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Mohammed El Amine BOUBEKEUR

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Devant le jury composé de :

Président :

Mr. Abderrahmene OUMANSOUR, Maître de Conférences A,
Université de Sidi Bel Abbès

Examineurs :

Mr. Mustapha YEBDRI, Professeur, Université Aboubakr
Belkaid de Tlemcen

Mr. Ghouti DJELLOULI, Professeur, Université de Saïda

Directeur de thèse :

Mr. Abdelkader LAKMECHE, Professeur, Université de
Sidi Bel Abbès

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Introduction

This thesis is devoted to discrete models describing the evolution of populations of different kinds. We are particularly interested in cases of food chains and epidemiological cases.

Most of works on mathematical modelling, of population dynamics considering continuous cases, when they represent numerically the data to obtain graphs describing the states of their models, go through the discretization to do programs for numerical simulations. Which motivate us to consider in our works discrete models in order to see if there are significant differences between them and continuous ones.

The first model studied here is a case of predatory prey discrete model, in which the prey is divided into juvenile and adult compartments, this choice is justified by the difference in access to the prey, our goal is to give mathematical analysis of the model and numerical simulations. More specifically, we consider the following system of difference equations;

$$\begin{aligned} N_1(n+1) &= N_1(n) \exp \left\{ B_2(n)(1 - \alpha_2(n)N_2(n)) \frac{N_2(n)}{N_1(n)} - d_1(n) - G(n)(1 - \alpha_1(n)N_1(n)) \right. \\ &\quad \left. - \frac{p_1(n)P(n)}{1 + p_1(n)h_1N_1(n)} \right\}, \\ N_2(n+1) &= N_2(n) \exp \left\{ G(n)(1 - \alpha_1(n)N_1(n)) \frac{N_1(n)}{N_2(n)} - \frac{p_2(n)P(n)}{1 + p_2(n)h_2N_2(n)} - d_2(n) \right\}, \\ P(n+1) &= P(n) \exp \left\{ \frac{p_1(n)N_1(n)}{1 + p_1(n)h_1N_1(n)} + \frac{p_2(n)N_2(n)}{1 + p_2(n)h_2N_2(n)} - D(n) \right\}. \end{aligned} \tag{1}$$

Where N_1 and N_2 represent respectively the juvenile and adult prey, the predator is represented by P , all the coefficients are positive, they are specified in chapter one. We study the existence of periodic solutions, our results are illustrated by simulations after doing theoretical study.

The second case concerns a discrete mathematical model for a food chain with the existence of the delay effect, in our numerical study we adapt the numerical method to the delay, to obtain representative simulations. The mathematical model studied here is obtained from the work of Zhung [72], it is a system governed by difference equations, more specifically we present the results of Zhung [72] of the following model

$$\begin{aligned} u_1(n+1) &= u_1(n) \exp[r_1(n) - d_1(n)u_1(n) - \frac{m_{12}(n)u_2(n)}{a_1(n)+b_1(n)u_1(n)+u_1^2(n)}], \\ u_2(n+1) &= u_2(n) \exp[\frac{m_{21}u_1(n-\tau)}{a_1(n)+b_1(n)u_1(n-\tau)+u_1^2(n-\tau)} - r_2(n) - d_2(n)u_2(n) - \frac{m_{23}(n)u_3(n)}{a_2(n)+b_2(n)u_2(n)+u_2^2(n)}], \\ u_3(n+1) &= u_3(n) \exp[\frac{m_{32}(n)u_2(n-\sigma)}{a_2(n)+b_2(n)u_2(n-\sigma)+u_2^2(n-\sigma)} - r_3(n) - d_3(n)u_3(n)], \end{aligned} \quad (2)$$

where all the coefficients are positive periodic sequences. the states u_1 , u_2 and u_3 represent respectively the the prey, the intermiate predator and the top predator populations. More details on the model are given in chapter two.

Our goal here is to give our numerical simulations for this model, to compare with those obtained by Zhung [72].

The third work is a discrete model case for an epidemiological model of HIV with treatment, the model is describing the evolution of the HIV when the population is under treatment. The model proposed here is the following

$$\begin{aligned} T_{n+1} &= T_n + rT_n \left(1 - \frac{T_n}{T_{max}}\right) - \beta T_n V_n \\ L_{n+1} &= L_n + \eta_l \beta T_n V_n - (a + d_1) L_n \\ I_{n+1} &= (1 - \eta_1) I_n + (1 - \eta_l) \beta T_n V_n + a L_n - d_2 I_n \\ V_{n+1} &= (1 - \eta_2) V_n + p d_2 I_n - d_3 V_n \end{aligned} \quad (3)$$

where the coefficient are positive and representing the different kind of cells; T_n normal, L_n latent and I_n infected one, V_n represents the virus at time n . The role of each parameter is described in chapter three.

We study the existence of equilibria and their stability to obtain informations about the behaviour of each population of the model, and we give numerical simulations to illustrate our results.

The works cited above constitute the chapters of these theses, we conclude it by some perspectives.

Periodic positive solutions of a discrete food chain predator-prey model

1.1 *Introduction*

In the last five decades mathematical ecology was investigated by many researchers, see [60] and [66], and the references cited therein. The stage structured population has drawn the attention of many scholars, see [1], [20], [42] and [73], and the references cited therein.

Abrams and Quince [1] have considered the stability analysis of the following stage structured density-independent prey growth where the juvenile prey population is vulnerable

$$\begin{aligned}
 \frac{dN_1}{dt} &= B_2N_2 - d_1N_1 - g(1 - \alpha_1N_1)N_1 - \frac{p_1N_1P}{1 + p_1hN_1} \\
 \frac{dN_2}{dt} &= GN_1 - d_2N_2 \\
 \frac{dP}{dt} &= P \left[\frac{ep_1N_1}{1 + p_1hN_1} - D \right]
 \end{aligned} \tag{1.1}$$

where N_1 and N_2 denote the population densities of juvenile and adult prey and P is that of the predator. The parameters d_1 , d_2 and D are the per capita death rates of juvenile prey, adult prey and predator respectively. Here the predation is limited to

juvenile prey with p_1 as per capita capture rate of juveniles by the predator, h is the corresponding handling time which includes the time spent for pursuing, capturing, killing, eating as well as digestion of each captured prey item. The handling time interval starts once the prey has been spotted, e is the conversion efficiency of ingested juvenile prey into new predator individuals (see [1]). The authors found out that an alternative equilibrium can emerge if the density-dependence on juvenile's growth rate and adult's birth rate is assumed, they assumed also that competition occurs in both prey stages for the following model

$$\begin{aligned}\frac{dN_1}{dt} &= B_2(1 - \alpha_2 N_2)N_2 - d_1 N_1 - G(1 - \alpha_1 N_1)N_1 - \frac{p_1 N_1 P}{1 + p_1 h N_1}, \\ \frac{dN_2}{dt} &= G(1 - \alpha_1 N_1)N_1 - d_2 N_2, \\ \frac{dP}{dt} &= P \left[\frac{e p_1 N_1}{1 + p_1 h N_1} - D \right].\end{aligned}\tag{1.2}$$

where α_1 and α_2 are the competition coefficients in both prey population stages. Later in [42], the authors investigated the following model

$$\begin{aligned}\frac{dN_1}{dt} &= B_2(1 - \alpha_2 N_2)N_2 - d_1 N_1 - G(1 - \alpha_1 N_1)N_1 - \frac{p_1 N_1 P}{1 + p_1 h N_1}, \\ \frac{dN_2}{dt} &= G(1 - \alpha_1 N_1)N_1 - d_2 N_2, \\ \frac{dP}{dt} &= P \left[\frac{p_1 N_1}{1 + p_1 h N_1} - D \right].\end{aligned}\tag{1.3}$$

and

$$\begin{aligned}\frac{dN_1}{dt} &= B_2(1 - \alpha_2 N_2)N_2 - d_1 N_1 - G(1 - \alpha_1 N_1)N_1, \\ \frac{dN_2}{dt} &= G(1 - \alpha_1 N_1)N_1 - d_2 N_2 - \frac{p_2 N_2 P}{1 + p_2 h N_2}, \\ \frac{dP}{dt} &= P \left[\frac{p_2 N_2}{1 + p_2 h N_2} - D \right].\end{aligned}\tag{1.4}$$

They obtained sufficient conditions under which the existence of a stable limit cycle. As their model incorporates non-periodic parameters.

Cushing [14] has already pointed out that nature and environment require some times the parameters to be periodic. One of the fundamental question in mathematical ecology is the existence of positive periodic solution *i.e.* the existence of cycle.

Other interested works on discrete population models for prey-predator can be found in the following papers [12], [41], [44] and [45].

Motivated by the work in [42], in this work we will discuss the existence of periodic positive solution of the following discrete time system where the predation is limited to both prey stages

$$\begin{aligned}
 N_1(n+1) &= N_1(n) \exp \left\{ B_2(n)(1 - \alpha_2(n)N_2(n)) \frac{N_2(n)}{N_1(n)} - d_1(n) - G(n)(1 - \alpha_1(n)N_1(n)) \right. \\
 &\quad \left. - \frac{p_1(n)P(n)}{1 + p_1(n)h_1N_1(n)} \right\}, \\
 N_2(n+1) &= N_2(n) \exp \left\{ G(n)(1 - \alpha_1(n)N_1(n)) \frac{N_1(n)}{N_2(n)} - \frac{p_2(n)P(n)}{1 + p_2(n)h_2N_2(n)} - d_2(n) \right\}, \\
 P(n+1) &= P(n) \exp \left\{ \frac{p_1(n)N_1(n)}{1 + p_1(n)h_1N_1(n)} + \frac{p_2(n)N_2(n)}{1 + p_2(n)h_2N_2(n)} - D(n) \right\}.
 \end{aligned} \tag{1.5}$$

Next we give conditions to obtain the existence of positive periodic solution.

1.2 Existence of positive periodic solution

Let X, Z be Banach spaces, $\Omega \subset X$ a bounded open set with closure $\bar{\Omega}$, and

$$L : \text{dom} L \subset X \rightarrow Z.$$

L will be called Fredholm mapping with index zero if:

- (i) L is linear and $\text{Im} L$ is closed in Z .
- (ii) $\ker L$ and $\text{coker} L$ have a finite dimension and

$$\dim \ker L = \dim \text{coker} L \quad (= \text{codim} \text{Im} L)$$

If L is a Fredholm mapping with index 0 and there exist continuous projectors

$$P : X \rightarrow X \quad \text{and} \quad Q : Z \rightarrow Z,$$

such that $\text{Im}P = \ker L$, $\text{Im}L = \ker Q = \text{Im}(I - Q)$, it follows that $L|_{\text{dom}L \cap \ker P} : (I - P)X \rightarrow \text{Im}L$ is invertible. We denote its inverse by K_P .

If Ω is an open bounded subset of X , the mapping N will be called L -compact on $\bar{\Omega}$ if the mapping $QN : \bar{\Omega} \rightarrow Z$ is continuous, $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \rightarrow X$ is compact, *i.e.* it is continuous and $K_P(I - Q)N(\bar{\Omega})$ is relatively compact, where $K_P : \text{Im}L \rightarrow \text{dom}L \cap \ker P$ is the inverse of the restriction L_P of L to $\text{dom}L \cap \ker P$, so that $LK_P = I$ and $K_PL = I - P$.

Since Q is isomorphic to $\ker L$, then there exist an isomorphic $J : \text{Im}Q \rightarrow \ker L$.

Lemma 1.2.1 ([16]). *(Continuation theorem) Let L be a Fredholm mapping of index zero and N be L -compact on $\bar{\Omega}$. Suppose*

(a) *for each $\lambda \in (0, 1)$, every solution x of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$,*

(b) *$QNx \neq 0$ for each $x \in \partial\Omega \cap \ker L$ and $\deg\{JQN, \Omega \cap \ker L, 0\} \neq 0$.*

Then the operator equation $Lx = Nx$ has at least one solution lying in $\text{dom}L \cap \bar{\Omega}$

We will use the following notations

$$I_\omega = \{0, 1, 2, \dots, \omega - 1\}, \quad \bar{f} = \frac{1}{\omega} \sum_{n=0}^{\omega-1} f(n),$$

$$f^l = \min_{n \in I_\omega} f(n), \quad f^u = \max_{n \in I_\omega} f(n),$$

where $\{f(n)\}$ is an ω -periodic sequence of real numbers defined for $n \in \mathbb{Z}$.

Lemma 1.2.2 ([65]). *Let $f : \mathbb{Z} \rightarrow \mathbb{R}$ be ω -periodic, i.e., $f(n + \omega) = f(n)$. Then for any fixed $n_1, n_2 \in I_\omega$, and any $n \in \mathbb{Z}$, one has*

$$f(n) \geq f(n_1) - \sum_{k=0}^{\omega-1} |f(k+1) - f(k)|$$

and

$$f(n) \leq f(n_2) + \sum_{k=0}^{\omega-1} |f(k+1) - f(k)|.$$

Theorem 1.2.1. *Assume that the following conditions are satisfied*

$$(a) \max\{\bar{R}, \bar{S}\} < \bar{D}, \text{ where } \bar{R} = \frac{1}{\omega} \sum_{n=0}^{\omega-1} \frac{p_1(n)e^{M_1}}{1 + p_1(n)h_1e^{M_1}} \quad \bar{S} = \frac{1}{\omega} \sum_{n=0}^{\omega-1} \frac{p_2(n)e^{M_2}}{1 + p_2(n)h_2e^{M_2}}$$

$$M_1 = \ln \left(\frac{\bar{d}_1 + \bar{G}}{G\alpha_1} \right) + 2\bar{d}_1\omega$$

$$M_2 = \ln \left(\frac{\bar{d}_1 + \bar{G}}{\alpha_1 d_2} \right) + 2\bar{d}_2\omega$$

$$(b) \deg\{JQN u, \Omega \cap \mathbb{R}^3, 0\} \neq 0, u \in \partial\Omega.$$

Then system (1.5) has at least one ω -periodic solution.

Proof. Let $l_3 := \{u = \{u(n)\} : u(n) \in \mathbb{R}^3, n \in \mathbb{Z}\}$.

For $a = (a_1, a_2, a_3) \in \mathbb{R}^3$, define $|a| := \max\{a_1, a_2, a_3\}$. Let $l^\omega(\subset l_3)$ denotes the subset of all ω periodic sequences equipped with usual supremum norm $\|\cdot\|$, it i.e.

$$\|u\| = \max_{n \in I_\omega} |u(n)| \quad \text{for } u = \{u(n) : n \in \mathbb{Z}\} \in l^\omega.$$

The space l^ω is a finite dimensional Banach space.

Let

$$l_0^\omega := \left\{ u = u(n) \in l^\omega : \sum_{n=0}^{\omega-1} u(n) = 0 \right\}$$

and

$$l_c^\omega := \{ u = u(n) \in l^\omega : u(n) = h \in \mathbb{R}^3, n \in \mathbb{Z} \}.$$

Then l_0^ω and l_c^ω are both closed linear subspaces of l^ω . Moreover, $l^\omega = l_0^\omega \oplus l_c^\omega$ and $\dim l_c^\omega = 3$. Put $N_1(n) = e^{u_1(n)}$, $N_2(n) = e^{u_2(n)}$ and $P(n) = e^{u_3(n)}$.

