N° d'ordre :

REPUBLIQUE ALGERIENNE DEMOCRATIQUE & POPULAIRE

MINISTERE DE L'ENSEIGNEMENT SUPERIEUR & DE LA RECHERCHE SCIENTIFIQUE

UNIVERSITE DJILLALI LIABES
FACULTE DES SCIENCES EXACTES
SIDI BEL ABBES

THESE DE DOCTORAT EN SCIENCES

Présentée par : **BENCHIRA Hayat**

Spécialité : MATHEMATIQUES
Option : EQUATIONS AUX DERIVEES PARTIELLES

Intitulée

Applications des méthodes variationnelles à quelques équations aux dérivées partielles du type elliptique

Soutenue le : 11/02/2021
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Dedicated

I perfect this height work of my parents, my haspand and my children for her help, hers motivation and her encouragements.

For my grand mother, me sister's, me brother's, son and thaughter.

File this work is a perfect for the affect and graduation when I love it.

Acknowledgement

I would like first, to send my warmest and sincere thanks to Madame **Atika Matallah**, I thank in it the researcher who gave me the opportunity to work in its direction, I thank in it the supervisor to the total commitment, the advice and indications always successful, finally I thank in it all the people who love me and colleague support and understanding without limits. all my gratitude goes to professor **Abbes Benaissa**, to have contributed to the enrichment of this thesis and to take part in my jury. I extend to professor **Ali Hakem**, the expression of my sincere thanks, for the honor it makes me chairing the jury of this thesis. I pray Madam Professor **Sara Litimien**, to find here the expression of all my gratitude, for the honor it makes me by accepting to be part of the jury that will examine this thesis. I extend to Madame **Safia Benmansour**, the expression of my sincere thanks and my whole gratitude, to have agreed to assess my work and be part of the jury. I thank warmly, Mr. **Mustapha Maghnafi**, to have agreed to participate in the jury will examine this manuscript.

In the end I would like to thank all people who helped me in the completion of this thesis.

Contents

Introduction

This thesis deals with the existence and multiplicity of solutions to the class of the following elliptic nonlocal problems

$$(\mathcal{P})_{\alpha,\beta,p,N}\begin{cases} -M(u)\left(\operatorname{div}\left(\frac{|\nabla u|^{p-2}}{|x|^{p\alpha}}\nabla u\right)+\mu\frac{|u|^{p-2}}{|x|^{p(\alpha+1)}}u\right)=\frac{|u|^{p^*-2}}{|x|^{p^*\beta}}u+\lambda f(x) & \text{in } \mathbb{R}^N \\ u\in W^{1,p}_{\alpha,\mu}\left(\mathbb{R}^N\right) \end{cases}$$

with $M(u)=\left(a\|u\|^p_{\alpha,\mu}+b\right)$, $N\geq 3$, $1<p<N$, $a>0$, $b\geq 0$, $0\leq\alpha<(N-p)/p$, $\alpha\leq\beta<\alpha+1$, $-\infty<\mu<\overline{\mu}:=[(N-(\alpha+1)p)/p]^p$, $f\not\equiv 0$, $\lambda\geq 0$ is a parameter, $p^*=pN/[N-p(1+\alpha-\beta)]$ is the critical Caffarelli-Kohn-Nirenberg exponent and

$$\|u\|^p_{\alpha,\mu}:=\int_{\mathbb{R}^N}\left(\frac{|\nabla u|^p}{|x|^{p\alpha}}-\mu\frac{|u|^p}{|x|^{p(\alpha+1)}}\right)dx$$

The problem $(\mathcal{P})_{\alpha,\beta,p,N}$ is related to the following well known Caffarelli-Kohn-Nirenberg inequality [16]

$$\left(\int_{\mathbb{R}^N}\frac{|u|^{p^*}}{|x|^{p^*\beta}}dx\right)^{1/p^*}\leq C_{\alpha,\beta,p,N}\|u\|^p_{\alpha,\mu}\text{ for all } u\in C_0^\infty\left(\mathbb{R}^N\right), \tag{1}$$

for some positive constant $C_{\alpha,\beta,p,N}$.

If $\beta=1$ in (1), then $p^*=p$, $C_{\alpha,\beta,p,N}=1/\overline{\mu}$ and we have the following weighted Hardy inequality

$$\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^{p(\alpha+1)}}dx\leq\frac{1}{\overline{\mu}}\int_{\mathbb{R}^N}\frac{|\nabla u|^p}{|x|^{p\alpha}}dx,\text{ for all } u\in C_0^\infty\left(\mathbb{R}^N\right).$$

We shall work with the space $W^{1,p}_{\alpha,\mu} := W^{1,p}_{\alpha,\mu}\left(\mathbb{R}^N\right)$ for $-\infty < \mu < \overline{\mu}$ endowed with the norm $\|.\|_{\alpha,\mu}$ which is equivalent to the norm $\|.\|_{\alpha,0}$, that is, we have the continuous embedding of $W^{1,p}_{\alpha,\mu}$ in $L^{p^*}\left(\mathbb{R}^N, |x|^{-p^*\beta}\right)$; where $L^{p^*}\left(\mathbb{R}^N, |x|^{-p^*\beta}\right)$ is the weighted $L^{p^*}\left(\mathbb{R}^N\right)$ space equipped with the norm

$$\|u\|_{L^{p^*}\left(\mathbb{R}^N,|x|^{-p^*\beta}\right)} = \left(\int_{\mathbb{R}^N} \frac{|u|^{p^*}}{|x|^{p^*\beta}}dx\right)^{1/p^*}.$$

We also use W^* to denote the dual space of $W^{1,p}_{\alpha,\mu}$.

