tag{2.1.1}$$

with $c = \frac{\|\varphi\|_\infty + 2d_2}{1-2d_1}$ *a positive constant. Then the problem* (2.0.1) *has at least one solution.*

Proof. Transform the problem (2.0.1) into a fixed point problem. Consider the operator $N : \Omega \to \mathcal{P}(\Omega)$, defined by

$$N(x)(t) = \left\{ h \in \Omega : h(t) = \begin{cases} \varphi(t), & \text{t} \in I \\ \varphi(0) - g(0, x(\nu)) + g(t, x_t(\nu)) + \int_0^t v(s)ds \ , & \text{t} \in J \end{cases} \right\},$$

with $v \in S_{F,x}$.

Note that a fixed point of N is a solution of (2.0.1).
We will show that N satisfies the assumptions of the nonlinear alternative of Leray-Schauder. The proof will be given in several steps.

Step 1: N is convex.
Let $h_1, h_2 \in N(x)$ with $x \in \Omega$, then there exist $v_1, v_2 \in S_{F,x}$, such that for every $t \in [-r, T]$, we have

$$h_i(t) = \begin{cases} \varphi(t), & \text{t} \in I \\ \varphi(0) - g(0, \varphi(\nu)) + g(t, x_t(\nu)) + \int_0^t v_i(s)ds \ , & \text{t} \in J, \end{cases}$$

with $i = 1.2$.
For every $\lambda \in [0, 1]$ we have:
if $t \in I$, then

$$\lambda h_1(t) + (1 - \lambda)h_2(t) = \varphi(t) \in N(x).$$

If $t \in J$, then

$$\lambda h_1(t) + (1 - \lambda)h_2(t) = \varphi(0) - g(0, \varphi(\nu)) + g(t, x_t(\nu)) + \int_0^t (\lambda v_1(s) + (1 - \lambda)v_2(s))ds.$$

Since F is a convex value then $S_{F,x}$ is convex and we have
$(\lambda v_1(s)+(1-\lambda)v_2(s)) \in S_{F,x}$ and therefore $\lambda h_1(t)+(1-\lambda)h_2(t) \in N(x)$, so N is convex.

Step 2: N is completely continuous.
(i) We show that N send the bounded into the bounded subset of Ω.
Let $B_q := \{x \in \Omega : \|x\|_\infty \leq q\}$. Clearly, the subset B_q is bounded of Ω. We take $x \in B_q$, and $h \in N(x)$, then there exist a function $v \in L^1(J, E)$, such that
$v(t) \in F(t, x_t, x_t(w))$, for every $t \in J$, with

$$h(t) = \begin{cases} \varphi(t), & \text{t} \in I \\ \varphi(0) - g(0, \varphi(\nu)) + g(t, x_t(\nu)) + \int_0^t v(s)ds \ , & \text{t} \in J. \end{cases}$$

For each $t \in J$ we have

$$\begin{aligned}\|h(t)\| &\leq \|\varphi(0) - g(0,\varphi(\nu))\| + \|g(t,x_t(\nu))\| + \|\int_0^t v(s)ds\| \\ &\leq \|\varphi\|_\infty + d_1\|\varphi\|_\infty + d_1\|x\|_\infty + 2d_2 + \psi(2q)\int_0^t p(s)ds \\ &\leq (1+d_1)\|\varphi\|_\infty + d_1 q + 2d_2 + \psi(2q)\|p\|_{L^1} := l_1.\end{aligned}$$

Hence for each $t \in [-r,T]$ we obtain

$$\|h\|_\infty \leq max(\|\varphi\|_\infty, l_1) := l.$$

Since l depends only on the constants d_1, d_2, q and the functions φ, p, ψ, so for every subset B_q bounded of Ω we have the subset $N(B_q)$ is bounded of Ω.
(ii) We show that N send the bounded into the equicontinuous subset of Ω.
Let $t_1, t_2 \in J$,with $t_1 < t_2$ and B_q be a bounded subset of Ω as in (i). Let $x \in B_q$, then for each $h \in N(x)$ we have

$$\|h(t_2) - h(t_1)\| \leq \|g(t_2, x_{t_2}(\nu)) - g(t_1, x_{t_1}(\nu))\| + \int_{t_1}^{t_2} p(s)\psi(2q)ds.$$

According to (ii) in (H_2) we will have the right-hand side tends to zero as $t_2 - t_1 \to 0$, implies the left-hand side tends to zero as $t_2 - t_1 \to 0$ so for every subset B_q bounded of Ω we have the subset $N(B_q)$ is equicontinuous of Ω.
(iii) Let $V \subset B_q$ such that $V \subset \overline{conv}(N(V))$. We have $V \subset \overline{conv}(N(V))$, bounded and equicontinuous, hence V is also, and therefore the function $t \mapsto \alpha(V(t))$, is continuous on J, with $V(t) = \{x(t) \in E : x \in V\}$. Let $V_t = \{x_t \in C(I,E) : x \in V\}$, by $(H_3) - (H_4)$ and the properties of the measure α, we have for every $t \in J$,

$$\alpha(\overline{conv}(N(V_t))) = \sup_{\theta \in I} \alpha(N(V_t(\theta)))$$

, with

$$\sup_{\theta \in I} \alpha(N(V_t(\theta))) = \alpha\{\varphi(0) - g(0,\varphi_0(\theta)) + g(t,V_t(\theta)) + \int_0^t v(s)ds : v(s) \in F(s,V_s,V_s(w))\},$$

and therefore

$$\begin{aligned}\sup_{\theta \in I} \alpha(V_t(\theta)) &\leq \alpha(g(t,V_t(\theta))) + \int_0^t \alpha\{v(s) : v(s) \in F(s,V_s,V_s(w))\}ds \\ &\leq M \sup_{\theta \in I} \alpha(V_t(\theta)) + \int_0^t (Q_1(s) \sup_{\theta \in I} \alpha(V_s(\theta)) + Q_2(s)\alpha(V_s(w)))ds.\end{aligned}$$

Set $Q(s) = Q_1(s) + Q_2(s)$, $\alpha(V) = \sup\{\alpha(V_t(\theta)) : \theta \in I\}$, hence we obtain the following estimate

$$\alpha(V) \leq (M + \|Q\|_{L^1})\alpha(V).$$

This implies that

$$[1-(M+\|Q\|_{L^1})]\alpha(V) \quad \leq \quad 0.$$

Using the condition $M+\|Q\|_{L^1}<1$ we obtain $\alpha(V)=0$, and therefore the subset V is relatively compact in Ω. So with (i), (ii) and (iii) the lemma conditions of Arzelá -Ascoli are satisfied, this implies that N is completely continuous.

Step 3: N has a closed graph.
Let $\{x_n\}_{n\in\mathbb{N}} \in \Omega$ be a sequence such that $x_n \to \widetilde{x} \in \Omega$, and let $h_n \in N(x_n)$ with $h_n \to \widetilde{h}$. We proof that $\widetilde{h} \in N(\widetilde{x})$. We have $h_n \in N(x_n)$ so there exist $v_n \in S_{F,x_n}$ such that

$$h_n(t)=\begin{cases} \varphi^n(t), & \text{t} \in I \\ \varphi^n(0)-g(0,\varphi^n(\nu))+g(t,x_t^n(\nu))+\displaystyle\int_0^t v_n(s)ds\ , & \text{t} \in J. \end{cases}$$

Now, just show that there exist $\widetilde{v} \in S_{F,\widetilde{x}}$, such that

$$\widetilde{h}(t)=\begin{cases} \widetilde{\varphi}(t), & \text{t} \in I \\ \widetilde{\varphi}(0)-g(0,\widetilde{\varphi}(\nu))+g(t,\widetilde{x}_t(\nu))+\displaystyle\int_0^t \widetilde{v}(s)ds\ , & \text{t} \in J. \end{cases}$$

It's clear that

$$\|(h_n(t)-\varphi^n(0)+g(0,\varphi^n(\nu))-g(t,x_t^n(\nu)))-(\widetilde{h}(t)-\widetilde{\varphi}(0)+g(0,\widetilde{\varphi}(\nu))-g(t,\widetilde{x}_t(\nu)))\|$$

tends to zero as $n \to +\infty$. Then we define the linear continuous operator
$\Gamma : L^1(J,E) \to C(J,E)$ by

$$\Gamma(v)(t)=\int_0^t v(s)ds.$$

So

$$\Gamma(v_n)(t)=\int_0^t v_n(s)ds=h_n(t)-\varphi^n(0)+g(0,\varphi^n(\nu))-g(t,x_t^n(\nu)),$$

implies

$$h_n(t)-\varphi^n(0)+g(0,\varphi^n(\nu))-g(t,x_t^n(\nu)) \in \Gamma(S_{F,x_n}).$$

By the Lemma 1.1.7, we obtain $\Gamma(S_{F,x})$ has a closed graph, and since

$$h_n(t)-\varphi^n(0)+g(0,\varphi^n(\nu))-g(t,x_t^n(\nu)) \to \widetilde{h}(t)-\widetilde{\varphi}(0)+g(0,\widetilde{\varphi}(\nu))-g(t,\widetilde{x}_t(\nu)).$$

Then

$$\widetilde{h}(t)-\widetilde{\varphi}(0)+g(0,\widetilde{\varphi}(\nu))-g(t,\widetilde{x}_t(\nu)) \in \Gamma(S_{F,\widetilde{x}}),$$

so

$$\int_0^t \widetilde{v}(s)ds \in \Gamma(S_{F,\widetilde{x}}),$$

this implies that

$$\widetilde{v}(t) \in S_{F,\widetilde{x}},$$

therefor

$$\widetilde{h} \in N(\widetilde{x}).$$

Consequently the multi-valued operator N has a closed graph.

Step 4: We proof that the subset $B_\lambda := \{x \in \Omega : x \in \lambda N(x), for\ some\ \lambda \in (0,1)\}$ is bounded. Let $x \in B_\lambda$, hence for each $t \in [-r, T]$ we have,

$$x(t) = \begin{cases} \lambda\varphi(t), & \text{t} \in I \\ \lambda(\varphi(0) - g(0,\varphi(\nu)) + g(t, x_t(\nu)) + \displaystyle\int_0^t v(s)ds) \ , & \text{t} \in J. \end{cases}$$

And we have the following estimates

$$\begin{aligned} \|x(t)\| &\leq \|\varphi(0) - g(0, x_0(\nu))\| + \|g(t, x_t(\nu))\| + \| \int_0^t v(s)ds\| \\ &\leq \|\varphi\|_\infty + 2d_1\|x\|_\infty + 2d_2 + \int_0^t p(s)\psi(\|x_s\| + \|x_s(w)\|)ds. \end{aligned}$$

This implies that

$$\|x\|_\infty \leq \frac{\|\varphi\|_\infty + 2d_2}{1 - 2d_1} + \frac{1}{1 - 2d_1}\int_0^t p(s)\psi(2\|x_s\|)ds.$$

We consider the function $\mu : [-r, T] \to \mathbb{R}^+$ define by

$$\mu(t) = \sup\{\|x(t)\| : t \in [-r, T]\}.$$

Hence

$$\mu(t) \leq \begin{cases} \|\varphi\|_\infty, & \text{t} \in I \\ \frac{\|\varphi\|_\infty + 2d_2}{1-2d_1} + \frac{1}{1-2d_1}\displaystyle\int_0^t p(s)\psi(2\|x_s\|)ds \ , & \text{t} \in J. \end{cases}$$

Let $t^\star \in [-r, T]$ such that $\mu(t) =\| x(t^\star) \|$, hence if $t^\star \in I$ we have

$$\mu(t) \leq \|\varphi\|_\infty.$$

If $t^\star \in J$, then

$$\mu(t) \leq \frac{\|\varphi\|_\infty + 2d_2}{1 - 2d_1} + \frac{1}{1 - 2d_1}\int_0^t p(s)\psi(2\mu(s))ds.$$

We take the right-hand side as a function $u(t)$, hence

$$u(0) = \frac{\|\varphi\|_\infty + 2d_2}{1 - 2d_1} := c,$$

and

$$\mu(t) \leq u(t) \; for \; each, \; t \in J.$$

Moreover we have

$$u^{'}(t) = \frac{1}{1 - 2d_1} p(t)\psi(2\mu(t)).$$

Since the function ψ is nondecreasing then

$$u^{'}(t) \leq \frac{1}{1 - 2d_1} p(t)\psi(2u(t)).$$

This implies that

$$\frac{u^{'}(t)}{\psi(2u(t))} \leq \frac{1}{1 - 2d_1} p(t),$$

so

$$\begin{aligned} \int_c^{u(t)} \frac{d\gamma}{\psi(2\gamma)} &\leq \frac{1}{1 - 2d_1} \int_0^t p(s)ds \\ &\leq \frac{1}{1 - 2d_1} \int_0^T p(s)ds. \end{aligned}$$

Using (2.1.1) to have that

$$\int_c^{u(t)} \frac{d\gamma}{\psi(2\gamma)} < \int_c^{\infty} \frac{d\gamma}{\psi(2\gamma)},$$

this implies that

$$u(t) < +\infty.$$

Hence there exists a constant $0 < \widetilde{b} < +\infty$ such that

$$u(t) \leq \widetilde{b}$$

so

$$\|x(t)\| \leq \widetilde{b} \; for \; each \; t \in J.$$

And Consequently we have

$$\|x\|_\infty \leq \max(\|\varphi\|_\infty, \widetilde{b}) := b \; on \; [-r, T].$$

With b depends only of the constants d_1, d_2, r, and the functions p, φ, ψ, and therefore B_λ, is bounded, so by nonlinear alternative of Leray-Schauder the operator N has at least one fixed point which is solution of the problem (2.0.1).

□

2.1.2 Non-convex case

For this case, we use the Filippov's theorem on a bounded interval.
Let $\psi \in C(I,E), h \in L^1(J,E)$, and let $y \in \Omega$ be a solution of the following differential problem with semi-linear equation:

$$\begin{cases} \frac{d}{dt}(y(t) - g(t, y_t(\nu))) = h(t) & \text{t} \in J \\ y(t) = \psi(t)\ , & \text{t} \in I. \end{cases}$$

We will consider the following assumptions:

(A_1) The multi-function $F : J \times C(I,E) \times E \to \mathcal{P}(E)$ has closed values, such that
(a) for all $u \in C(I,E), v \in E, t \mapsto F(t,u,v)$ is measurable,
(b) the map $\gamma : t \mapsto d(h(t), F(t, y_t, y_t(\omega)))$ is integrable.

(A_2) There exist a function $Q \in L^1(J, \mathbb{R}^+)$, such that

$$\|g(t,u) - g(t,v)\|_E \leq Q(t)\|u - v\|_E,$$

for all $u, v \in E$.

(A_3) There exist a function $p \in L^1(J, \mathbb{R}^+)$, and a strictly positive constant β such that

$$H_d(F(t,u_1,v_1); F(t,u_2,v_2)) \leq p(t)(\|u_1 - u_2\|_\infty + \|v_1 - v_2\|_E),$$

for all $t \in J$,and $u_i, v_i \in \overline{B}(y_t, \beta)$, with $i = 1.2$ and $\overline{B}(y_t, \beta)$ is the closed ball of $C(I,E)$.

(A_4) The functions p and Q satisfies the following condition,

$$2Q(t) + 2\int_0^t p(s)ds \leq 1 \text{ for every } t \in J.$$

We define the function ξ by

$$\xi(t) = \delta + 2p(t)\|x - y\|_\infty + \int_0^t \gamma(s)ds,$$

for each $t \in J$, with δ a strictly positive real. We have the following result

Theorem 2.1.2. *Assume that* (A_1)- (A_4) *and* $\xi(t) \leq \beta$, $\forall t \in J$ *hold. Then for all* $\varphi \in C(I,E)$ *with* $\|\varphi - \psi\|_\infty \leq \delta$,
the problem (2.0.1) *has at least one solution.*

Proof. Let $\varphi \in C(I,E)$ such that $\|\varphi - \psi\|_\infty \leq \delta$. We construct a sequence of function $(x^n(t))_{n\in\mathbb{N}}$ which will show that it is converging to some solution of problem (2.0.1) on the interval $[-r,T]$. Let $f_0 = h$ on J and $x^0(t) = y(t)$ for all $t \in [-r,T]$, i.e.

$$x^0(t) = \begin{cases} \psi(t), & \text{t} \in I \\ \psi(0) - g(0,y_0(\nu)) + g(t,y_t(\nu)) + \displaystyle\int_0^t f_0(s)ds\ , & \text{t} \in J. \end{cases}$$

Then we defined the multi-valued map $U_1 : J \to P(E)$, by

$$U_1(t) = F(t,x_t,x_t(\omega)) \cap B(h(t),\gamma(t)).$$

Since h and γ are measurable, Theorem 1.1.8 tells us that the ball $B(h(t),\gamma(t))$ is measurable. Moreover $F(t,x_t,x_t(\omega))$ is measurable and U_1 is nonempty.Indeed, since $v = 0$ is a measurable function, from lemma 1.1.4, there exists a function u which is a measurable selection of $F(t,x_t^0,x_t^0(\omega))$, and such that

$$\mid u(t) - h(t) \mid \leq d(h(t),F(t,x_t^0,x_t^0(\omega))) := \gamma(t),$$

so

$$\mid u(t) - h(t) \mid \leq \gamma(t),$$

hance

$$u(t) \in B(h(t),\gamma(t)) \ and \ u(t) \in F(t,x_t^0,x_t^0(\omega)),$$

this implies that

$$u(t) \in U_1(t).$$

We deduce that the intersection multi-valued operator $U_1(t)$ is measurable. By the Lemma 1.1.3, there exists a function $t \mapsto f_1(t)$ which a measurable selection for U_1. Consider

$$x^1(t) = \begin{cases} \varphi(t), & \text{t} \in I \\ \varphi(0) - g(0,x_0^0(\nu)) + g(t,x_t^0(\nu)) + \displaystyle\int_0^t f_1(s)ds\ , & \text{t} \in J. \end{cases}$$

Then for each $t \in J$ we have

$$\|x_t^1 - x_t^0\|_\infty \leq \|\varphi - \psi\|_\infty + 2Q(t)\|x_t - y_t\|_\infty + \int_0^t \mid f_1(s) - f_0(s) \mid ds,$$

this implies that

$$\|x_t^1 - x_t^0\|_\infty \leq \delta + 2Q(t)\|x_t - y_t\|_\infty + \int_0^t \gamma(s)ds := \xi(t) \leq \beta. \tag{2.1.2}$$

Then Lemma 1.1.6, tells us that $F(t,x_t^1,x_t^1(\omega)))$ is measurable. The ball $B(f_1(t),2p(t)\|x_t^1 - x_t^0\|_\infty)$ is also measurable by Theorem 1.1.8. Moreover the operator U_2 defined by

$$U_2(t) = F(t,x_t^1,x_t^1(\omega)) \cap B(f_1(t),2p(t)\|x_t^1 - x_t^0\|_\infty),$$

is measurable nonempty. Indeed, since f_1 is a measurable function, then by Lemma 1.1.4 we have the existence of a measurable selection u of $F(t, x_t^1, x_t^1(\omega))$, such that

$$\begin{aligned} \mid u(t) - f_1(t) \mid &\leq d(f_1(t), F(t, x_t^1, x_t^1(\omega))) \\ &\leq H_d(F(t, x_t^0, x_t^0(\omega)), F(t, x_t^1, x_t^1(\omega))) \\ &\leq 2p(t)\|x_t^1 - x_t^0\|_\infty, \end{aligned}$$

implies

$$u(t) \in B(f_1(t), 2p(t)\|x_t^1 - x_t^0\|_\infty) \text{ and } u(t) \in F(t, x_t^1, x_t^1(\omega)),$$

so

$$u(t) \in U_2(t).$$

Lemma 1.1.3 yields a measurable selection $f_2(t) \in U_2(t)$. Hence,

$$\mid f_2(t) - f_1(t) \mid \leq 2p(t)\|x_t^1 - x_t^0\|_\infty.$$

We define now x_t^2 by

$$x^2(t) = \begin{cases} \varphi(t), & \text{t} \in I \\ \varphi(0) - g(0, x_0^1(\nu)) + g(t, x_t^1(\nu)) + \displaystyle\int_0^t f_2(s)ds\ , & \text{t} \in J. \end{cases}$$

We obtain the following estimates:

$$\begin{aligned} \|x_t^2 - x_t^1\|_\infty &\leq 2Q(t)\|x_t^1 - x_t^0\|_\infty + \int_0^t \mid f_2(s) - f_1(s) \mid ds \\ &\leq 2Q(t)\|x_t^1 - x_t^0\|_\infty + \int_0^t 2p(s)\|x_s^1 - x_s^0\|_\infty ds \\ &\leq (2Q(t) + 2\int_0^t p(s)ds)\|x_t^1 - x_t^0\|_\infty. \end{aligned}$$

Set

$$r(t) = 2Q(t) + 2\int_0^t p(s)ds$$

and using (2.1.2) to have that

$$\|x_t^2 - x_t^1\|_\infty \leq r(t)\xi(t). \tag{2.1.3}$$

Defining now also the operator U_3 by

$$U_3(t) = F(t, x_t^2, x_t^2(\omega)) \cap B(f_2(t), 2p(t)\|x_t^2 - x_t^1\|_\infty).$$

Similarly with all the above we will have that: $U_3(t)$ is a measurable multi-valued operator with nonempty values, so there exists a measurable selection $f_3(t) \in U_3(t)$. This allows us to define

$$x^3(t) = \begin{cases} \varphi(t), & \text{t} \in I \\ \varphi(0) - g(0, x_0^2(\nu)) + g(t, x_t^2(\nu)) + \displaystyle\int_0^t f_3(s)ds\ , & \text{t} \in J. \end{cases}$$

For $t \in J$ we obtain the estimates:

$$\begin{aligned}
\|x_t^3 - x_t^2\|_\infty &\leq 2Q(t)\|x_t^2 - x_t^1\|_\infty + \int_0^t \mid f_3(s) - f_2(s) \mid ds \\
&\leq 2Q(t)\|x_t^2 - x_t^1\|_\infty + \int_0^t 2p(s)\|x_s^2 - x_s^1\|_\infty ds \\
&\leq (2Q(t) + 2\int_0^t p(s)ds)\|x_t^2 - x_t^1\|_\infty \\
&\leq r(t)\|x_t^2 - x_t^1\|_\infty.
\end{aligned}$$

We use (2.1.3) to have that

$$\|x_t^3 - x_t^2\|_\infty \leq (r(t))^2\xi(t). \tag{2.1.4}$$

The same manner be we define the operator U_4 by

$$U_4(t) = F(t, x_t^3, x_t^3(\omega)) \cap B(f_3(t), 2p(t)\|x_t^3 - x_t^2\|_\infty).$$

Arguing as for U_2, we can prove that U_4 is a measurable multi-valued operator with nonempty values, so there exists a measurable selection $f_4(t) \in U_4(t)$. Then we define x_t^4 by

$$x^4(t) = \begin{cases} \varphi(t), & \text{t} \in I \\ \varphi(0) - g(0, x_0^3(\nu)) + g(t, x_t^3(\nu)) + \int_0^t f_4(s)ds\ , & \text{t} \in J. \end{cases}$$

And we will obtain the estimate:

$$\begin{aligned}
\|x_t^4 - x_t^3\|_\infty &\leq 2Q(t)\|x_t^3 - x_t^2\|_\infty + \int_0^t \mid f_4(s) - f_3(s) \mid ds \\
&\leq 2Q(t)\|x_t^3 - x_t^2\|_\infty + \int_0^t 2p(s)\|x_s^3 - x_s^2\|_\infty ds \\
&\leq (2Q(t) + 2\int_0^t p(s)ds)\|x_t^3 - x_t^2\|_\infty \\
&\leq r(t)\|x_t^3 - x_t^2\|_\infty.
\end{aligned}$$

And using (2.1.4) to have that

$$\|x_t^4 - x_t^3\|_\infty \leq (r(t))^3\xi(t).$$

We repeat this processor for $n = 1, 2, 3, 4, ...$, we arrive at the following bound

$$\|x_t^n - x_t^{n-1}\|_\infty \leq (r(t))^{n-1}\xi(t). \tag{2.1.5}$$

By induction, suppose that (2.1.5) holds for some n and check (2.1.5) for $n + 1$. Let U_{n+1} defined by the following form

$$U_{n+1} := F(t, x_t^n, x_t^n(\omega)) \cap B(f_n(t), 2p(t)\|x_t^n - x_t^{n-1}\|_\infty).$$

As in the above arguing U_{n+1} is a nonempty measurable, then has a measurable selection $f_{n+1}(t) \in U_{n+1}(t)$, this allows us to define for $n \in \mathbb{N}$

$$x^{n+1}(t) = \begin{cases} \varphi(t), & \text{t} \in I \\ \varphi(0) - g(0, x_0^n(\nu)) + g(t, x_t^n(\nu)) + \displaystyle\int_0^t f_{n+1}(s)ds \ , & \text{t} \in J. \end{cases}$$

Therefore for each $t \in J$, we have

$$\begin{aligned} \|x_t^{n+1} - x_t^n\|_\infty &\leq 2Q(t)\|x_t^n - x_t^{n-1}\|_\infty + \int_0^t \mid f_{n+1}(s) - f_n(s) \mid ds \\ &\leq 2Q(t)\|x_t^n - x_t^{n-1}\|_\infty + \int_0^t 2p(s)\|x_t^n - x_t^{n-1}\|_\infty ds \\ &\leq (2Q(t) + 2\int_0^t p(s)ds)\|x_t^n - x_t^{n-1}\|_\infty \\ &\leq r(t)\|x_t^n - x_t^{n-1}\|_\infty. \end{aligned}$$

Using (2.1.5) to have that

$$\|x_t^{n+1} - x_t^n\|_\infty \leq (r(t))^n \xi(t).$$

Consequently, (2.1.5) holds true for all $n \in \mathbb{N}$. We deduce that (x_t^n) is a Cauchy sequence in the Banach space $\Omega := C([-r, T], E)$, since Ω is complete then (x_t^n) is converging to a limit function $x(t) \in \Omega$. Moreover, from the definition of U_{n+1}, we have

$$\mid f_{n+1}(t) - f_n(t) \mid \leq 2p(t)\|x_t^n - x_t^{n-1}\|_\infty \ for \ each \ t \in J.$$

Hence for almost every $t \in J$, $f_n(t)$ is also a Cauchy sequence in the Banach space E and then converging almost everywhere to some measurable function $f(t) \in E$. On the other hand we have by definition of (x_t^n), for each $t \in J$

$$x_n(t) - \varphi(0) + g(0, x_0^{n-1}(\nu)) - g(t, x_t^{n-1}(\nu)) = \int_0^t f_n(s)ds.$$

Using now,the fact that g is continuous in second variable, with the Lebesgue dominated convergence theorem, and passing to the limit as $n \to +\infty$ we have

$$\lim_{n\to+\infty}(x_n(t) - \varphi(0) + g(0, x_0^{n-1}(\nu)) - g(t, x_t^{n-1}(\nu))) = \lim_{n\to+\infty}\int_0^t f_n(s)ds),$$

so

$$x(t) - \varphi(0) + g(0, x_0(\nu)) - g(t, x_t(\nu))) = \int_0^t f(s)ds,$$

this implies that

$$x(t) = \varphi(0) - g(0, x_0(\nu)) + g(t, x_t(\nu))) + \int_0^t f(s)ds, \ for\ each\ t \in J$$

and $x(t) = \varphi(t)$, for every $t \in I$ hence

$$x(t) = \begin{cases} \varphi(t), & \text{t} \in I \\ \varphi(0) - g(0, x_0(\nu)) + g(t, x_t(\nu)) + \int_0^t f(s)ds\ , & \text{t} \in J, \end{cases}$$

where $f \in S_{F,x}$, therefore x is solution of the problem (2.0.1). □

Chapter 3

System of differential inclusions with delay

In this chapter, we will give some existence results of solutions to ensure that the solutions set is nonempty, next we will interested on the study of the topological and geometrical structure of this solution set. We use in this study: the fixed point theory (Nonlinear Alternative of Leray-Schauder) combining with the semi-group theory and the compactness measure. We consider the following problem

$$\begin{cases} \frac{d}{dt}(x(t) - g_1(t, x_t, y_t)) - A_1 x(t) \in F(t, x_t, y_t), & \text{t} \in J := [0, b] \\ \frac{d}{dt}(y(t) - g_2(t, x_t, y_t)) - A_2 y(t) \in G(t, x_t, y_t), & \text{t} \in J \\ x(t) = \varphi(t) \text{ and } y(t) = \psi(t), & \text{t} \in I := [-r, 0], \end{cases} \tag{3.0.1}$$

where A_1 and A_2 are infinitesimal generators of two strongly continuous semigroups $\{T_1(t), T_2(t)\}_{t\geq 0}$ on a Banach space E, with the operators A_i not necessarily bounded, $F, G : J \times C(I, E) \times C(I, E) \to \mathcal{P}_{cv}(E)$, are multi-valued maps, $g_i : J \times C(I, E) \times C(I, E) \to E$, are continuous vuxctions with $i = 1.2$, and $\varphi, \psi : I \to E$, are given continuous functions.