The system (1.5) can be reformulated as follows:

$$\begin{aligned} \Delta u_1(n) &= u_1(n+1) - u_1(n) \\ &= B_2(n)(1 - \alpha_2(n)e^{u_2(n)})e^{u_2(n)-u_1(n)} - d_1(n) - G(n)(1 - \alpha_1(n)e^{u_1(n)}) \\ &\quad - \frac{p_1(n)e^{u_3(n)}}{1 + p_1(n)h_1e^{u_1(n)}}, \\ \Delta u_2(n) &= u_2(n+1) - u_2(n) \\ &= G(n)(1 - \alpha_1(n)e^{u_1(n)})e^{u_1(n)-u_2(n)} - \frac{p_2(n)e^{u_3(n)}}{1 + p_2(n)h_2e^{u_2(n)}} - d_2(n), \\ \Delta u_3(n) &= u_3(n+1) - u_3(n) \\ &= -D(n) + \frac{p_1(n)e^{u_1(n)}}{1 + p_1(n)h_1e^{u_1(n)}} + \frac{p_2(n)e^{u_2(n)}}{1 + p_2(n)h_2e^{u_2(n)}}. \end{aligned}$$

In order to implant our problem into framework of continuation theorem, let us first define

$$l^\omega = \mathbb{X} = \mathbb{Z} = \{ u(n) = (u_1(n), u_2(n), u_3(n))^T \in \mathbb{R}^3, u(n+\omega) = u(n) \}$$

and

$$\|u\| = \|(u_1(n), u_2(n), u_3(n))^T\| = \max_{n \in I_\omega} |u_1(n)| + \max_{n \in I_\omega} |u_2(n)| + \max_{n \in I_\omega} |u_3(n)|$$

for any $u \in \mathbb{X}$ (or $\mathbb{Z}$). Then $\mathbb{X}$ and $\mathbb{Z}$ are Banach spaces with the norm $\|\cdot\|$.

Let

$$N \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{pmatrix} B_2(n)(1 - \alpha_2(n)e^{u_2(n)})e^{u_2(n)-u_1(n)} - d_1(n) - G(n)(1 - \alpha_1(n)e^{u_1(n)}) - \frac{p_1(n)e^{u_3(n)}}{1 + p_1(n)h_1e^{u_1(n)}} \\ G(n)(1 - \alpha_1(n)e^{u_1(n)})e^{u_1(n)-u_2(n)} - \frac{p_2(n)e^{u_3(n)}}{1 + p_2(n)h_2e^{u_2(n)}} - d_2(n) \\ -D(n) + \frac{p_1(n)e^{u_1(n)}}{1 + p_1(n)h_1e^{u_1(n)}} + \frac{p_2(n)e^{u_2(n)}}{1 + p_2(n)h_2e^{u_2(n)}} \end{pmatrix}$$

and $Lu = \Delta u(n)$, where $\Delta u(n) = (\Delta u_1(n), \Delta u_2(n), \Delta u_3(n))^T$.

According to Lemma 2.1 in [65], we have $\ker L = \mathbb{R}^3$, $\text{Im} L = \{z \in \mathbb{Z} : \sum_{n=0}^{\omega-1} z(n) = 0\}$

is closed in $\mathbb{Z}$ and $\dim \ker L = 3 = \text{codim}(\text{Im} L)$.

Therefore, L is a Fredholm mapping of index zero. Hence, there exist two continuous projectors $P : \mathbb{X} \rightarrow \mathbb{X}$ and $Q : \mathbb{Z} \rightarrow \mathbb{Z}$ such that $Px = \frac{1}{\omega} \sum_{n=0}^{\omega-1} x(n)$, $x \in \mathbb{X}$, $Qz =$

$$\frac{1}{\omega} \sum_{n=0}^{\omega-1} z(n), z \in \mathbb{Z}, \text{Im} P = \ker L \text{ and } \text{Im} L = \ker Q = \text{Im}(I - Q).$$

Furthermore, the generalized inverse of L , $K_P : L \rightarrow \ker P \cap \text{dom} L$, is given by

$$K_P(z) = \sum_{n=0}^{\omega-1} z(n) - \frac{1}{\omega} \sum_{n=0}^{\omega-1} (\omega - n)z(n).$$

Then $QN : \mathbb{X} \rightarrow \mathbb{Z}$ and $K_P(I - Q) : \text{Im} L \rightarrow \ker P \cap \text{dom} L$, read

$$QN(u, v) = \left(\frac{1}{\omega} \sum_{n=0}^{\omega-1} \Delta u(n), \frac{1}{\omega} \sum_{n=0}^{\omega-1} \Delta v(n) \right)^T.$$

That is

$$QNu = \begin{pmatrix} \frac{1}{\omega} \sum_{n=0}^{\omega-1} F_1(n) \\ \frac{1}{\omega} \sum_{n=0}^{\omega-1} F_2(n) \\ \frac{1}{\omega} \sum_{n=0}^{\omega-1} F_3(n) \end{pmatrix}$$

and

$$K_P(I - Q)Nu = (\Phi_1, \Phi_2, \Phi_3)^T$$

where

$$\begin{aligned} F_1(n) &= B_2(n)(1 - \alpha_2(n)e^{u_2(n)})e^{u_2(n)-u_1(n)} - d_1(n) - G(n)(1 - \alpha_1(n)e^{u_1(n)}) - \frac{p_1(n)e^{u_3(n)}}{1 + p_1(n)h_1e^{u_1(n)}}, \\ F_2(n) &= G(n)(1 - \alpha_1(n)e^{u_1(n)})e^{u_1(n)-u_2(n)} - \frac{p_2(n)e^{u_3(n)}}{1 + p_2(n)h_2e^{u_2(n)}} - d_2(n), \\ F_3(n) &= -D(n) + \frac{p_1(n)e^{u_1(n)}}{1 + p_1(n)h_1e^{u_1(n)}} + \frac{p_2(n)e^{u_2(n)}}{1 + p_2(n)h_2e^{u_2(n)}} \text{ and} \\ \Phi(u(n)) &= \sum_{n=0}^{\omega-1} \Delta u(n) - \frac{1}{\omega} \sum_{n=0}^{\omega-1} (\omega - n)\Delta u(n) - \left(\frac{n}{\omega} - \frac{\omega + 1}{2\omega}\right) \sum_{n=0}^{\omega-1} \Delta u(n). \end{aligned}$$

That is

$$K_P(I - Q)Nu = \begin{pmatrix} \sum_{n=0}^{\omega-1} F_1(n) \\ \sum_{n=0}^{\omega-1} F_2(n) \\ \sum_{n=0}^{\omega-1} F_3(n) \end{pmatrix} - \begin{pmatrix} \frac{1}{\omega} \sum_{n=0}^{\omega-1} (\omega - n)F_1(n) \\ \frac{1}{\omega} \sum_{n=0}^{\omega-1} (\omega - n)F_2(n) \\ \frac{1}{\omega} \sum_{n=0}^{\omega-1} (\omega - n)F_3(n) \end{pmatrix} + \begin{pmatrix} \left(\frac{n}{\omega} - \frac{\omega + 1}{2\omega}\right) \sum_{n=0}^{\omega-1} F_1(n) \\ \left(\frac{n}{\omega} - \frac{\omega + 1}{2\omega}\right) \sum_{n=0}^{\omega-1} F_2(n) \\ \left(\frac{n}{\omega} - \frac{\omega + 1}{2\omega}\right) \sum_{n=0}^{\omega-1} F_3(n) \end{pmatrix}$$

Obviously, $QN : X \rightarrow Y$ and $K_P(I - Q)N : X \rightarrow X$ are continuous with respect to n and they are mapping bounded continuous functions to bounded continuous functions. Since X is a finite dimensional Banach space, using the Ascoli-theorem, we see that $QN(\bar{\Omega})$ and $K_P(I - Q)N(\bar{\Omega})$ are relatively compact for any open bounded set $\Omega \subset \mathbb{X}$. Moreover, $QN(\bar{\Omega})$ is bounded. Then N is L -compact on $\bar{\Omega}$ for any open

bounded $\Omega \in \mathbb{X}$.

Now we reach the position to search for an appropriate open bounded subset Ω for the application of the continuation theorem, corresponding to the operator equation $Lu = \lambda Nu$, $\lambda \in (0, 1)$.

We have

$$\Delta u_1(n) = \lambda F_1(n), \quad (1.6)$$

$$\Delta u_2(n) = \lambda F_2(n), \quad (1.7)$$

$$\Delta u_3(n) = \lambda F_3(n). \quad (1.8)$$

Suppose that $u = u(n) \in \mathbb{X}$ is an arbitrary solution of system (1.6)-(1.8) for certain $\lambda \in (0, 1)$.

Summing (1.6)-(1.8) from 0 to $\omega - 1$, we obtain

$$\begin{aligned} \sum_{n=0}^{\omega-1} F_1(n) &= 0, \\ \sum_{n=0}^{\omega-1} F_2(n) &= 0, \\ \sum_{n=0}^{\omega-1} F_3(n) &= 0. \end{aligned}$$

That is

$$\sum_{n=0}^{\omega-1} H_1(n) = \sum_{n=0}^{\omega-1} d_1(n) = \bar{d}_1 \omega, \quad (1.9)$$

$$\sum_{n=0}^{\omega-1} H_2(n) = \sum_{n=0}^{\omega-1} d_2(n) = \bar{d}_2 \omega, \quad (1.10)$$

$$\sum_{n=0}^{\omega-1} H_3(n) = \sum_{n=0}^{\omega-1} D(n) = \bar{D} \omega \quad (1.11)$$

where

$$\begin{aligned} H_1(n) &= F_1(n) + d_1(n), \\ H_2(n) &= F_2(n) + d_2(n), \\ H_3(n) &= F_3(n) + D(n). \end{aligned}$$

From (1.6)-(1.11), it follows that

$$\begin{aligned} \sum_{n=0}^{\omega-1} |u_1(n+1) - u_1(n)| &= \lambda \sum_{n=0}^{\omega-1} |F_1(n)| \\ &< \sum_{n=0}^{\omega-1} d_1(n) + \sum_{n=0}^{\omega-1} |H_1(n)|, \\ \sum_{n=0}^{\omega-1} |u_2(n+1) - u_2(n)| &= \lambda \sum_{n=0}^{\omega-1} |F_2(n)| \\ &< \sum_{n=0}^{\omega-1} d_2(n) + \sum_{n=0}^{\omega-1} |H_2(n)|, \\ \sum_{n=0}^{\omega-1} |u_3(n+1) - u_3(n)| &= \lambda \sum_{n=0}^{\omega-1} |F_3(n)| \\ &< \sum_{n=0}^{\omega-1} D(n) + \sum_{n=0}^{\omega-1} |H_3(n)|, \end{aligned} \tag{1.12}$$

That is

$$\sum_{n=0}^{\omega-1} |u_1(n+1) - u_1(n)| < 2\bar{d}_1\omega, \tag{1.13}$$

$$\sum_{n=0}^{\omega-1} |u_2(n+1) - u_2(n)| < 2\bar{d}_2\omega, \tag{1.14}$$

$$\sum_{n=0}^{\omega-1} |u_3(n+1) - u_3(n)| < 2\bar{D}\omega. \tag{1.15}$$

Since $u(n) = (u_1(n), u_2(n), u_3(n))^T \in \mathbb{X}$, there exist $\zeta_i, \eta_i \in I_\omega$, such that

$$u_i(\eta_i) = \min_{n \in I_\omega} u_i(n), \quad u_i(\zeta_i) = \max_{n \in I_\omega} u_i(n), \quad i = 1, 2, 3. \quad (1.16)$$

From (1.9) and (1.16), we see that

$$u_1(\eta_1) \leq \ln \left(\frac{\bar{d}_1 + \bar{G}}{G\alpha_1} \right). \quad (1.17)$$

Using lemma 1.2.2, we obtain

$$u_1(n) \leq u_1(\eta_1) + \sum_{n=0}^{\omega-1} |u_1(n+1) - u_1(n)| \leq \ln \left(\frac{\bar{d}_1 + \bar{G}}{G\alpha_1} \right) + 2\bar{d}_1\omega = M_1. \quad (1.18)$$

From (1.10), (1.10) and (1.16), we obtain

$$u_2(\eta_2) \leq \ln \left(\frac{\bar{d}_1 + \bar{G}}{\alpha_1 d_2} \right). \quad (1.19)$$

Using lemma 1.2.2 and (1.19), we get

$$u_2(n) \leq u_2(\eta_2) + \sum_{n=0}^{\omega-1} |u_2(n+1) - u_2(n)| \leq \ln \left(\frac{\bar{d}_1 + \bar{G}}{\alpha_1 d_2} \right) + 2\bar{d}_2\omega = M_2. \quad (1.20)$$

From (1.11) and (1.16), we have

$$u_1(\zeta_1) \geq \ln \left(\frac{\bar{D} - \bar{S}}{\bar{p}_1} \right) \quad (1.21)$$

Using lemma 1.2.2 and (1.21), we get

$$u_1(n) \geq u_1(\zeta_1) - \sum_{n=0}^{\omega-1} |u_1(n+1) - u_1(n)| \geq \ln \left(\frac{\bar{D} - \bar{S}}{\bar{p}_1} \right) - 2\bar{d}_1\omega = M_3. \quad (1.22)$$

From (1.11) and (1.16), we see that

$$u_2(\zeta_2) \leq \ln \left(\frac{\bar{D} - \bar{R}}{\bar{p}_2} \right). \quad (1.23)$$

Using lemma 1.2.2 and (1.23), we get

$$u_2(n) \leq \ln \left(\frac{\bar{D} - \bar{R}}{\bar{p}_2} \right) - 2\omega \bar{d}_2 = M_4. \quad (1.24)$$

From (1.8) one can obtain

$$|u_3(n)| \leq |u_3(0)| + \lambda \left| \sum_{k=0}^{\omega-2} f(u_1(k), u_2(k)) \right| < |u_3(0)| + (\omega - 1)f^u. \quad (1.25)$$

Thus equation (1.25) leads to

$$u_3(\zeta_3) > -|u_3(0)| - (\omega - 1)f^u, \quad (1.26)$$

$$u_3(\eta_3) < |u_3(0)| + (\omega - 1)f^u. \quad (1.27)$$

It is easy to obtain the upper bound of $u_3(0)$ from either (1.9) or (1.10), in contrast the lower bound is obtained using both (1.9) and (1.11). From Lemma 1.2.2 and (1.26), we obtain

$$n_3(n) \geq u_3(\zeta_3) - \sum_{n=0}^{\omega-2} |u_3(n+1) - u_3(n)| \geq -|u_3(0)| - f^u(\omega - 1) - (2\bar{D})\omega = M_5, \quad (1.28)$$

$$n_3(n) \leq u_3(\eta_3) + \sum_{n=0}^{\omega-2} |u_3(n+1) - u_3(n)| \leq |u_3(0)| + f^u(\omega - 1) + (2\bar{D})\omega = M_6, \quad (1.29)$$

where $f^u = e^{M_1} + e^{M_2}$.

From (1.18) and (1.22) we have

$$\max_{n \in l_\omega} |u_1(n)| \leq \max\{|M_1|, |M_3|\} := \bar{M}_1.$$

From (1.20) and (1.24) we have

$$\max_{n \in l_\omega} |u_2(n)| \leq \max\{|M_2|, |M_4|\} := \bar{M}_2.$$

Thus from (1.28) we have

$$\max_{n \in l_\omega} |u_3(n)| \leq \max\{|M_5|, |M_6|\} := \bar{M}_3.$$

We have $M_i, \bar{M}_j (i = 1, 2, 3, 4, 5, 6)$ and $(j = 1, 2, 3)$ are independent of λ . Denote $M = \bar{M}_1 + \bar{M}_2 + \bar{M}_3 + \bar{M}_4$, where $\bar{M}_4$ is taken sufficiently large such that the unique solution of system (6)-(8) satisfies $\|(u_1^*, u_2^*, u_3^*)^T\| < M$.