This class of elliptic equations is called nonlocal because of the presence of the integral over the entire domain $\mathbb{R}^N$, which implies that the equation in $(\mathcal{P})_{\alpha,\beta,p,N}$ is no longer a pointwise identity. It is also called non-degenerate if $a > 0$ and $b \geq 0$, while it is named degenerate if $a = 0$ and $b > 0$.

In the regular case ($\alpha = \beta = \mu = 0$), $(\mathcal{P})_{0,0,p,N}$ is the elliptic version related to the stationary analog of the Kirchhoff equation introduced by Kirchhoff [31] as an extension of the classical D'Alembert wave equation for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the strings produced by transverse vibrations.

It is pointed out that regular nonlocal problems model several physical and also biological systems, so for their various motivations, it calls attention of many researchers.

The problem $(\mathcal{P})_{\alpha,\beta,p,N}$ has variational form. So, we should find solutions as critical points of the associated energy. In recent decades, a great attention was devoted to this approach, we precise that the presence of the critical Caffarelli-Kohn-Nirenberg exponent or the unboundedness of the domain are among the reasons to lack the compactness. Thus we cannot use the standard variational argument directly.

Here, we quote various results which we obtained.

The first chapter is devoted to the basic definitions and useful inequalities which we use frequently in this thesis.

In chapter two, we investigate the following elliptic problem with Hardy term and critical Sobolev exponent:

$$(\mathcal{P})_{0,0,2,3}\begin{cases} -\left(a\left\|u\right\|_{0,\mu}^{2}+b\right)\left[\Delta u+\mu\dfrac{u}{|x|^{2}}\right]=u^{5}+\lambda f\left(x\right) \quad \text{in } \mathbb{R}^{3} \\ u\in W_{0,\mu}^{1,2}. \end{cases}$$

Let V_{ε} solution of

$$\Delta V_{\varepsilon}+\mu\frac{V_{\varepsilon}}{|x|^{2}}=V_{\varepsilon}^{5} \quad \text{in } \mathbb{R}^{3}\backslash\left\{0\right\},$$

The main results of this chapiter are given in the following theorems.

Theorem 2.1 Let $a>0$, $b\geq 0$, $\mu<1/4$ and $f\in W^{*}\backslash\left\{0\right\}$. Then there exists a constant $\lambda_{1}>0$ such that problem $(\mathcal{P})_{0,0,2,3}$ has at least one solution for any $\lambda\in(0,\lambda_{1})$.

Theorem 2.2 Let $a>0$, $b\geq 0$, $\mu<1/4$ and $f\in W^{*}\backslash\left\{0\right\}$ such that $\int_{\mathbb{R}^{3}}f\left(x\right)V_{\varepsilon}dx\neq 0$. Then there exists a constant $\lambda_{2}>0$ with $\lambda_{2}\leq\lambda_{1}$ such that problem $(\mathcal{P})_{0,0,2,3}$ has at least two solutions for any $\lambda\in\left(0,\lambda_{2}\right)$.

To prove the existence of two distinct critical points of the associated energy functional, we first minimize the functional in a neighborhood of zero and use the Ekeland variational principle to find the first critical point which achieves a local minimum. Moreover, the level of this local minimum is negative. Next around the zero point, using the Mountain Pass Theorem we also obtain a critical point whose level is positive.

In chapter three, we generalize the results of the previous problem, more precisely, we study the existence of multiple solutions for the following elliptic Kirchhoff equation

$$(\mathcal{P})_{\alpha,\beta,2,N}\begin{cases} -\left(a\left\|u\right\|_{\alpha,\mu}^{2}+b\right)\left(\operatorname{div}\left(\frac{\nabla u}{|x|^{2\alpha}}\right)+\mu\frac{u}{|x|^{2(\alpha+1)}}\right)=\frac{|u|^{p^{*}-2}}{|x|^{p^{*}\beta}}u+\lambda f\left(x\right) & \text{in } \mathbb{R}^{N} \\ u\in W_{\alpha,\mu}^{1,2}\left(\mathbb{R}^{N}\right) \end{cases}$$

with the following assumptions.

(Hf) $f\in W^{*}\backslash\{0\}$ and $\int_{\mathbb{R}^{N}} f(x)\, V_{\varepsilon}dx\neq 0$.

$(H1)$ $3\leq N\leq 4$, $\beta-\alpha=1-\dfrac{N}{4}$, $0<a<(S_{\mu})^{-2}$, $b>0$.

$(H2)$ $N=3$, $\beta-\alpha<\dfrac{1}{4}$, $a>0$, $b=0$.

$(H3)$ $N=3$, $\beta-\alpha=0$, $a>0$, $b>0$.

The main result in this chapter is the following.