3.1 Existence of mild solutions

Before starting the first thing to do is to see what is the possible solution of our problem, for this we have the following definition.

Definition 3.1.1. *The function $(x, y) \in \Omega^2$ is said solution of problem (3.0.1) if for every $t \in [-r, b]$ we have*

$$x(t) = \begin{cases} \varphi(t), & t \in I \\ g_1(t, x_t, y_t) + T_1(t)[\varphi(0) - g_1(0, \varphi_0, \psi_0)] & \\ + \int_0^t T_1(t-s)[A_1 g_1(s, x_s, y_s) + v(s)]ds \ , & t \in J, \end{cases}$$

and

$$y(t) = \begin{cases} \psi(t), & t \in I \\ g_2(t, x_t, y_t) + T_2(t)[\psi(0) - g_2(0, \varphi_0, \psi_0)] & \\ + \displaystyle\int_0^t T_2(t-s)[A_2 g_2(s, x_s, y_s) + u(s)]ds \ , & t \in J. \end{cases}$$

With $\Omega = C([-r, b], E)$ and r, b are strictly positives constants. We introduce the following hypotheses:

(H_1) $F, G : J \times C(I, E) \times C(I, E) \to \mathcal{P}_{cv}(E)$ are multifunction with nonempty convex values, such that there exist $p_i \in L^1(J, \mathbb{R}^+)$ and there exist continuous and nondecreasing functions $\xi_i : [0, +\infty) \to (0, +\infty)$, $i = 1.2$ such that

$$\|F(t, x, y)\|_E \leq p_1(t)\xi_1(\|x\|_\infty + \|y\|_\infty) \text{ and } \|G(t, x, y)\|_E \leq p_2(t)\xi_2(\|x\|_\infty + \|y\|_\infty),$$

for every $t \in J$ and $x, y \in C(I, E)$.

(H_2) (i) There exists strictly positive constants $d_1 < \frac{1}{2}, d_2, d_3 < \frac{1}{2}$ *and* d_4, such that

$$\|g_1(t, x, y)\| \leq d_1(\|x\|_\infty + \|y\|_\infty) + d_2 \text{ and } \|g_2(t, x, y)\| \leq d_3(\|x\|_\infty + \|y\|_\infty) + d_4,$$

for every $t \in J$ and $x, y \in C(I, E)$,
(ii) the subset $\{t \mapsto g(t, u, v) : u, v \in B\}$ is equicontinuous for all subset bounded B of $C(I, E)$.

(H_3) There exist $Q_j \in L^1(J, \mathbb{R}^+)$ with $j = 1.2.3.4$ such that

$$\alpha(F(t, V_t, U_t)) \leq Q_1(t) \sup_{s \in I} \alpha(V_t(s)) + Q_2(t) \sup_{s \in I} \alpha(U_t(s)),$$

and

$$\alpha(G(t, V_t, U_t)) \leq Q_3(t) \sup_{s \in I} \alpha(V_t(s)) + Q_4(t) \sup_{s \in I} \alpha(U_t(s)),$$

for every $t \in J$ and for all subset bounded equicontinuous V_t, U_t of $C(I, E)$.

(H_4) $\alpha(g_1(t, V_t, U_t)) \leq d_1 \sup_{s \in I} \alpha(V_t(s)) + d_2 \sup_{s \in I} \alpha(U_t(s))$ *and* $\alpha(g_2(t, V_t, U_t)) \leq d_3 \sup_{s \in I} \alpha(V_t(s)) + d_4 \sup_{s \in I} \alpha(U_t(s))$, for every $t \in J$ and for all subset bounded equicontinuous V_t, U_t of $C(I, E)$.

(H_5) $1 > 2(d + \frac{dC_\beta b^{1-\beta}}{1-\beta} + M\|Q\|_{L^1})$.

Remark 3.1.1. *(a) The $T_i(t)$ are uniformly continuous semigroups, so there exists two constants $w_i \in \mathbb{R}$ and $\overline{M}_i > 0$ such that*

$$\|T_i(t)\|_{B(E)} \leq \overline{M}_i e^{w_i t}, \ \forall \ t > 0.$$

(b) As the A_i are infinitesimal generators of $T_i(t)$ that are uniformly continuous semigroups, so are the restrictions of semigroups analytic on the real positive half-line,

and thus by properties 1.5.6 we have the possibility to define $A_i^{\beta_i}$ *as are closed linear operators such that there exists constants* $C_{\beta_i} > 0$ *with* $\beta_i \in (0,1)$ *such that*

$$\|T_i(t)A_i\|_{B(E)} \simeq \|T_i(t)A_i^{\beta_i}\|_{B(E)} \leq \frac{C_{\beta_i}}{t^{\beta_i}},\ \forall t \in (0,b] \text{ with } i = 1.2.$$

But to simplify the calculus throughout the rest of this chapter set

$$\|T_i(t)\|_{B(E)} \leq M_i \text{ and } \frac{C_\beta}{t^\beta} = \max(\frac{C_{\beta_1}}{t^{\beta_1}}, \frac{C_{\beta_2}}{t^{\beta_2}}).$$

Now, we present our main result of this section.

Theorem 3.1.1. *Assume that assumptions* $(H_1) - (H_5)$ *hold, and if further*

$$\frac{(1-\beta)M}{1-\beta+2d(\beta+C_\beta b^{1-\beta}-1)}\int_0^b p(s)ds < \int_c^{+\infty}\frac{d\gamma}{\xi(\gamma)}. \tag{3.1.1}$$

Then the problem (3.0.1) *has at least one mild solution.*

Proof. Transform the problem (3.0.1) into a fixed point problem. Consider the operator $N : \Omega^2 \to \mathcal{P}(\Omega^2)$, defined by :$N(x,y) = \{(h_1,h_2) \in \Omega^2\}$, with

$$\begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} = \begin{cases} \begin{pmatrix} \varphi(t) \\ \psi(t) \end{pmatrix}, & \text{t} \in I \\ \begin{pmatrix} g_1(t,x_t,y_t) + T_1(t)[\varphi(0) - g_1(0,\varphi_0,\psi_0)] \\ + \int_0^t T_1(t-s)[A_1g_1(s,x_s,y_s) + v(s)]ds \\ g_2(t,x_t,y_t) + T_2(t)[\psi(0) - g_2(0,\varphi_0,\psi_0)] \\ + \int_0^t T_2(t-s)[A_2g_2(s,x_s,y_s) + u(s)]ds \end{pmatrix}, & \text{t} \in J, \end{cases}$$

where $v \in S_{F,(x,y)}$ and $u \in S_{G,(x,y)}$. It's clear that a fixed point of N is a solution of problem (3.0.1). First, we show that N satisfies the assumptions of Leray-Schauder nonlinear alternative. We will divide the proof into four steps.

Step 1: N is convex. Let $(h_1,h_2),(\overline{h}_1,\overline{h}_2) \in N(x,y)$ with $(x,y) \in \Omega^2$, then there exist $v,\overline{v} \in S_{F,(x,y)}$, and $u,\overline{u} \in S_{G,(x,y)}$, such that for every $t \in [-r,b]$, we have

$$\begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} = \begin{cases} \begin{pmatrix} \varphi(t) \\ \psi(t) \end{pmatrix}, & \text{t} \in I \\ \begin{pmatrix} g_1(t,x_t,y_t) + T_1(t)[\varphi(0) - g_1(0,\varphi_0,\psi_0)] + \\ \int_0^t T_1(t-s)[A_1g_1(s,x_s,y_s) + v(s)]ds \\ g_2(t,x_t,y_t) + T_2(t)[\psi(0) - g_2(0,\varphi_0,\psi_0)] + \\ \int_0^t T_2(t-s)[A_2g_2(s,x_s,y_s) + u(s)]ds \end{pmatrix}, & \text{t} \in J, \end{cases}$$

and

$$\begin{pmatrix} \overline{h}_1(t) \\ \overline{h}_2(t) \end{pmatrix} = \begin{cases} \begin{pmatrix} \varphi(t) \\ \psi(t) \end{pmatrix}, & t \in I \\ \begin{pmatrix} g_1(t,x_t,y_t) + T_1(t)[\varphi(0) - g_1(0,\varphi_0,\psi_0)] + \\ \int_0^t T_1(t-s)[A_1 g_1(s,x_s,y_s) + \overline{v}(s)]ds \\ g_2(t,x_t,y_t) + T_2(t)[\psi(0) - g_2(0,\varphi_0,\psi_0)] + \\ \int_0^t T_2(t-s)[A_2 g_2(s,x_s,y_s) + \overline{u}(s)]ds \end{pmatrix}, & t \in J. \end{cases}$$

For every $\lambda \in [0,1]$ we have:
if $t \in I$, then

$$\lambda \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} + (1-\lambda) \begin{pmatrix} \overline{h}_1(t) \\ \overline{h}_2(t) \end{pmatrix} = \begin{pmatrix} \varphi(t) \\ \psi(t) \end{pmatrix} \in N(x,y).$$

If $t \in J$, then

$$\lambda \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} + (1-\lambda) \begin{pmatrix} \overline{h}_1(t) \\ \overline{h}_2(t) \end{pmatrix} = \begin{pmatrix} g_1(t,x_t,y_t) + T_1(t)[\varphi(0) - g_1(0,\varphi_0,\psi_0)] + \\ \int_0^t T_1(t-s)[A_1 g_1(s,x_s,y_s)\lambda v(s) + (1-\lambda)\overline{v}(s)]ds \\ g_2(t,x_t,y_t) + T_2(t)[\psi(0) - g_2(0,\varphi_0,\psi_0)] + \\ \int_0^t T_2(t-s)[A_2 g_2(s,x_s,y_s)\lambda u(s) + (1-\lambda)\overline{u}(s)]ds \end{pmatrix}.$$

Since F and G have convex values then $S_{F,(x,y)}$ and $S_{G,(x,y)}$ are convex and we have $(\lambda v(s) + (1-\lambda)\overline{v}(s)) \in S_{F,(x,y)}$ and $(\lambda u(s) + (1-\lambda)\overline{u}(s)) \in S_{G,(x,y)}$, and therefore $\lambda(h_1(t), h_2(t)) + (1-\lambda)(\overline{h}_1(t), \overline{h}_2(t)) \in N(x,y)$, so N is convex.

Step 2: N is completely continuous.
(i) We show that N send the bounded into the bounded subset of Ω^2.
Let $B_q := \{(x,y) \in \Omega^2 : \|x\|_\infty + \|y\|_\infty \leq q\}$. Clearly, the subset B_q is bounded of Ω^2, we take $(x,y) \in B_q$, and $(h_1,h_2) \in N(x,y)$, then there exist functions $v, u \in L^1(J,E)$, such that
$v(t) \in F(t,x_t,y_t)$, and $u(t) \in G(t,x_t,y_t)$ for every $t \in J$, with

$$\begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} = \begin{cases} \begin{pmatrix} \varphi(t) \\ \psi(t) \end{pmatrix}, & t \in I \\ \begin{pmatrix} g_1(t,x_t,y_t) + T_1(t)[\varphi(0) - g_1(0,\varphi_0,\psi_0)] + \\ \int_0^t T_1(t-s)[A_1 g_1(s,x_s,y_s) + v(s)]ds \\ g_2(t,x_t,y_t) + T_2(t)[\psi(0) - g_2(0,\varphi_0,\psi_0)] + \\ \int_0^t T_2(t-s)[A_2 g_2(s,x_s,y_s) + u(s)]ds \end{pmatrix}, & t \in J. \end{cases}$$

By, (H_1), (H_2) and (b) of the Remark 3.1 to obtain the following estimates:
If $t \in I$, then

$$\|(h_1(t), h_2(t))\| = \|h_1(t)\| + \|h_2(t)\| \quad \leq \quad \|\varphi\|_\infty + \|\psi\|_\infty.$$

If $t \in J$, then

$$\begin{aligned}
\|h_1(t)\| &\leq \|g_1(t, x_t, y_t)\| + \|T_1(t)\|[\|\varphi(0) - g_1(0, \varphi_0, \psi_0)\|] \\
&+ \int_0^t \|T_1(t-s)A_1\|\|g_1(s, x_s, y_s)\|ds + \int_0^t \|T_1(t-s)\|\|v(s)\|ds \\
&\leq d_1 q + d_2 + M_1[\|\varphi\|_\infty + d_1(\|\varphi\|_\infty + \|\psi\|_\infty) + d_2] \\
&+ \int_0^t \frac{C_\beta}{(t-s)^\beta}[d_1 q + d_2]ds + M_1 \int_0^t p_1(s)\xi_1(q)ds.
\end{aligned}$$

In the same way we estimate h_2 as follows

$$\begin{aligned}
\|h_2(t)\| &\leq \|g_2(t, x_t, y_t)\| + \|T_2(t)\|[\|\psi(0) - g_2(0, \varphi_0, \psi_0)\|] \\
&+ \int_0^t \|T_2(t-s)A_2\|\|g_2(s, x_s, y_s)\|ds + \int_0^t \|T_2(t-s)\|\|u(s)\|ds \\
&\leq d_3 q + d_4 + M_2[\|\psi\|_\infty + d_3(\|\varphi\|_\infty + \|\psi\|_\infty) + d_4] \\
&+ \int_0^t \frac{C_\beta}{(t-s)^\beta}[d_3 q + d_4]ds + M_2 \int_0^t p_2(s)\xi_2(q)ds,
\end{aligned}$$

with

$$\int_0^t \frac{C_\beta}{(t-s)^\beta}ds = \lim_{w \to t} \int_0^w \frac{C_\beta}{(w-s)^\beta}ds.$$

So by a simple calculus we obtain

$$\int_0^t \frac{C_\beta}{(t-s)^\beta}ds = \frac{C_\beta}{1-\beta}t^{1-\beta}.$$

Now, let $M = \max(M_1, M_2)$, $d = \max(d_1, d_3)$, $\overline{d} = d_2 + d_4$, $\|p\|_{L^1} = \max(\|p_1\|_{L^1}, \|p_2\|_{L^1})$ and $\xi(q) = \xi_1(q) + \xi_2(q)$ and by addition we obtain

$$\begin{aligned}
\|h_1\|_\infty + \|h_2\|_\infty &\leq 2dq + \overline{d} + M[(1+2d)q + \overline{d} + \xi(q)\|p\|_{L^1}] + \frac{C_\beta}{1-\beta}b^{1-\beta}(2dq + \overline{d}) \\
&:= \widetilde{k}.
\end{aligned}$$

Finally we obtain

$$\|h_1\|_\infty + \|h_2\|_\infty \leq \max\{\|\varphi\|_\infty + \|\psi\|_\infty, \widetilde{k}\} := k.$$

Since k depends only on the constants $d, \overline{d}, q, M, b, \beta, C_\beta$ and the functions p, ξ, so for every subset B_q bounded of Ω^2 we have the subset $N(B_q)$ is bounded of Ω^2.

(ii) We will show that N send the bounded into the equicontinuous subset of Ω^2. Let $t_1, t_2 \in [-r, b]$, with $t_1 < t_2$ and B_q be a bounded subset of Ω^2 as in (i). Let $(x, y) \in B_q$, then we have

$$\begin{aligned}\|(h_1(t_2), h_2(t_2)) - (h_1(t_1), h_2(t_1))\| &= \|(h_1(t_2) - h_1(t_1), h_2(t_2) - h_2(t_1))\| \\ &= \|h_1(t_2) - h_1(t_1)\| + \|h_2(t_2) - h_2(t_1)\|.\end{aligned}$$

So if $t_1, t_2 \in I$ then

$$\|h_1(t_2) - h_1(t_1)\| + \|h_2(t_2) - h_2(t_1)\| \leq \|\varphi(t_2) - \varphi(t_1)\| + \|\psi(t_2) - \psi(t_1)\|.$$

We have by continuous of the functions φ and ψ

$$\|\varphi(t_2) - \varphi(t_1)\| + \|\psi(t_2) - \psi(t_1)\| \to 0 \ as \ t_1 \to t_2.$$

If $t_1, t_2 \in J$, then by (H_1), (H_2) with (b) in the Remark 3.1 we have

$$\begin{aligned}\|h_1(t_2) - h_1(t_1)\| &\leq \|g_1(t_2, x_{t_2}, y_{t_2}) - g_1(t_1, x_{t_1}, y_{t_1})\| \\ &+ \|T_1(t_2) - T_1(t_1)\|[\|x(0)\| + \|g_1(0, x_0, y_0)\|] \\ &+ \int_0^{t_1} \|(T_1(t_2 - s) - T_1(t_1 - s))A_1\|\|g_1(s, x_s, y_s)\|ds \\ &+ \int_{t_1}^{t_2} \|T_1(t_2 - s)A_1\|\|g_1(s, x_s, y_s)\|ds + \int_{t_1}^{t_2} \|T_1(t_2 - s)\|\|v(s)\|ds \\ &+ \int_0^{t_1} \|T_1(t_2 - s) - T_1(t_1 - s)\|\|v(s)\|ds \\ &\leq \|g_1(t_2, x_{t_2}, y_{t_2}) - g_1(t_1, x_{t_1}, y_{t_1})\| \\ &+ M_1\|T_1(t_2 - t_1) - Id\|[\|x(0)\| + d_1(\|x_0\| + \|y_0\|) + d_2] \\ &+ \|T_1(t_2 - t_1) - Id\| \int_0^{t_1} \|T_1(t_1 - s)A_1\|(d_1 q + d_2)ds \\ &+ \int_{t_1}^{t_2} \|T_1(t_2 - s)A_1\|(d_1 q + d_2)ds + M_1\xi_1(q) \int_{t_1}^{t_2} p_1(s)ds \\ &+ M_1\xi_1(q)\|T_1(t_2 - t_1) - Id\| \int_0^{t_1} p_1(s)ds \\ &\leq \|g_1(t_2, x_{t_2}, y_{t_2}) - g_1(t_1, x_{t_1}, y_{t_1})\| + (d_1 q + d_2) \int_{t_1}^{t_2} \frac{C_\beta}{(t_2 - s)^\beta} ds \\ &+ M_1\|T_1(t_2 - t_1) - Id\|[\|x(0)\| + d_1 q + d_2] \\ &+ (d_1 q + d_2)\|T_1(t_2 - t_1) - Id\| \int_0^{t_1} \frac{C_\beta}{(t_1 - s)^\beta} ds \\ &+ M_1\xi_1(q) \int_{t_1}^{t_2} p_1(s)ds + M_1\xi_1(q)\|T_1(t_2 - t_1) - Id\| \int_0^{t_1} p_1(s)ds.\end{aligned}$$

With $\int_0^{t_1} \frac{C_\beta}{(t_1-s)^\beta}ds = \frac{C_\beta}{1-\beta}(t_1)^{1-\beta}$ and $\int_{t_1}^{t_2} \frac{C_\beta}{(t_2-s)^\beta}ds = \frac{C_\beta}{1-\beta}(t_2-t_1)^{1-\beta}$.
And finally we obtain the following estimates

$$\begin{aligned}
\|h_1(t_2)-h_1(t_1)\| &\leq \|g_1(t_2,x_{t_2},y_{t_2})-g_1(t_1,x_{t_1},y_{t_1})\| + (d_1q+d_2)\frac{C_\beta}{1-\beta}(t_2-t_1)^{1-\beta} \\
&+ M_1\|T_1(t_2-t_1)-Id\|[\|x(0)\|+d_1q+d_2] \\
&+ (d_1q+d_2)\|T_1(t_2-t_1)-Id\|\frac{C_\beta}{1-\beta}b^{1-\beta} \\
&+ M_1\xi_1(q)\int_{t_1}^{t_2} p_1(s)ds + M_1\xi_1(q)\|T_1(t_2-t_1)-Id\|\int_0^{t_1} p_1(s)ds.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\|h_2(t_2)-h_2(t_1)\| &\leq \|g_2(t_2,x_{t_2},y_{t_2})-g_2(t_1,x_{t_1},y_{t_1})\| + (d_3q+d_4)\frac{C_\beta}{1-\beta}(t_2-t_1)^{1-\beta} \\
&+ M_2\|T_2(t_2-t_1)-Id\|[\|y(0)\|+d_3q+d_4] \\
&+ (d_3q+d_4)\|T_2(t_2-t_1)-Id\|\frac{C_\beta}{1-\beta}b^{1-\beta} \\
&+ M_2\xi_2(q)\int_{t_1}^{t_2} p_2(s)ds + M_2\xi_2(q)\|T_2(t_2-t_1)-Id\|\int_0^{t_1} p_2(s)ds.
\end{aligned}$$

We will use (ii) in hypotheses (H_2) on the functions g_i and the uniform continuity of $T_i(t)$ to see that the right-hand side tends to zero as $t_1 \to t_2$, this implies that

$$(\|h_1(t_2)-h_1(t_1)\| + \|h_2(t_2)-h_2(t_1)\|) \to 0 \ as \ t_1 \to t_2.$$

So for every subset B_q bounded of Ω^2 we have the subset $N(B_q)$ is equicontinuous of Ω^2.
(iii) Let $(V,U) \subset \overline{conv}(N(V,U))$, i.e. $V \subset \overline{conv}(N_1(V,U))$ and $U \subset \overline{conv}(N_2(V,U))$. We have $(V,U) \subset \overline{conv}(N(V,U))$, bounded and equicontinuous, hence (V,U) is also, and therefore the function $t \mapsto (\alpha(V(t)),\alpha(U(t)))$, is continuous on J. By $(H_3),(H_4)$ and the properties of the measure α, we have for every $t \in J$,

$$\alpha(\overline{conv}(N(V(t),U(t)))) = \alpha(N(V(t),U(t))) := (\alpha(N_1(V(t),U(t))),\alpha(N_2(V(t),U(t)))),$$

with $\alpha(N_1(V(t),U(t)) = \alpha\{g_1(t,V_t,U_t) + T_1(t)[\varphi(0)-g_1(0,\varphi_0,\psi_0)]$
$+ \int_0^t T_1(t-s)[A_1g_1(s,V_s,U_s)+v(s)]ds : v(s) \in F(s,V_s,U_s)\}$,
and $\alpha(N_2(V(t),U(t)) = \alpha\{g_2(t,V_t,U_t) + T_2(t)[\psi(0)-g_2(0,\varphi_0,\psi_0)]$
$+ \int_0^t T_2(t-s)[A_2g_2(s,V_s,U_s)+u(s)]ds : u(s) \in G(s,V_s,U_s)\}$.

Therefore

$$\begin{aligned}
\sup_{\theta\in I}\alpha(V_t(\theta)) &\leq \alpha(g_1(t,V_t,U_t)) + \int_0^t \|T_1(t-s)A_1\|\alpha(g_1(s,V_s,U_s))ds \\
&+ M_1\int_0^t \alpha\{v(s): v(s)\in F(s,V(s),U(s))\}ds \\
&\leq d_1\sup_{\theta\in I}\alpha(V_t(\theta)) + d_2\sup_{\theta\in I}\alpha(U_t(\theta)) + \int_0^t \frac{C_\beta}{(t-s)^\beta}(d_1\sup_{\theta\in I}\alpha(V_t(\theta)) + d_2\sup_{\theta\in I}\alpha(U_t(\theta)))ds \\
&+ M_1\int_0^t (Q_1(s)\sup_{\theta\in I}\alpha(V_s(\theta)) + Q_2(s)\sup_{\theta\in I}\alpha(U_s(\theta)))ds,
\end{aligned}$$

and

$$\begin{aligned}
\sup_{\theta\in I}\alpha(U_t(\theta)) &\leq \alpha(g_2(t,V_t,U_t)) + \int_0^t \|T_2(t-s)A_2\|\alpha(g_2(s,V_s,U_s))ds \\
&+ M_2\int_0^t \alpha\{u(s): u(s)\in G(s,V(s),U(s))\}ds \\
&\leq d_3\sup_{\theta\in I}\alpha(V_t(\theta)) + d_4\sup_{\theta\in I}\alpha(U_t(\theta)) + \int_0^t \frac{C_\beta}{(t-s)^\beta}(d_3\sup_{\theta\in I}\alpha(V_t(\theta)) + d_4\sup_{\theta\in I}\alpha(U_t(\theta)))ds \\
&+ M_2\int_0^t (Q_3(s)\sup_{\theta\in I}\alpha(V_s(\theta)) + Q_4(s)\sup_{\theta\in I}\alpha(U_s(\theta)))ds.
\end{aligned}$$

Set $M=\max(M_1,M_2)$, $d=\max(d_1,d_2,d_3,d_4)$,
$\|Q\|_{L^1}=\max(\|Q_1\|_{L^1},\|Q_2\|_{L^1},\|Q_3\|_{L^1},\|Q_4\|_{L^1})$, $(\alpha(V),\alpha(U)) = (\sup_{\theta\in I}\alpha(V_t(\theta)),\sup_{\theta\in I}\alpha(U_t(\theta))$
and by addition we obtain the following estimates

$$\alpha(V)+\alpha(U) \leq [2d + \frac{2dC_\beta b^{1-\beta}}{1-\beta} + 2M\|Q\|_{L^1}][\alpha(V)+\alpha(U)].$$

This implies that

$$[1-2(d+\frac{dC_\beta b^{1-\beta}}{1-\beta} + M\|Q\|_{L^1})][\alpha(V)+\alpha(U)] \leq 0.$$

Therefore with the condition (H_5), we obtain $\alpha(V)+\alpha(U)=0$, so $\alpha(V)=0$ and $\alpha(U)=0$, thus the subset (V,U) is relatively compact in E^2. So with (i), (ii) and (iii) the lemma conditions of Arzelá -Ascoli are satisfied, this implies that N is completely continuous.