Let

$$\Omega = \{u(n) = (u_1(n), u_2(n), u_3(n))^T \in X : \|u\| < M\}.$$

Clearly, Ω satisfies condition (a) of Lemma 2.1. If $u \in \partial\Omega \cap \ker L$, then u is a constant with $\|u\| = M$.

Hence, we have

$$\begin{pmatrix} g_1(u_1, u_2, u_3) \\ g_2(u_1, u_2, u_3) \\ g_3(u_1, u_2, u_3) \end{pmatrix} = \begin{pmatrix} (\bar{B}_2 - \bar{B}_2 \bar{\alpha}_2 e^{u_2}) e^{u_2 - u_1} - \bar{d}_1 - \bar{G} + \bar{G} \bar{\alpha}_1 e^{u_1} - \frac{1}{\omega} \sum_{n=0}^{\omega-1} \frac{p_1(n) e^{u_3}}{1 + p_1(n) h_1 e^{u_1}} \\ \bar{G} e^{u_1 - u_2} - \bar{G} \bar{\alpha}_1 e^{u_1} e^{u_1 - u_2} - \bar{d}_2 - \frac{1}{\omega} \sum_{n=0}^{\omega-1} \frac{p_1(n) e^{u_3}}{1 + p_2(n) h_2 e^{u_2}} \\ -\bar{D} + \frac{1}{\omega} \sum_{n=0}^{\omega-1} \frac{p_1(n) e^{u_1}}{1 + p_1(n) h_1 e^{u_1}} + \frac{1}{\omega} \sum_{n=0}^{\omega-1} \frac{p_2(n) e^{u_2}}{1 + p_2(n) h_2 e^{u_2}} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Taking $J = \text{Im} Q \rightarrow \ker L, (u, v)^T \rightarrow (u, v)^T$. We have

$$\deg\{JQNu, \Omega \cap \ker L, 0\} = \deg\{JQNu, \Omega \cap \mathbb{R}^3, 0\} = \text{sign} \left\{ \begin{vmatrix} \frac{\partial g_1}{\partial u_1} & \frac{\partial g_1}{\partial u_2} & \frac{\partial g_1}{\partial u_3} \\ \frac{\partial g_2}{\partial u_1} & \frac{\partial g_2}{\partial u_2} & \frac{\partial g_2}{\partial u_3} \\ \frac{\partial g_3}{\partial u_1} & \frac{\partial g_3}{\partial u_2} & \frac{\partial g_3}{\partial u_3} \end{vmatrix} \right\}.$$

In view of condition (b) of Theorem (1.2.1), we know that $\deg\{JQNu, \Omega \cap \mathbb{R}^3, 0\} \neq 0$. $\square$

Lemma 1.2.3. *If p_1 (resp. p_2) is sufficiently small, then $\deg\{JQNu, \Omega \cap \mathbb{R}^3, 0\} \neq 0$.*

Proof. If p_1 is sufficiently small, then

$$\begin{aligned} ccl \deg\{JQNu, \Omega \cap \ker L, 0\} &\simeq \frac{1}{\omega^2} (-\bar{B}_2(1 - \bar{\alpha}_2 e^{u_2})e^{u_2 - u_1} + \bar{G}\bar{\alpha}_1 e^{u_1}) \\ &\times \left(\sum_{n=0}^{\omega-1} \frac{p_2(n)e^{u_2}}{(1 + p_2(n)h_2 e^{u_2})^2} \sum_{n=0}^{\omega-1} \frac{p_2(n)e^{u_3}}{1 + p_2(n)h_2 e^{u_2}} \right) \end{aligned}$$

since JQN is continuous at 0, it follows that it is continuous at the neighborhood of 0, then $\deg\{JQNu, \Omega \cap \ker L, 0\} \neq 0$.

The same proof is adopted for p_2 sufficiently small. $\square$

Corollary 1.2.1. *If p_1 and p_2 are constants then Theorem 1.2.1 holds true, with*

$$\bar{D} > \max\{\tilde{R}, \tilde{S}\}, \text{ where } \tilde{R} = \frac{p_1 e^{M_1}}{1 + p_1 h_1 e^{M_1}} \text{ and } \tilde{S} = \frac{p_2 e^{M_2}}{1 + p_2 h_2 e^{M_2}}.$$

1.3 Numerical simulations

In this section, we use some computer simulations to show the feasibility our previous results. Let's consider the following four periodic system

$$\begin{aligned}
N_1(n+1) &= N_1(n) \exp \left\{ \left(1.81 + 0.01 \sin \left(\frac{\pi n}{2} \right) \right) \left(1 - \left(1.3 + 0.02 \cos \left(\frac{\pi n}{2} \right) \right) N_2(n) \right) \frac{N_2(n)}{N_1(n)} \right. \\
&\quad - \left(0.19 + 0.01 \sin \left(\frac{\pi n}{2} \right) \right) - \left(0.8 + 0.02 \sin \left(\frac{\pi n}{2} \right) \right) \\
&\quad \left. - \left(1 - \left(0.5 + 0.01 \cos \left(\frac{\pi n}{2} \right) \right) N_2(n) \right) - \frac{\left(1 + 0.01 \sin \left(\frac{\pi n}{2} \right) \right) P(n)}{1 + \left(1 + 0.01 \sin \left(\frac{\pi n}{2} \right) \right) N_1(n)} \right\}, \\
N_2(n+1) &= N_2(n) \exp \left\{ \left(0.8 + 0.02 \sin \left(\frac{\pi n}{2} \right) \right) \left(1 - \left(0.04 + 0.01 \cos \left(\frac{\pi n}{2} \right) \right) N_1(n) \right) \frac{N_1(n)}{N_2(n)} \right. \\
&\quad - \frac{\left(0.7 + 0.01 \cos \left(\frac{\pi n}{2} \right) \right) P(n)}{1 + \left(2 + 0.01 \sin \left(\frac{\pi n}{2} \right) \right) \left(0.7 + 0.01 \cos \left(\frac{\pi n}{2} \right) \right) N_2(n)} \\
&\quad \left. - \left(0.31 + 0.01 \sin \left(\frac{\pi n}{2} \right) \right) \right\}, \\
P(n+1) &= P(n) \exp \left\{ \frac{\left(1 + 0.01 \sin \left(\frac{\pi n}{2} \right) \right) N_1(n)}{1 + \left(1 + 0.01 \sin \left(\frac{\pi n}{2} \right) \right) N_1(n)} \right. \\
&\quad + \frac{\left(0.7 + 0.01 \cos \left(\frac{\pi n}{2} \right) \right) N_2(n)}{1 + \left(2 + 0.01 \sin \left(\frac{\pi n}{2} \right) \right) \left(0.7 + 0.01 \cos \left(\frac{\pi n}{2} \right) \right) N_2(n)} \\
&\quad \left. - \left(0.34 + 0.02 \cos \left(\frac{\pi n}{2} \right) \right) \right\}.
\end{aligned}$$

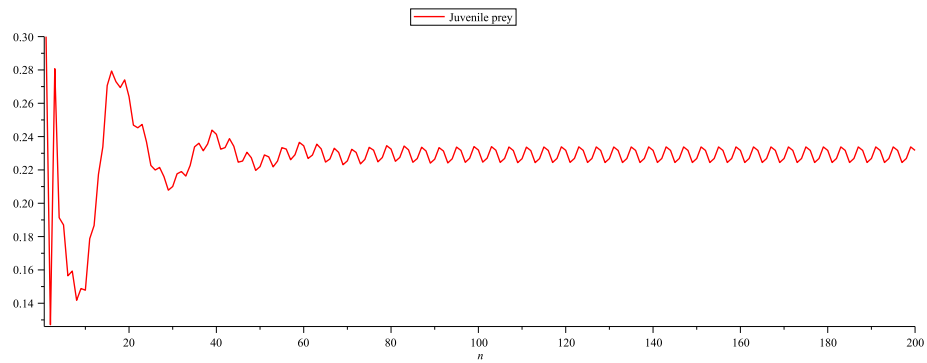


Figure 1.1: Juvenile prey curve for initial condition $(N_1(0), N_2(0), P(0)) = (0.30, 0.64, 0.80)$.

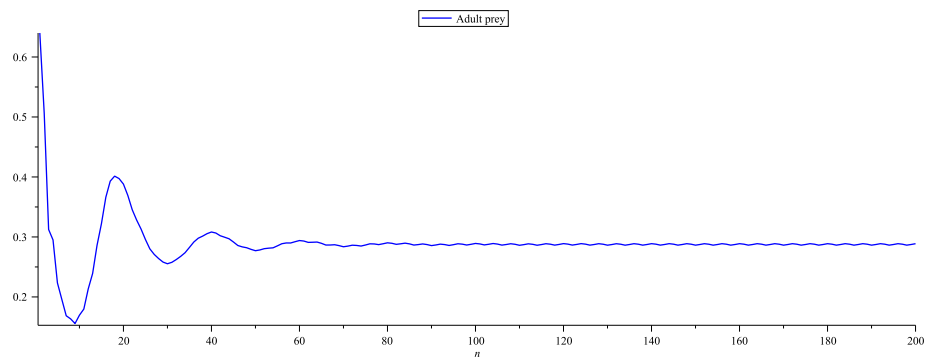


Figure 1.2: Adult prey curve for initial condition $(N_1(0), N_2(0), P(0)) = (0.30, 0.64, 0.80)$.

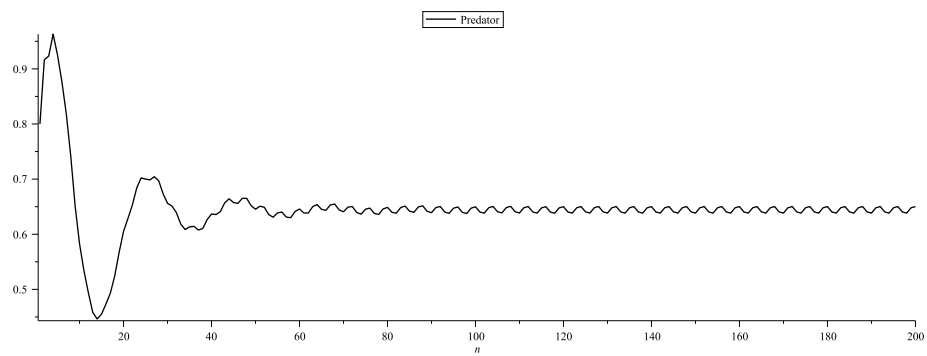


Figure 1.3: Predator curve for initial condition $(N_1(0), N_2(0), P(0)) = (0.30, 0.64, 0.80)$.

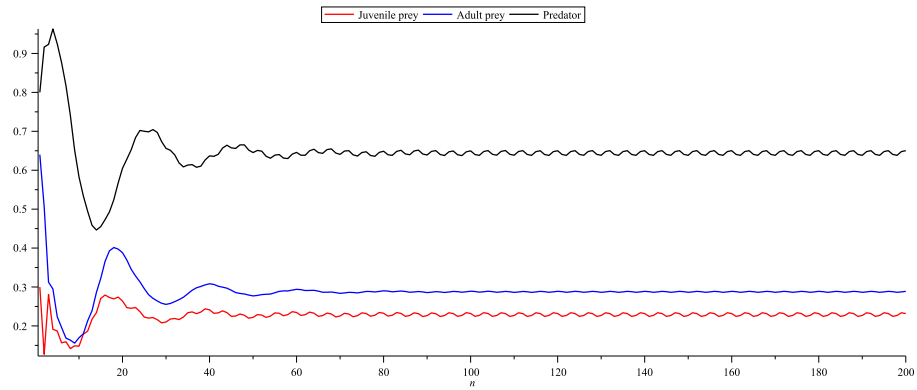


Figure 1.4: Predator curve (black line), adult prey curve (blue line) and juvenile curve (red line) for initial condition $(N_1(0), N_2(0), P(0)) = (0.30, 0.64, 0.80)$.

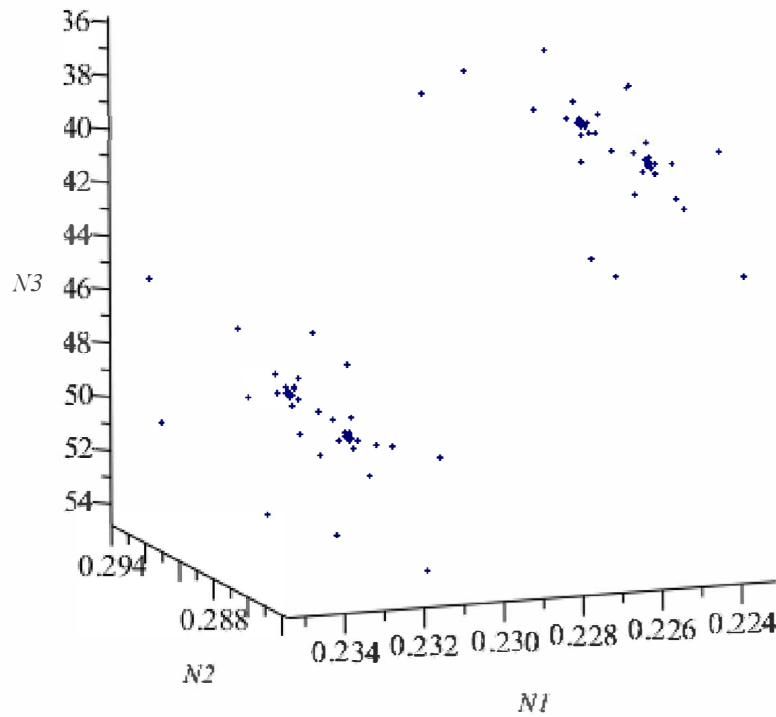


Figure 1.5: Phase portrait of trajectory from initial condition $(N_1(0), N_2(0), P(0)) = (0.30, 0.64, 0.80)$.

1.4 *Concluding remarks*

In this chapter we have considered a discrete predator-prey mathematical model describing the interaction between predator and two levels of prey population constituted by juveniles and adults. Our aim is to obtain periodic positive solution of the model under study. We have obtained sufficient conditions to have positive periodic solution of the stated model, our results are illustrated by numerical simulations. In future work, we are planning to consider the case of delay and bifurcation analysis.

Chapter 2

Periodic solutions for a food chain system with functional response

2.1 Introduction

In this chapter we present the main results obtained in [71], where the author considers a discrete models for food chain system, which become an important area in population dynamics. Many works have considered the problem of the behavior of discrete models using results on difference equations, particularly some models have been extensively studied in [15, 42, 55, 58, 60].

The mathematical model studied here is governed by difference equations, more specifically we give the study of the following model

$$\begin{aligned} u_1(n+1) &= u_1(n) \exp[r_1(n) - d_1(n)u_1(n) - \frac{m_{12}(n)u_2(n)}{a_1(n)+b_1(n)u_1(n)+u_1^2(n)}], \\ u_2(n+1) &= u_2(n) \exp[\frac{m_{21}u_1(n-\tau)}{a_1(n)+b_1(n)u_1(n-\tau)+u_1^2(n-\tau)} - r_2(n) - d_2(n)u_2(n) - \frac{m_{23}(n)u_3(n)}{a_2(n)+b_2(n)u_2(n)+u_2^2(n)}], \\ u_3(n+1) &= u_3(n) \exp[\frac{m_{32}(n)u_2(n-\sigma)}{a_2(n)+b_2(n)u_2(n-\sigma)+u_2^2(n-\sigma)} - r_3(n) - d_3(n)u_3(n)], \end{aligned} \tag{2.1}$$

where all the coefficients are positive periodic sequences.