Theorem 3.1 Suppose that f satisfies (Hf) and assume that one of the hypotheses (Hi) holds for $i=\overline{1,3}$, then, there exists a constant $\lambda_{*}>0$ such that the problem $(\mathcal{P})_{\alpha,\beta,2,N}$ has at least two nontrivial solutions in $W_{\alpha,\mu}^{1,2}$ for any $\lambda\in\left]0,\lambda_{*}\right[$.

The last chapter is centered on a singular elliptic quasilinear problem, we study the existence, the nonexistence and multiplicity of solutions for the following Kirchhoff type problem:

$$(\mathcal{P})_{\alpha,\beta,p,N}\begin{cases} -\left(a\left\|u\right\|_{\alpha,\mu}^{p}+b\right)\left(\operatorname{div}\left(\frac{|\nabla u|^{p-2}}{|x|^{p\alpha}}\nabla u\right)+\mu\frac{|u|^{p-2}}{|x|^{p(\alpha+1)}}u\right)=\frac{|u|^{p^{*}-2}}{|x|^{p^{*}\beta}}u+\lambda f\left(x\right) & \text{in } \mathbb{R}^{N} \\ u\in W_{\alpha,\mu}^{1,p}\left(\mathbb{R}^{N}\right) \end{cases}$$

When $\lambda=0$, we show that the nonexistence of solutions for the above problem are related to N, α, β, a, and b. When $\lambda>0$, by using Ekeland's variational principle and the Mountain Pass Theorem, we show the existence of a first solution with negative energy and the existence of a second solution with positive energy.

Chapter 1

Preliminaries

In this chapter, we will introduce and state without proofs some important materials needed in the proof of our results (see [4] , [13] , [16] , [24]).

1.1 Critical point and critical value

Let X be a Banach space, X^* the dual of X and $I : X \to \mathbb{R}$.

Definition 1.1 *A function I is called Fréchet differentiable at $u \in X$ if there exists a bounded linear operator $A \in X^*$ such that*

$$\lim_{\|v\|_X \to 0} \frac{|I(u+v) - I(u) - \langle A, v \rangle|}{\|v\|_X} = 0.$$

If there exists a such operator A, it is unique, so we write $I'(u) = A$ and call it the Fréchet derivative of I at u. A function I that is Fréchet differentiable for any point of X is said to be C^1 if the function I' is continuous.

Definition 1.2 *We call that $u \in X$ is a critical point of I if $I'(u) = 0$, otherwise u is called a regular point.*

Let $c \in \mathbb{R}$, we say that c is a critical value of I if there exists a critical point u in X such that $I(u) = c$, otherwise c is called regular.

1.2 Palais-Smale sequence, Palais-Smale condition

The notion of critical point can be defined as a local minima, but in general it needs that a certain compactness property holds, for example $I(u) = \exp(-u)$, the value $c = 0$ is never attained. For this we will require the so called Palais-Smale condition to be satisfied by I. For this, we introduce the following definitions.

Definition 1.3 *We call a sequence $(u_n) \in X$ a Palais-Smale sequence on X if $I(u_n) \to c$ and $\|I'(u_n)\|_{X^*} \to 0$ as $n \to +\infty$.*

We can now with the help of the above definition define the Palais-Smale condition.

Definition 1.4 *Let $c \in \mathbb{R}$. We say that I satisfies the Palais-Smale condition at level c ($(PS)_c$ for short), if any Palais-Smale sequence contains a convergent subsequence in X.*

Let us observe that if $I \in C^1(X, \mathbb{R})$ satisfies the Palais-Smale condition, any point of accumulation $\overline{u}$ of a Palais-Smale sequence u_n, is a critical point of I. We have implicitly $I'(\overline{u}) = 0$ and $I(\overline{u}) = c$.

1.3 Mountain Pass Theorem and Ekeland's variational principle

A powerful tool for proving the existence of a critical point of a functional, is given by the following theorem.

Theorem 1.5 [24] *Let (V,d) be a complete metric space, and $I : V \to (-\infty,+\infty]$ a lower semicontinuous functional, not identically equal to $+\infty$ $(I \neq +\infty)$ which is bounded from below* $(\inf_V I > -\infty)$. *Then, for all $\varepsilon > 0$, there exists $\gamma_\varepsilon \in V$ such that*

$$\inf_{\gamma\in V} I(\gamma) \leq I(\gamma_\varepsilon) \leq \inf_{\gamma\in V} I(\gamma) + \varepsilon,$$

$$I(\gamma_\varepsilon) < I(\gamma) + \varepsilon d(\gamma,\gamma_\varepsilon), \ \forall \gamma \in V, \ \text{such that } \gamma \neq \gamma_\varepsilon.$$

Corollary 1.6 [24] *If V is a Banach space and $I \in C^1(V,\mathbb{R})$ is bounded from below, then there exists a minimizing sequence (u_n) for I in V such that*

$$I(u_n) \to \inf_V I \ \text{and } I'(u_n) \to 0 \quad \text{in } V^* \text{ as } n \to +\infty.$$

Remark 1.7 *The Theorem* 1.5 *and the Corollary* 1.6 *show the possibility of finding minimizing sequences under certain conditions on the functional.*

Theorem 1.8 [4] *Let $I \in C^1(X,\mathbb{R})$ satisfying (PS) condition. Assume that*

(1) $I(0) = 0$,

(2) there exists two numbers ρ and α such that $I(u) \geq \alpha$ for every $u \in X$ with $\|u\|_X = \rho$,

(3) there exists $v \in X$ such that $I(v) < \alpha$ and $\|v\|_X \geq \rho$.