Step 3: N is a closed graph.
Let $\{(x^n,y^n)\}_{n\in\mathbb{N}}\in\Omega^2$ be a sequence such that
$(x^n,y^n)\to(\widetilde{x},\widetilde{y})\in\Omega^2$, and let $(h_1^n,h_2^n)\in N(x^n,y^n)$ with $(h_1^n,h_2^n)\to(\widetilde{h_1},\widetilde{h_2})$. We need

to show that $(\widetilde{h}_1, \widetilde{h}_2) \in N(\widetilde{x}, \widetilde{y})$. We have $(h_1^n, h_2^n) \in N(x^n, y^n)$ so there exist $(v_n, u_n) \in S_{F,(x^n,y^n)} \times S_{G,(x^n,y^n)}$ such that

$$\begin{pmatrix} h_1^n \\ h_2^n \end{pmatrix} = \begin{cases} \begin{pmatrix} \varphi^n(t) \\ \psi^n(t) \end{pmatrix}, & \text{t} \in I \\ \begin{pmatrix} g_1(t, x_t^n, y_t^n) + T_1(t)[\varphi^n(0) - g_1(0, \varphi_0^n, \psi_0^n)] + \\ \int_0^t T_1(t-s)[A_1 g_1(s, x_s^n, y_s^n) + v_n(s)]ds \\ g_2(t, x_t^n, y_t^n) + T_2(t)[\psi^n(0) - g_2(0, \varphi_0^n, \psi_0^n)] + \\ \int_0^t T_2(t-s)[A_2 g_2(s, x_s^n, y_s^n) + u_n(s)]ds \end{pmatrix}, & \text{t} \in J. \end{cases}$$

Now, it remains to show that there exist $(\widetilde{v}, \widetilde{u}) \in S_{F,(\widetilde{x},\widetilde{y})} \times S_{G,(\widetilde{x},\widetilde{y})}$, such that

$$\begin{pmatrix} \widetilde{h}_1 \\ \widetilde{h}_2 \end{pmatrix} = \begin{cases} \begin{pmatrix} \widetilde{\varphi}(t) \\ \widetilde{\psi}(t) \end{pmatrix}, & \text{t} \in I \\ \begin{pmatrix} g_1(t, \widetilde{x}_t, \widetilde{y}_t) + T_1(t)[\widetilde{\varphi}(0) - g_1(0, \widetilde{\varphi}_0, \widetilde{\psi}_0)] + \\ \int_0^t T_1(t-s)[A_1 g_1(s, \widetilde{x}_s, \widetilde{y}_s) + \widetilde{v}(s)]ds \\ g_2(t, \widetilde{x}_t, \widetilde{y}_t) + T_2(t)[\widetilde{\psi}(0) - g_2(0, \widetilde{\varphi}_0, \widetilde{\psi}_0)] + \\ \int_0^t T_2(t-s)[A_2 g_2(s, \widetilde{x}_s, \widetilde{y}_s) + \widetilde{u}(s)]ds \end{pmatrix}, & \text{t} \in J. \end{cases}$$

For simplify the formulas set

$$\begin{pmatrix} w_1^n(t) \\ w_2^n(t) \end{pmatrix} = \begin{pmatrix} g_1(t, x_t^n, y_t^n) + T_1(t)[\varphi^n(0) - g_1(0, \varphi_0^n, \psi_0^n)] \\ + \int_0^t T_1(t-s) A_1 g_1(s, x_s^n, y_s^n)ds \\ g_2(t, x_t^n, y_t^n) + T_2(t)[\psi^n(0) - g_2(0, \varphi_0^n, \psi_0^n)] \\ + \int_0^t T_2(t-s) A_2 g_2(s, x_s^n, y_s^n)ds \end{pmatrix},$$

$$\begin{pmatrix} \widetilde{w}_1(t) \\ \widetilde{w}_2(t) \end{pmatrix} = \begin{pmatrix} g_1(t, \widetilde{x}_t, \widetilde{y}_t) + T_1(t)[\widetilde{\varphi}(0) - g_1(0, \widetilde{\varphi}_0, \widetilde{\psi}_0)] \\ + \int_0^t T_1(t-s) A_1 g_1(s, \widetilde{x}_s, \widetilde{y}_s)ds \\ g_2(t, \widetilde{x}_t, \widetilde{y}_t) + T_2(t)[\widetilde{\psi}(0) - g_2(0, \widetilde{\varphi}(0), \widetilde{\psi}_0)] \\ + \int_0^t T_2(t-s) A_2 g_2(s, \widetilde{x}_s, \widetilde{y}_s)ds. \end{pmatrix},$$

and

$$\begin{pmatrix} \Delta_1(t) \\ \Delta_2(t) \end{pmatrix} = \begin{pmatrix} (h_1^n - w_1^n)(t) \\ (h_2^n - w_2^n)(t) \end{pmatrix} - \begin{pmatrix} (\widetilde{h}_1 - \widetilde{w}_1)(t) \\ (\widetilde{h}_2 - \widetilde{w}_2)(t) \end{pmatrix}.$$

Then, we have

$$\| \begin{pmatrix} \Delta_1(t) \\ \Delta_2(t) \end{pmatrix} \| = \|(h_1^n - w_1^n)(t) - (\widetilde{h}_1 - \widetilde{w}_1)(t)\| + \|(h_2^n - w_2^n)(t) - (\widetilde{h}_2 - \widetilde{w}_2)(t)\|.$$

It is clear that this last quantity tends to zero as $n \to +\infty$. Then we defined the linear continuous operator $\Gamma : (L^1(J,E))^2 \to (C(J,E))^2$ by

$$\Gamma(v,u)(t) = (\Gamma_1(v)(t), \Gamma_2(u)(t)),$$

with

$$(\Gamma_1(v)(t), \Gamma_2(u)(t)) = \left(\int_0^t T_1(t-s)v(s)ds, \int_0^t T_2(t-s)u(s), ds \right).$$

Clearly

$$(\Gamma_1(v_n)(t), \Gamma_2(u_n)(t)) = ((h_1^n - w_1^n(t)), (h_2^n - w_2^n(t))).$$

By the Lemma 1.1.7 we have $(\Gamma_1(S_{F,(x^n,y^n)}), \Gamma_2(S_{G,(x^n,y^n)}))$ has a closed graph, and since

$$((h_1^n - w_1^n(t)), (h_2^n - w_2^n(t))) \to ((\widetilde{h}_1 - \widetilde{w}_1(t)), (\widetilde{h}_2 - \widetilde{w}_2(t))).$$

Then

$$((\widetilde{h}_1 - \widetilde{w}_1)(t), (\widetilde{h}_2 - \widetilde{w}_2)(t)) \in (\Gamma_1(S_{F,(\widetilde{x},\widetilde{y})}), \Gamma_2(S_{G,(\widetilde{x},\widetilde{y})})),$$

so

$$\left(\int_0^t T_1(t-s)\widetilde{v}(s)ds, \int_0^t T_2(t-s)\widetilde{u}(s), ds \right) \in \left(\Gamma_1(S_{F,(\widetilde{x},\widetilde{y})}), \Gamma_2(S_{G,(\widetilde{x},\widetilde{y})}) \right),$$

hence

$$(\widetilde{v}(t), \widetilde{u}(t)) \in (S_{F,(\widetilde{x},\widetilde{y})}, S_{G,(\widetilde{x},\widetilde{y})})$$

this implies that

$$(\widetilde{h_1}, \widetilde{h_2}) \in N(\widetilde{x}, \widetilde{y}).$$

Consequently the multi-valued operator N has a closed graph.

Step 4: We show that the subset
$B_\lambda = \{(x,y) \in \Omega^2 : (x,y) \in \lambda N(x,y), for\ some\ \lambda \in (0,1)\}$ is bounded. Let $(x,y) \in B_\lambda$, hence for each $t \in [-r,b]$ we have

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{cases} \lambda \begin{pmatrix} \varphi(t) \\ \psi(t) \end{pmatrix}, & \text{t} \in I \\ \lambda \begin{pmatrix} g_1(t, x_t, y_t) + T_1(t)[\varphi(0) - g_1(0, \varphi_0, \psi_0)] + \\ \int_0^t T_1(t-s)[A_1 g_1(s, x_s, y_s) + v(s)]ds \\ g_2(t, x_t, y_t) + T_2(t)[\psi(0) - g_2(0, \varphi_0, \psi_0)] + \\ \int_0^t T_2(t-s)[A_2 g_2(s, x_s, y_s) + u(s)]ds \end{pmatrix}, & \text{t} \in J. \end{cases}$$

By (H_1), (H_2) and (b) of the Remark 3.1 we have:
if $t \in I$ then

$$\|(x(t), y(t))\| = \|x(t)\| + \|y(t)\| \quad \leq \quad \|\varphi\|_\infty + \|\psi\|_\infty.$$

If $t \in J$ then

$$\begin{aligned}
\|x(t)\| &\leq \|g_1(t, x_t, y_t)\| + \|T_1(t)\|[\|\varphi(0) - g_1(0, \varphi_0, \psi_0)\|] \\
&+ \int_0^t \|T_1(t-s)A_1\|\|g_1(s, x_s, y_s)\|ds + \int_0^t \|T_1(t-s)\|\|v(s)\|ds \\
&\leq d_1(\|x\| + \|y\|) + d_2 + M_1[\|\varphi\|_\infty + d_1(\|\varphi\|_\infty + \|\psi\|_\infty) + d_2] \\
&+ \int_0^t \frac{C_\beta}{(t-s)^\beta} ds[d_1(\|x\| + \|y\|) + d_2] + M_1 \int_0^t p_1(s)\xi_1(\|x_s\| + \|y_s\|)ds \\
&\leq d_1(\|x\| + \|y\|) + d_2 + M_1[\|\varphi\|_\infty + d_1(\|\varphi\|_\infty + \|\psi\|_\infty) + d_2] \\
&+ \frac{C_\beta}{1-\beta} b^{1-\beta}[d_1(\|x\| + \|y\|) + d_2] + M_1 \int_0^t p_1(s)\xi_1(\|x_s\| + \|y_s\|)ds
\end{aligned}$$

In the same way we estimate $y(t)$ as follows

$$\begin{aligned}
\|y(t)\| &\leq \|g_2(t, x_t, y_t)\| + \|T_2(t)\|[\|\psi(0) - g_2(0, \varphi_0, \psi_0)\|] \\
&+ \int_0^t \|T_2(t-s)A_2\|\|g_2(s, x_s, y_s)\|ds + \int_0^t \|T_2(t-s)\|\|u(s)\|ds \\
&\leq d_3(\|x\| + \|y\|) + d_4 + M_2[\|\psi\|_\infty + d_3(\|\varphi\|_\infty + \|\psi\|_\infty) + d_4] \\
&+ \int_0^t \frac{C_\beta}{(t-s)^\beta} ds[d_3(\|x\| + \|y\|) + d_4] + M_2 \int_0^t p_2(s)\xi_2(\|x_s\| + \|y_s\|)ds \\
&\leq d_3(\|x\| + \|y\|) + d_4 + M_2[\|\psi\|_\infty + d_3(\|\varphi\|_\infty + \|\psi\|_\infty) + d_4] \\
&+ \frac{C_\beta}{1-\beta} b^{1-\beta}[d_3(\|x\| + \|y\|) + d_4] + M_2 \int_0^t p_2(s)\xi_2(\|x_s\| + \|y_s\|)ds.
\end{aligned}$$

Set $M = \max(M_1, M_2)$, $d = \max(d_1, d_3)$, $\overline{d} = d_2 + d_4$,
$p(t) = \max\{p_1(t), p_2(t); \forall t \in J\}$ and $\xi(t) = \xi_1(t) + \xi_2(t); \forall t \in [0, +\infty)$ and by addition we obtain

$$\begin{aligned}
\|x(t)\| + \|y(t)\| &\leq 2d(\|x_t\| + \|y_t\|) + \overline{d} + M[(1+2d)(\|\varphi\|_\infty + \|\psi\|_\infty) + \overline{d}] \\
&+ \frac{C_\beta}{1-\beta} b^{1-\beta}[2d(\|x_s\| + \|y_s\|) + \overline{d}] + M \int_0^t p(s)\xi(\|x_s\| + \|y_s\|)ds.
\end{aligned}$$

This implies that

$$\begin{aligned}\|x(t)\| + \|y(t)\| &\leq \frac{M[(1+2d)(\|\varphi\|_\infty + \|\psi\|_\infty) + \overline{d}] + \overline{d}\frac{C_\beta}{1-\beta}b^{1-\beta}}{1 - 2d(1 - \frac{C_\beta}{1-\beta}b^{1-\beta})} \\ &+ \frac{M}{1 - 2d(1 - \frac{C_\beta}{1-\beta}b^{1-\beta})}\int_0^t p(s)\xi(\|x_s\| + \|y_s\|)ds \\ &\leq \frac{(1-\beta)M[(1+2d)(\|\varphi\|_\infty + \|\psi\|_\infty) + \overline{d}] + \overline{d}C_\beta b^{1-\beta}}{1 - \beta + 2d(\beta + C_\beta b^{1-\beta} - 1)} \\ &+ \frac{(1-\beta)M}{1 - \beta + 2d(\beta + C_\beta b^{1-\beta} - 1)}\int_0^t p(s)\xi(\|x_s\| + \|y_s\|)ds.\end{aligned}$$

We consider the function $\mu : [-r, b] \to \mathbb{R}^+$ defined by

$$\mu(t) = \sup\{\|x(t)\| + \|y(t)\| : t \in [-r, b]\}.$$

Hence

$$\mu(t) \leq \begin{cases} \|\varphi\|_\infty + \|\psi\|_\infty, & \text{t} \in I \\ \frac{(1-\beta)M[(1+2d)(\|\varphi\|_\infty + \|\psi\|_\infty) + \overline{d}] + \overline{d}C_\beta b^{1-\beta}}{1 - \beta + 2d(\beta + C_\beta b^{1-\beta} - 1)} & \\ + \frac{(1-\beta)M}{1 - \beta + 2d(\beta + C_\beta b^{1-\beta} - 1)}\int_0^t p(s)\xi(\|x_s\| + \|y_s\|)ds. \ , & \text{t} \in J. \end{cases}$$

Let $t^\star \in [-r, b]$ such that $\mu(t) = \|x(t^\star)\| + \|y(t^\star)\|$, so we have

$$\mu(t) \leq \begin{cases} \|\varphi\|_\infty + \|\psi\|_\infty, & \text{t} \in I \\ \frac{(1-\beta)M[(1+2d)(\|\varphi\|_\infty + \|\psi\|_\infty) + \overline{d}] + \overline{d}C_\beta b^{1-\beta}}{1 - \beta + 2d(\beta + C_\beta b^{1-\beta} - 1)} & \\ + \frac{(1-\beta)M}{1 - \beta + 2d(\beta + C_\beta b^{1-\beta} - 1)}\int_0^t p(s)\xi(\mu(s))ds. \ , & \text{t} \in J. \end{cases}$$

For every $t \in J$ we take the right-hand side as a function $\nu(t)$, hence

$$\nu(0) = \frac{(1-\beta)M[(1+2d)(\|\varphi\|_\infty + \|\psi\|_\infty) + \overline{d}] + \overline{d}C_\beta b^{1-\beta}}{1 - \beta + 2d(\beta + C_\beta b^{1-\beta} - 1)} := c,$$

and

$$\mu(t) \ \leq \ \nu(t) \ for \ each, \ t \in J.$$

Moreover we have

$$\nu^{'}(t) = \frac{(1-\beta)M}{1 - \beta + 2d(\beta + C_\beta b^{1-\beta} - 1)}p(t)\xi(\mu(t)).$$

We use the growth of the function ξ to obtain the following estimate

$$\nu^{'}(t) \leq \frac{(1-\beta)M}{1-\beta+2d(\beta+C_\beta b^{1-\beta}-1)}p(t)\xi(\nu(t)),$$

implies

$$\frac{\nu^{'}(t)}{\xi(\nu(t))} \leq \frac{(1-\beta)M}{1-\beta+2d(\beta+C_\beta b^{1-\beta}-1)}p(t).$$

By passing to the integral we obtain

$$\begin{aligned}\int_{\nu(0)}^{\nu(t)} \frac{d\gamma}{\xi(\gamma)} &\leq \frac{(1-\beta)M}{1-\beta+2d(\beta+C_\beta b^{1-\beta}-1)}\int_0^t p(s)ds \\ &\leq \frac{(1-\beta)M}{1-\beta+2d(\beta+C_\beta b^{1-\beta}-1)}\int_0^b p(s)ds.\end{aligned}$$

Using (3.1.1) to have

$$\int_c^{\nu(t)} \frac{d\gamma}{\xi(\gamma)} < \int_c^{+\infty} \frac{d\gamma}{\xi(\gamma)}.$$

So $\nu(t) < +\infty$ therefore, there exist a positive constant k such that $\nu(t) \leq k$, this implies that

$$\|x\|_\infty + \|y\|_\infty \leq \max(\|\varphi\|_\infty + \|\psi\|_\infty, k)$$

that's finished, so the subset B_λ is bounded. And therefore the multi-values operator N satisfies all the conditions of Leray-Schauder nonlinear alternative, so he admits at least one fixed point which is mild solution of problem (3.0.1). □

Theorem 3.1.2. *Under the conditions of above theorem, the solution sets of* (3.0.1)*, is compact.*

Proof. Let $S(\varphi,\psi)$, be a solutions sets of the problem (3.0.1).

$S(\varphi,\psi)$ is a compact set in Ω^2.
Let the sequence $\{(x^n,y^n)\}_{n\in\mathbb{N}} \subset S(\varphi,\psi)$, then there exist a sequence $(v_n,u_n) \in S_{F,(x^n,y^n)} \times S_{G,(x^n,y^n)}$ such that

$$\begin{pmatrix} x^n(t) \\ y^n(t) \end{pmatrix} = \begin{cases} \begin{pmatrix} \varphi(t) \\ \psi(t) \end{pmatrix}, & \text{t} \in I \\ \begin{pmatrix} g_1(t,x_t^n,y_t^n) + T_1(t)[\varphi(0) - g_1(0,\varphi_0,\psi_0)] \\ + \int_0^t T_1(t-s)[A_1 g_1(s,x_s^n,y_s^n) + v_n(s)]ds \\ g_2(t,x_t^n,y_t^n) + T_2(t)[\psi(0) - g_2(0,\varphi_0,\psi_0)] \\ + \int_0^t T_2(t-s)[A_2 g_2(s,x_s^n,y_s^n) + u_n(s)]ds \end{pmatrix}, & \text{t} \in J. \end{cases}$$

Let $B_S := \{(x^n, y^n) \in S(\varphi, \psi), n \in \mathbb{N}\}$; then from earlier the Step 4 and (ii) in Step 2 of the proof of Theorem 3.1.1, we conclude that B_S is bounded and equicontinuous. Then from the Arzelá -Ascoli theorem we can conclude that B_S is compact, so there exist a subsequence $(x_k^n, y_k^n) \subset S(\varphi, \psi)$ with $(x_k^n, y_k^n) \to (x, y)$. The other hand we have the sequence (v_n, u_n) is integrally bounded and E is reflexive. From the Dunford-Pettis theorem [88] there exist a subsequence, still denoted (v_n^k, u_n^k) which converges weakly to an element $(v(\cdot), u(\cdot)) \in (L^1)^2$. By Mazur's Lemma [70] we obtain the existence of $(a_i^n, c_i^n) \geq (0, 0), i = 1, ..., n$ such that $\Sigma_{i=1}^n a_i^n = 1$ and $\Sigma_{i=1}^n c_i^n = 1$ and the sequence of convex combinations $(\rho_n(\cdot), \tau_n(\cdot)) = (\Sigma_{i=1}^n a_i^n v_i(\cdot), \Sigma_{i=1}^n c_i^n u_i(\cdot))$ converge strongly to $(v, u) \in (L^1)^2$. And since F and G takes convex values, using Lemma 1.1.13, we obtain that

$$\begin{aligned} v(t) &\in \bigcap_{n\geq 1} \overline{\{\rho_n(t)\}}, a.e.\ t \in J \\ &\subset \bigcap_{n\geq 1} \overline{co}\{v_k(t),\ k \geq n\} \\ &\subset \bigcap_{n\geq 1} \overline{co}\{\bigcup_{k\geq n} F(t, x_t^k, y_t^k)\} \\ &= \overline{co}(\limsup_{k\to\infty} F(t, x_t^k, y_t^k)), \end{aligned}$$

so

$$v(t) \in \overline{co}(\limsup_{k\to\infty} F(t, x_t^k, y_t^k)), \tag{3.1.2}$$

and

$$\begin{aligned} u(t) &\in \bigcap_{n\geq 1} \overline{\{\tau_n(t)\}}, a.e.\ t \in J \\ &\subset \bigcap_{n\geq 1} \overline{co}\{u_k(t),\ k \geq n\} \\ &\subset \bigcap_{n\geq 1} \overline{co}\{\bigcup_{k\geq n} G(t, x_t^k, y_t^k)\} \\ &= \overline{co}(\limsup_{k\to\infty} G(t, x_t^k, y_t^k)), \end{aligned}$$

this implies that

$$u(t) \in \overline{co}(\limsup_{k\to\infty} G(t, x_t^k, y_t^k)). \tag{3.1.3}$$

Since F and G are *u.s.c.* with compacts values, then by Lemma 1.1.12, we have

$$\limsup_{k\to\infty} F(t, x_t^k, y_t^k) = F(t, x_t, y_t),\ \ for\ a.e\ t \in J,$$

and

$$\limsup_{k\to\infty} G(t, x_t^k, y_t^k) = G(t, x_t, y_t),\ \ for\ a.e\ t \in J.$$

This with (3.1.2) and (3.1.3) implies that $(v(t), u(t)) \in F(t, x_t, y_t) \times G(t, x_t, y_t)$. Now, it remains to show that $(x, y) \in S(\varphi, \psi)$. We consider the set

$$\begin{pmatrix} z(t) \\ w(t) \end{pmatrix} = \begin{cases} \begin{pmatrix} \varphi(t) \\ \psi(t) \end{pmatrix}, & \text{t} \in I \\ \begin{pmatrix} g_1(t, x_t, y_t) + T_1(t)[\varphi(0) - g_1(0, \varphi_0, \psi_0)] + \\ \int_0^t T_1(t-s)[A_1 g_1(s, x_s, y_s) + v(s)]ds \\ g_2(t, x_t, y_t) + T_2(t)[\psi(0) - g_2(0, \varphi_0, \psi_0)] + \\ \int_0^t T_2(t-s)[A_2 g_2(s, x_s, y_s) + u(s)]ds \end{pmatrix}, & \text{t} \in J. \end{cases}$$

Using the Remark 3.1.1, with the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned} \|x^n - z\|_\infty &\leq \|g_1(t, x_t^n, y_t^n) - g_1(t, x_t, y_t)\| \\ &+ \int_0^t \|T_1(t-s)A_1\| \|g_1(s, x_s^n, y_s^n) - g_1(s, x_s, y_s)\| ds \\ &+ \int_0^t \|T_1(t-s)\| \|v_n(s) - v(s)\| ds \\ &\leq \|g_1(t, x_t^n, y_t^n) - g_1(t, x_t, y_t)\| + \int_0^t \frac{C_\beta}{(t-s)^\beta} \|g_1(s, x_s^n, y_s^n) - g_1(s, x_s, y_s)\| ds \\ &+ M_1 \int_0^t \|v_n(s) - v(s)\| ds, \end{aligned}$$

and

$$\begin{aligned} \|y^n - w\|_\infty &\leq \|g_2(t, x_t^n, y_t^n) - g_2(t, x_t, y_t)\| \\ &+ \int_0^t \|T_2(t-s)A_2\| \|g_2(s, x_s^n, y_s^n) - g_2(s, x_s, y_s)\| ds \\ &+ \int_0^t \|T_2(t-s)\| \|u_n(s) - u(s)\| ds \\ &\leq \|g_2(t, x_t^n, y_t^n) - g_2(t, x_t, y_t)\| + \int_0^t \frac{C_\beta}{(t-s)^\beta} \|g_2(s, x_s^n, y_s^n) - g_2(s, x_s, y_s)\| ds \\ &+ M_2 \int_0^t \|u_n(s) - u(s)\| ds. \end{aligned}$$

It's clear that by the convergence of (v_n, u_n) to (v, u) the right-hand side tends to

zero as $n \to \infty$, this implies that $(x^n, y^n) \to (z, w)$ as $n \to \infty$, therefore

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{cases} \begin{pmatrix} \varphi(t) \\ \psi(t) \end{pmatrix}, & \text{t} \in I \\ \begin{pmatrix} g_1(t, x_t, y_t) + T_1(t)[\varphi(0) - g_1(0, \varphi_0, \psi_0)] + \\ \int_0^t T_1(t-s)[A_1 g_1(s, x_s, y_s) + v(s)]ds \\ g_2(t, x_t, y_t) + T_2(t)[\psi(0) - g_2(0, \varphi_0, \psi_0)] + \\ \int_0^t T_2(t-s)[A_2 g_2(s, x_s, y_s) + u(s)]ds \end{pmatrix}, & \text{t} \in J. \end{cases}$$

So $(x, y) \in S(\varphi, \psi)$ and therefore $S(\varphi, \psi)$ is compact. □

For the next we will prove that $S(\varphi, \psi)$ is an $R_\delta - set$, hence an acyclic space. Consider the first-order single-values problem:

$$\begin{cases} \frac{d}{dt}(x(t) - g_1(t, x_t, y_t)) - A_1 x(t) = f_1(t, x_t, y_t), & \text{t} \in J \\ \frac{d}{dt}(y(t) - g_2(t, x_t, y_t)) - A_2 y(t) = f_2(t, x_t, y_t), & \text{t} \in J \\ x(t) = \varphi(t) \text{ and } y(t) = \psi(t), & \text{t} \in I, \end{cases} \tag{3.1.4}$$

where $f_i : J \times E \times E \longrightarrow E$ are given functions.