To explore the periodic solutions of differential equation and difference equation models, coincidence degree theory is a common method. Inspired by the works of Stefan

Hilger [27], to unify the continuous and discrete dynamic systems, we consider the following dynamic system on time scales,

$$\begin{aligned} x^\Delta(t) &= r_1(t) - d_1(t)e^{x(t)} - \frac{m_{12}(t)e^{y(t)}}{a_1(t)+b_1(t)e^{x(t)}+e^{2x(t)}}, \\ y^\Delta(t) &= -r_2(t) + \frac{m_{21}(t)e^{x(t-\tau)}}{a_1(t)+b_1(t)e^{x(t-\tau)}+e^{2x(t-\tau)}} - d_2(t)e^{y(t)} - \frac{m_{23}(t)e^{z(t)}}{a_2(t)+b_2(t)e^{y(t)}+e^{2y(t)}}, \\ z^\Delta(t) &= -r_3(t) + \frac{m_{32}(t)e^{y(t-\sigma)}}{a_2(t)+b_2(t)e^{y(t-\sigma)}+e^{2y(t-\sigma)}} - d_3(t)e^{z(t)}, \end{aligned} \quad (2.2)$$

where $t \in \mathbb{T}$ and $\mathbb{T}$ is a time scale that is unbounded above, $x^\Delta(t)$ is the delta derivative of x at t , see [9]. All the coefficients are positive ω -periodic functions. Set $u_1(t) = e^{x(t)}$, $u_2(t) = e^{y(t)}$, $u_3(t) = e^{z(t)}$, then (2.2) can be reduced to (2.1) when $\mathbb{T} = \mathbb{Z}$.

The aim of this chapter is to study the periodicity of three-species food chain system on time scales.

There are several papers on periodicity in dynamic systems on time scales by using the coincidence degree theory, see [7, 8, 50, 72].

2.2 Preliminaries

To reach the aim of this work, some useful lemma about time scales and the continuation theorem of the coincidence degree theory are presented in this section, for more details on time scales see [16, 64].

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers $\mathbb{R}$. We assume that the time scale $\mathbb{T}$ is unbounded above and below. The following results are from [16, 64].

Lemma 2.2.1 ([64]). *Let $t_1, t_2 \in I_\omega$ and $t \in \mathbb{T}$. If $g : \mathbb{T} \rightarrow \mathbb{R} \in C_{rd}(\mathbb{T})$ is ω -periodic,*

then

$$\begin{aligned} g(t) &\leq g(t_1) + \frac{1}{2} \int_k^{k+\omega} |g^\Delta(s)| \Delta s, \\ g(t) &\geq g(t_2) - \frac{1}{2} \int_k^{k+\omega} |g^\Delta(s)| \Delta s, \end{aligned}$$

where the constant factor $1/2$ is the best possible.

Some notations are indicated, to be used in this chapter.

Let $\mathbb{T}$ be ω -periodic, that is, $t \in \mathbb{T}$ implies $t + \omega \in \mathbb{T}$,

$$\begin{aligned} k &= \min\{\mathbb{R}^+ \cap \mathbb{T}\}, \quad I_\omega = [k, k + \omega] \cap \mathbb{T}, \quad g^L = \inf_{t \in \mathbb{T}} g(t), \\ g^M &= \sup_{t \in \mathbb{T}} g(t), \quad \bar{g} = \frac{1}{\omega} \int_{I_\omega} g(s) \Delta s = \frac{1}{\omega} \int_k^{k+\omega} g(s) \Delta s, \end{aligned}$$

where $g \in C_{rd}(\mathbb{T})$ is an ω -periodic real function, i.e., $g(t + \omega) = g(t)$ for all $t \in \mathbb{T}$.

The main tool used here is the Mawhin's continuation theorem, it is stated in the following lemma.

Lemma 2.2.2 ([16]). *Let L be a Fredholm mapping of index zero and N be L -compact on $\bar{\Omega}$. Suppose*

- (a) *for each $\lambda \in (0, 1)$, every solution u of $Lu = \lambda Nu$ is such that $u \notin \partial\Omega$,*
- (b) *$QNu \neq 0$ for each $u \in \partial\Omega \cap \ker L$ and the Brouwer degree $\deg\{JQN, \Omega \cap \ker L, 0\} \neq 0$.*

Then the operator equation $Lu = Nu$ has at least one solution lying in $\text{Dom } L \cap \bar{\Omega}$.

2.3 Main results

Theorem 2.3.1. *If $m_{32}^L e^{L_2} > r_3^M (a_2^M + b_2^M e^{M_2} + e^{2M_2})$ holds, where*

$$L_2 = \ln \frac{a_2^L r_3^L}{m_{32}^M} - \frac{m_{21}^M \omega}{b_1^L}$$

and

$$M_2 = \ln \frac{r_1^M (a_1^M + \frac{b_1^M r_1^M}{d_1^L} + \frac{2r_1^M}{d_1^L})}{m_{12}^L} + \frac{m_{21}^M \omega}{b_1^L},$$

then (2.2) has at least one ω -periodic solution.

Proof. Let

$$X = Z = \left\{ (x, y, z)^T \in C(\mathbb{T}, \mathbb{R}^3) : x(t+\omega) = x(t), y(t+\omega) = y(t), z(t+\omega) = z(t), \forall t \in \mathbb{T} \right\},$$

$$\|(x, y, z)^T\| = \max_{t \in I_\omega} |x(t)| + \max_{t \in I_\omega} |y(t)| + \max_{t \in I_\omega} |z(t)|, \quad (x, y, z)^T \in X \quad (\text{or in } Z).$$

Then X and Z are both Banach spaces when they are endowed with the above norm

$$\|\cdot\|.$$

Let

$$N \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix},$$

where

$$\begin{aligned} N_1 &= r_1(t) - d_1(t)e^{x(t)} - \frac{m_{12}(t)e^{y(t)}}{a_1(t) + b_1(t)e^{x(t)} + e^{2x(t)}}, \\ N_2 &= -r_2(t) + \frac{m_{21}(t)e^{x(t-\tau)}}{a_1(t) + b_1(t)e^{x(t-\tau)} + e^{2x(t-\tau)}} - d_2(t)e^{y(t)} - \frac{m_{23}(t)e^{z(t)}}{a_2(t) + b_2(t)e^{y(t)} + e^{2y(t)}}, \\ N_3 &= -r_3(t) + \frac{m_{32}(t)e^{y(t-\sigma)}}{a_2(t) + b_2(t)e^{y(t-\sigma)} + e^{2y(t-\sigma)}} - d_3(t)e^{z(t)}, \end{aligned}$$

$$L \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x^\Delta \\ y^\Delta \\ z^\Delta \end{bmatrix}, \quad P \begin{bmatrix} x \\ y \\ z \end{bmatrix} = Q \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{\omega} \int_k^{k+\omega} x(t) \Delta t \\ \frac{1}{\omega} \int_k^{k+\omega} t(t) \Delta t \\ \frac{1}{\omega} \int_k^{k+\omega} z(t) \Delta t \end{bmatrix}.$$

Obviously, $\ker L = \mathbb{R}^3$, $\text{Im } L = \{(x, y, z)^T \in Z : \bar{x} = \bar{y} = \bar{z} = 0, t \in \mathbb{T}\}$, $\dim \ker L = 3 = \text{codim Im } L$. Since $\text{Im } L$ is closed in Z , then L is a Fredholm mapping of index zero. It is easy to show that P and Q are continuous projections such that $\text{Im } P = \ker L$ and $\text{Im } L = \ker Q = \text{Im}(I - Q)$. Furthermore, the generalized inverse (of L) $K_P : \text{Im } L \longrightarrow \ker P \cap \text{Dom } L$ exists and is given by

$$K_P \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \int_k^t x(s) \Delta s - \frac{1}{\omega} \int_k^{k+\omega} \int_k^t x(s) \Delta s \Delta t \\ \int_k^t y(s) \Delta s - \frac{1}{\omega} \int_k^{k+\omega} \int_k^t y(s) \Delta s \Delta t \\ \int_k^t z(s) \Delta s - \frac{1}{\omega} \int_k^{k+\omega} \int_k^t z(s) \Delta s \Delta t \end{bmatrix}.$$

Thus

$$QN \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{\omega} \int_k^{k+\omega} (r_1(t) - d_1(t)e^{x(t)} - \frac{m_{12}(t)e^{y(t)}}{a_1(t) + b_1(t)e^{x(t)} + e^{2x(t)}}) \Delta t \\ \frac{1}{\omega} \int_k^{k+\omega} (-r_2(t) + \frac{m_{21}(t)e^{x(t-\tau)}}{a_1(t) + b_1(t)e^{x(t-\tau)} + e^{2x(t-\tau)}} - d_2(t)e^{y(t)} - \frac{m_{23}(t)e^{z(t)}}{a_2(t) + b_2(t)e^{y(t)} + e^{2y(t)}}) \Delta t \\ \frac{1}{\omega} \int_k^{k+\omega} (-r_3(t) + \frac{m_{32}(t)e^{y(t-\sigma)}}{a_2(t) + b_2(t)e^{y(t-\sigma)} + e^{2y(t-\sigma)}} - d_3(t)e^{z(t)}) \Delta t \end{bmatrix}$$

and

$$K_P(I-Q)N \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \int_k^t x(s) \Delta s - \frac{1}{\omega} \int_k^{k+\omega} \int_k^t x(s) \Delta s \Delta t - \left(t - k - \frac{1}{\omega} \int_k^{k+\omega} (t - k) \Delta t \right) \bar{x} \\ \int_k^t y(s) \Delta s - \frac{1}{\omega} \int_k^{k+\omega} \int_k^t y(s) \Delta s \Delta t - \left(t - k - \frac{1}{\omega} \int_k^{k+\omega} (t - k) \Delta t \right) \bar{y} \\ \int_k^t z(s) \Delta s - \frac{1}{\omega} \int_k^{k+\omega} \int_k^t z(s) \Delta s \Delta t - \left(t - k - \frac{1}{\omega} \int_k^{k+\omega} (t - k) \Delta t \right) \bar{z} \end{bmatrix}.$$

Clearly, QN and $K_P(I - Q)N$ are continuous. According to the Arzela-Ascoli theo-

rem, we can show that $K_P(I - Q)N(\bar{\Omega})$ is compact for any open bounded set $\Omega \subset X$ and $QN(\bar{\Omega})$ is bounded. Thus, N is L -compact on $\bar{\Omega}$.

An appropriate open bounded subset Ω for the application of the continuation theorem, Lemma 2.2.2, should be found.

For the operator equation $Lu = \lambda Nu$, where $\lambda \in (0, 1)$, we have

$$\begin{aligned} u_1^\Delta(t) &= \lambda(r_1(t) - d_1(t)e^{x(t)} - \frac{m_{12}(t)e^{y(t)}}{a_1(t)+b_1(t)e^{x(t)}+e^{2x(t)}}), \\ u_2^\Delta(t) &= \lambda(-r_2(t) + \frac{m_{21}(t)e^{x(t-\tau)}}{a_1(t)+b_1(t)e^{x(t-\tau)}+e^{2x(t-\tau)}} - d_2(t)e^{y(t)} - \frac{m_{23}(t)e^{z(t)}}{a_2(t)+b_2(t)e^{y(t)}+e^{2y(t)}}), \\ u_3^\Delta(t) &= \lambda(-r_3(t) + \frac{m_{32}(t)e^{y(t-\sigma)}}{a_2(t)+b_2(t)e^{y(t-\sigma)}+e^{2y(t-\sigma)}} - d_3(t)e^{z(t)}). \end{aligned} \quad (2.3)$$

Assume that $(u_1, u_2, u_3)^T \in X$ is a solution of system (2.3) for a certain $\lambda \in (0, 1)$. Integrating (2.3) on both sides from k to $k + \omega$, we obtain

$$\begin{aligned} \int_k^{k+\omega} [d_1(t)e^{x(t)} + \frac{m_{12}(t)e^{y(t)}}{a_1(t)+b_1(t)e^{x(t)}+e^{2x(t)}}] \Delta t &= \bar{r}_1 \omega, \\ \int_k^{k+\omega} [r_2(t) + d_2(t)e^{y(t)} + \frac{m_{23}(t)e^{z(t)}}{a_2(t)+b_2(t)e^{y(t)}+e^{2y(t)}}] \Delta t &= \int_k^{k+\omega} \frac{m_{21}(t)e^{x(t-\tau)}}{a_1(t)+b_1(t)e^{x(t-\tau)}+e^{2x(t-\tau)}} \Delta t, \\ \int_k^{k+\omega} [r_3(t) + d_3(t)e^{z(t)}] \Delta t &= \int_k^{k+\omega} \frac{m_{32}(t)e^{y(t-\sigma)}}{a_2(t)+b_2(t)e^{y(t-\sigma)}+e^{2y(t-\sigma)}} \Delta t. \end{aligned} \quad (2.4)$$

Since $(x, y, z)^T \in X$, there exist $\xi_i, \eta_i \in I_\omega$, $i = 1, 2, 3$, such that

$$\begin{cases} x(\xi_1) = \min_{t \in I_\omega} \{x(t)\}, & x(\eta_1) = \max_{t \in I_\omega} \{x(t)\}, \\ y(\xi_2) = \min_{t \in I_\omega} \{y(t)\}, & y(\eta_2) = \max_{t \in I_\omega} \{y(t)\}, \\ z(\xi_3) = \min_{t \in I_\omega} \{z(t)\}, & z(\eta_3) = \max_{t \in I_\omega} \{z(t)\}. \end{cases} \quad (2.5)$$

From (2.3) and (2.4), we have

$$\begin{aligned} \int_k^{k+\omega} |x^\Delta(t)|\Delta t &\leq 2\bar{r}_1\omega, \\ \int_k^{k+\omega} |y^\Delta(t)|\Delta t &\leq 2\frac{m_{21}^M\omega}{b_1^L} \text{ and} \\ \int_k^{k+\omega} |z^\Delta(t)|\Delta t &\leq 2\frac{m_{32}^M\omega}{b_2^L}. \end{aligned}$$

By the first equations of (2.4) and (2.5), we have

$$d_1(\xi_1)e^{x(\xi_1)} < r_1(\xi_1).$$

That is,

$$x(\xi_1) < \ln \frac{r_1^M}{d_1^L}.$$

From the second equation of (2.4), we have

$$r_2(\eta_2) < \frac{m_{21}(\eta_2)e^{x(\eta_2-\tau)}}{a_1(\eta_2)}$$

and

$$x(\eta_1) \geq x(\eta_2 - \tau) > \ln \frac{r_2^L a_1^L}{m_{21}^M}.$$

According to Lemma 2.2.1, we have

$$\begin{aligned} x(t) &\leq x(\xi_1) + \frac{1}{2} \int_k^{k+\omega} |x^\Delta(t)|\Delta t \leq \ln \frac{r_1^M}{d_1^L} + \bar{r}_1\omega := M_1 \text{ and} \\ x(t) &\geq x(\eta_1) - \frac{1}{2} \int_k^{k+\omega} |x^\Delta(t)|\Delta t \geq \ln \frac{r_2^L a_1^L}{m_{21}^M} - \bar{r}_1\omega := L_1. \end{aligned}$$