Define

$$\Gamma := \{\gamma \in C([0,1],\ X),\ \gamma(0) = 0,\ \gamma(1) = v\},$$

then

$$c := \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} I(u) \geq \alpha$$

is a critical value.

Remark 1.9 *The Mountain Pass Theorem of Ambrosetti and Rabinowitz has been frequently applied in order to establish the existence of critical points for functionals.*

1.4 Useful inequalities and Sobolev embedding

Theorem 1.10 *(Sobolev-Gagliardo-Nirenberg)*

Let $1 \leq p < N$, Sobolev embedding gives

$$W^{1,p}\left(\mathbb{R}^N\right) \hookrightarrow L^{p^*}\left(\mathbb{R}^N\right)$$

where $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{N}$. Moreover there exists a constant $C = C(p,N)$ such that

$$\|u\|_{p^*} \leq C \|\nabla u\|_{L^p(\mathbb{R}^N)}, \quad \forall u \in W^{1,p}\left(\mathbb{R}^N\right).$$

Corollary 1.11 *Let $1 \leq p < N$, then*

$$W^{1,p}\left(\mathbb{R}^N\right) \hookrightarrow L^q\left(\mathbb{R}^N\right), \quad \forall q \in [p, p^*]$$

with continuous embedding.

Theorem 1.12 [13] *Let $m \geq 1$ and $1 \leq p < \infty$. We have*

$$\text{if } \frac{1}{p} - \frac{m}{N} > 0, \text{ then } W^{m,p}\left(\mathbb{R}^N\right) \hookrightarrow L^q\left(\mathbb{R}^N\right) \text{ where } \frac{1}{q} = \frac{1}{p} - \frac{m}{N},$$

Corollary 1.13 *If* $\frac{1}{p} - \frac{m}{N} = 0$, *then* $W^{m,p}\left(\mathbb{R}^N\right) \hookrightarrow L^q\left(\mathbb{R}^N\right), \ \forall q \in [p, +\infty[$,

if $\frac{1}{p} - \frac{m}{N} < 0$, *then* $W^{m,p}\left(\mathbb{R}^N\right) \hookrightarrow L^\infty\left(\mathbb{R}^N\right)$,

with continuous embedding.

1.4.1 Some integral inequalities

We will give here some important integral inequalities. These inequalities play an important role in applied mathematics and also, they are very useful in our next chapters.

Theorem 1.14 [13] *Let* q *and* q' *such that* $1 < q,\ q' < \infty$ *and* $\frac{1}{q} + \frac{1}{q'} = 1$. *If* $f \in L^q$ *and* $g \in L^{q'}$, *then*

$$fg \in L^1 \text{ and } \int |fg|\, dx \leq \left(\int |f|^q\, dx\right)^{\frac{1}{q}} \left(\int |g|^{q'}\, dx\right)^{\frac{1}{q'}}.$$

Lemma 1.15 [13] *Let* $0 \leq s \leq 1$. *Then*

$$\|u\|_{L^r} \leq \|u\|_{L^t}^s \|u\|_{L^q}^{1-s},$$

valid for $u \in L^q$ *with* $1 \leq t \leq r \leq q$, $\frac{1}{r} = \frac{s}{t} + \frac{1-s}{q}$

Theorem 1.16 [16] *(Caffarelli-Kohn-Nirenberg inequality)*

$$\left(\int_{\mathbb{R}^N} \frac{|u|^{p^*}}{|x|^{p^*\beta}} dx\right)^{1/p^*} \leq C_{\alpha,\beta} \left(\int_{\mathbb{R}^N} \frac{|\nabla u|^p}{|x|^{p\alpha}} dx\right)^{1/p} \text{ for all } u \in C_0^\infty\left(\mathbb{R}^N\right), \qquad (1)$$

for some positive constant $C_{\alpha,\beta}$.

If $\beta = \alpha + 1$ *in* (1), *then* $p^* = p$, $C_{\alpha,\beta} = \frac{1}{\mu} = \left[\frac{p}{N - (\alpha+1)p}\right]^p$ *and we have the following weighted Hardy inequality*

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{p(\alpha+1)}} dx \leq \frac{1}{\mu} \int_{\mathbb{R}^N} \frac{|\nabla u|^p}{|x|^{p\alpha}} dx, \text{ for all } u \in C_0^\infty\left(\mathbb{R}^N\right).$$

1.4.2 Some algebraic inequalities

Since our study is based on some known algebraic inequalities, we want to recall few of them here.

Lemma 1.17 [13] *For all* $a,\ b \in \mathbb{R}^+$, *we have*

$$ab \leq \tau a^2 + \frac{1}{4\tau} b^2$$

where τ *is any positive constant.*

Lemma 1.18 [13] *For all* $a,\ b \in \mathbb{R}^+$, *the following inequality holds*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

1.5 Modes of convergence

Definition 1.19 [13] *If* (x_n) *is a sequence in* X, *then* x_n *converges weakly to* x *if*

$$\langle f, x_n \rangle \to \langle f, x \rangle \text{ as } n \to +\infty \text{ for all } f \in X^*,$$

we write, $x_n \rightharpoonup x$.