Denote by $S_f(\varphi, \psi)$ the set of all solutions of problem (3.1.4). We will state and prove an Aronszajn-type result for this problem, we need to add the following assumptions:

(c_1) $f_i : J \times C(I, E) \times C(I, E) \longrightarrow E$ are Carathéodory functions.

(c_2) There exists functions $p_i \in L^1(J, \mathbb{R}^+)$, and continuous nondecreasing functions $\xi_i : [0, \infty) \longrightarrow (0, \infty)$ such that

$$\|f_i(t, x, y)\| \leq p_i(t)\xi_i(\|x\| + \|y\|) \text{ for a.e. } t \in J \text{ and each } x \in E, \ i = 1.2$$

with

$$\frac{(1-\beta)M}{1 - \beta + 2d(\beta + C_\beta b^{1-\beta} - 1)} \int_0^b p(s)ds < \int_c^{+\infty} \frac{d\gamma}{\xi(\gamma)}.$$

Then, our result is given by

Theorem 3.1.3. *Assume that assumptions $(c_1), (c_2)$ and $(H_1) - (H_3)$ hold. Then the set $S_f(\varphi, \psi)$ is an $R_\delta - set$, hence an acyclic space.*

Proof. We will Transform the problem 3.1.4 into a fixed point problem. Consider the operator $K : \Omega^2 \to \Omega^2$, defined by :$K(x,y) = (K_1, K_2)$, with

$$\begin{pmatrix} K_1(x,y)(t) \\ K_2(x,y)(t) \end{pmatrix} = \begin{cases} \begin{pmatrix} \varphi(t) \\ \psi(t) \end{pmatrix}, & \text{t} \in I \\ \begin{pmatrix} g_1(t,x_t,y_t) + T_1(t)[\varphi(0) - g_1(0,\varphi_0,\psi_0)] + \\ \int_0^t T_1(t-s)[A_1 g_1(s,x_s,y_s) + f_1(s,x_s,y_s)]ds \\ g_2(t,x_t,y_t) + T_2(t)[\psi(0) - g_2(0,\varphi_0,\psi_0)] + \\ \int_0^t T_2(t-s)[A_2 g_2(s,x_s,y_s) + f_2(s,x_s,y_s)]ds \end{pmatrix}, & \text{t} \in J. \end{cases}$$

The fixed point set of K noted $FixK = S_f(\varphi,\psi)$. From Theorem 3.1.1, we know that $S_f(\varphi,\psi)$ is nonempty and by (c_2) combined with the Step 4 in Theorem 3.1.1, there exist $\widetilde{M} > 0$ such that For every $(x,y) \in S_f(\varphi,\psi)$.

$$\|x\|_\infty + \|y\|_\infty < \widetilde{M}. \tag{3.1.5}$$

Define the functions $\widetilde{f}_i$ and $\widetilde{g}_i$ with $i = 1.2$, by

$$\widetilde{f}_i(t,x,y) = \begin{cases} f_i(t,x,y), & if\ \|x\|_\infty + \|y\|_\infty \le \widetilde{M} \\ f_i(t, \frac{\widetilde{M}x}{2\|x\|}, \frac{\widetilde{M}y}{2\|y\|}), & if\ \|x\|_\infty + \|y\|_\infty > \widetilde{M}, \end{cases}$$

and

$$\widetilde{g}_i(t,x,y) = \begin{cases} g_i(t,x,y), & if\ \|x\|_\infty + \|y\|_\infty \le \widetilde{M} \\ g_i(t, \frac{\widetilde{M}x}{2\|x\|}, \frac{\widetilde{M}y}{2\|y\|}), & if\ \|x\|_\infty + \|y\|_\infty > \widetilde{M}. \end{cases}$$

Since f_i is L^1-Carathéodory, the functions $\widetilde{f}_i$ is Carathéodory and is integrably bounded by (c_2). So there exists $h_i \in L^1(J,\mathbb{R}^+)$ such that

$$\|\widetilde{f}_i(t,x,y)\| \le h_i(t)\ with\ i = 1.2. \tag{3.1.6}$$

Consider the modified problem

$$\begin{cases} \frac{d}{dt}(x(t) - \widetilde{g}_1(t,x_t,y_t)) - A_1 x(t) = \widetilde{f}_1(t,x_t,y_t), & \text{t} \in J \\ \frac{d}{dt}(y(t) - \widetilde{g}_2(t,x_t,y_t)) - A_2 y(t) = \widetilde{f}_2(t,x_t,y_t), & \text{t} \in J \\ x(t) = \varphi(t)\ and\ y(t) = \psi(t), & \text{t} \in I. \end{cases} \tag{3.1.7}$$

We can easily prove that $S_f(\varphi,\psi) = S_{\widetilde{f}}(\varphi,\psi) = Fix\widetilde{K}$, where $\widetilde{K} : \Omega^2 \to \Omega^2$ is defined

by $\widetilde{K}(x,y) = (\widetilde{K}_1(x,y), \widetilde{K}_2(x,y))$ with

$$\begin{pmatrix} \widetilde{K}_1(x,y)(t) \\ \widetilde{K}_2(x,y)(t) \end{pmatrix} = \begin{cases} \begin{pmatrix} \varphi(t) \\ \psi(t) \end{pmatrix}, & \text{t} \in I \\ \begin{pmatrix} \widetilde{g}_1(t,x_t,y_t) + T_1(t)[\varphi(0) - \widetilde{g}_1(0,\varphi_0,\psi_0)] + \\ \int_0^t T_1(t-s)[A_1\widetilde{g}_1(s,x_s,y_s) + \widetilde{f}_1(s,x_s,y_s)]ds \\ \widetilde{g}_2(t,x_t,y_t) + T_2(t)[\psi(0) - \widetilde{g}_2(0,\varphi_0,\psi_0)] + \\ \int_0^t T_2(t-s)[A_2\widetilde{g}_2(s,x_s,y_s) + \widetilde{f}_2(s,x_s,y_s)]ds \end{pmatrix}, & \text{t} \in J. \end{cases}$$

By the inequalities (3.1.6) and (3.1.5), we deduce that

$$\begin{aligned} \|\widetilde{K}_1(x,y)\|_\infty &\leq d_1(\|x_t\|_\infty + \|y_t\|_\infty) + d_2 + M_1[\|\varphi\|_\infty + d_1(\|\varphi\|_\infty + \|\psi\|_\infty) + d_2] \\ &+ \int_0^t \|T_1(t-s)A_1\|[d_1(\|x_s\|_\infty + \|y_s\|_\infty + d_2]ds + M_1\|h_1\|_{L^1} \\ &\leq d_1\widetilde{M} + d_2 + M_1[\|\varphi\|_\infty + d_1\widetilde{M} + d_2] + \frac{C_\beta b^{1-\beta}}{1-\beta}(d_1\widetilde{M} + d_2) + M_1\|h_1\|_{L^1}, \end{aligned}$$

and

$$\begin{aligned} \|\widetilde{K}_2(x,y)\|_\infty &\leq d_3(\|x_t\|_\infty + \|y_t\|_\infty) + d_4 + M_2[\|\psi\|_\infty + d_3(\|\varphi\|_\infty + \|\psi\|_\infty) + d_4] \\ &+ \int_0^t \|T_2(t-s)A_2\|[d_3(\|x_s\|_\infty + \|y_s\|_\infty) + d_4]ds + M_2\|h_2\|_{L^1} \\ &\leq d_3\widetilde{M} + d_4 + M_2[\|\psi\|_\infty + d_3\widetilde{M} + d_4] + \frac{C_\beta b^{1-\beta}}{1-\beta}(d_3\widetilde{M} + d_4) + M_2\|h_2\|_{L^1}. \end{aligned}$$

Let $M = \max(M_1, M_2)$, $d = \max(d_1, d_3)$, $\overline{d} = d_2 + d_4$,
$\|h(t)\|_{L^1} = \|h_1\|_{L^1} + \|h_2\|_{L^1}$ and by addition we obtain

$$\begin{aligned} \|\widetilde{K}_1(x,y)\|_\infty + \|\widetilde{K}_2(x,y)\|_\infty &\leq 2d\widetilde{M} + \overline{d} + M[(1+2d)\widetilde{M} + \overline{d}] + \frac{C_\beta b^{1-\beta}}{1-\beta}(2d\widetilde{M} + \overline{d}) \\ &+ M\|h(t)\|_{L^1} := R. \end{aligned}$$

Then $\widetilde{K}$ is uniformly bounded. We can show that $\widetilde{K} : \Omega^2 \to \Omega^2$ is compact which allows us to define the compact perturbation of the identity $\widetilde{B}(x,y) = (x,y) - \widetilde{K}(x,y)$ which is a proper map. From the compactness of $\widetilde{K}$ and Lemma 1.2.4, we can easily prove that all conditions of Theorem 1.2.3 are satisfied. Therefore the solutions set $S_{\widetilde{f}}(\varphi,\psi) = \widetilde{B}^{-1}(0,0)$ is an $R_\delta - set$, hence an acyclic space by Lemma 1.2.2. □

3.2 Example

Now, we introduce an example to which Theorem 3.1.1 can be applied.
Let B an open of $\mathbb{R}^n$ $(n \geq 1)$, with smooth boundary ∂B, be a measurable space. We

consider the differential inclusion system

$$\begin{cases} \frac{d}{dt}u_t(x) + \Delta u \in M_1(t, u(t-1,x), v(t-1,x)), & \text{t} \in J := [0,T] \\ \frac{d}{dt}v_t(x) + \Delta v \in M_2(t, u(t-1,x), v(t-1,x)), & \text{t} \in J \\ u_t(x) = \varphi_t(x) \text{ and } v_t(x) = \psi_t(x), & \text{t} \in I := [-1,0]. \end{cases} \tag{3.2.1}$$

Set $E = L^2(B)$ which is a separable Banach space, and let us define $A : D(A) \subset L^2(B) \to L^2(B)$ by $Au = -\Delta u$, $\forall u \in D(A)$, where

$$D(A) = \{u \in L^2(B) : u \in H_0^1(B), \Delta u \in L^2(B)\}$$

which is dense in $L^2(B)$, so from Hille-Yosida's theorem the operator A genered a strongly continuous semi-group, and satisfies (b) in the remark 3.1.1.
Let $M_i : [0,b] \times C(I,E) \times C(I,E) \to \mathcal{P}(E)$ be are multivalued maps defined by

$$M_i(t, u(t-1), v(t-1)) = \{h_i \in E : h_i(x) \in \widetilde{m}_i(t, u(t-1), v(t-1)), a.e.(u,v) \in B^2\},$$

where $\widetilde{m}_i : J \times \mathbb{R}^2 \to \mathcal{P}(\mathbb{R})$ are defined by

$$\widetilde{m}_i(t, u(t-1), v(t-1)) = [(m_1)_i(t, u(t-1), v(t-1)), (m_2)_i(t, u(t-1), v(t-1))],$$

and $(m_1)_i, (m_2)_i : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ can be defined as:

$$(m_1)_i(t, u(t-1), v(t-1)) = \lim_{(u',v') \to (u,v)} \inf m_i(t, u', v')$$

and

$$(m_2)_i(t, u(t-1), v(t-1)) = \lim_{(u',v') \to (u,v)} \sup m_i(t, u', v'),$$

with $i = 1.2$ where $m_i : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies:

($\widetilde{H}_1$) There exist continuous functions $a_i, b_i \in L^1(J, \mathbb{R}_+^\star)$, with a_i/b_i is a monotone function, such that

$$|m_i(t,u,v)| \leq a_i(t)|u| + b_i(t)|v|,$$

($\widetilde{H}_2$) $\|a(t)\| + \|b(t)\| < \frac{1-\beta-2C_\beta b^{1-\beta}}{2M(1-\beta)}$.

This allows us to show that the multivalued maps M_i satisfies the hypotheses $(H_1), (H_3), (H_5)$. Thus the conclusion follows from Theorem 3.1.1.

Chapter 4

Fractional calculus

This chapter will consist of three sections. In first section we cite some definitions and results of fractional integral calculus within the meaning of Riemann-Liouville. In second section we cite some definitions and results of fractional derivative calculus exactly the Riemann-Liouville fractional derivative and Caputo fractional derivative and its important properties. While in section 4.3 we wish to establish some nonlinear integral inequalities that can be used in the study of fractional differential equations and inclusions. The proofs in this last section are based on Young's and Holder's inequalities.

Before starting, we need to define the Euler Gamma and Bêta functions which plays a very important role in the theory of fractional calculus and we will proofed their important properties.

Definition 4.0.1. *The Euler Gamma function* $\Gamma(z)$ *is defined by following integral*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

Proposition 4.0.1. *the following properties are holds.*
(1) $\Gamma(z+1) = z\Gamma(z)$,
(2) $\Gamma(1) = \Gamma(2) = 1$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, *and for each* $n \in \mathbb{N}^\star$ $\Gamma(\frac{1}{2}+n) = \frac{1\times3\times5\times\ldots\times(2n-1)}{2^n}\sqrt{\pi}$
(3) $\Gamma(n+1) = n!$, $\forall n \in \mathbb{N}$,
(4) $\Gamma(z)$ *is a monotonic and strictly decreasing function for* $0 < z \leq 1$.

Proof. The property (1) is the most important which can be proved by an integration by parts. For $0 < a < b$ we have

$$\Gamma(z+1) = \int_a^b e^{-t} t^z dt = [-t^z e^{-t}]_a^b + z\int_a^b e^{-t} t^{z-1} dt = \frac{a^z}{e^a} - \frac{b^z}{e^b} + z\Gamma(z).$$

If $b \to +\infty$ and $a \to 0$, then $\Gamma(z+1) = z\Gamma(z)$. For the proof of (2) we have $\Gamma(1) = \int_a^b e^{-t} dt = [-e^{-t}]_a^b = e^{-a} - e^{-b}$, if $b \to +\infty$ and $a \to 0$ then $\Gamma(1) = 1$, and using the

property (1) to get that $\Gamma(2) = \Gamma(1+1) = 1 \times \Gamma(1) = 1$, and for $\Gamma(\frac{1}{2})$, we have by definition $\Gamma(\frac{1}{2}) = \int_0^{\infty} e^{-t}t^{-\frac{1}{2}}dt$, for $0 < a < b < +\infty$, using the change of variable $x = t^{\frac{1}{2}}$ then, $dt = 2t^{\frac{1}{2}}dx$, so $\int_a^b e^{-t}t^{-\frac{1}{2}}dt = 2\int_{\sqrt{a}}^{\sqrt{b}} e^{-x^2}dx$, and if $a \to 0, b \to +\infty$, we obtain $\Gamma(\frac{1}{2}) = 2\int_0^{+\infty} e^{-x^2}dx$. Now we will calculate the integral $I = \int_0^{+\infty} e^{-x^2}dx$, we have

$$I^2 = \int_0^{+\infty}\int_0^{+\infty} e^{-(x^2+y^2)}dxdy.$$

By using polar coordinates we get

$$I^2 = \int_0^{+\infty}\int_0^{\frac{\pi}{2}} e^{-r^2}rd\theta dr = \frac{\pi}{4},$$

therefor $I = \frac{\sqrt{\pi}}{2}$, so $\Gamma(\frac{1}{2}) = 2I = \sqrt{\pi}$.

We now use property (1) with the proof by induction we have for $n = 1$,

$$\Gamma(\frac{1}{2}+1) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi},$$

for $n = 2$,

$$\Gamma(\frac{1}{2}+2) = \Gamma(\frac{1}{2}+1+1) = \Gamma(\frac{3}{2}+1) = \frac{3}{2}\Gamma(\frac{1}{2}+1) = \frac{3}{2}\times\frac{1}{2}\sqrt{\pi} = \frac{1\times 3}{2^2}\sqrt{\pi},$$

suppose that for all $n \geq 3$ the property $\Gamma(\frac{1}{2}+n) = \frac{1\times 3\times\ldots\times(2n-1)}{2^n}\sqrt{\pi}$ is true, therefore for $n+1$ we will have

$$\Gamma(\frac{1}{2}+n+1) = (\frac{1}{2}+n)\Gamma(\frac{1}{2}+n) = \frac{1\times 3\times\ldots\times(2n-1)\times[2(n+1)-1]}{2^{n+1}}\sqrt{\pi}.$$

We also use the proof by induction on (3), for $n = 0$ we have $\Gamma(0+1) = \Gamma(1) = 1 = 0!$, suppose that for all $n \geq 1$ the property $\Gamma(n) = (n-1)!$ is true and we will show that $\Gamma(n+1) = n!$ is also, it is clear that $\Gamma(n+1) = n\Gamma(n) = n(n-1)! = n!$, so (3) is true. For the proof of (4) we take $t_1, t_2 \in (0, +\infty)$ such that $t_1 < t_2$, if $0 < z \leq 1$ then $s = 1 - z \geq 0$, so $t_1^s < t_2^s \Rightarrow t_1^{-s} > t_2^{-s}$ and we have also $t_1 < t_2 \Rightarrow e^{t_1} < e^{t_2} \Rightarrow e^{-t_1} > e^{-t_2}$, therefore $e^{-t_1}t_1^{-s} > e^{-t_2}t_2^{-s} \Rightarrow e^{-t_1}t_1^{z-1} > e^{-t_2}t_2^{z-1}$, and by passage to the integral we know that this last inequality remains true, i.e. that $\Gamma(z)(t_1) > \Gamma(z)(t_2)$, so the result (4). □

Remark 4.0.1. *(a) From property (1) and by induction we can easily deduce that for all* $n \in \mathbb{N}^{\star}$

$$\Gamma(x) = \frac{\Gamma(x+n)}{x(x+1)...(x+n-1)}$$

this relation allows to define $\Gamma(x)$ *for negative real* x *such that* $-n < x < -n+1$ *In this case there for* $n = [\alpha]+1$ *we have* $0 < n-\alpha < 1$*, which means that* $-n < -\alpha < -n+1$ *and according to the above convention, we will have*

$$\Gamma(-\alpha) = \frac{\Gamma(n-\alpha)}{-\alpha(-\alpha+1)...(-\alpha+n-1)}.$$

(b) The Gamma function extended the factorial to real values, this by property (3), of the same property we can deduce that it is extended the law of composition C_n^p to real values because for all $n, p \in \mathbb{N}$ with $p \leq n$

$$C_n^p = \frac{n!}{p!(n-p)!} = \frac{n}{p(n-p)} \frac{\Gamma(n)}{\Gamma(n)\Gamma(n-p)}.$$

Definition 4.0.2. *The Bêta function is defined by*

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt, \ Re(p) > 0, \ Re(q) > 0.$$

Proposition 4.0.2. *(1) The Bêta function is related to the Gamma function by the following relation*

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \ Re(p) > 0, \ Re(q) > 0,$$

(2) The Bêta function is symmetric, i.e.

$$B(p,q) = B(q,p), \ Re(p) > 0, \ Re(q) > 0.$$

4.1 Fractional integral

The integral of order 1 with lower limit $a \in \mathbb{R}$ for a function $f \in L^1([a, +\infty), \mathbb{R})$ is defined by

$$I_{a^+}^1 f(t) = \int_a^t f(s)ds.$$

And the integral of order 1 with upper limit $b \in \mathbb{R}$ for a function $f \in L^1((-\infty, b], \mathbb{R})$ is defined by

$$I_{b^-}^1 f(t) = \int_t^b f(s)ds.$$

For a second primitive we will have

$$I_{a^+}^2 f(t) = \int_a^t \left(\int_a^s f(\tau)d\tau \right) ds, \ and \ I_{b^-}^2 f(t) = \int_t^b \left(\int_s^b f(\tau)d\tau \right) ds.$$

According to Fubini's theorem we reduce this double integral to a simple integral as follows

$$I_{a^+}^2 f(t) = \int_a^s f(\tau)d\tau \int_a^t ds = \frac{1}{1!} \int_a^s (s-\tau)f(\tau)d\tau,$$

and

$$I_{b^-}^2 f(t) = \int_s^b f(\tau)d\tau \int_t^b ds = \frac{1}{1!} \int_s^b (s-\tau)f(\tau)d\tau.$$

By a simple iteration to order n we obtain

$$I^n_{a^+}f(t) = \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1}f(s)ds, \text{ and } I^n_{b^-}f(t) = \frac{1}{(n-1)!}\int_t^b (t-s)^{n-1}f(s)ds.$$

Since $\Gamma(n) = (n-1)!, \forall n \in \mathbb{N}^\star$, we obtain

$$I^n_{a^+}f(t) = \frac{1}{\Gamma(n)}\int_a^t (t-s)^{n-1}f(s)ds, \text{ and } I^n_{b^-}f(t) = \frac{1}{\Gamma(n)}\int_t^b (t-s)^{n-1}f(s)ds.$$

We will use the fact that the Gamma function extends the factorial to real values to have the notion of Riemann-Liouville fractional integral as the following.

Definition 4.1.1. *The Riemann-Liouville fractional integral $I^\beta_{a^+}f$ and $I^\beta_{b^-}f$ of order $\beta \in \mathbb{C}$ with $Re(\beta) > 0$ are defined by*

$$I^\beta_{a^+}f(t) = \frac{1}{\Gamma(\beta)}\int_a^t (t-s)^{\beta-1}f(s)ds, \ t > s > a,$$

and

$$I^\beta_{b^-}f(t) = \frac{1}{\Gamma(\beta)}\int_t^b (s-t)^{\beta-1}f(s)ds, \ t < s < b.$$

Provided the right-hand side is pointwise defined on $[a, \infty)$. For simplify the writing, we will denote in the suite $I^\beta_{a^+}$ by I^β. Particular case $I^0 f(t) = f(t)$, (i.e I^0 the identity operator).

The Riemann-Liouville fractional integral has the following properties which play an important role in the study of fractional equation.

Proposition 4.1.1.

$$I^\beta[I^\alpha f(t)] = I^{\beta+\alpha}f(t) = I^\alpha(I^\beta f(t)), \ Re(\beta) > 0, \ Re(\alpha) > 0.$$

Proof. By definition we have

$$\begin{aligned} I^\beta[I^\alpha f(t)] &= \frac{1}{\Gamma(\beta)}\int_a^t (t-s)^{\beta-1}(I^\alpha f)(s)ds \\ &= \frac{1}{\Gamma(\beta)\Gamma(\alpha)}\int_a^t\int_a^s (t-s)^{\beta-1}(s-u)^{\alpha-1}f(u)duds, \end{aligned}$$

where $f \in C([a,b])$. So from the Fubini's theorem we will change the order of integra-

tion, by the variable $s = u + r(t-u)$, we obtain

$$\begin{aligned} I^{\beta}[I^{\alpha}f(t)] &= \frac{1}{\Gamma(\beta)\Gamma(\alpha)}\int_a^t f(u)\left[\int_u^t (t-s)^{\beta-1}(s-u)^{\alpha-1}ds\right]du \\ &= \frac{1}{\Gamma(\beta)\Gamma(\alpha)}\int_a^t f(u)\left[(t-u)^{\beta+\alpha-1}\int_0^1 r^{\alpha-1}(1-r)^{\beta-1}ds\right]du \\ &= \frac{B(\beta,\alpha)}{\Gamma(\beta)\Gamma(\alpha)}\int_a^t (t-u)^{\beta+\alpha-1}f(u)du \\ &= \frac{\Gamma(\beta)\Gamma(\alpha)}{\Gamma(\beta+\alpha)\Gamma(\beta)\Gamma(\alpha)}\int_a^t (t-u)^{\beta+\alpha-1}f(u)du \\ &= \frac{1}{\Gamma(\beta+\alpha)}\int_a^t (t-u)^{\beta+\alpha-1}f(u)du \\ &= I^{\beta+\alpha}f(t). \end{aligned}$$

□

Definition 4.1.2. *The fractional substantial integral of order ν with lower limit 0 for a function $f \in L^1[0,+\infty)$ is defined as*

$$I_s^{\nu}f(t) = \frac{1}{\Gamma(\nu)}\int_0^t (t-r)^{\nu-1}e^{-\beta(t-r)}f(r)dr,\ \nu > 0.$$

The next properties can be referenced in [16, 73, 79]

Lemma 4.1.2. *[79] Let $f(t)$ be continuous on $[a,\infty)$, and $\nu > 0$. Then for all $t > a$,*

$$\lim_{\nu\to 0} I_s^{\nu}f(t) = f(t).$$

So we can put $I_s^0 f(t) = f(t)$.