From the first equations of (2.4) and (2.5), we obtain

$$\frac{m_{12}(\xi_1)e^{y(\xi_1)}}{a(\xi_1) + b_1(\xi_1)e^{x(\xi_1)} + e^{2x(\xi_1)}} < r_1(\xi_1) \text{ and}$$

$$y(\xi_2) < y(\xi_1) < \ln \frac{r_1^M(a_1^M + \frac{b_1^M r_1^M}{d_1^L} + \frac{2r_1^M}{d_1^L})}{m_{12}^L}.$$

Then

$$y(t) \leq y(\xi_2) + \frac{1}{2} \int_k^{k+\omega} |y^\Delta(t)| \Delta t < \ln \frac{r_1^M(a_1^M + \frac{b_1^M r_1^M}{d_1^L} + \frac{2r_1^M}{d_1^L})}{m_{12}^L} + \frac{m_{21}^M \omega}{b_1^L} := M_2.$$

From the third equation of (2.4), we have

$$\frac{m_{32}(\xi_3)}{a_2(\xi_3)} e^{y(\xi_3 - \sigma)} > r_3(t),$$

this reduces to

$$y(\eta_2) \geq y(\xi_3 - \sigma) > \ln \frac{a_2^L r_3^L}{m_{32}^M}.$$

Then

$$y(t) \geq y(\eta_2) - \frac{1}{2} \int_k^{k+\omega} |y^\Delta(t)| \Delta t \geq \ln \frac{a_2^L r_3^L}{m_{32}^M} - \frac{m_{21}^M \omega}{b_1^L} := L_2.$$

According to the first equation of (2.4), we have

$$\frac{m_{32}(\xi_3)}{b_2(\xi_3)} > d_3(\xi_3) e^{z(\xi_3)}.$$

Then

$$z(t) \leq z(\xi_3) + \frac{1}{2} \int_k^{k+\omega} |z^\Delta(t)| \Delta t \leq \ln \frac{m_{32}^M}{b_2^L d_3^L} + \frac{m_{32}^M \omega}{b_2^L} := M_3.$$

We have

$$d_3^M e^{z(\eta_3)} > \frac{m_{32}^L e^{L_2}}{a_2^M + b_2^M e^{M_2} + e^{2M_2}} - r_3^M.$$

Then

$$z(t) \geq z(\eta_3) - \frac{1}{2} \int_k^{k+\omega} |z^\Delta(t)| \Delta t \geq \ln \frac{\frac{m_{32}^L e^{L_2}}{a_2^M + b_2^M e^{M_2} + e^{2M_2}} - r_3^M}{d_3^M} - \frac{m_{32}^M \omega}{b_2^L} := L_3.$$

Therefore,

$$\begin{aligned} \max_{t \in [k, k+\omega]} |x(t)| &\leq \max\{|M_1|, |L_1|\} := R_1, \\ \max_{t \in [k, k+\omega]} |y(t)| &\leq \max\{|M_2|, |L_2|\} := R_2 \text{ and} \\ \max_{t \in [k, k+\omega]} |z(t)| &\leq \max\{|M_3|, |L_3|\} := R_3. \end{aligned}$$

That is R_1, R_2 and R_3 are independent of λ .

Let $R = R_1 + R_2 + R_3 + R_0$, where R_0 is taken sufficiently large such that for the equations

$$\begin{aligned} \bar{r}_1 - \bar{d}_1 e^x - \frac{1}{\omega} \int_\kappa^{\kappa+\omega} \frac{m_{12}(t) e^y}{a_1(t) + b_1(t) e^x + e^{2x}} \Delta t &= 0, \\ -\bar{r}_2 + \frac{1}{\omega} \int_\kappa^{\kappa+\omega} \frac{m_{21}(t) e^x}{a_1(t) + b_1(t) e^x + e^{2x}} \Delta t - \bar{d}_2 e^y - \frac{1}{\omega} \int_\kappa^{\kappa+\omega} \frac{\bar{m}_{23} e^z}{a_2(t) + b_2(t) e^y + e^{2y}} \Delta t &= 0, \\ -\bar{r}_3 + \frac{1}{\omega} \int_\kappa^{\kappa+\omega} \frac{m_{32}(t) e^y}{a_2(t) + b_2(t) e^y + e^{2y}} \Delta t - \bar{d}_3 e^z &= 0, \end{aligned} \tag{2.6}$$

every solution $(x^*, y^*, z^*)^T$ of (2.6) satisfies $\|(x^*, y^*, z^*)^T\| < R$. Now, we define $\Omega = \{(x, y, z)^T \in X : \|(x, y, z)^T\| < R\}$. Then it is clear that Ω satisfies the requirement (a) of Lemma 2.2.2. If $(u_1, u_2, u_3)^T \in \partial\Omega \cap \ker L = \partial\Omega \cap \mathbb{R}^3$, then $(x, y, z)^T$ is a constant vector in $\mathbb{R}^3$ with $\|(x, y, z)^T\| = |x| + |y| + |z| = R$, so we have

$$QN \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Using the assumption of Theorem 2.3.1 and the definition of topological degree, we deduce that $\deg(JQN, \Omega \cap \ker L, 0) \neq 0$. That is Ω satisfies all requirements of Lemma 2.2.2, therefore, system (2.2) has at least one ω -periodic solution in $\text{Dom } L \cap \bar{\Omega}$. $\square$

2.4 Numerical simulations

In this section, we give some numerical simulations to illustrate the theoretical results obtained in section 2.3., the values of the parameters are:

$$r_1(n) = 0.9 + 0.01 \sin \frac{\pi n}{2},$$

$$r_2(n) = 0.04 + 0.01 \cos \frac{\pi n}{2},$$

$$r_3(n) = 0.21 + 0.01 \sin \frac{\pi n}{2},$$

$$d_1(n) = 0.2 + 0.02 \cos \frac{\pi n}{2},$$

$$d_2(n) = 0.04 + 0.01 \cos \frac{\pi n}{2},$$

$$d_3(n) = 0.23 + 0.02 \cos \frac{\pi n}{2},$$

$$a_1(n) = 0.5 + 0.02 \sin \frac{\pi n}{2},$$

$$a_2(n) = 1,$$

$$b_1(n) = 0.04 + 0.01 \cos \frac{\pi n}{2},$$

$$b_2(n) = 0.7(0.7 + 0.01 \cos \frac{\pi n}{2}),$$

$$\tau = 1,$$

$$\sigma = 1,$$

$$m_{12}(n) = 0.09 + 0.01 \sin \frac{\pi n}{2},$$

$$m_{21}(n) = 0.5 + 0.02 \sin \frac{\pi n}{2},$$

$$m_{23}(n) = 0.7 + 0.01 \cos \frac{\pi n}{2} \text{ and}$$

$$m_{32}(n) = 0.01(1 + 0.01 \cos \frac{\pi n}{2}).$$

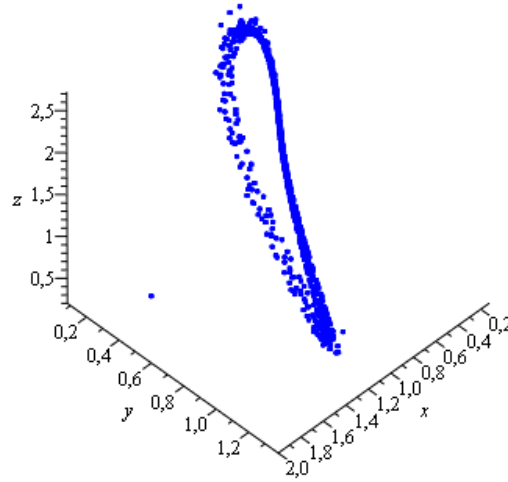


Figure 2.1: Phase portrait for food chain system (2.1) of trajectory from initial condition $(u_1(0), u_2(0), u_3(0)) = (2, 0.6, 1)$, $(u_1(n), u_2(n), u_3(n)) = (x(n), y(n), z(n))$.

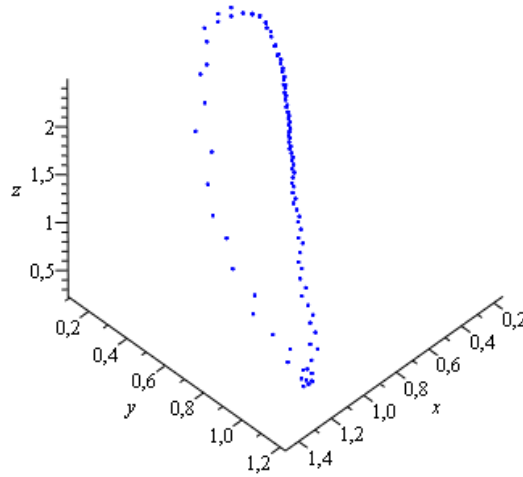


Figure 2.2: Phase portrait for food chain system (2.1) of trajectory from initial condition $(u_1(0), u_2(0), u_3(0)) = (2, 0.6, 1)$, $(u_1(n), u_2(n), u_3(n)) = (x(n), y(n), z(n))$. Here we plot the last hundred points.

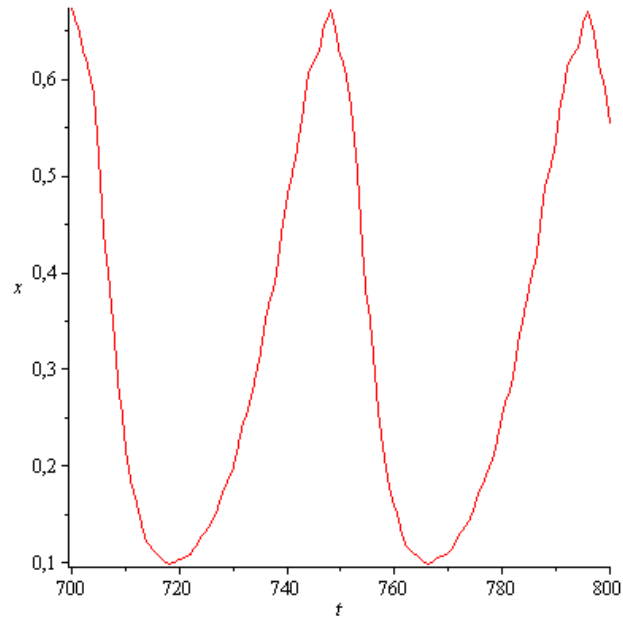


Figure 2.3: Prey curve for initial condition $(u_1(0), u_2(0), u_3(0)) = (2, 0.6, 1)$, $t = n$ and $u_1(n) = x(n)$.

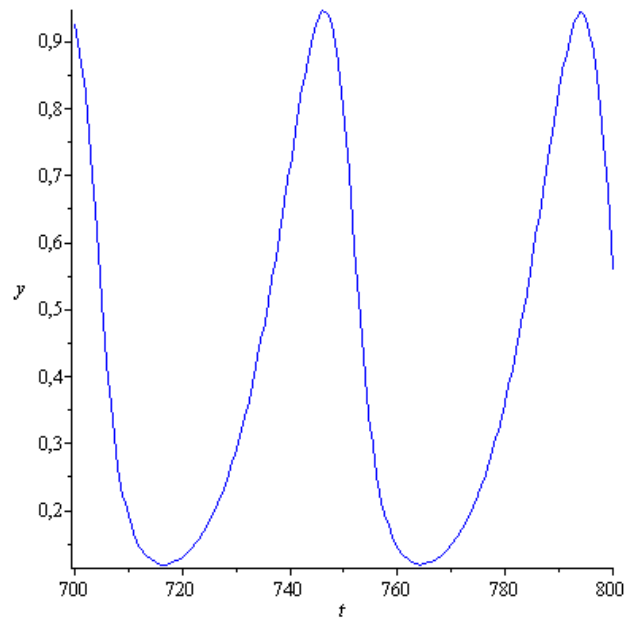


Figure 2.4: Intermediate predator curve for initial condition $(u_1(0), u_2(0), u_3(0)) = (2, 0.6, 1)$, $t = n$ and $u_2(n) = y(n)$.

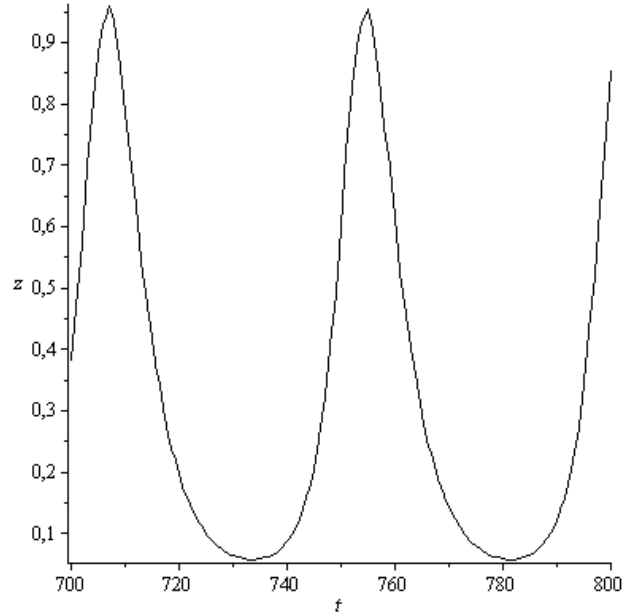


Figure 2.5: Top predator curve for initial condition $(u_1(0), u_2(0), u_3(0)) = (2, 0.6, 1)$, $t = n$ and $u_3(n) = z(n)$.

2.5 Conclusion

In this chapter, we have presented the theoretical work of [71], it is a study of a three species food chain model. The model unifies the food chain system with Monod-Haldane functional response and time delay governed by dynamical equations on time scales and their discrete analogues in form of difference equations. By the mean of Mawhin's continuation theorem of coincidence degree theory, the existence of periodic solutions is proved under some sufficient conditions on the parameters of the model. Moreover, we have given our simulations for this model which is the aim of this chapter.

On stability of a discrete HIV model with latent CD4+ cells incorporating ART

3.1 *Introduction*

Many of mathematical studies were dedicated to the analysis of HIV infection since its first occurrence in 1981. HIV infection and is characterized by a severe reduction in CD4+ T cells, which means an infected person develops a very weak immune system and becomes vulnerable to contracting life-threatening infections. Treatment with HIV medicines (called antiretroviral therapy or ART) is recommended for everyone with HIV. ART are medications that treat HIV. The drugs do not kill or cure the virus. However, when taken in combination they can prevent the growth of the virus. When the virus is slowed down, so is HIV disease. When a virus is present in the body but exists in a resting (latent) state without producing more virus, this state is called Viral Latency. A latent viral infection usually does not cause any noticeable symptoms [47] and can last a long period of time before becoming active and causing symptoms. HIV is capable of viral latency, as seen in the reservoirs of latent HIV-infected cells [5] that persist in a person's body despite antiretroviral therapy (ART). Resting CD4 cells (or other cells) that are infected with HIV but not actively pro-

ducing HIV. Latent HIV reservoirs are established during the earliest stage of HIV infection. Although antiretroviral therapy (ART) can reduce the level of HIV in the blood to an undetectable level, latent reservoirs of HIV continue to survive. When a latently infected cell is reactivated, the cell begins to produce HIV again. For this reason, ART cannot cure HIV infection.