Definition 1.20 [13] *Let (u_n) be a sequence in X.*

Then u_n converges strongly to u in X if and only if

$$\lim_{n \to +\infty} \|u_n - u\|_X = 0.$$

Chapter 2

Elliptic Kirchhoff problem with Sobolev exponent

2.1 Introduction

In this chapter, we are concerned with the existence and multiplicity of solutions to the following Kirchhoff problem with the critical Sobolev exponent

$$(\mathcal{P}_1)\begin{cases} -M\left(\|u\|_{\mu}^{2}\right)\left[\Delta u+\mu\dfrac{u}{|x|^{2}}\right]=u^{5}+\lambda f\left(x\right) \quad \text{in } \mathbb{R}^{3} \\ u\in H_{\mu}\left(\mathbb{R}^{3}\right) \end{cases}$$

where $M\left(t\right)=at+b$, a and b are two positive constants, λ is a positive parameter, $-\infty<\mu<1/4$,

$$\|u\|_{\mu}^{2}:=\int_{\mathbb{R}^{3}}\left(|\nabla u|^{2}-\mu\frac{|u|^{2}}{|x|^{2}}\right)dx$$

is the norm in $H_{\mu}\left(\mathbb{R}^{3}\right)$ and f belongs to $H_{\mu}^{-1}\left(\mathbb{R}^{3}\right)$, ($H_{\mu}^{-1}\left(\mathbb{R}^{3}\right)$ is the dual of $H_{\mu}\left(\mathbb{R}^{3}\right)$).

Such problems are frequently called nonlocal because the function M contains an integral over the domain $\mathbb{R}^3$ which implies that the equation in $(\mathcal{P}_1)$ is no longer a pointwise identity.

The original one-dimensional Kirchhoff equation was first introduced by Kirchhoff [31] in 1883, he take into account the changes in length of the strings produced by transverse vibrations.

The problem $(\mathcal{P}_1)$ is also related to the stationary analogue of the following evolutionary higher order problem which can been considered as an extension of the classical D'Alembert wave equation for free vibrations of elastic strings :

$$\begin{cases} u_{tt} - \left(a\displaystyle\int_{\Omega} |\nabla u|^2\, dx + b\right) \Delta u = h\left(x, u\right), & \text{in } \Omega \times (0, T)\,, \\ u = 0 & \text{on } \partial\Omega \times (0, T)\,, \\ u\left(x, 0\right) = u_0\left(x\right),\ u_t\left(x, 0\right) = u_1\left(x\right). \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain $(N \geq 1)$, T is a positive constant, u_0, u_1 are given functions. In such problems, u denotes the displacement, $h\left(x, u\right)$ the external force, b is the initial tension and a is related to the intrinsic properties of the strings (such as Young's modulus).

It is well known that the Kirchhoff type problem has mechanical and biological motivations, for example when an elastic string with fixed ends is subjected to transverse vibrations. They also serve as model in biological systems where u describes a process depending on the average of itself as population density. The presence of the nonlocal term makes the theorical study of these problems so difficult, then they have attracted the attention of many researchers in particular after the work of Lions [33], where a functional analysis approach was proposed to attack them.

In recent years, the existence and multiplicity of solutions for stationary problems of Kirchhoff type were also investigated in some papers, via variational methods like the Ekeland variational principle and the Mountain Pass Theorem. Some interesting results in bounded domains can be found in ([5] , [7] , [9] , [12] , [34]).

In the regular case and in the unbounded domain $\mathbb{R}^N$, some earlier classical investigations of the following Kirchhoff equations

$$(\mathcal{P}_{V,g}) \left\{ -M \left(\int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right) \Delta u + V(x) u = g(x,u), \text{ in } \mathbb{R}^3 \right.$$

have been done, where $N \geq 3$, $M(t) = at + b$, $a > 0$, b is a positive constants, $V \in C\left(\mathbb{R}^N, \mathbb{R}\right)$ and $g \in C\left(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}\right)$ is subcritical and satisfies sufficient conditions to ensure the boundedness of any Palais-Smale or Cerami sequence. Such problems become more complicated since the Sobolev embedding $\left(H^1\left(\mathbb{R}^N\right), \|.\|_\mu\right) \hookrightarrow \left(L^p\left(\mathbb{R}^N\right), |.|_p\right)$ is not compact for all $p \in [2, 2_*(N)]$, where $\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx\right)^{\frac{1}{2}}$ is the standard norm in $H^1\left(\mathbb{R}^N\right)$, $|u|_p = \left(\int_{\mathbb{R}^N} |u|^p \, dx\right)^{\frac{1}{p}}$ is the norm in $L^p\left(\mathbb{R}^N\right)$ and $2_*(N)$ is the critical Sobolev exponent.