Lemma 4.1.3. *Let $f(t)$ be continuous on $[a,\infty)$, and $\mu,\nu > 0$. Then for all $t > a$,*

$$I_s^{\mu}[I_s^{\nu}f(t)] = I_s^{\mu+\nu}f(t) = I_s^{\nu}[I_s^{\mu}f(t)].$$

4.2 Fractional derivative

Definition 4.2.1. *The Riemann-Liouville fractional derivative of order β with $0 \le n-1 < \beta < n$ and lower limit a for a function f is given by*

$${}^{RL}D^{\beta}f(t) = \frac{1}{\Gamma(n-\beta)}\frac{d^n}{dt^n}\int_a^t (t-s)^{n-\beta-1}f(s)ds,\ t > a,\ 0 \le n-1 < \beta < n.$$

The particular case for $\beta = 0$ and $\beta = 1$ we have

$${}^{RL}D^0 f(t) = f(t) \text{ and } {}^{RL}D^n f(t) = f^n(t).$$

Definition 4.2.2. *For $0 < \beta < 1$ and $t \in (a, b)$ the Riemann-Liouville fractional derivative of order β with lower limit a for a function f is given by*

$$ {}^{RL}_{a^+}D^{\beta}_t f(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_a^t (t-s)^{-\beta} f(s) ds. $$

We can reduce the case general where $n-1 < \beta < n$ in this case particular, we pos $\alpha = \beta - n + 1$, then $0 < \alpha < 1$ and we obtain

$$ {}^{RL}_{a^+}D^{\beta}_t f(t) = {}^{RL}_{a^+}D^{\alpha+n-1}_t f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-s)^{-\alpha} f(s) ds. $$

Lemma 4.2.1. *Let $f \in C([a, b])$ and $n-1 < \beta < n$ then $f(t) \in I^{\beta}([a, b])$ if and only if $I^{n-\beta} f \in C^n([a, b])$ and $(\frac{d^k}{dt^k} I^{n-\beta} f)_{t=a} = 0$; $k = 0; 1; 2; \ldots; n-1$.*

Theorem 4.2.2. *(1) If $n-1 < \beta < n$, then,*

$$ {}^{RL}_{a}D^{\beta}_t (I^{\beta} f(t) = f(t) \text{ for each function } f \in C([a, b]). $$

(2) If $n-1 < \beta < n$, and $f \in C^n([a, b])$ and moreover if f verified the condition of Lemma 4.2.1 then,

$$ I^{\beta}({}^{RL}_{a}D^{\beta}_t f(t)) = f(t). $$

If no

$$ I^{\beta}({}^{RL}_{a}D^{\beta}_t f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\beta-k-1}}{\Gamma(\beta-k)} \frac{d^{n-k-1}}{dt^{n-k-1}} (I^{n-\beta} f(t)). $$

Then in particular: if $0 < \beta < 1$ then

$$ I^{\beta}({}^{RL}_{a}D^{\beta}_t f(t)) = f(t) - \frac{(t-a)^{\beta-1}}{\Gamma(\beta)} (I^{1-\beta} f(t)), $$

and for $\beta = n$ we have

$$ I^{\beta}({}^{RL}_{a}D^{\beta}_t f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{f^k(a)(t-a)^k}{k!}. $$

Definition 4.2.3. *The Caputo fractional derivative of order β with $0 \leq n-1 < \beta < n$ for a function f is given by*

$$ {}^{c}_{a}D^{\beta}_t f(t) = \frac{1}{\Gamma(n-\beta)} \int_a^t (t-s)^{n-\beta-1} f^{(n)}(s) ds =, \ t > a. $$

${}^{c}_{a}D^{\beta}_t f(t)$ is called also the Caputo fractional differential operator of order β.

Remark 4.2.1. *For $n = 1$ we have $0 < \beta < 1$ and*

$$ {}_a^c D_t^\beta f(t) = \frac{1}{\Gamma(1-\beta)} \int_a^t (t-s)^{-\beta} f'(s) ds, \ t > a. $$

Definition 4.2.4. *The Caputo fractional derivative of order β with $0 \leq n-1 < \beta < n$ and lower limit a for a function f is given by*

$$ {}_{a^+}^c D_t^\beta f(t) = \frac{1}{\Gamma(n-\beta)} \int_a^t (t-s)^{n-\beta-1} f^{(n)}(s) ds, \ t > a. $$

Definition 4.2.5. *The Caputo fractional derivative of order β with $0 \leq n-1 < \beta < n$ and upere limit b for a function f is given by*

$$ {}_{b^-}^c D_t^\beta f(t) = \frac{1}{\Gamma(n-\beta)} \int_t^b (t-s)^{n-\beta-1} f^{(n)}(s) ds, \ t < b. $$

We use only ${}_a^c D_t^\beta f(t)$ instead of ${}_{a^+}^c D_t^\beta f(t)$ in the following properties.

Theorem 4.2.3. *For $n-1 \leq \beta \leq n$, if $f \in C^n([a,b])$ then the differential operator ${}_a^c D_t^\beta f(t)$ exist p. p on $[a,b]$ and:*
If $\beta \notin \mathbb{N}$ the Caputo fractional derivative can be presented by

$$ {}_a^c D_t^\beta f(t) = \frac{1}{\Gamma(n-\beta)} \int_a^t (t-s)^{n-\beta-1} f^{(n)}(s) ds = I^{n-\beta} f^{(n)}(t), \ t > a, $$

and in the particulars cases: if $n = 1$ i.e. $0 < \beta < 1$, and $f \in C([a,b])$ we have

$$ {}_a^c D_t^\beta f(t) = \frac{1}{\Gamma(1-\beta)} \int_a^t (t-s)^{-\beta} f'(s) ds = I^{1-\beta} f'(t), \ t > a, $$

if $\beta = n$ and $f \in C^n([a,b])$ then

$$ {}_a^c D_t^\beta f(t) = f^{(n)}(t), \ t > a. $$

Let $n-1 < \beta < n, n \in \mathbb{N}, \beta \in \mathbb{R}$ and $f(t)$ a function such that ${}_a^c D_t^\beta f(t)$ exist. Then, following properties hold.

Proposition 4.2.4.

$$ \lim_{\beta \to n} ({}_a^c D_t^\beta f(t)) = f^{(n)}(t), $$

and

$$ \lim_{\beta \to n-1} ({}_a^c D_t^\beta f(t)) = f^{(n-1)}(t) - f^{(n-1)}(0). $$

For $\lambda \in \mathbb{R}$ and functions f, g such that ${}_a^c D_t^\beta f(t)$ and ${}_a^c D_t^\beta g(t)$ exist, we have

$$ {}_a^c D_t^\beta (\lambda f(t) + g(t)) = \lambda {}_a^c D_t^\beta f(t) + {}_a^c D_t^\beta g(t), $$

i.e. the Caputo fractional differential operator is linear.
For $n, m \in \mathbb{N} : n-1 < \beta < n$ *the composition* ${}_a^c D_t^\beta f(t)$ *with* ${}^{RL}D^m f(t)$ *is non-commutative in general, i.e.*

$$ {}_a^c D_t^\beta ({}^{RL}D^m f(t)) \neq^{RL} D^m ({}_a^c D_t^\beta f(t)).$$

Note that if the function f satisfied $f^{(m)}(0) = 0$ for $m = 0, 1, 2, ..., n-1$, then the Riemann-Liouville and Caputo fractional derivatives coincides,
i.e. ${}^cD^\beta f(t) =^{RL} D^\beta f(t)$. In general the Riemann-Liouville and Caputo fractional derivatives do not coincide

The link between the Caputo fractional derivative and the Riemann-Liouville fractional derivative is given by the following theorem

Theorem 4.2.5. *Let* $Re(\beta) > 0$ *with* $n-1 < Re(\beta) < n$, $n \in \mathbb{N}^\star$ *and let* f *a function such that* ${}_a^c D_t^\beta f(t)$ *and* ${}^{RL}D^m f(t)$ *exists then,*

$$ {}_a^c D_t^\beta f(t) =^{RL} D^m f(t) - \sum_{k=0}^{n-1} \frac{f^k(0)}{\Gamma(k-\beta)+1} t^{k-\beta}.$$

Proof. Using the Taylor limited development for the function f in $t = 0$ we obtain

$$\begin{aligned} f(t) &= f(0) + \frac{f^{'}(0)}{1!}t + \frac{f^{''}(0)}{2!}t^2 + ... + \frac{f^{(n-1)}(0)}{(n-1)!}t^{n-1} + R_{n-1} \\ &= \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{\Gamma(k+1)} t^k + R_{n-1}, \end{aligned}$$

with

$$R_{n-1} = \int_0^t \frac{f^{(n)}(s)}{(n-1)!}(t-s)^{n-1} ds,$$

we use the property of integration to order n to obtain

$$R_{n-1} = \frac{1}{\Gamma(n)} \int_0^t f^{(n)}(s)(t-s)^{n-1} ds = I^n f^{(n)}(t).$$

We consider $f(t) = t^\alpha$, we have

$$I_a^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} s^\alpha ds,$$

let $s = tr$ so we obtain

$$\begin{aligned} I_a^\beta f(t) &= \frac{1}{\Gamma(\beta)} \int_0^1 (t-tr)^{\beta-1} t(tr)^\alpha dr \\ &= \frac{1}{\Gamma(\beta)} \int_0^1 t^{\beta-1}(1-r)^{\beta-1} t^{\alpha+1} r^\alpha dr \\ &= \frac{1}{\Gamma(\beta)} \int_0^1 t^{\alpha+\beta} r^{(\alpha+1)-1}(1-r)^{\beta-1} dr \\ &= \frac{B(\alpha+1,\beta)}{\Gamma(\beta)} t^{\alpha+\beta}, \end{aligned}$$

with the relation $B(\alpha+1,\beta) = \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}$ we obtain

$$I_a^\beta f(t) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}.$$

And by definition we have

$$\begin{aligned} {}^{RL}D^\beta f(t) &= \frac{d^n}{dt^n}(I_a^{n-\beta} f(t)) \\ &= \frac{d^n}{dt^n}(\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+n-\beta+1)} t^{\alpha+n-\beta}) \\ &= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+n-\beta+1)} \frac{d^n}{dt^n}(t^{\alpha+n-\beta}) \\ &= \frac{(n+\alpha-\beta)(n+\alpha-\beta-1)...(\alpha-\beta+1)\Gamma(\alpha+1)}{(n+\alpha-\beta)(n+\alpha-\beta-1)...(\alpha-\beta+1)\Gamma(\alpha-\beta+1)} t^{\alpha-\beta}, \end{aligned}$$

so

$${}^{RL}D^\beta t^\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\beta+1)} t^{\alpha-\beta}.$$

Using the linearity of the Riemann-Liouville operator we obtain

$$\begin{aligned} {}^{RL}D^\beta f(t) &= {}^{RL}D^\beta (\sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{\Gamma(k+1)} t^k + R_{n-1}) \\ &= (\sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{\Gamma(k+1)} t^k) {}^{RL}D^\beta t^k + {}^{RL}D^\beta R_{n-1} \\ &= \sum_{k=0}^{n-1} \frac{f^{(k)}(0)\Gamma(k+1)}{\Gamma(k+1)\Gamma(k-\beta+1)} t^{k-\beta} + {}^{RL}D^\beta I^n f^{(n)}(t) \\ &= \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{\Gamma(k-\beta+1)} t^{k-\beta} + I^{n-\beta} f^{(n)}(t) \\ &= \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{\Gamma(k-\beta+1)} t^{k-\beta} + {}^{C}D^\beta f(t). \end{aligned}$$

Therefore

$$^cD^\beta f(t) =^{RL} D^\beta f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{\Gamma(k-\beta+1)} t^{k-\beta}.$$

□

Note that if $f^{(k)}(0) = 0$ for $k = 0, 1, 2, ..., n-1$, we have

$$_a^cD_t^\beta f(t) =^{RL} D^m f(t).$$

Remark 4.2.2. *For $f(t) = c$ with c a constant, we have*

$$_a^cD_t^\beta f(t) = 0, \ t > a.$$

Definition 4.2.6. *The Caputo fractional substantial derivative D_s^μ with $0 < \mu < 1$ for a function f is given by*
$D_s^\mu f(t) = D_s^m[I_s^\nu f(t)]$, $\nu = m - \mu$, m is the smallest integer which exceeds μ, where $I_s^\nu f(t)$ is given by Definition 4.1.2, β is a positive constant and

$$D_s^m = (\frac{\partial}{\partial t} + \beta)^m = (D+\beta)^m = (D+\beta)(D+\beta)...(D+\beta).$$

Lemma 4.2.6. *Let $f(t)$ be (m-1)-time continuously differentiable on (a, ∞), such that its m-times derivative be integrable on any finite subinterval of $[a, \infty)$ and $\nu > 0$. Then for all $t > a$,*

$$D_s^\mu[I_s^\nu f(t)] = f(t).$$

4.3 Nonlinear integral inequalities

In this section we will give one of our main results in this thesis, it is exactly some generalizations of the nonlinear integral inequalities of Henry type.

Theorem 4.3.1. *Let $0 < \alpha < 1$, $k_0, k_1, k_2 > 0$. For $i = 1, 2, ..., n$, let $f_i, \lambda_i, \gamma_i,$ and g be non-negative functions that are locally integrable on $I = [0, T]$, and let $\psi_i : [0, \infty) \to [0, \infty)$ be nondecreasing continuous functions. If $u(t)$ is a non-negative continuous function on I satisfying:*

$$u(t) \le k_0 + \sum_{i=1}^{n} k_1 \int_0^t (t-s)^{\alpha-1} g(s)\gamma_i(s)u(s)ds + \sum_{i=1}^{n} k_2 \int_0^t (t-s)^{\alpha-1} f_i(s)\lambda_i(s)\psi_i(u(s))ds$$

then

$$u(t) \le \Psi^{-1}\left(\Psi(\widetilde{k}) + \int_0^t m(s)ds\right),$$

where $\frac{1}{p}+\frac{1}{q}=1$,

$$m(t)=\max\left(\sum_{i=1}^{n}\frac{1}{q}(\gamma_i(t)g(t))^q,\sum_{i=1}^{n}\frac{1}{q}(\lambda_i(t)f_i(t))^q\right),\ \widetilde{k}=k_0+\frac{(k_1^p+k_2^p)nT^{p(\alpha-1)+1}}{p(p(\alpha-1)+1)},$$

$$\Psi(z)=\int_{k_0}^{z}\frac{dy}{y^q+\widetilde{\psi}(y)},\ \widetilde{\psi}(y)=\max\{\psi_i^q(y),i=1.2...n\},$$

Ψ^{-1} *is the inverse function of* Ψ *and for every* $t\in[0,T]$,

$$\Psi(\widetilde{k})+\int_0^t m(s)ds\in Dom(\Psi^{-1}),$$

where $Dom(\Psi^{-1})$ *is the domain of* Ψ^{-1}.

Proof. By Young's inequality, we see that

$$k_1(t-s)^{\alpha-1}g(s)\gamma_i(s)u(s)\leq\frac{k_1^p}{p}(t-s)^{p(\alpha-1)}+\frac{1}{q}(\gamma_i(s)g(s)u(s))^q,$$

and

$$k_2(t-s)^{\alpha-1}f_i(s)\lambda_i(s)\psi_i(u(s))\leq\frac{k_2^p}{p}(t-s)^{p(\alpha-1)}+\frac{1}{q}(\lambda_i(s)f_i(s))^q(\psi_i(u(s)))^q$$

then,

$$\begin{aligned}u(t)&\leq k_0+\sum_{i=1}^{n}\left(\frac{k_1^p}{p}\int_0^t(t-s)^{p(\alpha-1)}+\frac{1}{q}\int_0^t(\gamma_i(s)g(s)u(s))^q ds\right)\\&+\sum_{i=1}^{n}\left(\frac{k_2^p}{p}\int_0^t(t-s)^{p(\alpha-1)}+\frac{1}{q}\int_0^t(\lambda_i(s)f_i(s))^q(\psi_i(u(s)))^q ds\right),\end{aligned}$$

therefore

$$\begin{aligned}u(t)&\leq k_0+\frac{(k_1^p+k_2^p)nT^{p(\alpha-1)+1}}{p(p(\alpha-1)+1)}+\sum_{i=1}^{n}\frac{1}{q}\int_0^t(\gamma_i(s)g(s)u(s))^q ds\\&+\sum_{i=1}^{n}\frac{1}{q}\int_0^t(\lambda_i(s)f_i(s))^q(\psi_i(u(s)))^q ds.\end{aligned}$$

Thus

$$u(t)\leq\widetilde{k}+\sum_{i=1}^{n}\frac{1}{q}\int_0^t(\gamma_i(s)g(s)u(s))^q ds+\sum_{i=1}^{n}\frac{1}{q}\int_0^t(\lambda_i(s)f_i(s))^q(\psi_i(u(s)))^q ds.$$

Setting

$$v(t) = \widetilde{k} + \sum_{i=1}^{n} \frac{1}{q} \int_0^t (\gamma_i(s)g(s)u(s))^q ds + \sum_{i=1}^{n} \frac{1}{q} \int_0^t (\lambda_i(s)f_i(s))^q (\psi_i(u(s)))^q ds,$$

we see that

$$v'(t) = \sum_{i=1}^{n} \frac{1}{q} (\gamma_i(t)g(t)u(t))^q + \sum_{i=1}^{n} \frac{1}{q} (\lambda_i(t)f_i(t))^q (\psi_i(u(t)))^q, \text{ and } v(0) = \widetilde{k}.$$

Since ψ_i are nondecreasing functions and $u(t) \leq v(t)$ we then obtain

$$v'(t) \leq \sum_{i=1}^{n} \frac{1}{q} (\gamma_i(t)g(t)v(t))^q + \sum_{i=1}^{n} \frac{1}{q} (\lambda_i(t)f_i(t))^q (\psi_i(v(t)))^q, \tag{4.3.1}$$

so

$$v'(t) \leq m(t)(v^q(t) + \widetilde{\psi}(v(t))),$$

where

$$m(t) = \max\left(\sum_{i=1}^{n} \frac{1}{q} (\gamma_i(t)g(t))^q, \sum_{i=1}^{n} \frac{1}{q} (\lambda_i(t)f_i(t))^q \right) \text{ and } \widetilde{\psi}(v) = \max\{\psi_i^q(v), i = 1.2...n\}.$$

Therefore,

$$\frac{v'(t)}{v^q(t) + \widetilde{\psi}(v(t))} \leq m(t), \quad t \in [0, T].$$

Integrating both sides of the above inequality from 0 to t, we obtain

$$\int_0^t \frac{v'(s)}{v^q(s) + \widetilde{\psi}(v(s))} ds \leq \int_0^t m(s) ds.$$

Then, we get

$$\int_{v(0)}^{v(t)} \frac{dx}{x^q + \widetilde{\psi}(x)} \leq \int_0^t m(s) ds,$$

or

$$\Psi(v(t)) = \int_{k_0}^{v(t)} \frac{dx}{x^q + \widetilde{\psi}(x)} = \int_{k_0}^{\widetilde{k}} \frac{dx}{x^q + \widetilde{\psi}(x)} + \int_{\widetilde{k}}^{v(t)} \frac{dx}{x^q + \widetilde{\psi}(x)} = \Psi(\widetilde{k}) + \int_{\widetilde{k}}^{v(t)} \frac{dx}{x^q + \widetilde{\psi}(x)},$$

since $v(0) = \widetilde{k}$. Thus, it follows that

$$\Psi(v(t)) \leq \Psi(\widetilde{k}) + \int_0^t m(s) ds,$$

and hence

$$v(t)) \leq \Psi^{-1}\left(\Psi(\widetilde{k}) + \int_0^t m(s)ds\right).$$

Since $u(t) \leq v(t)$ we obtain

$$u(t)) \leq \Psi^{-1}\left(\Psi(\widetilde{k}) + \int_0^t m(s)ds\right),$$

which prove the theorem. □

In the particular case where $g(t) \equiv 0$, we have the following corollary to Theorem 4.3.1.

Corollary 4.3.2. *Let $k_0, k_1 > 0$, $q > \frac{1}{\alpha}$, and the functions f_i, λ_i, ψ_i, and u be as in Theorem 4.3.1. If*

$$u(t) \leq k_0 + \sum_{i=1}^{n} k_1 \int_0^t (t-s)^{\alpha-1} f_i(s)\lambda_i(s)\psi_i(u(s))ds,$$

then,

$$u(t) \leq \Psi^{-1}\left(\Psi(\bar{k}) + \sum_{i=1}^{n} \frac{1}{q}\int_0^t (f_i(s)\lambda_i(s))^q ds\right),$$

where

$$\Psi(z) = \int_{k_0}^{z} \frac{dy}{\widetilde{\psi}(y)},\ \widetilde{\psi}(y) = \max\{\psi_i^q(y), i = 1.2...n\},\ \bar{k} = k_0 + \frac{nk_1^p T^{p(\alpha-1)+1}}{p(p(\alpha-1)+1)},$$

Ψ^{-1} is the inverse function of Ψ and for every $t \in [0,T]$,

$$\Psi(\bar{k}) + \sum_{i=1}^{n} \frac{1}{q}\int_0^t (f_i(s)\lambda_i(s))^q ds \in Dom(\Psi^{-1}).$$

Proof. Proceeding exactly as in the proof of theorem 4.3.1, we again arrive at the inequality 4.3.1. Define the function $\widetilde{\psi}$ by

$$\widetilde{\psi}(x) = \max\{\psi_i^q(x) : i = 1, 2, ..., n\},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Then,

$$v'(t) \leq \sum_{i=1}^{n} \frac{1}{q}(f_i(t)\lambda_i(t))^q ds\widetilde{\psi}(v(t))$$

which implies that

$$\frac{v'(t)}{\widetilde{\psi}(v(t))} \leq \sum_{i=1}^{n} \frac{1}{q}(f_i(t)\lambda_i(t))^q ds.$$

Integrating both sides of the above inequality from 0 to t, we obtain

$$\int_{v(0)}^{v(t)} \frac{dy}{\widetilde{\psi}(y)} \leq \sum_{i=1}^{n} \frac{1}{q} \int_0^t (f_i(s)\lambda_i(s))^q ds.$$

Therefore

$$v(t) \leq \Psi^{-1}\left(\Psi(\bar{k}) + \sum_{i=1}^{n} \frac{1}{q} \int_0^t (f_i(s)\lambda_i(s))^q ds\right).$$

In view of the fact that $u(t) \leq v(t)$ this proves the corollary. □

If $\alpha = 1$ and $f_i(t) \equiv 1$ for $i = 1, 2, ..., n$ in Corollary 4.3.2, we have the following version of the result of Agarwal et al.([5], Theorem 2.1).

Corollary 4.3.3. *Let $k_0, k_1 > 0$ and the functions λ_i, ψ_i, and u be as in Theorem 4.3.1. If*

$$u(t) \leq k_0 + \sum_{i=1}^{n} k_1 \int_0^t \lambda_i(s)\psi_i(u(s))ds,$$

then

$$u(t) \leq \Psi_n^{-1}\left(\Psi_n(k_{n-1}) + \int_0^t \lambda_n(s)ds\right),$$

where

$$\Psi_i(z) = \int_{z_i}^{z} \frac{dy}{\psi_i(y)},\ z > z_i > 0,\ i = 1, 2, ..., n,$$

the constants k_i are given by

$$k_i = \Psi_i^{-1}[\Psi_i(k_{i-1}) + \|\lambda_i\|_{T_1}],$$

and $T_1 \in I$ is the largest number such that

$$\|\lambda_i\|_{T_1} := \int_0^{T_1} \lambda_i(s)ds \leq \int_{k_{i-1}}^{\infty} \frac{dy}{\psi_i(y)} ds.$$

For $n = 1$, $g \equiv 0$, and $f_1 \equiv \lambda_1 \equiv 1$ in Theorem 4.3.1, we obtain the following fractional Bihari type inequality on bounded intervals.

Corollary 4.3.4. *Let $k_0, k_1 > 0$, $q > \frac{1}{\alpha}$ and the functions ψ and u be as in Theorem 4.3.1, such that*

$$u(t) \leq k_0 + k_1 \int_0^t (t-s)^{\alpha-1}\psi(u(s))ds.$$

Then, for every $t \in [0, T]$ we have

$$u(t) \leq \Phi^{-1}\left(\Phi(\bar{k}) + \frac{t}{q}\right), \tag{4.3.2}$$

where $p\left(1-\frac{1}{q}\right)=1$,

$$\Phi(z)=\int_{k_0}^{z}\frac{dz}{(\psi(z))^q},\quad \bar{k}=k_0+\frac{k_1^pT^{p(\alpha-1)+1}}{p(p(\alpha-1)+1)}$$

and

$$\Phi(\bar{k})+\frac{t}{q}\in Dom(\Phi^{-1}),\quad \textit{for every } t\in[0,T].$$

If $n=1$ and $\lambda_1(t)\equiv 1$ in Corollary 4.3.2, then by using Hölder inequality, we give the other generalization of above inequality.

Corollary 4.3.5. *Let* $k_0,k_1>0$, *the functions* f,ψ *and* u *be as in Theorem 4.3.1,* $p\left(1-\frac{1}{q}\right)=1$ *and* $q>\frac{1}{\alpha}$. *If*

$$u(t)\le k_0+k_1\int_0^t(t-s)^{\alpha-1}f(s)\psi(u(s))ds,$$

then

$$u(t)\le\left[\Psi^{-1}\left(\frac{2^qk_1^qT^{q(p(\alpha-1)+1)}}{(p(\alpha-1)+1)^q}\int_0^t f^q(s)ds\right)\right]^{\frac{1}{q}}\quad t\in[0,T],\tag{4.3.3}$$

where

$$\Psi(z)=\int_{2^qk^q}^{z}\frac{dx}{(\psi(\sqrt[q]{x}))^q},\quad z\ge 2^qk^q.$$

Next, for convenience, we introduce a class of functions to be used to obtain new fractional Bihari type inequalities.

Definition 4.3.1. *A function* $\psi:[0,\infty)\to[0,\infty)$ *is said to belong to the class* H *if:*

(H_1) ψ *is continuous, nondecreasing and positive for every* $z>0$.

(H_2) *There exist* $\phi:[0,\infty)\to[0,\infty)$ *continuous function such that*

$$\psi(\gamma z)\le\phi(\gamma)\psi(z),\quad \textit{for every } z\ge 0,\ \gamma>0.$$

For additional details on this class of functions, see [32].
By using Young's inequality we give a generalization of Theorem 4.3.1.