3.2 Discrete time HIV model

We consider the following model, which is generalized from the model proposed in [43] by including the fact that ART does not affect the latent cells and by using the forward Euler method. The variables are described in Table 3.1 and the parameters are defined in Table 3.2.

$$\begin{aligned}
 T_{n+1} &= T_n + rT_n \left(1 - \frac{T_n}{T_{max}}\right) - \beta T_n V_n \\
 L_{n+1} &= L_n + \eta_l \beta T_n V_n - (a + d_1) L_n \\
 I_{n+1} &= (1 - \eta_1) I_n + (1 - \eta_l) \beta T_n V_n + a L_n - d_2 I_n \\
 V_{n+1} &= (1 - \eta_2) V_n + p d_2 I_n - d_3 V_n
 \end{aligned} \tag{3.1}$$

Table 3.1: Variables in the model

Population	Definition
T_n	Healthy CD4+ T-cells at the state n
L_n	Latently infected CD4+ T-cells at the state n
I_n	Productively infected CD4+ T-cells at the state n
V_n	Free infectious virus at the state n

Table 3.2: Parameters in the model

Parameter	Definition
r	Logistic growth rate of healthy CD4+ T-cells
T_{max}	Carrying capacity of healthy CD4+ T-cells
β	Infection rate of healthy CD4+ T-cells by free infectious virus
η_1	Drug efficacy of ART on the productively infected CD4+ T-cells ($0 \leq \eta_1 \leq 1$)
η_2	Drug efficacy of ART on the free infectious virus ($0 \leq \eta_2 \leq 1$)
η_l	Fraction of infections leading to latently infected CD4+ T-cells
d_1	Death rate of latently infected CD4+ T-cells
d_2	Death rate of productively infected CD4+ T-cells
d_3	Clearance rate of free virus
a	Activation rate from latently to productively infected CD4+ T-cells
p	Average number of free virus released by a productively infected CD4+

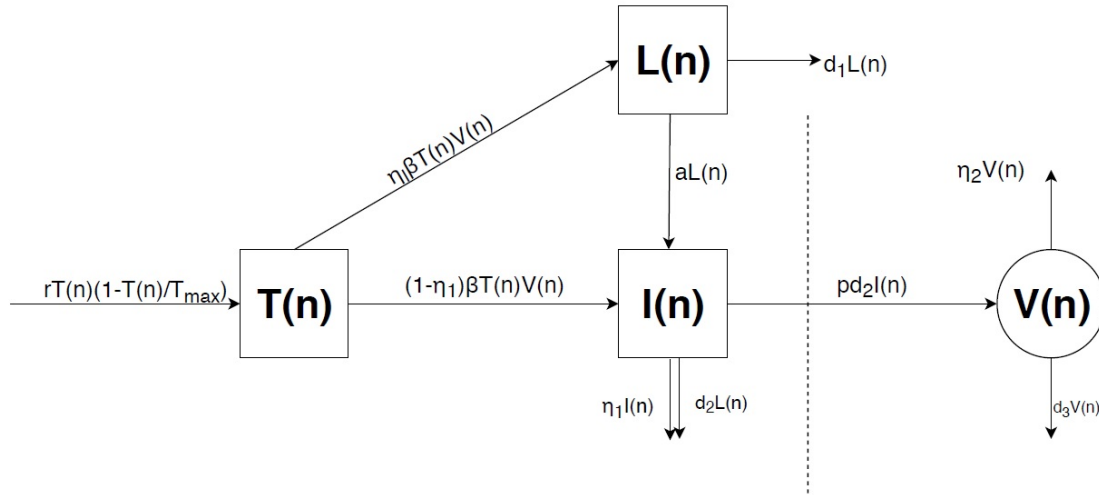


Figure 3.1: HIV model with logistic growth and latent infection incorporating ART

If ART is 100% successful, then $\eta_1 = \eta_2 = 1$, and the system 3.1 reduces to

$$\begin{aligned}
T_{n+1} &= T_n + rT_n \left(1 - \frac{T_n}{T_{max}}\right) - \eta_l \beta T_n V_n \\
L_{n+1} &= L_n + \eta_l \beta T_n V_n - (a + d_1) L_n \\
I_{n+1} &= I_n + aL_n - d_2 I_n \\
V_{n+1} &= V_n - d_3 V_n
\end{aligned} \tag{3.2}$$

3.3 Preliminary results

3.3.1 Jury Stability Test[46]

Assume that the characteristic equation is as follows,

$$P(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n$$

where $a_0 > 0$.

The Jury table is given by

Row	z^0	z^1	z^2	z^3	z^4	$\dots$	z^n
1	a_n	a_{n-1}	a_{n-2}	$\dots$	$\dots$	$\dots$	a_0
2	a_0	a_1	a_2	$\dots$	$\dots$	$\dots$	a_n
3	b_{n-1}	b_{n-2}	$\dots$	$\dots$	$\dots$	b_0	
4	b_0	b_1	$\dots$	$\dots$	$\dots$	b_{n-1}	
5	c_{n-2}	c_{n-2}	$\dots$	$\dots$	c_0		
6	c_0	c_1	$\dots$	$\dots$	c_{n-2}		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			
$2n - 3$	q_2	q_1	q_0				

where

$$\begin{aligned}
 b_k &= \begin{vmatrix} a_n & a_{n-1-k} \\ a_0 & a_{k+1} \end{vmatrix}, \quad k = 0, 1, 2, \dots, n-1, \\
 c_k &= \begin{vmatrix} b_{n-1} & b_{n-2-k} \\ b_0 & b_{k+1} \end{vmatrix}, \quad k = 0, 1, 2, \dots, n-2, \\
 &\vdots \\
 q_k &= \begin{vmatrix} p_3 & p_{2-k} \\ p_0 & p_{k+1} \end{vmatrix}, \quad k = 0, 1, 2.
 \end{aligned}$$

This system will be stable if:

1. $|a_n| < a_0$
2. $P(Z)|_{z=1} > 0$
3. $(-1)^n P(Z)|_{z=-1} > 0$
- 4.

$$\begin{aligned}
 |b_{n-1}| &> |b_0| \\
 |c_{n-2}| &> |c_0| \\
 &\vdots \\
 |q_2| &> |q_0|.
 \end{aligned}$$

3.3.2 Global well-posedness

Proposition 3.3.1. *The system (3.1) with initial conditions (T_0, L_0, I_0, V_0) has unique bounded solution sequence $\{T_n\}$, $\{L_n\}$, $\{I_n\}$ and $\{V_n\}$ in $\mathbb{R}_+^4$ defined for all $n \geq 0$.*

Proof. Recall that a discrete time system $u_{n+1} - u_n = F(u)$ on $\mathbb{R}_+^m$ is called quasi-positive if the condition

$$u \geq 0, \quad u_k = 0 \Rightarrow F_k(u) \geq 0$$

is valid for all $k = 0, 1, \dots, m$ where $F = (F_1, F_2, \dots, F_m)$.

Let $F : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be given by $F(T_n, L_n, I_n, V_n) = (F_1, F_2, F_3, F_4)$ where

$$\begin{cases} F_1 = rT_n \left(1 - \frac{T_n}{T_{max}}\right) - \beta T_n V_n, \\ F_2 = \eta_l \beta T_n V_n - (a + d_1) L_n, \\ F_3 = -\eta_1 I_n + (1 - \eta_l) \beta T_n V_n + a L_n - d_2 I_n, \\ F_4 = -\eta_2 V_n + p d_2 I_n - d_3 V_n. \end{cases}$$

For $n \geq 0$ and $(T_n, L_n, I_n, V_n) \in \mathbb{R}^4$ it follows that $F_1 \geq 0$ when $T_n = 0$, $F_2 = \eta_l \beta T_n V_n \geq 0$ when $L_n = 0$, $F_3 = (1 - \eta_l) \beta T_n V_n + a L_n \geq 0$ when $I_n = 0$ and $F_4 = p d_2 I_n \geq 0$ when $V_n = 0$. Thus, there exists a unique nonnegative solution to (3.1) in $\mathbb{R}_+^4$ for $n \geq 0$. From the three equations we get with $S_n = P(T_n + L_n + I_n) + V_n$,

$$\begin{aligned} S_{n+1} - S_n &= pr \left(1 - \frac{T_n}{T_{max}}\right) - p d_1 L_n - p \eta_1 I_n - (\eta_2 + d_3) V_n \\ &\leq pr - p \left(\frac{pr}{T_{max}} T_n + d_1 L_n + \eta_1 I_n \right) - \eta_2 V_n \\ &\leq pr - \min \left(\frac{r}{T_{max}}, d_1, \eta_1, \eta_2 \right) S_n \end{aligned}$$

From Lemma 2.2 in [46], for $\epsilon > 0$, there exists a number $N_0 \in \mathbb{N}$ such that any solution sequence $\{S_n\}$ with initial value $S_0 > 0$ satisfies $S_n \leq M + \epsilon$ for all $n > N_0$. Hence we obtain the boundedness of the solutions. Thus the existence of a unique nonnegative bounded solution is proved. $\square$

3.4 Existence and stability of steady states

Theorem 3.4.1. *the the system (3.1) has at least three equilibria:*

- A trivial equilibrium point $E_0 = (0, 0, 0, 0)$ which is biologically uninterested to study.
- A disease free equilibrium $E_1 = (T_{max}, 0, 0, 0)$

- An infected equilibrium point $E_2 = (T^*, L^*, I^*, V^*)$ which exists if and only if

$T^* < T_{max}$, where

$$\begin{aligned} T^* &= \frac{(d_3 + \eta_2)(d_2 + \eta_1)(d_1 + a)}{pd_2(a + d_1(1 - \eta_l))\beta}, \\ V^* &= \frac{r}{\beta} \left(1 - \frac{T^*}{T_{max}}\right), \\ L^* &= \frac{r\eta_l}{a + d_1} \left(1 - \frac{T^*}{T_{max}}\right) T^* \text{ and} \\ I^* &= \frac{[a + d_1(1 - \eta_l)] r \left(1 - \frac{T^*}{T_{max}}\right) T^*}{(a + d_1)(\eta_1 + d_2)}. \end{aligned}$$

Proof. If $T_n = 0$ then $L_n = I_n = V_n = 0$.

If $T_n = T_{max}$ then $L_n = I_n = V_n = 0$.

If $T_n \neq 0$ and $T_n \neq T_{max}$, then for $T^* < T_{max}$ we obtain

$$\begin{aligned} V^* &= \frac{r}{\beta} \left(1 - \frac{T^*}{T_{max}}\right), \\ L^* &= \frac{\eta_l r}{a + d_1} \left(1 - \frac{T^*}{T_{max}}\right)^*, \\ I^* &= \frac{r((1 - \eta_l)d_1 + a)}{(a + d_1)(\eta_1 + d_2)} \left(1 - \frac{T^*}{T_{max}}\right) T^* \text{ and} \\ T^* &= \frac{(\eta_2 + d_3)(a + d_1)(\eta_1 + d_2)}{pd_2((1 - \eta_l)d_1 + a)\beta}. \end{aligned}$$

□

Theorem 3.4.2. *Let*

$$\begin{aligned} T^{2*} &:= \frac{(1 - a - d_1)(1 - \eta_1 - d_2)(1 - \eta_2 - d_3) - 1}{pd_2\beta((1 - d_1)(1 - \eta_l) - a)}, \\ T^{3*} &:= \frac{(2 - a - d_1)(2 - \eta_1 - d_2)(2 - \eta_2 - d_3)}{pd_2(a - (2 - d_1)(1 - \eta_l))\beta}, \\ T^{4*} &:= \frac{-C_1 + \sqrt{C_1^2 - 4C_0C_2}}{2C_2}, \\ T^{5*} &:= \frac{-B_1 + \sqrt{B_1^2 - 4B_0B_2}}{2B_2}. \end{aligned}$$

1. The trivial equilibrium E_0 is unstable.
2. The disease free equilibrium E_1 is stable if and only if $0 < r < 2$, $\eta_l > 1 - \frac{a}{2 - d_1}$ and $T^{3*} < T_{max} < \min(T^*, T^{2*}, T^{4*}, T^{5*})$.

3. The infected equilibrium E_2 is stable if and only if

(a) $|a_4| < a_0 = 1$,

(b) $F(1) > 0$ and $(-1)^4 F(-1) > 0$, and

(c) $|b_3| > |b_0|$ and $|c_2| > |c_0|$ where

$$a_1 = -R - D_1 - D_2 - D_3,$$

$$a_2 = RD_1 + D_2D_3 + (D_2 + D_3)(R + D_1) - pd_2(1 - \eta_l)\beta T^*,$$

$$a_3 = -RD_1(D_2 + D_3) - D_2D_3(R + D_1) + pd_2\beta(1 - \eta_l)(R + D_1)T^* - pd_2\beta a\eta_l T^* + pd_2\beta^2(1 - \eta_l)T^*V^*,$$

$$a_4 = RD_1D_2D_3 - pd_2\beta(1 - \eta_l)RD_1T^* + pd_2\beta a\eta_l RT^* - pd_2\beta^2(1 - \eta_l)D_1T^*V^*,$$

$$b_3 = \begin{vmatrix} a_4 & a_0 \\ a_0 & a_4 \end{vmatrix}, \quad b_2 = \begin{vmatrix} a_4 & a_1 \\ a_0 & a_3 \end{vmatrix}, \quad b_1 = \begin{vmatrix} a_4 & a_2 \\ a_0 & a_2 \end{vmatrix}, \quad b_0 = \begin{vmatrix} a_4 & a_3 \\ a_0 & a_1 \end{vmatrix},$$

$$c_2 = \begin{vmatrix} b_3 & b_0 \\ b_0 & b_3 \end{vmatrix}, \quad c_1 = \begin{vmatrix} b_3 & b_1 \\ b_0 & b_2 \end{vmatrix}, \quad \text{and } c_0 = \begin{vmatrix} b_3 & b_2 \\ b_0 & b_1 \end{vmatrix}.$$

Proof. At point E_i , $i = \overline{0, 2}$, the Jacobian matrix of system (3.1) is given by

$$J(E_i) = \begin{pmatrix} 1 + r \left(1 - \frac{2T_i}{T_{max}}\right) - \beta V_i & 0 & 0 & -\beta T_i \\ \eta_l \beta V_i & 1 - a - d_1 & 0 & \eta_l \beta T_i \\ (1 - \eta_l) \beta V_i & a & 1 - \eta_l - d_2 & (1 - \eta_l) \beta T_i \\ 0 & 0 & pd_2 & 1 - \eta_2 - d_3 \end{pmatrix}.$$

Denote

$$R = 1 + r \left(1 - \frac{2T_i}{T_{max}}\right) - \beta V_i,$$

$$D_1 = 1 - a - d_1,$$

$D_2 = 1 - \eta_1 - d_2$, and

$D_3 = 1 - \eta_2 - d_3$.

1. In case of trivial equilibrium $E_0 = (0, 0, 0, 0)$, the Jacobian matrix of system (3.1) is given by

$$J(E_0) = \begin{pmatrix} 1+r & 0 & 0 & 0 \\ 0 & 1-a-d_1 & 0 & 0 \\ 0 & a & 1-\eta_1-d_2 & 0 \\ 0 & 0 & pd_2 & 1-\eta_2-d_3 \end{pmatrix}.$$

The eigenvalues of $J(E_0)$ are $1+r$, $1-a-d_1$, $1-\eta_1-d_2$ and $1-\eta_2-d_3$. Since $|1+r| > 1$ then E_0 is unstable.

2. In case of disease free equilibrium $E_1 = (T_{max}, 0, 0, 0)$, the Jacobian matrix of system (3.1) is given by

$$J(E_1) = \begin{pmatrix} R & 0 & 0 & -\beta T_{max} \\ 0 & D_1 & 0 & \eta_l \beta T_{max} \\ 0 & a & D_2 & (1-\eta_l) \beta T_{max} \\ 0 & 0 & pd_2 & D_3 \end{pmatrix}$$

where $R = 1 - r$.