To overcome the lack of compactness of the Sobolev embedding, many authors imposed some conditions on the potential function $V(x)$ for example in [41], Wu used the following assumption:

$(*)$ $\inf V(x) \geq c > 0$ and for all $d > 0$, $meas \left\{x \in \mathbb{R}^N : V(x) \leq d < 1\right\} < \infty$ to show the existence of nontrivial solutions to $(\mathcal{P}_{V,g})$. On the other hand, Chena and Li in [21] studied $(\mathcal{P}_{V,g})$ where $g(x,u) = h(x,u) + k(x)$, h satisfies the Ambrosetti-Rabinowitz type condition, $k \in L^2(\mathbb{R}^3)$ and V verifies $(*)$. They proved the existence of multiple solutions by using Ekeland's variational principle and the Mountain Pass Theorem. Recently, Li and al. [17] studied $(\mathcal{P}_{V,g})$ where $V \equiv 0$, they proved the existence of a constant $a_0 > 0$ such that $(\mathcal{P}_{0,g})$ admits a positive solution for all $a \in (0, a_0)$.

However, from the results mentioned above, there are very few existence results for singular nonlocal type problems (when $\mu > 0$) in particular for those who contain singularity in the diverge operator. This is a more difficult and interesting situation comparing with the regular case (when $\mu = 0$). Moreover, the main difficulties in such problem appear in the fact that for nonlocal problems with critical exponent, to overcome the lack of compactness, we need to determine a good level of the Palais-Smale and have to verify that the critical value is contained in the range of this level.

Let V_ε solution of

$$\Delta V_\varepsilon + \mu \frac{V_\varepsilon}{|x|^2} = V_\varepsilon^5 \quad \text{in } \mathbb{R}^3 \setminus \{0\} .$$

The main results of this chapter are given in the following theorems.

Theorem 2.1 *Let $a > 0$, $b \geq 0$, $\mu < 1/4$, and $f \in H_\mu^{-1}(\mathbb{R}^3) \setminus \{0\}$. Then there exists a constant $\lambda_1 > 0$ such that problem $(\mathcal{P}_1)$ has at least one solution for any $\lambda \in (0, \lambda_1)$.*

Theorem 2.2 *Let $a > 0$, $b \geq 0$, $\mu < 1/4$, and $f \in H_\mu^{-1}(\mathbb{R}^3) \setminus \{0\}$ such that $\int_{\mathbb{R}^3} f(x) V_\varepsilon dx \neq 0$. Then there exists a constant $\lambda_2 > 0$ with $\lambda_2 \leq \lambda_1$ such that problem $(\mathcal{P}_1)$ has at least two solutions for any $\lambda \in (0, \lambda_2)$.*

Here we give a brief sketch of the way how we get two distinct critical points of the energy functional. First, we minimize the functional in a neighborhood of zero and use the Ekeland variational principle to find the first critical point which achieves a local minimum. Moreover, the level of this local minimum is negative. Next around the zero point, using the Mountain Pass Theorem we also obtain a critical point whose level is positive.

2.2 Auxiliary results

To start this section, we need to introduce the following notations.

$\|.\|_*$ denotes the norm in $H_\mu^{-1}(\mathbb{R}^3)$, B_ρ is the ball centred at 0 and of radius ρ, and $\circ_n(1)$ denotes $\circ_n(1) \to 0$ as $n \to +\infty$.

Define the constant

$$S_\mu := \inf\left\{\int_{\mathbb{R}^3}\left(|\nabla u|^2 - \mu\frac{u^2}{|x|^2}\right)dx : \ u \in H_\mu\left(\mathbb{R}^3\right), \ \int_{\mathbb{R}^3} u^6 dx = 1\right\},$$

It is well known that the embedding $H_\mu(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ is continuous but not compact and S_μ is achieved by a family of functions

$$V_\varepsilon(x) := \frac{\left[12\varepsilon\left(\frac{1}{4}-\mu\right)\right]^{\frac{1}{4}}}{\left[\varepsilon|x|^{1-2\sqrt{\frac{1}{4}-\mu}} + |x|^{1+2\sqrt{\frac{1}{4}-\mu}}\right]^{\frac{1}{2}}}, \ \varepsilon > 0,$$

see [40]. Moreover, there holds

$$\Delta V_\varepsilon + \mu\frac{V_\varepsilon}{|x|^2} = V_\varepsilon^5 \quad \text{in } \mathbb{R}^3\backslash\{0\},$$

and

$$\|V_\varepsilon\|_\mu^2 = \int_{\mathbb{R}^3} V_\varepsilon^6 dx = (S_\mu)^{\frac{3}{2}}.$$

Since our approach is variational, we define the energy functional associated to the problem $(\mathcal{P}_1)$ by:

$$I_\lambda(u) = \frac{a}{4}\|u\|_\mu^4 + \frac{b}{2}\|u\|_\mu^2 - \frac{1}{6}\int_{\mathbb{R}^3} u^6 dx - \lambda\int_{\mathbb{R}^3} f(x)\, u\, dx, \quad \text{for all } u \in H_\mu\left(\mathbb{R}^3\right).$$

It is clear that I_λ is well defined in $H_\mu(\mathbb{R}^3)$ and belongs to $C^1(H_\mu(\mathbb{R}^3),\mathbb{R})$.