Theorem 4.3.6. *Let* $0<\alpha<1$, $q>\frac{1}{\alpha}$, *and assume that for* $i=1,2,...,n$, *the functions* f,λ_i,γ_i,h *and* g *are non-negative and locally integrable,* f *and* g *are increasing, and* $(fg)/h$ *is decreasing on* $I=[0,T]$. *Additionally, let* $\psi_i\in H$, $i=1,2,...,n$, *with corresponding multiplier functions* ϕ_i. *If* $u(t)$ *is a non-negative continuous function on* I *satisfying*

$$u(t)\le h(t)+\sum_{i=1}^{n}\int_0^t(t-s)^{\alpha-1}\left(g(t)\gamma_i(s)u(s)+f(t)\lambda_i(s)\psi_i(u(s))\right)ds\tag{4.3.4}$$

then

$$u(t) \leq h(t)\Psi^{-1}\left(\Psi(\widetilde{k}) + \int_0^t m(s)ds\right),$$

where

$$m(t) = \max\left(\sum_{i=1}^n \frac{1}{q}(\gamma_i(t)g(t))^q, \sum_{i=1}^n \frac{1}{q}\left(\frac{f(t)\lambda_i(t)\phi_i(h(t))}{h(t)}\right)^q\right),\ \widetilde{k} = 1+\frac{2nT^{p(\alpha-1)+1}}{p(p(\alpha-1)+1)},$$

$$\Psi(z) = \int_1^z \frac{dy}{y^q + \widetilde{\psi}(y)},\ \widetilde{\psi}(y) = \max\{\psi_i^q(y), i = 1, 2, ..., n\}.$$

Proof. By 4.3.4 we have

$$\begin{aligned} u(t) &\leq h(t) + g(t)\sum_{i=1}^n \int_0^t (t-s)^{\alpha-1}\gamma_i(s)u(s)ds \\ &+ f(t)\sum_{i=1}^n \int_0^t (t-s)^{\alpha-1}\lambda_i(s)\psi_i(u(s))ds, \end{aligned}$$

then

$$\begin{aligned} \frac{u(t)}{f(t)g(t)} &\leq \frac{h(t)}{g(t)f(t)} + \sum_{i=1}^n \int_0^t (t-s)^{\alpha-1}\frac{\gamma_i(s)}{f(s)}u(s)ds \\ &+ \sum_{i=1}^n \int_0^t (t-s)^{\alpha-1}\frac{\lambda_i(s)}{g(s)}\psi_i(u(s))ds, \end{aligned}$$

which implies that

$$\begin{aligned} \frac{u(t)}{h(t)} &\leq 1 + \sum_{i=1}^n \int_0^t (t-s)^{\alpha-1}\frac{g(s)\gamma_i(s)}{h(s)}u(s)ds \\ &+ \sum_{i=1}^n \int_0^t (t-s)^{\alpha-1}\frac{f(s)\lambda_i(s)}{h(s)}\psi_i(u(s))ds \\ &\leq 1 + \sum_{i=1}^n \int_0^t (t-s)^{\alpha-1}g(s)\gamma_i(s)\frac{u(s)}{h(s)}ds \\ &+ \sum_{i=1}^n \int_0^t (t-s)^{\alpha-1}\frac{f(s)\lambda_i(s)\phi_i(h(s))}{h(s)}\psi_i\left(\frac{u(s)}{h(s)}\right)ds. \end{aligned}$$

Set $z = \frac{u(t)}{h(t)}$ to see that this last inequality is equivalent to

$$\begin{aligned} z(t) &\leq 1 + \sum_{i=1}^n \int_0^t (t-s)^{\alpha-1}g(s)\gamma_i(s)z(s)ds \\ &+ \sum_{i=1}^n \int_0^t (t-s)^{\alpha-1}\frac{f(s)\lambda_i(s)\phi_i(h(s))}{h(s)}\psi_i(z(s))ds. \end{aligned}$$

Only we replace f_i by $\frac{f(s)\phi_i(h(s))}{h(s)}$ in Theorem 4.3.1, we obtain

$$z(t) \leq \Psi^{-1}\left(\Psi(\widetilde{k}) + \int_0^t m(s)ds\right).$$

Therefore,

$$u(t) \leq h(t)\Psi^{-1}\left(\Psi(\widetilde{k}) + \int_0^t m(s)ds\right),$$

which completes the proof. □

For $n = 1$, $g \equiv 0$, and $f(t) \equiv k$ we next have a another fractional Bihari type inequality on a bounded interval.

Corollary 4.3.7. *Let $u, \lambda : [0,\infty) \to [0,\infty)$ are two continuous functions and $\psi \in H$ with corresponding multiplier function ϕ on $[0,\infty)$ and $h(t) > 0$ is a monotonic, nondecreasing and continuous function on $[0,\infty)$, and $k > 0$ be a constant with $q > \frac{1}{\alpha}$. If*

$$u(t) \leq h(t) + k\int_0^t (t-s)^{\alpha-1}\lambda(s)\psi(u(s))ds,\ t \in [0,T], \tag{4.3.5}$$

then

$$u(t) \leq h(t)\Psi^{-1}\left(\Psi(\bar{k}) + \frac{1}{q}\int_0^t \left(\frac{\phi(h(s))\lambda(s)}{h(s)}\right)^q ds\right), \quad t \in [0,T],$$

where

$$\Psi(z) = \int_1^z \frac{du}{(\psi(u))^q}, \quad p\left(1-\frac{1}{q}\right) = 1, \quad \bar{k} = 1 + \frac{k^p T^{p(\alpha-1)+1}}{p(p(\alpha-1)+1)},$$

and

$$\Psi(\bar{k}) + \frac{1}{q}\int_0^t \left(\frac{\phi(h(s))f(s)}{h(s)}\right)^q ds \in Dom(\Psi^{-1}), \quad t \in [0,T].$$

Proof. From 4.3.5 we have

$$u(t) \leq h(t) + k\int_0^t (t-s)^{\alpha-1}\lambda(s)\psi(u(s))ds,\ t \in [0,T],$$

the fact that h is nondecreasing, then

$$\begin{aligned}\frac{u(t)}{h(t)} &\leq 1 + k\int_0^t (t-s)^{\alpha-1}\frac{\lambda(s)}{h(s)}\psi\left(\frac{h(s)u(s)}{h(s)}\right)ds \\ &\leq 1 + k\int_0^t (t-s)^{\alpha-1}\frac{\phi(h(s))\lambda(s)}{h(s)}\psi\left(\frac{u(s)}{h(s)}\right)ds.\end{aligned}$$

Applying Young's inequality gives

$$\frac{u(t)}{h(t)} \leq 1+\frac{k^p}{p}\int_0^t (t-s)^{p(\alpha-1)}ds+\frac{1}{q}\int_0^t \left(\frac{\phi(h(s))\lambda(s)}{h(s)}\right)^q \left(\psi\left(\frac{u(s)}{h(s)}\right)\right)^q ds.$$

This implies that

$$\frac{u(t)}{h(t)} \leq 1+\frac{k^p T^{p(\alpha-1)+1}}{p(p(\alpha-1)+1)}+\frac{1}{q}\int_0^t \left(\frac{\phi(h(s))\lambda(s)}{h(s)}\right)^q \left(\psi\left(\frac{u(s)}{h(s)}\right)\right)^q ds.$$

Define

$$v(t)=1+\frac{k^p T^{p(\alpha-1)+1}}{p(p(\alpha-1)+1)}+\frac{1}{q}\int_0^t \left(\frac{\phi(h(s))\lambda(s)}{h(s)}\right)^q \left(\psi\left(\frac{u(s)}{h(s)}\right)\right)^q ds.$$

It follows that

$$v'(t)=\frac{1}{q}\left(\frac{\phi(h(t))\lambda(t)}{h(t)}\right)^q \left(\psi\left(\frac{u(t)}{h(t)}\right)\right)^q \quad and \quad v(0)=\bar{k}.$$

Since ψ is nondecreasing function, then

$$v'(t)\leq \frac{1}{q}\left(\frac{\phi(h(t))\lambda(t)}{h(t)}\right)^q (\psi(v(t)))^q.$$

By integration from 0 to t, we obtain

$$\int_{\bar{k}}^{v(t)} \frac{dz}{(\psi(z))^q} \leq \frac{1}{q}\int_0^t \left(\frac{\phi(h(s))\lambda(s)}{h(s)}\right)^q ds.$$

Consequently, it follows that

$$v(t)\leq \Psi^{-1}\left(\Psi(\bar{k})+\frac{1}{q}\int_0^t \left(\frac{\phi(h(s))\lambda(s)}{h(s)}\right)^q ds\right), \quad t\in[0,T].$$

Hence

$$u(t)\leq h(t)\Psi^{-1}\left(\Psi(\bar{k})+\frac{1}{q}\int_0^t \left(\frac{\phi(h(s))\lambda(s)}{h(s)}\right)^q ds\right), \quad t\in[0,T],$$

which completes the proof. □

As another result in this same spirit, we have the following theorem.

Theorem 4.3.8. *Let $0<\alpha<1$, $q>\frac{1}{\alpha}$ and assume that u, λ_i, $i=1,2,...,n$, and h are non-negative functions, that are locally integrable on $I=[0,T]$, and let $\psi_i \in H$ with corresponding multiplier functions ϕ_i, moreover assume that the function h is nondecreasing and satisfies the condition*

$(\widetilde{H})$ for each $t \in I$ there exist a continuous function χ such that $h(z(t)) \leq \frac{\chi(t)}{z(t)}$.
If

$$u(t) \leq h(u(t)) + \sum_{i=1}^{n} \int_0^t (t-s)^{\alpha-1} \lambda_i(s) \frac{u(s)}{\phi_i(h(u(s)))} \psi_i(u(s)) ds, \tag{4.3.6}$$

then

$$u(t) \leq \left[\chi(t) \Psi^{-1} \left(\Psi(\widetilde{k}) + \int_0^t m(s) ds \right) \right]^{\frac{1}{2}},$$

where

$$m(t) = \max \left(\frac{n}{pq}, \sum_{i=1}^{n} \frac{1}{q^2} \lambda_i^{q^2}(t) \right), \ \widetilde{k} = 1 + \frac{nT^{p(\alpha-1)+1}}{p(p(\alpha-1)+1)},$$

$$\Psi(z) = \int_1^z \frac{dy}{y^{pq} + \widetilde{\psi}(y)}, \ \widetilde{\psi}(y) = \max\{\psi_i^{q^2}(y), i = 1.2...n\}.$$

Proof. From 4.3.6 and the fact that ψ_i satisfies (H_2) and h is increasing, we obtain

$$\frac{u(t)}{h(u(t))} \leq 1 + \sum_{i=1}^{n} \int_0^t (t-s)^{\alpha-1} \lambda_i(s) \frac{u(s)}{h(u(s))} \psi_i \left(\frac{u(s)}{h(u(s))} \right) ds.$$

Set $z = \frac{u(t)}{h(u(t))}$ to see that this last inequality is equivalent to

$$z(t) \leq 1 + \sum_{i=1}^{n} \int_0^t (t-s)^{\alpha-1} \lambda_i(s) z(s) \psi_i(z(s)) ds.$$

By Young's inequality, we see that

$$\begin{aligned} z(t) &\leq 1 + \sum_{i=1}^{n} \left(\frac{1}{p} \int_0^t (t-s)^{p(\alpha-1)} + \frac{1}{q} \int_0^t \lambda_i^q(s) z^q(s) \psi_i^q(z(s)) \right) ds \\ &\leq 1 + \frac{nT^{p(\alpha-1)+1}}{p(p(\alpha-1)+1)} + \sum_{i=1}^{n} \frac{1}{q} \left(\frac{1}{p} \int_0^t z(s)^{pq} ds + \frac{1}{q} \int_0^t \lambda_i^{q^2}(s) \psi_i^{q^2}(z(s)) \right) ds \end{aligned}$$

this implies that

$$z(t) \leq \widetilde{k} + \sum_{i=1}^{n} \frac{1}{pq} \int_0^t z^{pq}(s) ds + \sum_{i=1}^{n} \frac{1}{q^2} \int_0^t \lambda_i^{q^2}(s) \psi_i^{q^2}(z(s)) ds.$$

Letting

$$v(t) = \widetilde{k} + \sum_{i=1}^{n} \frac{1}{pq} \int_0^t z^{pq}(s) ds + \sum_{i=1}^{n} \frac{1}{q^2} \int_0^t \lambda_i^{q^2}(s) \psi_i^{q^2}(z(s)) ds,$$

we define the functions m and $\widetilde{\psi}$ as follows

$$m(t) = \max\left(\frac{n}{pq}, \sum_{i=1}^{n} \frac{1}{q^2}\lambda_i^{q^2}(t)\right) \text{ and } \widetilde{\psi}(t) = \max\{\psi_i^{q^2}(t),\ i = 1, 2, ..., n\}.$$

Thus, with the nondecreasing of ψ_i we obtain

$$\begin{aligned} v'(t) &= \frac{n}{pq} z^{pq}(t) + \sum_{i=1}^{n} \frac{1}{q^2}\lambda_i^{q^2}(t)\psi_i^{q^2}(z(t)) \\ &\leq m(t)\left(v^{pq}(t) + \widetilde{\psi}(v(t))\right). \end{aligned}$$

Hence

$$\frac{v'(t)}{v^{pq}(t) + \widetilde{\psi}(v(t))} \leq m(t).$$

By integration from 0 to t, we obtain

$$\int_{v(0)}^{v(t)} \frac{dx}{x^{pq} + \widetilde{\psi}(x)} \leq \int_0^t m(s)ds.$$

With $v(0) = \widetilde{k}$ we have

$$\Psi(v(t)) \leq \Psi(\widetilde{k}) + \int_0^t m(s)ds,$$

which implies that

$$v(t) \leq \Psi^{-1}\left(\Psi(\widetilde{k}) + \int_0^t m(s)ds\right).$$

Since $\frac{u(t)}{h(u(t))} = z(t) \leq v(t)$ and $(\widetilde{H})$ holds, we finally obtain

$$\begin{aligned} u(t) &\leq h(u(t))\Psi^{-1}\left(\Psi(\widetilde{k}) + \int_0^t m(s)ds\right) \\ &\leq \frac{\chi(t)}{u(t)}\Psi^{-1}\left(\Psi(\widetilde{k}) + \int_0^t m(s)ds\right). \end{aligned}$$

Thus,

$$u^2(t) \leq \chi(t)\Psi^{-1}\left(\Psi(\widetilde{k}) + \int_0^t m(s)ds\right).$$

Consequently, it follows that

$$u(t) \leq \left[\chi(t)\Psi^{-1}\left(\Psi(\widetilde{k}) + \int_0^t m(s)ds\right)\right]^{\frac{1}{2}}.$$

This completes the proof of the theorem. □

Chapter 5

Fractional Differential Inclusions with Delay

In this chapter we wish to apply some of the inequalities obtained in section 4.3 of chapter 4 and the new Leray-Schauder type fixed point theorem obtained in section 1.4 of chapter 1 in the convex case, and the Filippov theorem in the non-convex case to proving the existence of solutions to the Cauchy problem for the fractional delay differential inclusions

$$\begin{cases} {}^cD^\beta x(t) \in F(t, x_t) \ , & \text{t} \in J, \\ x(t) = \varphi(t) \ , & \text{t} \in I, \end{cases} \tag{5.0.1}$$

where: $0 < \beta \leq 1$, $J := [0, T]$, $I := [-r, 0]$, r, T are strictly positive constants and $F : J \times C(I, E) \to \mathcal{P}_{cp,cv}(E)$ is a multivalued function.
For any function x defined on $[-r, T]$ and any $t \in J$ we denote by x_t the element of $C(I, E)$ defined by

$$x_t(\theta) = x(t + \theta), \ \theta \in I.$$

Let $\Omega := C([-r, T], E)$. We begin by recalling the definition of the solution of the problem (5.0.1).

Definition 5.0.1. *The function $x \in \Omega$ is solution of the problem (5.0.1) if there exists one function $v \in S^q_{F,x}$, such that*

$$x(t) = \begin{cases} \varphi(t), & t \in I \\ \varphi(0) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} v(s) ds \ , & t \in J. \end{cases}$$

5.1 Convex case

We will need to make use of the following assumptions in our results in this section.

(H_1) The multivalued map $F(t, \cdot)$ has a strong measurable selection for $x \in C(I, E)$.

(H_2) The map $F(t, \cdot)$ is upper hemicontinuous for almost all $t \in J$.

(H_3) There exists a function $\psi \in H$ and $f \in L^q(J, \mathbb{R}^+)$ with $q > \frac{1}{\beta}$ such that

$$\|F(t,x)\|_E \leq f(t)\psi(\|x\|_\infty)$$

for every $t \in J$ and $x \in C(I, E)$.

(H_4) There exists $g \in L^q(J, \mathbb{R}^+)$ such that for all bounded $D \subset C(I, E)$, we have

$$\alpha(F(t, D)) \leq g(t) \sup_{\theta \in J_0} \alpha(D(\theta)), \ for\ a.e.t \in J,$$

where $D(\theta) = \{\phi(\theta) : \phi \in D\}$.

In the following proposition we establish some properties of the selections functions operator.

Proposition 5.1.1. *If E is a reflexive space and the multifunction F satisfies $(H_1) - (H_3)$ for $1 < q < \infty$, then the selections functions operator $S_F^q : C([-r, T], E) \to L^q([-r, T], E)$ given by*

$$S_F^q(x) = \{v \in L^q(J, E) : v(t) \in F(t, x_t)\} \ a.e.t \in J$$

has nonempty convex weakly compact values and is a weakly upper semicontinuous multivalued operator.

Proof. The convexity of S_F^q follows immediately from the convexity of F. Let $x \in C([-r, T], E)$, there exists a step sequence x_n converging uniformly to x in $C([-r, T], E)$. By conditions (H_1) and (H_3), there is a sequence v_n of strong measurable selections of F such that

$$v_n(t) \in F(t, (x_n)_t) \ a.e.\ t \in J$$

and

$$\|v_n(t)\| \leq p(t)\psi(\rho) \ a.e. \ \in J,$$

where

$$\|x_n\|_\infty \leq \rho \ for\ all\ n \in \mathbb{N}.$$

Hence,

$$v_n \in S_F^q(x_n) \ for\ all\ n \in \mathbb{N},$$

and (v_n) is bounded sequence in $L^q(J, E)$. Since E is a reflexive space, by the duality Theorem ([35] V.1.1), the space $L^q(J, E)$ is also reflexive. By the Eberlein-Smulian Theorem (see Brezis [25]), there exists a subsequence, still denoted by v_k, that converges weakly to $v \in L^q(J, E)$. Additionally, we have

$$v(t) \in \cap_{n\geq 1}\overline{\{v_k(t) : k \geq n\}} \subset \cap_{n\geq 1}\overline{co}\{v_k(t) : k \geq n\}, \ a.e.\ t \in J. \tag{5.1.1}$$

Since $F(t,\cdot)$ is upper hemicontinuous,

$$\limsup_{n\to\infty}\sigma(x^\star,F(t,(x_n)_t))\leq\sigma(x^\star,F(t,x_t)),\ \ for\ all\ x^\star\in E^\star. \tag{5.1.2}$$

Since $v_n(t)\in F(t,(x_n)_t)$ *a.e.* $t\in J$ it follows that

$$v_n(t)\in\overline{co}B\left(F(t,(x_n)_t),\frac{1}{n}\right)\ \ a.e.\ t\in J. \tag{5.1.3}$$

By (5.1.1) - (5.1.3), and applying the Lebesgue dominated convergence theorem, we obtain

$$v(t)\in F(t,x_t),\ \ a.e.\ t\in J.$$

Therefore, $S_F^q(x)\neq\emptyset$ and $S_F^q(x)\in P_{wcp,cv}(C([-r,T],E))$.
Let $\{(x_n,v_n)\}_{n\in\mathbb{N}}\in Gr(S_F^q)$ such that (x_n) converges to $x\in C([-r,T],E)$. Then, for each $n\in\mathbb{N}$, we have

$$v_n(t)\in(F(t,(x_n)_t),\ \ a.e\ t\in J.$$

Applying a similar argument, we can prove that the Nemytskii operator is weakly upper semicontinuous. □

Now consider the operator $N:C([-r,T],E)\to P(C([-r,T],E))$ defined by

$$N(x)(t)=\left\{\ h\in\Omega:h(t)=\left\{\begin{array}{ll}\varphi(t), & \mathrm{t}\in J_0,\\ \varphi(0)+\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}v(s)ds\ , & \mathrm{t}\in J,\end{array}\right.\right\}, \tag{5.1.4}$$

where $v\in S_F^q(x)$. Notice that N can be written in the form

$$N(x)=L(x)+(KoS_F^q)(x),\ \ x\in C([-r,T],E), \tag{5.1.5}$$

where $L:C([-r,T],E)\to C([-r,T],E)$ is defined by

$$L(x(t))=\left\{\begin{array}{ll}\varphi(t), & \mathrm{t}\in I,\\ \varphi(0)\ , & \mathrm{t}\in J,\end{array}\right.$$

and $K:L^q(J,E)\to C([-r,T],E)$ by

$$K(v(t))=\left\{\begin{array}{ll}0, & \mathrm{t}\in I,\\ \frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}v(s)ds\ , & \mathrm{t}\in J.\end{array}\right.$$

Next, we establish an important property of the operator $N(x)$ defined in (5.1.4).

Proposition 5.1.2. *If conditions $(H_1)-(H_4)$ hold, then the operator N is u.s.c. and α-condensing.*

Proof. We will divide the proof into several steps. we begin by showing that N is u.s.c. and has nonempty convex values.

Step 1: $N(\cdot) \in P_{cv}(E)$. Since L is single valued, K is a bounded linear operator, and S_F^q has nonempty convex values, for each $x \in C([-r,T],E)$ we have $N(x) \in P_{cv}(E)$.

Step 2: N is u.s.c. We first show that N maps bounded sets into bounded subsets of $C([-r,T],E)$. Let $B_\rho := \{x \in C([-r,T],E) : \|x\|_\infty \leq \rho\}$. For $x \in B_\rho$ and $h \in N(x)$, there exists $v \in S_{F,x}^q$ such that

$$h(t) = \begin{cases} \varphi(t), & \text{t} \in I, \\ \varphi(0) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} v(s) ds \ , & \text{t} \in J. \end{cases}$$

Then, by (H_3) and Holder's inequality

$$\begin{aligned} \|h(t)\| &\leq \|\varphi(0)\| + \|\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} v(s) ds\| \\ &\leq \|\varphi\|_\infty + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s) \psi(\|x_s\|) ds \\ &\leq \|\varphi\|_\infty + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s) \psi(\|x\|_\infty) ds \\ &\leq \|\varphi\|_\infty + \frac{\psi(\rho)}{\Gamma(\beta)} \left(\int_0^t (t-s)^{p\beta-p} ds \right)^{\frac{1}{p}} \|f\|_{L^q}. \end{aligned}$$

Hence,

$$\|h\|_\infty \leq \|\varphi\|_\infty + \frac{T^{\beta-1+\frac{1}{P}} \psi(\rho)}{\Gamma(\beta)} \|f\|_{L^q}.$$

Next, we wish to show that N maps bounded sets into equicontinuous subsets of $C([-r,T],E)$. Let $t_1, t_2 \in J$ with $t_1 < t_2$ and let B_ρ be a bounded subset of $C([-r,T],E)$. If $x \in B_\rho$, then for each $h \in N(x)$, we have

$$\|h(t_2) - h(t_1)\| \leq \frac{\psi(\rho)}{\Gamma(\beta)} \int_{t_1}^{t_2} (t_2 - s)^{\beta-1} f(s) ds + \frac{\psi(\rho)}{\Gamma(\beta)} \int_0^{t_1} |(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}| f(s) ds.$$

From the Holder and Biernacki inequalities, it follows that

$$\begin{aligned}
\|h(t_2)-h(t_1)\| &\leq \frac{p\psi(\rho)(t_2-t_1)^{\beta-1+\frac{1}{p}}}{(p\beta-p+1)\Gamma(\beta)}\|f\|_{L^q} \\
&+ \frac{\psi(\rho)}{\Gamma(\beta)}\left(\int_0^{t_1}\left((t_1-s)^{\beta-1}-(t_2-s)^{\beta-1}\right)^p ds\right)^{\frac{1}{p}}\|f\|_{L^q} \\
&\leq \frac{p\psi(\rho)(t_2-t_1)^{\beta-1+\frac{1}{p}}}{(p\beta-p+1)\Gamma(\beta)}\|f\|_{L^q} \\
&+ \frac{\psi(\rho)}{\Gamma(\beta)}\left(\int_0^{t_1}\left((t_1-s)^{p\beta-p}-(t_2-s)^{p\beta-p}\right) ds\right)^{\frac{1}{p}}\|f\|_{L^q} \\
&\leq \frac{p\psi(\rho)(t_2-t_1)^{\beta-1+\frac{1}{p}}}{(p\beta-p+1)\Gamma(\beta)}\|f\|_{L^q} \\
&+ \frac{\psi(\rho)\left(t_1^{p\beta-p+1}+(t_2-t_1)^{p\beta-p+1}-t_2^{p\beta-p+1}\right)^{\frac{1}{p}}}{(p\beta-p+1)\Gamma(\beta)}\|f\|_{L^q}.
\end{aligned}$$

The left-hand side tends to zero as $t_2 - t_1 \to 0$, so $N(B_\rho)$ is equicontinuous in $C([-r,T],E)$.