The characteristic equation for system (3.1) is given by

$$P(\lambda) = (\lambda - R)F(\lambda),$$

where

$$F(\lambda) = \lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3, \quad (3.3)$$

$$A_1 = -D_1 - D_2 - D_3,$$

$A_2 = D_1D_2 + D_1D_3 + D_2D_3 - pd_2(1 - \eta_l)\beta T_{max}$, and

$A_3 = -D_1D_2D_3 + pd_2\beta(D_1(1 - \eta_l) - a\eta_l)T_{max}$.

The eigenvalues of Jacobian matrix $J(E_1)$ of system (3.1) are R and the roots of F .

From the Jury criteria, if we can prove:

$$(a) \quad |A_3| = |-D_1D_2D_3 + pd_2\beta(D_1(1 - \eta_l) - a\eta_l)T_{max}| < A_0 = 1,$$

$$(b) \quad F(1) > 0 \text{ and } (-1)^3F(-1) > 0, \text{ and}$$

$$(c) \quad |A_3^2 - 1| > |A_3A_1 - A_2| \text{ or } -|A_3^2 - 1| < A_3A_1 - A_2 < |A_3^2 - 1|.$$

Then the norm of the three roots of (3.10) is less than one. Noting that

- (a) Since $0 < a < 1$, $0 < d_i < 1$ for $i = \overline{1, 3}$ and $0 < \eta_k < 1$ for $k = \overline{1, 2}$ then $-1 < D_i < 1$ for $i = \overline{1, 3}$ and we have $|A_3| < 1$ if and only if $D_1D_2D_3 - 1 < pd_2\beta((1 - d_1)(1 - \eta_l) - a)T_{max} < D_1D_2D_3 + 1$. That is

$$\begin{cases} 0 < T_{max} < T^{2*} & \text{if } \eta_l > 1 - \frac{a}{(1-d_1)}, \\ T_{max} > 0 & \text{if } \eta_l = 1 - \frac{a}{(1-d_1)}, \\ 0 < T_{max} < T^{2*} & \text{if } \eta_l < 1 - \frac{a}{(1-d_1)}, \end{cases} \quad (3.4)$$

$$\text{where } T^{2*} := \frac{(1-a-d_1)(1-\eta_1-d_2)(1-\eta_2-d_3)-1}{pd_2\beta((1-d_1)(1-\eta_l)-a)}$$

- (b) We have

$$\begin{aligned} F(1) &= 1 + A_1 + A_2 + A_3 \\ &= (1 - D_1)(1 - D_2)(1 - D_3) - pd_2((1 - D_1)(1 - \eta_l) + a\eta_l)\beta T_{max} \\ &= pd_2(d_1(1 - \eta_l) + a)\beta \left(\frac{(a+d_1)(\eta_1+d_2)(\eta_2+d_3)}{pd_2(d_1(1-\eta_l)+a)\beta} - T_{max} \right) \\ &= pd_2(d_1(1 - \eta_l) + a)\beta (T^* - T_{max}) > 0 \end{aligned}$$

if and only if

$$T_{max} < T^*. \quad (3.5)$$

$$\begin{aligned} F(-1) &= -1 + A_1 - A_2 + A_3 \\ &= -(1 + D_1)(1 + D_2)(1 + D_3) + pd_2((1 + D_1)(1 - \eta_l) - a\eta_l)\beta T_{max} \\ &= (2 - a - d_1)(2 - \eta_1 - d_2)(2 - \eta_2 - d_3) \\ &\quad - pd_2(a - (2 - d_1)(1 - \eta_l))\beta T_{max} < 0 \end{aligned}$$

if and only if

$$T_{max} > T^{3*} \text{ and } \eta_l > 1 - \frac{a}{(2 - d_1)} \quad (3.6)$$

$$\text{where } T^{3*} := \frac{(2-a-d_1)(2-\eta_1-d_2)(2-\eta_2-d_3)}{pd_2(a-(2-d_1)(1-\eta_l))\beta}.$$

(c) Since $-1 < A_3 < 1$ then

$$\begin{aligned} |A_3^2 - 1| &= 1 - A_3^2 \\ &= 1 - [-D_1D_2D_3 + pd_2\beta((1 - d_1)(1 - \eta_l) - a)T_{max}]^2 \\ &= 1 - D_1^2D_2^2D_3^2 + 2D_1D_2D_3pd_2\beta((1 - d_1)(1 - \eta_l) - a)T_{max} \\ &\quad - p^2d_2^2\beta^2((1 - d_1)(1 - \eta_l) - a)^2T_{max}^2 \end{aligned}$$

and

$$\begin{aligned} A_3A_1 - A_2 &= (-D_1D_2D_3 + pd_2\beta((1 - d_1)(1 - \eta_l) - a)T_{max})A_1 \\ &\quad - D_1D_2 - D_1D_3 - D_2D_3 + pd_2(1 - \eta_l)\beta T_{max} \\ &= D_2D_3(D_1^2 - 1) + D_1D_3(D_2^2 - 1) + D_1D_2(D_3^2 - 1) \\ &\quad + pd_2\beta[((1 - d_1)(1 - \eta_l) - a)A_1 + (1 - \eta_l)]T_{max}. \end{aligned}$$

Then we have

$A_3A_1 - A_2 < |A_3^2 - 1|$ is equivalent to

$$P(T_{max}) = C_2T_{max}^2 + C_1T_{max} + C_0 < 0$$

where

$$C_2 = p^2 d_2^2 \beta^2 ((1 - d_1)(1 - \eta_l) - a)^2 > 0,$$

$$C_1 = p d_2 \beta [(1 - \eta_l) + (A_1 - 2D_1 D_2 D_3)((1 - d_1)(1 - \eta_l) - a)] \text{ and}$$

$$C_0 = D_2 D_3 (D_1^2 - 1) + D_1 D_3 (D_2^2 - 1) + D_1 D_2 (D_3^2 - 1) - 1 + D_1^2 D_2^2 D_3^2.$$

it's obviously to see that $C_0 < 0$ for $0 < D_i < 1$ (resp. $-1 < D_i < 0$).

Suppose now that there exists $1 \leq k \leq 3$ such that $0 < D_k < 1$ and $-1 < D_i < 0$ (resp. $-1 < D_k < 0$ and $0 < D_i < 1$) for $i \neq k$ so we can write

$$\begin{aligned} C_0 &= D_j D_l (1 + D_j D_l) D_k^2 - (1 - D_j D_l)(D_j + D_l) D_k - D_j D_l - 1, \\ j &\neq k \neq l \neq j. \text{ Then, } C_0 < 0 \text{ if and only if } 0 < D_k < \\ &\frac{(1 - D_j D_l)(D_j + D_l) + \sqrt{\Delta}}{2 D_j D_l (1 + D_j D_l)} \text{ (resp. } \frac{(1 - D_j D_l)(D_j + D_l) - \sqrt{\Delta}}{2 D_j D_l (1 + D_j D_l)} < D_k < 0) \text{ where } \Delta = \\ &(1 - D_1 D_2)^2 (D_1 + D_2)^2 + 4 D_1 D_2 (1 + D_1 D_2)^2. \text{ Since } \frac{(1 - D_j D_l)(D_j + D_l) + \sqrt{\Delta}}{2 D_j D_l (1 + D_j D_l)} > 1 \\ &\text{(resp. } \frac{(1 - D_j D_l)(D_j + D_l) - \sqrt{\Delta}}{2 D_j D_l (1 + D_j D_l)} < -1) \text{ for } -1 < D_j < 0 \text{ and } -1 < D_l < 0 \text{ (resp.} \\ &0 < D_j < 1 \text{ and } 0 < D_l < 1) \text{ then } C_0 < 0 \text{ for } 0 < D_k < 1, -1 < D_j < 0 \\ &\text{and } -1 < D_l < 0 \text{ (resp. } -1 < D_k < 0, 0 < D_j < 1 \text{ and } 0 < D_l < 1). \\ &\text{Therefore,} \end{aligned}$$

$$A_3 A_1 - A_2 < |A_3^2 - 1| \text{ if and only if } T_{max} < T^{4*} \quad (3.7)$$

where

$$T^{4*} := \frac{-C_1 + \sqrt{C_1^2 - 4C_0 C_2}}{2C_2}.$$

The inequality $-|A_3^2 - 1| < A_3 A_1 - A_2$ is equivalent to

$$P(T_{max}) = B_2 T_{max}^2 + B_1 T_{max} + B_0 < 0$$

where

$$B_2 = p^2 d_2^2 \beta^2 ((1 - d_1)(1 - \eta_l) - a)^2 > 0,$$

$$B_1 = -pd_2\beta[(1 - \eta_l) + (A_1 + 2D_1D_2D_3)((1 - d_1)(1 - \eta_l) - a)] \text{ and}$$

$$B_0 = -(1 - D_1D_2)(1 - D_1D_3)(1 - D_2D_3) < 0.$$

Therefore,

$$A_3A_1 - A_2 < |A_3^2 - 1| \text{ if and only if } T_{max} < T^{5*} \quad (3.8)$$

where

$$T^{5*} := \frac{-B_1 + \sqrt{B_1^2 - 4B_0B_2}}{2B_2}$$

Noting that

$$|R| < 1 \Leftrightarrow 0 < r < 2. \quad (3.9)$$

From (3.4)-(3.9), the eigenvalues of jacobian matrix $J(E_1)$ are less than 1 if and only if $0 < r < 2$, $\eta_l > 1 - \frac{a}{2-d_1}$ and

$$T^{3*} < T_{max} < \min(T^*, T^{2*}, T^{4*}, T^{5*}).$$

3. In case of infected equilibrium $E_2 = (T^*, L^*, I^*, V^*)$, the Jacobian matrix of system (3.1) is given by

$$J(E_2) = \begin{pmatrix} R & 0 & 0 & -\beta T^* \\ \eta_l \beta V^* & D_1 & 0 & \eta_l \beta T^* \\ (1 - \eta_l) \beta V^* & a & D_2 & (1 - \eta_l) \beta T^* \\ 0 & 0 & pd_2 & D_3 \end{pmatrix}$$

where

$$R = 1 + r \left(1 - \frac{2T^*}{T_{max}} \right) - \beta V^*.$$

The characteristic equation for system (3.1) is given by

$$P(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4, \quad (3.10)$$

$$a_1 = -R - D_1 - D_2 - D_3,$$

$$a_2 = RD_1 + D_2D_3 + (D_2 + D_3)(R + D_1) - pd_2(1 - \eta_l)\beta T^*,$$

$$a_3 = -RD_1(D_2 + D_3) - D_2D_3(R + D_1) + pd_2\beta(1 - \eta_l)(R + D_1)T^* - pd_2\beta a\eta_l T^* + pd_2\beta^2(1 - \eta_l)T^*V^* \text{ and}$$

$$a_4 = RD_1D_2D_3 - pd_2\beta(1 - \eta_l)RD_1T^* + pd_2\beta a\eta_l RT^* - pd_2\beta^2(1 - \eta_l)D_1T^*V^*.$$

The eigenvalues of Jacobian matrix $J(E_2)$ of system (3.1) are the roots of P .

From the Jury criteria, if we can prove:

$$(a) \quad |a_4| < a_0 = 1,$$

$$(b) \quad F(1) > 0 \text{ and } (-1)^4 F(-1) > 0, \text{ and}$$

$$(c) \quad |b_3| > |b_0| \text{ and } |c_2| > |c_0| \text{ where}$$

$$b_3 = \begin{vmatrix} a_4 & a_0 \\ a_0 & a_4 \end{vmatrix}, \quad b_2 = \begin{vmatrix} a_4 & a_1 \\ a_0 & a_3 \end{vmatrix}, \quad b_1 = \begin{vmatrix} a_4 & a_2 \\ a_0 & a_2 \end{vmatrix}, \quad b_0 = \begin{vmatrix} a_4 & a_3 \\ a_0 & a_1 \end{vmatrix},$$

$$c_2 = \begin{vmatrix} b_3 & b_0 \\ b_0 & b_3 \end{vmatrix}, \quad c_1 = \begin{vmatrix} b_3 & b_1 \\ b_0 & b_2 \end{vmatrix}, \quad \text{and } c_0 = \begin{vmatrix} b_3 & b_2 \\ b_0 & b_1 \end{vmatrix}.$$

Then the norm of the three roots of (3.10) is less than one. Noting that

□

3.5 Numerical simulation

In this section, we carry out numerical simulations to explore the cell and viral dynamics of system 3.1 and to show how antiretroviral therapy impact the T-cell count and viral load.

Table 3.3: List of parameters [74]

Parameter	Unit	Data1	Data2	Data3
r	day^{-1}	0.1	0.03	0.1
T_{max}	μl^{-1}	1500	1500	1500
β	$\mu l^{-1} day^{-1}$	10^{-4}	10^{-4}	10^{-4}
η_1		0.7	0.4	0.3
η_2		0.8	0.5	0.4
η		0.02	0.001	0.5
d_1	day^{-1}	0.04	0.004	0.2
d_2	day^{-1}	1	1	0.8
d_3	day^{-1}	20	3	15
a	day^{-1}	0.1	0.01	0.3
p	Virons/Cell	1000	200	500

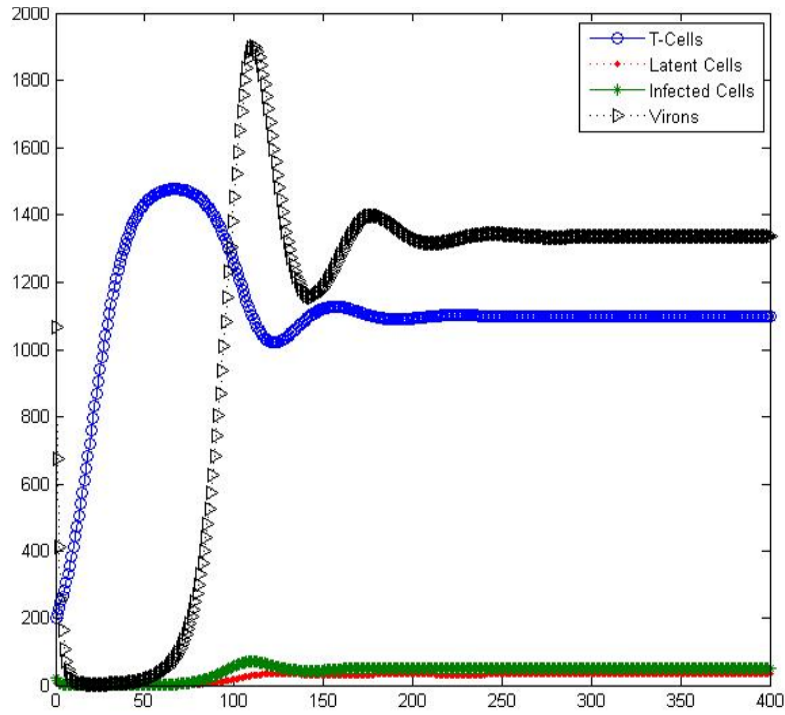


Figure 3.2: For the parameters see the Data1 values of Table 3.3 with the initial conditions $(T_0, L_0, I_0, V_0 = 200.0396, 4.2248, 21.3308, 1066.5411)$. see [74] for more details.