$u \in H_\mu(\mathbb{R}^3)$ is said to be a weak solution of problem $(\mathcal{P}_1)$ if it satisfies

$$\left(a\|u\|_\mu^2+b\right)\int_{\mathbb{R}^3}\left(\nabla u\nabla v-\mu\frac{uv}{|x|^2}\right)dx-\int_{\mathbb{R}^3}\left(u^5v-\lambda f(x)v\right)dx=0,\quad \forall v\in H_\mu(\mathbb{R}^3).$$

In order to prove our main results, we give the following lemmas.

Lemma 2.3 *Let $(u_n)\subset H_\mu(\mathbb{R}^3)$ be a Palais-Smale sequence of I_λ for some $c\in\mathbb{R}$. Then*

$$u_n \rightharpoonup u \text{ in } H_\mu\left(\mathbb{R}^3\right),$$

for some u with $I_\lambda'(u)=0$.

Proof. We have

$$c+o_n(1)=I_\lambda(u_n) \text{ and } o_n(1)=\langle I_\lambda'(u_n),u_n\rangle,$$

then

$$\begin{aligned} c+o_n(1) &= I_\lambda(u_n)-\frac{1}{6}\langle I_\lambda'(u_n),u_n\rangle \\ &= a\frac{1}{12}\|u\|_\mu^4+b\frac{1}{3}\|u\|_\mu^2-\lambda\frac{5}{6}\int_{\mathbb{R}^3}f(x)u_n dx \\ &\geq \frac{a}{12}\|u_n\|_\mu^4+\frac{b}{3}\|u_n\|_\mu^2-\lambda\frac{5}{6}\|f\|_*\|u_n\|_\mu. \end{aligned}$$

Hence (u_n) is bounded in $H_\mu(\mathbb{R}^3)$. Up to a subsequence if necessary, we obtain

$$u_n \rightharpoonup u \text{ in } H_\mu\left(\mathbb{R}^3\right),$$

$$u_n \to u \text{ a.e. in } \mathbb{R}^3,$$

$$u_n \rightharpoonup u \text{ in } L^6\left(\mathbb{R}^3\right).$$

and

$$\int_{\mathbb{R}^3} f(x)\, u_n dx \to \int_{\mathbb{R}^3} f(x)\, u dx$$

thus $\langle I'_\lambda (u_n), \varphi \rangle = 0$ for all $\varphi \in C_0^\infty (\mathbb{R}^3)$, which means that $I'_\lambda (u) = 0$.

This completes the proof of Lemma 2.3. ■

Lemma 2.4 *Let $f \in W^* \setminus \{0\}$ Then there exist positive numbers δ_1, ρ_1 and λ_1 such that for all $\lambda \in]0, \lambda_1[$ we have*

(i) $I_\lambda (u) \geq \delta_1 > 0$, *with* $\|u\|_\mu = \rho_1$,

(ii) $I_\lambda (u)|_{B_{\rho_1}} \geq -\frac{1}{2} \left(\lambda^{\frac{3}{4}} \|f\|_* \right)^2$.

Proof. Let $u \in H_\mu (\mathbb{R}^3) \setminus \{0\}$ and $\rho = \|u\|_\mu$. We have by the definition of S_μ

$$\begin{aligned} I_\lambda(u) &\geq \frac{a}{4}\rho^4 + \frac{b}{2}\rho^2 - \frac{(S_\mu)^{-3}}{6}\rho^6 - \lambda \|f\|_* \rho, \\ &\geq \frac{a}{4}\rho^4 + \frac{b}{2}\rho^2 - \frac{(S_\mu)^{-3}}{6}\rho^6 - \left(\lambda^{\frac{3}{4}} \|f\|_* \right) \left(\lambda^{\frac{1}{4}} \rho \right), \end{aligned}$$

by the elementary inequality

$$AB \leq \frac{A^2}{2} + \frac{B^2}{2}$$

we have that

$$\begin{aligned} I_\lambda(u) &\geq \frac{a}{4}\rho^4 - \frac{(S_\mu)^{-3}}{6}\rho^6 + \frac{b}{2}\rho^2 - \frac{1}{2} \left(\lambda^{\frac{3}{4}} \|f\|_* \right)^2 - \frac{1}{2} \left(\lambda^{\frac{1}{4}} \rho \right)^2, \\ &\geq \frac{a}{4}\rho^4 + \frac{b - \lambda^{\frac{1}{2}}}{2}\rho^2 - \frac{(S_\mu)^{-3}}{6}\rho^6 - \frac{1}{2} \left(\lambda^{\frac{3}{4}} \|f\|_* \right)^2. \\ &\geq \frac{a}{4}\rho^4 - \frac{(S_\mu)^{-3}}{6}\rho^6 - \frac{1}{2} \left(\lambda^{\frac{3}{4}} \|f\|_* \right)^2, \text{ for all } \lambda \leq b^2. \end{aligned}$$