Step 3: N is a condensing operator for a suitable measure of noncompactness α. Given a bounded subset $D \in P_b([-r,T],E)$, let $mod_c(D)$ be the modulus of equicontinuity of the set of functions D, i.e.,

$$mod_c(D) = \lim_{\delta\to 0}\sup_{x\in D}\max_{|t_2-t_1|\leq\delta}|x(t_2)-x(t_1)|.$$

It is well known (see, for example, ([57], example 2.1.2)) that $mod_c(D)$ defines a measure of noncompactness in $C([-r,T],E)$ that satisfies all of the properties in lemma 1.3.2 except regularity. Given the Hausdorff's measure of noncompactness χ, let $\overline{\alpha}$ be the real measure of noncompactness defined on bounded subset of $C([-r,T],X)$ by

$$\overline{\alpha}(D) = \sup_{t\in[0,T]} e^{-\tau\int_0^t f^q(s)ds}\chi(D(t)),$$

where $\tau > 0$ chosen such that

$$q^\star = \frac{2T^{\beta-1+\frac{1}{p}}}{\sqrt[p]{q\tau(p\beta-p+1)}\Gamma(\beta)} < 1. \tag{5.1.6}$$

Then we define the measure of noncompactness α on bounded subsets of $C([-r,T],E)$ by

$$\alpha(D) = \max_{D\in\Delta(C([-r,T],E))}(\overline{\alpha}(D), mod_c(D)),$$

where $\Delta(C([-r,T],E))$ is the collection of all countable subsets of B. The measure of noncompactness α is monotone, regular, and nonsingular (see ([57], example 2.1.4)). To show that N is α-condensing, let $B \subset C([-r,T],E)$ be a bounded set in $C([-r,T],E)$ such that

$$\alpha(B) \leq \alpha(N(B)). \tag{5.1.7}$$

To show that B is relatively compact, let $\{x_n : n \in \mathbb{N}\} \subset B$. From (5.1.5), each h_n in $N(x_n)$ can be represented as

$$h_n = L(x_n) + K(v_n), \ with \ v_n \in S_{F(x_n)}. \tag{5.1.8}$$

Moreover, (5.1.7) yields

$$\alpha(\{h_n : n \in \mathbb{N}\}) \geq \alpha(\{x_n : n \in \mathbb{N}\}). \tag{5.1.9}$$

From condition (H_4), for a.e. $t \in J$,

$$\begin{aligned}
\chi(\{v_n(t) : n \in \mathbb{N}\}) &\leq \chi(F(t, \{x_n(t)\}_{n=1}^{\infty})) \\
&\leq g(t)\chi(\{(x_t)_n\}_{n=1}^{\infty}) \\
&\leq g(t) \sup_{0\leq s\leq t} \chi(\{(x_s)_n\}_{n=1}^{\infty}) \\
&\leq e^{\tau \int_0^t g^q(s)ds} g(t)\overline{\alpha}(\{x_n\}_{n=1}^{\infty}).
\end{aligned}$$

Then

$$\chi(\{v_n(t) : n \in \mathbb{N}\}) \leq e^{\tau \int_0^t g^q(s)ds} g(t)\overline{\alpha}(\{x_n\}_{n=1}^{\infty}). \tag{5.1.10}$$

It is clair that $s \mapsto (t-s)^{\beta-1}g(s) \in L^q((0,t))$. Then, Lemma 1.3.8 implies

$$\begin{aligned}
\chi(\{K(v_n)(t)\}_{n=1}^{\infty}) &\leq \overline{\alpha}(\{x_n\}_{n=1}^{\infty}) \frac{2}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s) e^{\tau \int_0^s g^q(r)dr} ds \\
&\leq \overline{\alpha}(\{x_n\}_{n=1}^{\infty}) \frac{2}{\Gamma(\beta)} \left(\int_0^t (t-s)^{p\beta-p} ds \right)^{\frac{1}{p}} \left(\int_0^t g^q(s) e^{q\tau \int_0^s g^q(r)dr} \right)^{\frac{1}{q}} ds \\
&\leq \overline{\alpha}(\{x_n\}_{n=1}^{\infty}) \frac{2t^{\beta-1+\frac{1}{p}}}{\sqrt[p]{(p\beta-p+1)}\Gamma(\beta)} \left(\int_0^t g^q(s) e^{q\tau \int_0^s g^q(r)dr} \right)^{\frac{1}{q}} ds \\
&\leq \overline{\alpha}(\{x_n\}_{n=1}^{\infty}) \frac{2t^{\beta-1+\frac{1}{p}}}{\sqrt[p]{q\tau(p\beta-p+1)}\Gamma(\beta)} e^{\tau \int_0^t g^q(s)ds}.
\end{aligned}$$

Therefore,

$$\overline{\alpha}(\{x_n\}_{n=1}^{\infty}) \leq \overline{\alpha}(\{h_n\}_{n=1}^{\infty}) = \sup_{t\in J} e^{-\tau \int_0^t g^q(s)ds} \chi(\{h_n(t)\}_{n=1}^{\infty}) \leq q^{\star}\overline{\alpha}(\{x_n\}_{n=1}^{\infty}). \tag{5.1.11}$$

Using (5.1.6), we have

$$\overline{\alpha}(\{x_n\}_{n=1}^{\infty}) = 0. \tag{5.1.12}$$

Moreover, $\overline{\alpha}(x_n) = 0$ implies that $\chi(\{x_n(t)\}) = 0$ for a.e. $t \in [-r, T]$. In turn, (5.1.10) implies that

$$\chi(\{v_n(t)\}) = 0 \ for \ a.e. \ t \in [-r, T].$$

Hence, (5.1.8) implies that $\chi(\{h_n\}_{n=1}^{\infty}) = 0$. By an argument similar to the one we used to prove that N maps bounded sets into equicontinuous sets, we can prove that the set $\{h_n\}$ is equicontinuous, and so $mod_c(B) = 0$. It follows that $\alpha(\{h_n\}_{n=1}^{\infty}) = 0$, which by (5.1.9) implies that $\overline{\alpha}(\{x_n\}_{n=1}^{\infty}) = 0$. Thus, B is relatively compact. By the Arzela-Ascoli theorem, N is completely continuous, from which we have that N is α-condensing.

Step 4: N has a closed graph. Let $\{x_n\}_{n\in\mathbb{N}} \in C([-r, T], E)$ be a sequence such that $\{x_n\}_{n\in\mathbb{N}}$ converges to $\widetilde{x}$, $h_n \in N(x_n)$, and $\{h_n\}_{n\in\mathbb{N}}$ converges to $\widetilde{h}$. We need to show that $\widetilde{h} \in N(\widetilde{x})$. So for each $n \in \mathbb{N}$ there exist $v_n \in S_F^q(x_n)$ such that

$$h_n(t) = L(x_n(t)) + K(v_n(t)), \ t \in [-r, T].$$

Similar to the proof of proposition 5.1.1, we can conclude that there exists a subsequence of v_n converging weakly to v in L^q and satisfying

$$v(t) \in F(t, \widetilde{x}_t), \ a.e. \ t \in J.$$

Since K is a continuous linear operator,

$$K(v_n(t)) \to K(v(t)) \ as \ n \to \infty.$$

On the other hand, the continuity of L implies that

$$L(x_n(t)) \to L(\widetilde{x}(t) \ as \ n \to \infty.$$

Therefore,

$$h_n(t) \to L(\widetilde{x}(t) + K(v(t)) \ as \ n \to \infty.$$

This implies

$$h = L(\widetilde{x}) + K(v) \in N(\widetilde{x}).$$

Hence, the multivalued operator N has a closed graph. In view of Steps 1-4, the proof of the proposition is complete. □

We are now ready to give our main existence result for problem (5.0.1).

Theorem 5.1.3. *Assume that* (H_1)- (H_4) *hold.Then the problem (5.0.1) has at least one solution on* $[-r, T]$ *moreover the solution set of this problem is compact.*

Proof. It is clear that the solutions of (5.0.1) correspond to the set $FixN$ of fixed point of the multivalued operator N defined in Proposition 5.1.2. By Propositions 5.1.1 and

5.1.2, $N(\cdot) \in P_{cv,wcp}(C([-r,T],E))$ and it is u.s.c., admissible, and α-condensing. Let $x \in C([-r,T],E)$ be such that

$$x \in \lambda N(x) \ for \ some \ 0 < \lambda < 1.$$

Then,

$$\|x(t)\| \leq \|\varphi\|_\infty + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s)\psi(\|x_s\|)ds.$$

Let $\mu(t) = \sup_{-r\leq s\leq t} \|x(s)\|$; then

$$\mu(t) \leq \|\varphi\|_\infty + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s)\psi(\mu(s))ds.$$

From Corollary 4.3.5

$$u(t) \leq \left[\Psi^{-1}\left(\frac{2^q k_1^q T^{q(p(\alpha-1)+1)}}{(p(\alpha-1)+1)^q} \int_0^t f^q(s)ds\right)\right]^{\frac{1}{q}} \quad t \in [0,T],$$

where

$$\Psi(z) = \int_{2^q k^q}^z \frac{dx}{(\psi(\sqrt[q]{x}))^q}, \quad z \geq 2^q k^q.$$

Hence,

$$\|x\|_\infty \leq \left[\Psi^{-1}\left(\frac{2^q k_1^q T^{q(p(\alpha-1)+1)}}{(p(\alpha-1)+1)^q}\|f\|_{L^q}\right)\right]^{\frac{1}{q}} := M^\star,$$

and so the set

$$M = \{x \in C([-r,T],E) : x \in \lambda N(x), \lambda \in (0,1)\}$$

is bounded. As a consequence of Lemma 1.4.2, N has a fixed point x in $C([-r,T],E)$ that is a solution to problem (5.0.1). Finally, since $FixN$ is bounded, by Lemma 1.4.4, $FixN$ is also compact. This proves the theorem. □

Remark 5.1.1. *In all our results in Section 4.3 and in the main theorem in Section 5.1 (see condition H_3, we have asked that $q > \frac{1}{\beta}$. It would be of interest to see what might be proved in the case where $q \leq \frac{1}{\beta}$.*

5.2 Non-convex case

In this case we use the Filippov's theorem on a bounded interval.
Let $\psi \in C(I,E), h \in L^1(J,E)$, and let $y \in \Omega$ be a solution of the following differential problem with semi-linear equation

$$\begin{cases} {}^cD^\beta y(t) = h(t) \ , & \mathrm{t} \in J \\ y(t) = \psi(t) \ , & \mathrm{t} \in I. \end{cases}$$

We will consider the following assumptions

(A_1) The multi-function $F : J \times C(I,E) \to P(E)$ has closed values, such that
(a) for all $u \in E, t \mapsto F(t,u)$ is measurable,
(b) the map $\gamma : t \mapsto d(h(t), F(t,y_t))$ is integrable.

(A_2) There exists a function $g \in L^q(J,\mathbb{R}^+)$, and a strictly positive real ρ such that

$$H_d(F(t,u);F(t,v)) \leq g(t)\|u-v\|_\infty,$$

for all $t \in J$,and $u, v \in \overline{B}(y_t,\rho)$, with $\overline{B}(y_t,\rho)$ is the closed ball of $C(I,E)$.

(A_3) The function g satisfied the following condition,

$$\frac{T^{p(\beta-1)+1}}{\Gamma^p(\beta)p(p(\beta-1)+1)} + \frac{1}{q}\|g\|_{L^q} \leq 1 \text{ for every } t \in J.$$

We define the function ξ by

$$\xi(t) = \delta + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\gamma(s)ds.$$

For each $t \in J$, with δ a strictly positive real.

We have the following result

Theorem 5.2.1. *Assume that* (A_1)- (A_4) *hold. Moreover if*

$$\xi(t) \leq \rho.$$

Then, for all $\varphi \in C(I,E)$ *with* $\|\varphi-\psi\|_\infty \leq \delta$,
the problem (5.0.1) has at least one solution on $[-r,T]$.

Proof. Let $\varphi \in C(I,E)$ such that $\|\varphi-\psi\|_\infty \leq \delta$. We construct a sequence of function $(x^n(t))_{n\in\mathbb{N}}$ which will show that it is converging to some solution of problem (5.0.1) on the interval $[-r,T]$. Let $f_0 = h$ on J and $x^0(t) = y(t)$ for all $t \in [-r,T]$, i.e.

$$x^0(t) = \begin{cases} \psi(t), & \text{t} \in I \\ \psi(0) + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}f_0(s)ds\ , & \text{t} \in J. \end{cases}$$

Then we defined the multi-valued map $U_1 : J \to P(E)$, by

$$U_1(t) = F(t,x_t) \cap B(h(t),\gamma(t)).$$

Since h and γ are measurable, Theorem 1.1.8 tells us that the ball $B(h(t),\gamma(t))$ is measurable. Moreover $F(t,x_t,x_t(\omega))$ is measurable and U_1 is nonempty.Indeed, since

$v = 0$ is a measurable function, from lemma 1.1.4 there exists a function u_1 which is a measurable selection of $F(t, x_t^0)$, and such that

$$\mid u_1(t) - h(t) \mid \leq d(h(t), F(t, x_t^0)) := \gamma(t),$$

this implies that

$$\mid u_1(t) - h(t) \mid \leq \gamma(t),$$

so

$$u_1(t) \in B(h(t), \gamma(t)) \; and \; u_1(t) \in F(t, x_t^0),$$

therefore

$$u_1(t) \in U_1(t).$$

We deduce that the intersection multi-valued operator $U_1(t)$ is measurable. By the Lemma 1.1.3, there exists a function $t \mapsto f_1(t)$ which a measurable selection for U_1. Consider

$$x^1(t) = \begin{cases} \varphi(t), & \text{t} \in I \\ \varphi(0) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f_1(s) ds \ , & \text{t} \in J. \end{cases}$$

Then for each $t \in J$ we have

$$\begin{aligned} \|x_t^1 - x_t^0\|_\infty &\leq \|\varphi - \psi\|_\infty + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \mid f_1(s) - f_0(s) \mid ds, \\ &\leq \delta + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \gamma(s) ds := \xi(t) \leq \rho. \end{aligned}$$

Then Lemma 1.1.6, tells us that $F(t, x_t^1))$ is measurable. The ball $B(f_1(t), g(t)\|x^1 - x^0\|_\infty)$ is also measurable by Theorem 1.1.8. Moreover the operator U_2 defined by

$$U_2(t) = F(t, x_t^1) \cap B(f_1(t), g(t)\|x^1 - x^0\|_\infty),$$

is measurable nonempty. Indeed, since f_1 is a measurable function, then by Lemma 1.1.4 we have the existence of a measurable selection u_2 of $F(t, x_t^1)$, such that

$$\begin{aligned} \mid u_2(t) - f_1(t) \mid &\leq d(f_1(t), F(t, x_t^1)) \\ &\leq H_d(F(t, x_t^0), F(t, x_t^1)) \\ &\leq g(t)\|x^1 - x^0\|_\infty, \end{aligned}$$

so

$$u_2(t) \in B(f_1(t), g(t)\|x^1 - x^0\|_\infty) \; and \; u_2(t) \in F(t, x_t^1),$$

hence

$$u_2(t) \in U_2(t).$$

Lemma 1.1.3 yields a measurable selection $f_2(t) \in U_2(t)$. So,

$$\mid f_2(t) - f_1(t) \mid \leq g(t)\|x^1 - x^0\|_\infty.$$

We define now x_t^2 by

$$x^2(t) = \begin{cases} \varphi(t), & \text{t} \in I \\ \varphi(0) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f_2(s) ds \ , & \text{t} \in J. \end{cases}$$

We obtain the following estimates:

$$\begin{aligned} \|x^2 - x^1\|_\infty &\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \mid f_2(s) - f_1(s) \mid ds \\ &\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s) \|x_s^1 - x_s^0\| ds \\ &\leq \frac{1}{\Gamma(\beta)} \left(\frac{T^{p(\beta-1)+1}}{p(p(\beta-1)+1)} + \frac{1}{q} \|g\|_{L^q} \right) \|x^1 - x^0\|_\infty. \end{aligned}$$

So

$$\|x^2 - x^1\|_\infty \leq \lambda(t)\xi(t), \tag{5.2.1}$$

where

$$\lambda(t) = \frac{1}{\Gamma(\beta)} \left(\frac{T^{p(\beta-1)+1}}{p(p(\beta-1)+1)} + \frac{1}{q} \|g\|_{L^q} \right).$$

Defining now also the operator U_3 by

$$U_3(t) = F(t, x_t^2) \cap B(f_2(t), g(t)\|x^2 - x^1\|_\infty).$$

Similarly with all the above we will have that: $U_3(t)$ is a measurable multi-valued operator with nonempty values, so there exists a measurable selection $f_3(t) \in U_3(t)$. This allows us to define

$$x^3(t) = \begin{cases} \varphi(t), & \text{t} \in I \\ \varphi(0) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f_3(s) ds \ , & \text{t} \in J. \end{cases}$$

For $t \in J$ we obtain the estimates:

$$\begin{aligned} \|x^3 - x^2\|_\infty &\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \mid f_3(s) - f_2(s) \mid ds \\ &\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s) \|x_s^2 - x_s^1\| ds \\ &\leq (\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s) ds) \|x^2 - x^1\|_\infty \\ &\leq \lambda(t) \|x^2 - x^1\|_\infty. \end{aligned}$$

We use (5.2.1) to have that

$$\|x^3 - x^2\|_\infty \leq (\lambda(t))^2 \xi(t). \tag{5.2.2}$$

The same manner be we define the operator U_4 by

$$U_4(t) = F(t, x_t^3) \cap B(f_3(t), g(t)\|x^3 - x^2\|_\infty).$$

Arguing as for U_2, we can prove that U_4 is a measurable multi-valued operator with nonempty values, so there exists a measurable selection $f_4(t) \in U_4(t)$. Then we define x_t^4 by

$$x^4(t) = \begin{cases} \varphi(t), & \text{t} \in I \\ \varphi(0) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f_4(s) ds \ , & \text{t} \in J. \end{cases}$$

And we will obtain the estimate:

$$\begin{aligned} \|x^4 - x^3\|_\infty &\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \mid f_4(s) - f_3(s) \mid ds \\ &\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s) \|x_s^3 - x_s^2\| ds \\ &\leq (\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s) ds) \|x^3 - x^2\|_\infty \\ &\leq \lambda(t) \|x^3 - x^2\|_\infty. \end{aligned}$$

And using (5.2.2) to have that

$$\|x^4 - x^3\|_\infty \leq (\lambda(t))^3 \xi(t).$$

We repeat this processor for $n = 1, 2, 3, 4, ...$, we arrive at the following bound

$$\|x^n - x^{n-1}\|_\infty \leq (\lambda(t))^{n-1} \xi(t). \tag{5.2.3}$$

By induction, suppose that (5.2.3) holds for some n and check (5.2.3) for $n+1$. Let U_{n+1} defined by the following form

$$U_{n+1} := F(t, x_t^n) \cap B(f_n(t), g(t)\|x^n - x^{n-1}\|_\infty).$$

As in the above arguing U_{n+1} is a nonempty measurable, then has a measurable selection $f_{n+1}(t) \in U_{n+1}(t)$, this allows us to define for $n \in \mathbb{N}$

$$x^{n+1}(t) = \begin{cases} \varphi(t), & \text{t} \in I \\ \varphi(0) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f_{n+1}(s) ds \ , & \text{t} \in J. \end{cases}$$

Therefore for each $t \in J$, we have

$$\begin{aligned} \|x^{n+1} - x^n\|_\infty &\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \mid f_{n+1}(s) - f_n(s) \mid ds \\ &\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s) \|x_s^n - x_s^{n-1}\| ds \\ &\leq (\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s) ds) \|x^n - x^{n-1}\|_\infty \\ &\leq \lambda(t) \|x^n - x^{n-1}\|_\infty. \end{aligned}$$

Using (5.2.3) to have that

$$\|x^{n+1} - x^n\|_\infty \leq (\lambda(t))^n \xi(t).$$

Consequently, (5.2.3) holds true for all $n \in \mathbb{N}$. We deduce that (x_t^n) is a Cauchy sequence in the Banach space $\Omega := C([-r, T], E)$, since Ω is complete then (x_t^n) is converging to a limit function $x(t) \in \Omega$. Moreover, from the definition of U_{n+1}, we have

$$\mid f_{n+1}(t) - f_n(t) \mid \leq g(t)\|x^n - x^{n-1}\|_\infty \; for \; each \; t \in J.$$

Hence for almost every $t \in J$, $f_n(t)$ is also a Cauchy sequence in the Banach space E and then converging almost everywhere to some measurable function $f(t) \in E$. On the other hand we have by definition of (x_t^n), for each $t \in J$

$$x_n(t) - \varphi(0) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f_n(s) ds).$$

Using now,the fact that $f_n \to f$, with the Lebesgue dominated convergence theorem, and passing to the limit as $n \to +\infty$ we have

$$\lim_{n \to +\infty} (x_n(t) - \varphi(0) \quad = \quad \lim_{n \to +\infty} \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f_n(s) ds),$$

so

$$x(t) - \varphi(0) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s) ds,$$

this implies that

$$x(t) = \varphi(0) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s) ds, \; for \; each \; t \in J,$$

and $x(t) = \varphi(t)$, for every $t \in I$ so

$$x(t) = \begin{cases} \varphi(t), & \text{t} \in I \\ \varphi(0) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s) ds \; , & \text{t} \in J. \end{cases}$$

With $f \in S_{F,x}$, therefore $x(t)$ is solution of the problem (5.0.1) on $[-r, T]$. □

Chapter 6

Fractional lattice system with delay

Main objective in this chapter, is to analyze the existence of solutions of fractional differential equations with the Caputo fractional substantial derivative, for following fractional lattice system with delay

$$\begin{cases} D_s^{\alpha}[u_i(t) - g_i(t, u_i(t-\eta(t)))] - Au_i(t) = F_i(t, u_{it}, u_{it,\tau}) \ , & \text{t} \geq 0, \\ u_i(t) = \varphi_i(t) \ , & \text{t} \in I, \end{cases} \tag{6.0.1}$$

where:
X be a separable Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$,
$F_i(t, u_{it}, u_{it,\tau}) = -f_i(t, u_{it}, u_i(t-\tau(t, u_{it}, \sigma)))$, $Au_i(t) = u_{i-1}(t) + 2u_i(t) - \lambda u_i(t)$
$\lambda \in \mathbb{R}$, $0 < \alpha < 1$, $i \in \mathbb{Z}$, $J = [0, T], I = [-h, 0]$,
$f_i : J \times C(I, X) \times X \to X$, are given functions,
$g_i : J \times X \to X$ are given continuous functions in its two variables,
$\varphi : I \to X$, $\tau : J \times C(I, X) \times \Sigma \to [0, h]$ and $\eta : J \to [0, h]$ are given continuous functions.
With Σ a parameter space with norm $|\cdot|_{\Sigma}$. For any function u_i defined on $[-h, T]$ and any $t \in J$ we denote by u_{it} the element of $C(I, X)$ defined by

$$u_{it}(\theta) = u_i(t + \theta), \ \theta \in I.$$

Here $u_{it}(\cdot)$ represents the history of the state from time $t - h$, up to the present time t.

For simplify the calculus let $C([-h, T], X) = C_T$ and $\nu = -\eta(t)$, and $\omega = -\tau(t, u_t, \sigma)$, hence $\nu, \omega \in [-h, 0]$, then we obtain $u(t - \eta(t)) = u(t + \nu) = u_t(\nu)$, and $u(t - \tau(t, u_t, \sigma)) = u(t + \omega) = u_t(\omega)$ this implies that, $g_i(t, u(t - \eta(t))) = g_i(t, u_t(\nu))$, and $f_i(t, u_t, u(t - \tau(t, u_t, \sigma))) = f_i(t, u_t, u_t(\omega))$.

6.1 Existence of solutions

For this study, We consider the following delay differential equations with Caputo fractional substantial tim derivative in the separable Hilbert space X.

$$\begin{cases} D_s^{\alpha}[u(t) - g(t, u_t(\nu))] = f(t, u_t, u_t(\omega)) \ , & \text{t} \geq 0, \\ u(t) = \varphi(t) \ , & \text{t} \in I, \end{cases} \tag{6.1.1}$$

where the nonlinear term f is weakly continuous in bounded sets.

Definition 6.1.1. *A function $u \in C_T$ is called a solution of the initial value problem 6.1.1 if $u(t) = \varphi(t)$ for $t \in I$ with $\varphi \in C(I, X)$, and, for $t \in J$, $u(t)$ satisfies the integral equation*

$$u(t) = [\varphi(0) - g(0, \varphi(\nu))]e^{-\beta t} + g(t, u_t(\nu)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} e^{-\beta(t-r)} f(r, u_r, u_r(\gamma))dr.$$

Introduce the following hypotheses:

(H_1) The function $f : J \times C(I, X) \times C(I, X) \to X$ is continuous, and there exist $M_1 > 0$ such that

$$\|f(t, u, v) - f(t, \widetilde{u}, \widetilde{v})\|_X \leq M_1(\|u - \widetilde{u}\|_{\infty} + \|v - \widetilde{v}\|_{\infty})$$

for every $t \in J$, u, $\widetilde{u}$, v, $\widetilde{v} \in C(I, X)$.

(H_2) The function $f(t, 0, 0)$ is $L^{1+\frac{1}{\alpha}}$ locally integrable, and there exists $b > 0$ such that

$$\int_0^t (t-r)^{\alpha-1} e^{-\beta(t-r)} \|f(r, 0, 0)\| dr \leq b, \ \forall t \geq 0.$$

(H_3) (i) There exist constants $0 < d_1 < 1$ and $d_2 \geq 0$ such that

$$\|g(t, u)\|_X \leq d_1 \|u\|_{\infty} + d_2, \ \text{ for each } \ t \in J, \, u \in C(I, X),$$

(ii) the subset $\{t \mapsto g(t, u) : u \in B\}$ is equicontinuous for all subset bounded B of C_T.

Our result is given by

Theorem 6.1.1. *Assume that (H_1)- (H_3) hold. Then the problem* (6.1.1) *has at least one solution on $[-h, T]$, moreover the solution set of this problem is bounded.*

Proof. Transform the problem (6.1.1) into a fixed point problem. Consider the operator $S_\varphi : C_T \to C_T$ defined by

$$S_\varphi(u)(t) = \begin{cases} \varphi(t), & \text{t} \in I \\ (\varphi(0) - g(0, \varphi(\nu)))e^{-\beta t} + g(t, u_t(\nu)) & \\ +\frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} e^{-\beta(t-r)} f(r, u_r, u_r(\gamma))dr \ , & \text{t} \in J. \end{cases}$$

Since X is a separable space, there exists a sequence such that the subset $\{e_1, e_2, ..., e_n, ...\}$ is dense in X, and by the Gram-Schmidt procedure, we can obtain an orthonormal subset of X, still denoted by $\{e_1, e_2, ..., e_n, ...\}$. Taking $X_n = \{e_1, e_2, ..., e_n\}$, and $P_n : X \to X_n$ an orthonormal projector, defined as $P_n u = u_n$, and $P_n \varphi = \varphi_n$, we can introduce the mapping $S_{n,\varphi} : C_T \to C_T$ as, for any $\varphi_n \in C(I, X)$,

$$S_{n,\varphi}(u)_n(t) = \begin{cases} \varphi_n(t), & \text{t} \in I \\ (\varphi_n(0) - g(0, \varphi_n(\nu)))e^{-\beta t} + g(t, u_n(t+\nu)) & \\ +\frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} e^{-\beta(t-r)} P_n f(r, u_{nr}, u_{nr}(\gamma))dr \ , & \text{t} \in J. \end{cases}$$

For any positive constant ρ, denote by

$$A_\rho := \left\{ u_n \in C_T : u_n(t) = \varphi_n(t), t \in I, and \ \sup_{t \in J} \|u_n(t)\| \leq \rho \right\}.$$

Note that a fixed point of $S_{n,\varphi}$ is a solution of (6.1.1).
We will show that $S_{n,\varphi}$ satisfies the assumptions of Leray-Schauder's fixed point theorem. The proof will be given in several steps.

Step 1: We will show that for any $\varphi_n \in C(I, X)$ the operator $S_{n,\varphi}$ maps the bounded subset $A_\rho \subset C_T$ into another bounded set $A_{\rho'} \subset C_T$, i. e, for all $\rho > 0$, there exists a positive constant ρ' such that for each $u_n \in A_\rho$ we have $\sup_{t \in J} \|S_{n,\varphi} u_n(t)\| \leq \rho'$. Let $u_n \in A_\rho$ be arbitrary. From the definition of $S_{n,\varphi}$ and the hypotheses $(H_1) - (H_3)$ we have for all $t \in J$,

$$\|S_{n,\varphi} u_n(t)\| \leq (1+d_1)\|\varphi_n\|_\infty + d_1\rho + 2d_2 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} e^{-\beta(t-r)} \|f(r, u_{nr}, u_{nr}(\gamma))\| dr,$$

by (H_1) and (H_2) to see that

$$\begin{aligned} \|f(r, u_{nr}, u_{nr}(\gamma))\| &= \|f(r, u_{nr}, u_\gamma) - f(r, 0, 0) + f(r, 0, 0)\| \\ &\leq |f(r, u_{nr}, u_\gamma) - f(r, 0, 0)\| + \|f(r, 0, 0)\| \\ &\leq 2M_1 \|u_n\|_\infty + \|f(r, 0, 0)\|, \end{aligned}$$

we replace in the above inequality and we obtain

$$\|S_{n,\varphi} u_n(t)\| \leq (1 + d_1)\|\varphi_n\|_\infty + d_1\rho + 2d_2 + \frac{b}{\Gamma(\alpha)} + \frac{2M_1\rho T^\alpha}{\Gamma(\alpha+1)} := \rho', \ \forall t \in J.$$

Hence, $S_{n,\varphi}$ maps the bounded set A_ρ into another bounded set $A_{\rho'}$.