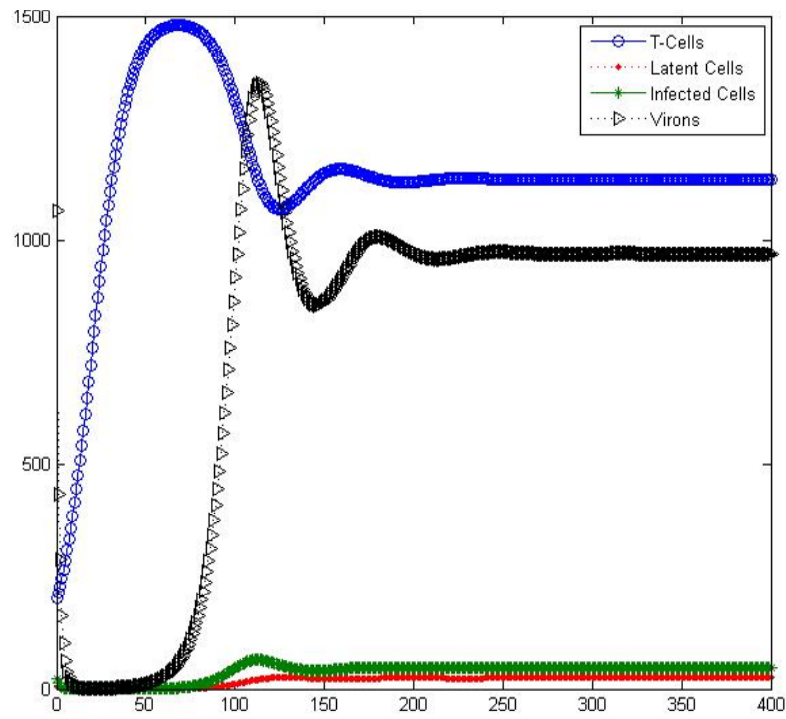


Figure 3.3: For the parameters see the Data2 values of Table 3.3 with the initial conditions $(T_0, L_0, I_0, V_0 = 200.0396, 4.2248, 21.3308, 1066.5411)$

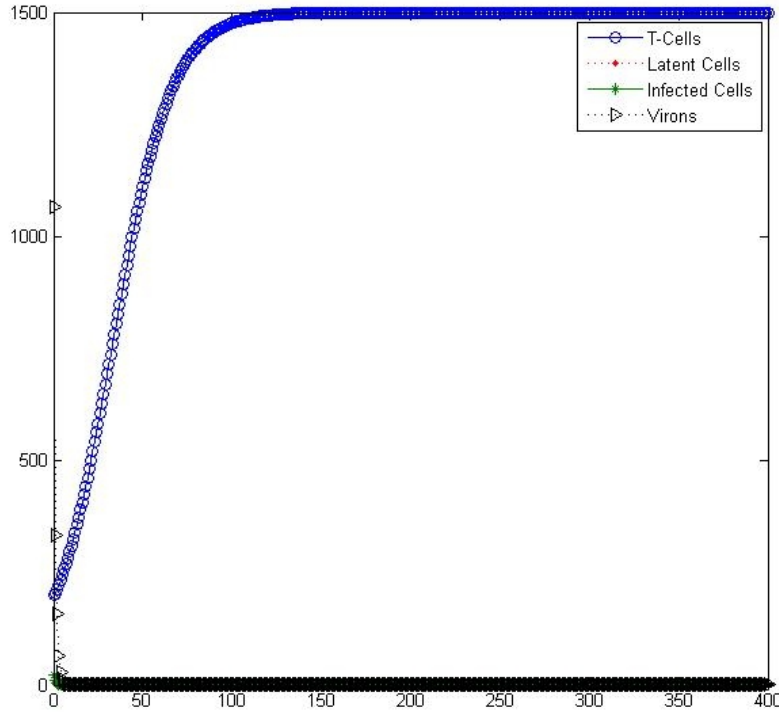


Figure 3.4: For the parameters see the Data3 values of Table 3.3 with the initial conditions $(T_0, L_0, I_0, V_0 = 200.0396, 4.2248, 21.3308, 1066.5411)$

With the increase of the combination drug therapy, the number of the healthy CD4+ T-cells becomes larger, while the number of the latently infected CD4+ T-cells, productively infected CD4+ T-cells and infectious virions become smaller and vanish.

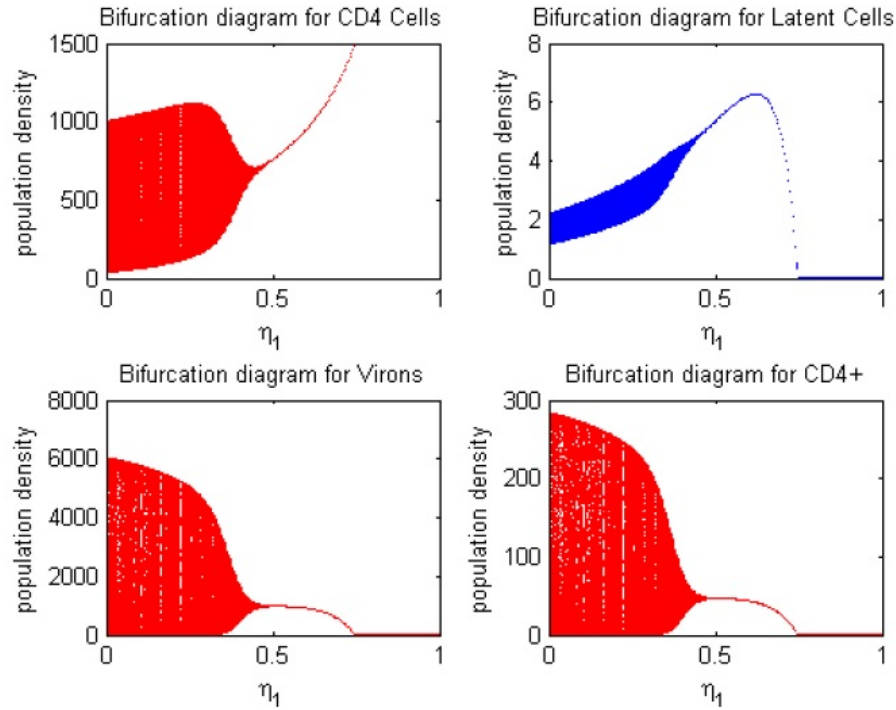


Figure 3.5: Bifurcation Diagram of system 3.1 using Data1 of table 3.3 and the initial condition $(T_0, L_0, I_0, V_0 = 200.0396, 4.2248, 21.3308, 1066.5411)$ with respect to r

This diagram shows that the system 3.1 undergoes a chaotic behavior during the therapy when the drug concentration $\eta_1 < 0.8$, we reach high level of treatment and the virons are eradicated when $\eta_1 \geq 0.8$ and $\eta_2 = 0.7$.

Now we examine the effect of the logistic growth parameter r on the progression of system 3.1 cc

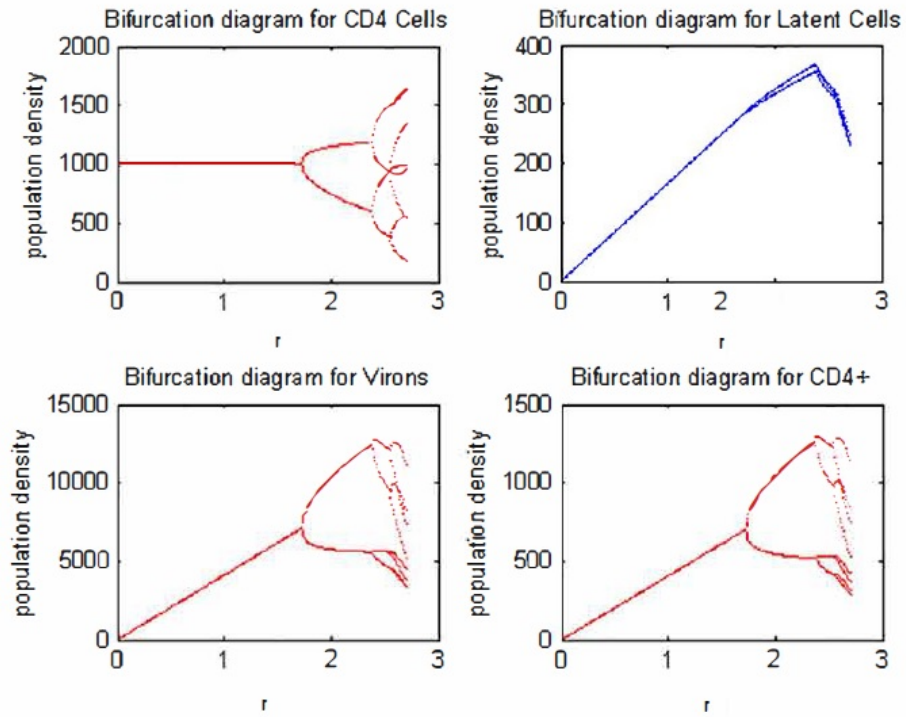


Figure 3.6: Bifurcation Diagram of system 3.1 using Data1 of table 3.3 and $p = 80$ with respect to r and $(T_0, L_0, I_0, V_0 = 200.0396, 4.2248, 21.3308, 1066.5411)$

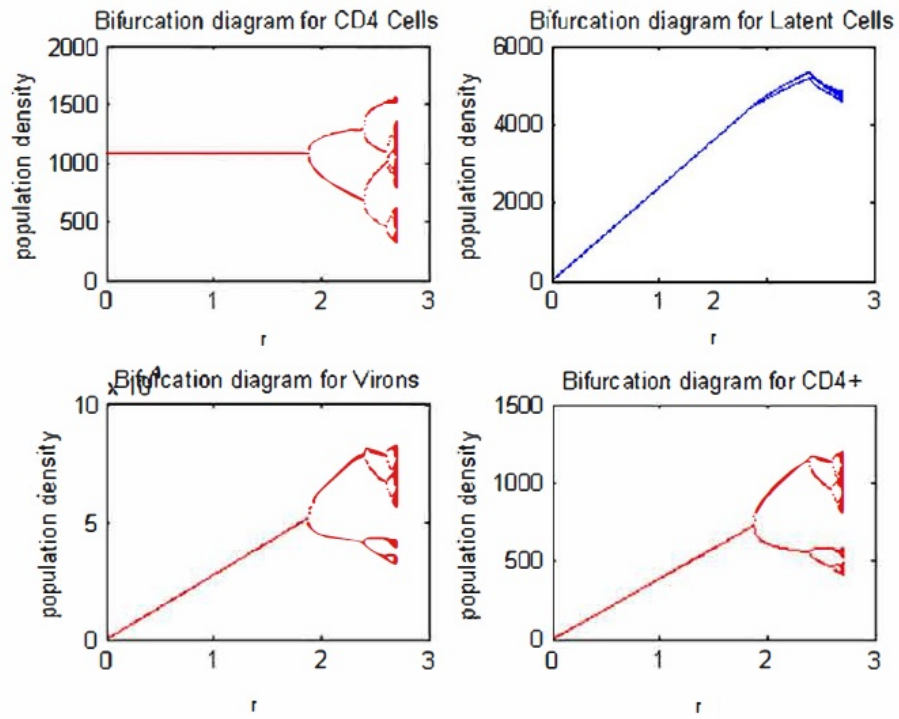


Figure 3.7: Bifurcation Diagram of system 3.1 using Data2 of table 3.3 with respect to r and $(T_0, L_0, I_0, V_0 = 200.0396, 4.2248, 21.3308, 1066.5411)$

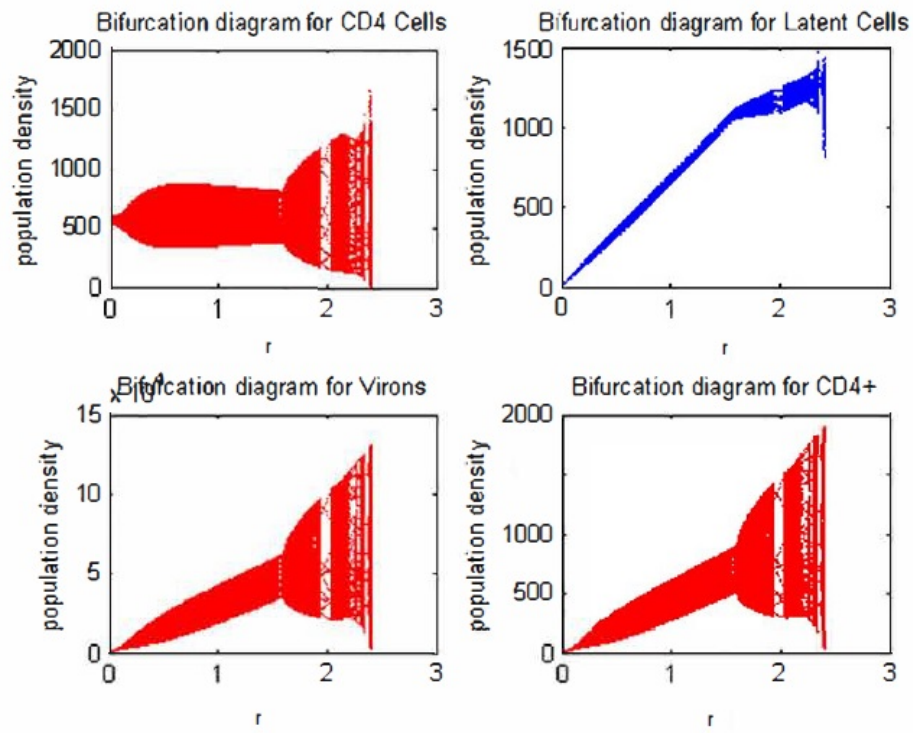


Figure 3.8: Bifurcation Diagram of system 3.1 using Data3 of table 3.3 and $p = 230$ with respect to r and $(T_0, L_0, I_0, V_0 = 200.0396, 4.2248, 21.3308, 1066.5411)$

3.6 Conclusion

In this chapter we have developed a new discrete mathematical model for HIV diseases with treatment. We have done mathematical analysis of the our model, we have proved the well posedness of the model and the boundedness of the positive solutions. We have studied the existence and the local stability of the steady states using the Jury criteria. In fact we have proved that the model admits at most three steady states, the trivial E_0 and the free disease E_1 steady states which exist always, and the endemic s steady states E^* which exists for carrying capacity T_{max} less some threshold T^* . We have found thresholds for the local stability of each steady state cited above. To illustrate our theoretical study, we have done some numerical simulations, we have noted that for some cases we obtain periodic phenomena which predicts that eventually there are bifurcations for some values of the parameters of the model. This question will be considered in a future works.

Perspectives

The works of these theses have considered a difference equations for a discrete models on population dynamics, our aims are to give theoretical and numerical analysis of the models studied here, to deduce conclusions on the behaviors of the different populations of the model under study.

First, we have considered a food chain model constituted by a system of three difference equations, representing a prey and a predator, with a prey separated into juvenile and adult subpopulations due to the difference in the predation. We have obtained a sufficient conditions for the existence of a positive periodic solutions, using continuation theorem based on the degree theory. We have treated the discrete mathematical model numerically to find what is hiding under simulations can be quite useful to explore the complete behavior of the model under study. Numerical simulations require building algorithms to ensure that the curves are well established. In the second chapter, we have presented the theoretical analysis of the work in [71], and we have established numerical simulations to illustrate the theoretical results.

In last chapter, we have developed a discrete model for HIV with treatment, we have studied the existence of steady states and their stability, we have end this chapter by numerical simulations to illustrate our theoretical results.

In the future, we plan to consider more realistic models for food chain and epidemiological discrete models by considering the cases of different kind of delay and the

cases of structured discrete models containing space and age.

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