Step 2: $S_{n,\varphi}$ is continuous on C_T.
Let $u_n^k \in C_T$ be a sequence such that $u_n^k \to u_n$ in C_T as $k \to \infty$. We can prove that, for any $t \in J$, as k goes to ∞,

$$\|u_{nt}^k - u_{nt}\|_\infty = \sup_{\theta \in I} \|u_n^k(t+\theta) - u_n(t+\theta)\| \leq \sup_{t \in J} \|u_n^k(t) - u_n(t)\| \to 0.$$

Thus, for every $t \in J$, we have

$$\begin{aligned}
\|S_{n,\varphi}u_n^k(t) - S_{n,\varphi}u_n(t)\| &\leq \|g(t, u_{nt}^k(\nu)) - g(t, u_{nt}(\nu))\| \\
&+ \frac{2M_1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} e^{-\beta(t-r)} \|u_n^k(s) - u_n(s)\| ds \\
&\leq \|g(t, u_{nt}^k(\nu)) - g(t, u_{nt}(\nu))\| \\
&+ \frac{2M_1}{\Gamma(\alpha)} \|u_{nt}^k - u_{nt}\|_{C_T} \int_0^t (t-r)^{\alpha-1} e^{-\beta(t-r)} ds \\
&\leq \|g(t, u_{nt}^k(\nu)) - g(t, u_{nt}(\nu))\| + \frac{2M_1 T^\alpha}{\Gamma(\alpha+1)} \|u_{nt}^k - u_{nt}\|_{C_T}.
\end{aligned}$$

Hence, by continuous of g in its two variables we have $\|g(t, u_{nt}^k(\nu)) - g(t, u_{nt}(\nu))\| \to 0$ as $k \to \infty$ and since $\|u_{nt}^k - u_{nt}\|_{C_T} \to 0$, as $k \to \infty$, then we obtain

$$\|S_{n,\varphi}u_n^k(t) - S_{n,\varphi}u_n(t)\| \to 0 \ as \ k \to \infty.$$

This proved that the operator $S_{n,\varphi}$ is continuous on C_T.

Step 3: $S_{n,\varphi}$ is completely continuous. Since X_n has a finite dimension n, it suffices to prove that the family of continuous operators $\{S_{n,\varphi}u_n : u_n \in A_\rho\}$ is uniformly bounded and equicontinuous of C_T on J. As proved in Step 1, for any $u_n \in A_\rho$, we find $\|S_{n,\varphi}u_n(t)\|_{C_T} \leq \rho'$, which implies that $\{S_{n,\varphi}u_n : u_n \in A_\rho\}$ is uniformly bounded. On the other hand, for any $t_1, t_2 \in J$ with $t_1 < t_2$, we have

$$\|S_{n,\varphi}u_n(t_2) - S_{n,\varphi}u_n(t_1)\| \leq Q_1 + Q_2 + Q_3 + Q_4.$$

With

$$\begin{aligned}
Q_1 &= \|\varphi_n(0) - g(0, \varphi_n(\nu))\| |e^{-\beta t_2} - e^{-\beta t_1}| \\
&\leq [(1+d_1)\|\varphi_n\| + 2d_2] e^{-\beta t_1} (1 - e^{-\beta(t_2 - t_1)}) \to 0 \ as \ t_1 \to t_2,
\end{aligned}$$

$$Q_2 = \|g(t_2, u_{nt_2}(\nu)) - g(t_1, u_{nt_1}(\nu))\| \to 0 \ by \ (ii) \ in \ (H_3),$$

$$\begin{aligned}
Q_3 &= \frac{1}{\Gamma(\alpha)}\int_0^{t_1}[(t_1-r)^{\alpha-1}e^{-\beta(t_1-r)}-(t_2-r)^{\alpha-1}e^{-\beta(t_2-r)}]\|f(r,u_{nr},u_{nr}(\gamma))\|dr\\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}e^{-\beta(t_1-r)}[(t_1-r)^{\alpha-1}-(t_2-r)^{\alpha-1}e^{-\beta(t_2-t_1)}]\|f(r,u_{nr},u_{nr}(\gamma))-f(r,0,0)\|dr\\
&+ \frac{1}{\Gamma(\alpha)}\int_0^{t_1}e^{-\beta(t_1-r)}[(t_1-r)^{\alpha-1}-(t_2-r)^{\alpha-1}e^{-\beta(t_2-t_1)}]\|f(r,0,0)\|dr\\
&\leq \frac{2M_1\rho}{\Gamma(\alpha+1)}[t_1^{\alpha}+e^{-\beta(t_2-t_1)}((t_2-t_1)^{\alpha}-t_2^{\alpha})]\\
&+ \frac{1}{\Gamma(\alpha)}\int_0^{t_1}e^{-\beta(t_1-r)}[(t_1-r)^{\alpha-1}-(t_2-r)^{\alpha-1}e^{-\beta(t_2-t_1)}]\|f(r,0,0)\|dr,
\end{aligned}$$

let $L_1>0$ a constant such that $(t_1-r)^{\alpha-1}\leq(t_2-r)^{\alpha-1}e^{-\beta(t_2-t_1)}+L_1(t_2-t_1)(t_1-r)^{\alpha-1}$. Hence

$$[(t_1-r)^{\alpha-1}-(t_2-r)^{\alpha-1}e^{-\beta(t_2-t_1)}]e^{-\beta(t_1-r)}\leq L_1(t_2-t_1)(t_1-r)^{\alpha-1}e^{-\beta(t_1-r)}$$

this implies that

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_0^{t_1}e^{-\beta(t_1-r)}[(t_1-r)^{\alpha-1}-(t_2-r)^{\alpha-1}e^{-\beta(t_2-t_1)}]\|f(r,0,0)\|dr\\
&\leq \frac{L_1(t_2-t_1)}{\Gamma(\alpha)}\int_0^{t_1}(t_1-r)^{\alpha-1}e^{-\beta(t_1-r)}\|f(r,0,0)\|dr\\
&\leq \frac{L_1b}{\Gamma(\alpha)}(t_2-t_1).
\end{aligned}$$

Thus

$$Q_3 \leq \frac{2M_1\rho}{\Gamma(\alpha+1)}[t_1^{\alpha}+e^{-\beta(t_2-t_1)}((t_2-t_1)^{\alpha}-t_2^{\alpha})]+\frac{L_1b}{\Gamma(\alpha)}(t_2-t_1)\to 0\ as\ t_1\to t_2,$$

$$\begin{aligned}
Q_4 &= \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-r)^{\alpha-1}e^{-\beta(t_2-r)}\|f(r,u_{nr},u_{nr}(\gamma))\|dr\\
&\leq \frac{2M_1\rho}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha}+\frac{1}{\Gamma(\alpha)}\left(\frac{(t_2-t_1)^{p(\alpha-1)+1}}{p(\alpha-1)+1}\right)^{\frac{1}{p}}\left(\int_{t_1}^{t_2}\|f(r,0,0)\|^q dr\right)^{\frac{1}{q}}.
\end{aligned}$$

This implies that

$$Q_4\to 0\ as\ t_1\to t_2.$$

The conclusion that

$$\|S_{n,\varphi}u_n(t_2)-S_{n,\varphi}u_n(t_1)\|\to 0\ as\ t_1\to t_2.$$

Therefore $\{S_{n,\varphi}u_n : u_n\in A_\rho\}$ is equicontinuous on J, and by consequently $S_{n,\varphi}A_\rho$ is relatively compact in X_n.

So from Step 1, Step 2, Step 3 and the Arzelá -Ascoli theorem, we obtain that $S_{n,\varphi} : C([-h,T];X_n) \to C([-h,T];X_n)$ is continuous and completely continuous.

Step 4: We proof that subset $B_\lambda = \{u_n \in C_T : u_n = \lambda N_n(u_n), for\ some\ \lambda \in (0,1)\}$ is bounded.

Let $u_n \in B_\lambda$ Hence for each $t \in [-h,T]$ we have,

$$u_n(t) = \begin{cases} \lambda\varphi_n(t), & \text{t} \in I \\ \lambda(\varphi_n(0) - g(0,\varphi_n(\nu)) + g(t,u_{nt}(\nu)) \\ +\frac{1}{\Gamma(\alpha)}\int_0^t (t-r)^{\alpha-1}e^{-\beta(t-r)}P_nf(r,u_{nr},u_{nr}(\gamma))dr)\ , & \text{t} \in J. \end{cases}$$

By (H_1)- (H_3) we have the following estimates

$$\begin{aligned} \|u_n(t)\| &\leq (1+d_1)\|\varphi_n\|_{C_0} + d_1\|u_n(t)\| + 2d_2 \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-r)^{\alpha-1}e^{-\beta(t-r)}\|f(r,u_{nr},u_{nr}(\gamma))\|dr, \end{aligned}$$

this implies that

$$\begin{aligned} \|u_n(t)\| &\leq \frac{(1+d_1)\|\varphi\|_{C_0}+2d_2}{1-d_1} + \frac{1}{(1-d_1)\Gamma(\alpha)}\int_0^t (t-r)^{\alpha-1}e^{-\beta(t-r)}\|f(r,0,0)\|dr \\ &+ \frac{1}{(1-d_1)\Gamma(\alpha)}\int_0^t (t-r)^{\alpha-1}e^{-\beta(t-r)}\|f(r,u_{nr},u_{nr}(\gamma)) - f(r,0,0)\|dr \\ &\leq \frac{(1+d_1)\|\varphi\|_{C_0}+2d_2}{1-d_1} + \frac{b}{(1-d_1)\Gamma(\alpha)} \\ &+ \frac{2M_1}{(1-d_1)\Gamma(\alpha+1)}\int_0^t (t-r)^{\alpha-1}\|u_n(r)\|dr. \end{aligned}$$

By the Gronwall's lemma we obtain

$$\|u_n(t)\|_\infty \leq \left[\frac{(1+d_1)\|\varphi\|_{C_0}+2d_2}{1-d_1} + \frac{b}{(1-d_1)\Gamma(\alpha)}\right] e^{\frac{2M_1T^\alpha}{(1-d_1)\Gamma(\alpha+1)}}.$$

Therefore B_λ, is bounded, so by nonlinear alternative of Leray-Schauder the operator $S_{n,\varphi}$ admits at least one fixed point u_n which is a local solution of the problem (6.1.1) in $C([-h,T];X_n)$. □

Lemma 6.1.2. *Under the assumptions of Theorem 6.1.1, for any $\varphi \in C(I,X)$, problem* (6.1.1) *has at least one solution in $C([-h,T];X)$.*

Proof. In Theorem 6.1.1, we have proved the existence of local solution of problem (6.1.1) in $C([-h,T];X_n)$. By taking the limit, we prove the existence of local solution of problem (6.1.1) in $C([-h,T];X)$. For this, we form the approximating equations:

$$\begin{cases} D_s^\alpha(u_n(t) - g(t,u_{nt}(\nu))) = P_nf(t,u_{nt},u_{nt}(\gamma))\ , & \text{t} \geq 0 \\ u_n(t) = \varphi_n(t)\ , & \text{t} \in I. \end{cases} \tag{6.1.2}$$

Similarly as in Theorem 6.1.1, there exists a local solution u_n to problem (6.1.2) on $[-h, T]$ such that there exists a positive constant b withe

$$\sup_{t\in[-h,T]} \|u_n(t)\| \leq b. \tag{6.1.3}$$

Since X is a separable Hilbert space, from (6.1.3) we can conclude that for any $t \in [0, T]$, $\{u_n(t)\}_{n\geq 1}$ is weakly convergent to $u(t)$ in X. Moreover for any $t_m, t \in I$ with $t_m \to t$, we have when n, m go to ∞,

$$\|u_n(t_m) - u(t)\| \leq \|\varphi_n(t_m) - \varphi(t_m)\| + \|\varphi(t_m) - \varphi(t)\| \to 0.$$

On the other hand, for any $t_m, t \in J$ with $t_m \to t$, and any $v \in X^*$, we have as n, m go to ∞,

$$|\langle u_n(t_m) - u(t), v\rangle| \leq |\langle u_n(t_m) - u(t_m), v\rangle| + |\langle u(t_m) - u(t), v\rangle| \to 0.$$

Thus $\{u_{n,t}\}_{n\geq 1}$ is weakly convergent to u_t in $C(I, X)$.

Now we prove that $u(\cdot)$ is a solution of (6.1.1). For this we need to pass to the limit in the following integral

$$u_n(t) := \begin{cases} \varphi_n(t), & \text{t} \in I \\ \varphi_n(0) - g(0, \varphi_n(\nu)) + g(t, u_{nt}(\nu)) & \\ +\frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} e^{-\beta(t-r)} P_n f(r, u_{nr}, u_{nr}(\gamma)) dr \ , & \text{t} \in J. \end{cases} \tag{6.1.4}$$

Since the function f is Lipschitz continuous, then for any $t \in J$, we obtain

$$f(t, u_{n,t}, u_{n,t}(\gamma)) \to f(t, u_t, u_t(\gamma)), \ \textit{weakly in } X. \tag{6.1.5}$$

Using the Riesz representation theorem, we obtain that for any $v \in X^*$, there exists a unique $w \in X$ such that $\langle u, v\rangle = \langle u, w\rangle$ for all $u \in X$, and therefore

$$\begin{aligned} & |\langle P_n f(t, u_{nt}, u_{nt}(\gamma)), v\rangle - \langle f(t, u_{nt}, u_{nt}(\gamma)), v\rangle| \\ = \ & |\langle P_n f(t, u_{nt}, u_{nt}(\gamma)) - f(t, u_{nt}, u_{nt}(\gamma)), w\rangle| \\ = \ & |\langle f(t, u_{nt}, u_{nt}(\gamma)), (Id - P_n)w\rangle| \leq 2b\|(Id - P_n)w\| \to 0, \ n \to \infty, \end{aligned}$$

so

$$|\langle P_n f(t, u_{nt}, u_{nt}(\gamma)), v\rangle - \langle f(t, u_{nt}, u_{nt}(\gamma)), v\rangle| \to 0, \ \textit{as } n \to \infty. \tag{6.1.6}$$

Then by (6.1.5) , (6.1.6) and Lebesgue's dominated convergence theorem we obtain for any $v \in X^*$,

$$\left\langle \int_0^t (t-r)^{\alpha-1} e^{-\beta(t-r)} P_n f(r, u_{nr}, u_{nr}(\gamma)) dr, v \right\rangle \to \left\langle \int_0^t (t-r)^{\alpha-1} e^{-\beta(t-r)} f(r, u_r, u_r(\gamma)) dr, v \right\rangle.$$

This with (6.1.4) imply that, for any $v \in X^*$,

$$\begin{aligned}\langle u(t), v\rangle &= \langle [u(0) - g(0, u_0(\nu))]e^{-\beta t}, v\rangle + \langle g(t, u_t(\nu)), v\rangle \\ &+ \frac{1}{\Gamma(\alpha)}\left\langle \int_0^t (t-r)^{\alpha-1} e^{-\beta(t-r)} f(r, u_r, u_r(\gamma))dr, v \right\rangle,\end{aligned}$$

which means that

$$u(t) = [u(0) - g(0, u_0(\nu))]e^{-\beta t} + g(t, u_t(\nu)) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-r)^{\alpha-1} e^{-\beta(t-r)} f(r, u_r, u_r(\gamma))dr$$

for all $t \in J$. □

Theorem 6.1.3. *(Global existence and uniqueness and continuous dependence on the initial values of solutions of* (6.1.1)*) Assume that* (H_1)- (H_3) *hold, further if* g *is* $L-$*Lipschitz. Then,*
(i) for any $\varphi \in C(I, X)$*, the problem* (6.1.1) *has a unique global solution* $u(\cdot, \varphi)$ *on* $[-h, T]$*,*
(ii) the solution $u(\cdot, \varphi)$ *depends continuously on* φ*, i.e.,*

$$\limsup_{\varphi \to \psi} \|u(t, \varphi) - v(t, \psi)\|_\infty = 0.$$

Proof. **Part 1:** From Lemma 6.1.2, we know that (6.1.1) has a local solution on $[-h, T]$. So we need to prove that the local solution is a global one. Namely, we prove that the local solution will not blow up on any bounded interval. By a similar procedure as we used in Step 4, we can prove that for any solution $u(t)$, we have for all $t \geq 0$,

$$\|u_t\|_\infty \leq \left[\frac{(1+d_1)\|\varphi\|_{C_0} + 2d_2}{1-d_1} + \frac{b}{(1-d_1)\Gamma(\alpha)}\right] e^{\frac{2M_1 t^\alpha}{(1-d_1)\Gamma(\alpha+1)}},$$

which implies that the local solution $u(t)$ will not blow up on any bounded interval$[0, T]$. Then it is actually a global solution on $[-h, \infty)$.

Finally, we show that the global solution is unique. Let $u(t), v(t)$ be solutions of (6.1.1) with the same initial value φ. Then for $t \in J$,

$$\begin{aligned}\|u(t) - v(t)\| &\leq \|g(t, u_t(\nu)) - g(t, v_t(\nu))\| \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-r)^{\alpha-1} e^{-\beta(t-r)} \|f(r, u_r, u_r(\nu)) - f(r, v_r, v_r(\nu))\| dr.\end{aligned}$$

Replacing t by $t+\theta$ in this last inequality, note that $\|u(t+\theta) - v(t+\theta)\| = 0$ when $t+\theta \leq 0$ with $\theta \in I$. On the other hand, taking $p = 1+\alpha,\ q = 1+\frac{1}{\alpha}$, then by *Hölder·s* inequality, we have, for $t+\theta \geq 0$,

$$\|u_t - v_t\|_\infty \leq L\|u_t - v_t\|_\infty + \frac{2M_1 T^{\alpha^2}}{\alpha^2 \Gamma(\alpha)}\left(\int_0^{t+\theta} e^{-q\beta(t+\theta-r)} \|u_r - v_r\|_\infty^q dr\right)^{\frac{1}{q}},$$

so

$$\|u_t - v_t\|_\infty \leq k \left(\int_0^{t+\theta} e^{-q\beta(t+\theta-r)} \|u_r - v_r\|_\infty^q dr \right)^{\frac{1}{q}},$$

with $k = \frac{2M_1 T^{\alpha^2}}{(1-L)\alpha^2 \Gamma(\alpha)}$, this implies that

$$\|u_t - v_t\|_\infty^q \leq k^q \int_0^{t+\theta} e^{-q\beta(t+\theta-r)} \|u_r - v_r\|_\infty^q dr.$$

Thus

$$e^{q\beta t}\|u_t - v_t\|_\infty^q \leq k^q e^{-q\beta\theta} \int_0^t e^{q\beta r} \|u_r - v_r\|_\infty^q dr \ \forall t \in J.$$

By the Gronwall inequality, we obtain $\|u_t - v_t\|^q = 0$, which implies that $\|u_t - v_t\|_\infty = 0$ for all $t \in J$. Thus the global solution is unique.

Part 2: Let $\varphi,\ \psi \in C(I, X)$. Since $u(\cdot, \varphi)$ and $v(\cdot, \psi)$ are solutions of (6.1.1), the conditions $(H_1) - (H_3)$ implies that

$$\begin{aligned} \|u(t,\varphi) - v(t,\psi)\| &\leq e^{-\beta t}[\|\varphi(0) - \psi(0)\| + L\|\psi(\nu) - \varphi(\nu)\|] + L\|u_t(\nu,\varphi) - v_t(\nu,\psi)\| \\ &+ \frac{2M_1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} e^{-\beta(t-r)} \|u_r(\cdot,\varphi) - v_r(\cdot,\psi)\|_\infty dr. \end{aligned}$$

Replacing t by $t + \theta$ with $t + \theta \in [-h, T]$, so in the last inequality, if $t + \theta \in I$ then,

$$\|u(t+\theta,\varphi) - v(t+\theta,\psi)\| = \|\varphi(t+\theta) - \psi(t+\theta)\| \leq \|\varphi - \psi\|_\infty \to 0 \ as \ \varphi \to \psi,$$

if $t + \theta \in J$ then,

$$\begin{aligned} \|u(t+\theta,\varphi) - v(t+\theta,\psi)\| &\leq e^{-\beta(t+\theta}(1+L)\|\varphi - \psi\|_\infty + L\|u_t(\cdot,\varphi) - v_t(\cdot,\psi)\|_\infty \\ &+ \frac{2M_1}{\Gamma(\alpha)} \int_0^{t+\theta} (t+\theta-r)^{\alpha-1} e^{-\beta(t\theta-r)} \|u_r(\cdot,\varphi) - v_r(\cdot,\psi)\|_\infty dr. \end{aligned}$$

Taking $p = 1 + \alpha,\ q = 1 + \frac{1}{\alpha}$, then by *Hölder's* inequality, we obtain

$$e^{q\beta t}\|u_t(\cdot,\varphi) - v_t(\cdot,\psi)\|_\infty^q \leq (\widetilde{k})^q e^{-q\beta\theta}\|\varphi - \psi\|_\infty^q + 2^q k^q \int_0^t e^{q\beta r}\|u_r(\cdot,\varphi) - v_r(\cdot,\psi)\|_\infty^q dr,$$

with $\widetilde{k} = \frac{2(1+L)}{1-L}$ and k as in the **Part 1**.
By the Gronwall lemma, we conclude that, for all $t \in J$,

$$\|u_t(\cdot,\varphi) - v_t(\cdot,\psi)\|_\infty^q \leq (\widetilde{k})^q \|\varphi - \psi\|_\infty^q e^{2^q k^q e^{-q\beta\theta} t - q\beta(t+\theta)} \to 0,\ as \ \varphi \to \psi,$$

which implies

$$\lim_{\varphi \to \psi} sup \|u(t,\varphi) - v(t,\psi)\|_\infty = 0.$$

□

6.1 Existence of solutions

Now we will apply our abstract theory developed in the previous section to the fractional lattice system (6.0.1).

Setting of the problem.
Let $\ell^2 = \{v = (v_i)_{i\in\mathbb{Z}} : \sum_{i\in\mathbb{Z}} v_i^2 < \infty\}$ be the separable Hilbert space with norm $\|v\| = \left(\sum_{i\in\mathbb{Z}} v_i^2\right)^{\frac{1}{2}}$ and scalar product $\langle w, v\rangle = \sum_{i\in\mathbb{Z}} w_i v_i$, and let $\ell^\infty = \{v = (v_i)_{i\in\mathbb{Z}} : \sup_{i\in\mathbb{Z}} |v_i| < \infty\}$ be the Banach space with norm $\|v\|_\infty = \sup_{i\in\mathbb{Z}} |v_i|$. Further, we use the notation $X = \ell^2$, $C_T = C([-h, T]; \ell^2)$ and $C_{T,\infty} = C([-h, T]; \ell^\infty)$, with the norms

$$\|u\|_{C_T} = \sup_{t\in[-h,T]} \|u(t)\|, \text{ and } \|u\|_{C_{T,\infty}} = \sup_{t\in[-h,T]} \|u(t)\|_\infty.$$

By simple calculus, we can verify that $C_T \subset C_{T,\infty}$ with $\|u\|_{C_T} \geq \|u\|_{C_{T,\infty}}$ for all $u \in C_T$.

We consider the system (6.0.1) on the phase space C_T. We assume the following conditions:

(C_1) The operator $f : [0,\infty)\times C(I,X)\times C(I,X) \to X$ given by the rule $(f(t, u_t, v_t))_i = f_i(t, u_{it}, u_{it})$, $i \in \mathbb{Z}$, is well defined, bounded and continuous.

(C_2) There exists a constant $L_f > 0$ such that

$$\|f(t,\varphi,\bar{\varphi}) - f(t,\psi,\bar{\psi})\|_X \leq L_f(\|\varphi-\psi\|_\infty + \|\bar{\varphi}-\bar{\psi}\|_\infty)$$

for all $\varphi, \psi, \bar{\varphi}, \bar{\psi} \in C(I, X)$ and $t \geq 0$.

(C_3) The function $f(t, 0, 0)$ is L^1 locally integrable, and there exists $b > 0$ such that

$$\int_0^t (t-r)^{\alpha-1} e^{-\beta(t-r)} \|f(r,0,0)\| dr \leq b, \ \forall t \geq 0.$$

We establish the existence of solution for problem (6.0.1). For this we rewrite it in abstract form. We define the operator $N : X \to X$ by

$$(Nu)_i = -u_{i-1} - 2u_i, \ i \in \mathbb{Z}.$$

We also define the operator $F : [0,\infty) \times C(I,X) \times C(I,X) \to X$ as $F(t,u,v) = -Nu(0) - f(t,u,v) - \lambda u(0)$. Thus problem (6.0.1) can be rewritten as

$$\begin{cases} D_s^\alpha(u(t) - g(t, u_t(\nu))) = F(t, u_t, u_t(\gamma)) \ , & \text{t} \geq 0 \\ u(t) = \varphi(t) \ , & \text{t} \in I. \end{cases} \tag{6.1.7}$$

Theorem 6.1.4. *Assume that* (C_1)- (C_3) *hold. Then for any* $\varphi \in C(I, X)$*, the problem* (6.0.1) *has a unique solution on* $[-h, \infty)$.

Proof. We would like to mention that condition (C_1) holds if all the component functions $f_i,\ i \in \mathbb{Z}$, are globally Lipschitz with the same Lipschitz constant, while condition (C_2) holds provided that each component function $f_i,\ i \in \mathbb{Z}$ is Lipschitz continuous with a constant L_{f_i} and $L_f = \left(\sum_{i\in\mathbb{Z}} L_{f_i}^2\right)^{\frac{1}{2}} < \infty$. And condition (C_3) holds if it holds for each f_i with constant L_i, and $b = \sum_{i\in\mathbb{Z}} L_i < \infty$. With all this, it is not difficult to check that $F(t, u_t, u_t(\gamma))$ satisfies $(H_1) - (H_3)$. Thus the conclusion follows from Theorem 6.1.1. □

Conclusion

The main objectives of this thesis were to investigate the existence of solutions for some classes of differential inclusions and equations with delay. The topological and geometrical structures of the solution sets are also presented. We have obtained some generalizations for the fractional nonlinear integral of Gronwall-Bihari-Henry inequality, as well as the realization of a new version for the multi-valued Leray-Schauder type fixed point theorem in strictly star-shaped sets.

We plan to extend the results presented in this thesis by considering impulsive semilinear fractional differential equations and inclusions.

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