Now, we consider the function

$$\Psi(\rho) = \frac{a}{4}\rho^4 - \frac{(S_\mu)^{-3}}{6}\rho^6.$$

It is easy to see that

$$\Psi(\rho) \geq 0 \text{ for all } \rho \leq \rho_1$$

$$\max_{\rho \geq 0} \Psi(\rho) = \Psi(\rho_1) = \frac{1}{12}(S_\mu)^6 a^3 \geq 0$$

with

$$\rho_1 = \left[a (S_\mu)^3\right]^{\dfrac{1}{2}}.$$

So, for $\|u\|_\mu = \rho_1$ and $\lambda \leq \left(\frac{1}{12\|f\|_*^2}\right)^{\frac{2}{3}} (S_\mu)^4 a^2$, we have

$$\begin{aligned}
I_\lambda(u) &\geq \Psi(\rho_1) - \frac{1}{2}\left(\lambda^{\frac{3}{4}}\|f\|_*\right)^2 \\
&\geq \frac{1}{2}\Psi(\rho_1) + \frac{1}{2}\Psi(\rho_1) - \frac{1}{2}\left(\lambda^{\frac{3}{4}}\|f\|_*\right)^2 \\
&\geq \frac{1}{2}\Psi(\rho_1) \\
&\geq \frac{1}{24}(S_\mu)^6 a^3.
\end{aligned}$$

Therefore,

$$I_\lambda(u) \geq -\frac{1}{2}\left(\lambda^{\frac{3}{4}}\|f\|_*\right)^2, \text{ for } \|u\|_\mu \leq \rho_1,$$

and then we can choose δ_1, ρ_1 and λ_1 such that

$$\begin{aligned}
\delta_1 &= \frac{1}{24}(S_\mu)^6 a^3 \\
\rho_1 &= \left[a^3 (S_\mu)\right]^{\frac{1}{2}} \\
\lambda_1 &= \min\left\{\left(\frac{1}{12\|f\|_*^2}\right)^{\frac{2}{3}} (S_\mu)^4 a^2, b^2\right\},
\end{aligned}$$

the conclusion holds. ■

Lemma 2.5 *Let $(u_n) \subset H_\mu(\mathbb{R}^3)$ be a $(PS)_c$ sequence of I_λ for some $c \in \mathbb{R}$ such that $u_n \rightharpoonup u$ in $H_\mu(\mathbb{R}^3)$, then*

$$\text{either } u_n \to u \quad \text{or } c \geq I_\lambda(u) + C_*,$$

where

$$\begin{aligned} C_* &= \frac{a}{12}\left(a\left(S_\mu\right)^3+\left(a^2\left(S_\mu\right)^6+4b\left(S_\mu\right)^3\right)^{\frac{1}{2}}\right)^2 \\ &\quad +\frac{b}{6}\left(a\left(S_\mu\right)^3+\left(a^2\left(S_\mu\right)^6+4b\left(S_\mu\right)^3\right)^{\frac{1}{2}}\right). \end{aligned}$$

Proof. By the proof of Lemma 2.3 we have (u_n) is a bounded sequence in $H_\mu\left(\mathbb{R}^3\right)$.

Furthermore, if we write $v_n = u_n - u$, we derive that $v_n \rightharpoonup 0$ in $H\left(\mathbb{R}^3\right)$. Then by Brezis-Lieb Lemma [14] we have

$$\begin{cases} \|u_n\|_\mu^2 = \|v_n\|_\mu^2 + \|u\|_\mu^2 + o_n(1), \\ \displaystyle\int_{\mathbb{R}^3} u_n^6 dx = \int_{\mathbb{R}^3} v_n^6 dx + \int_{\mathbb{R}^3} u^6 dx + o_n(1). \end{cases} \tag{2.1}$$

By (2.1), we obtain

$$o_n(1) = a\|v_n\|_\mu^4 + b\|v_n\|_\mu^2 + 2a\|v_n\|_\mu^2\|u\|_\mu^2 - \int_{\mathbb{R}^3} v_n^6 dx \tag{2.2}$$

and

$$c + o_n(1) = I_\lambda(u) + a\frac{1}{4}\|v_n\|_\mu^4 + \frac{b}{2}\|v_n\|_\mu^2 + \frac{a}{2}\|v_n\|_\mu^2\|u\|_\mu^2 - \frac{1}{6}\int_{\mathbb{R}^3} v_n^6 dx.$$

Therefore,

$$c + o_n(1) = I_\lambda(u) + a\frac{1}{12}\|v_n\|_\mu^4 + \frac{b}{3}\|v_n\|_\mu^2 + \frac{a}{6}\|v_n\|_\mu^2\|u\|_\mu^2. \tag{2.3}$$

Using the hypothesis that (v_n) is bounded in $H_\mu\left(\mathbb{R}^3\right)$, there exists $l > 0$ such that $\|v_n\|_\mu \to l > 0$, then by (2.2) and the Sobolev inequality we obtain

$$l^2 \geq S\left[\left(bl^2 + al^4\right)\right]^{1/3}.$$

Therefore,

$$S^{-3}l^6 - bl^2 - al^4 \geq 0,$$

this implies that

$$l^2 \geq \frac{aS^3 + \sqrt{a^2 S^6 + 4bS^3}}{2}.$$

From the above inequality and (2.3), we conclude

$$\begin{aligned} c &\geq I_\lambda(u) + \frac{a}{12} l^4 + \frac{b}{3} l^2 \\ &\geq I_\lambda(u) + \frac{a}{48} \left(a (S_\mu)^3 + \left(a^2 (S_\mu)^6 + 4b (S_\mu)^3 \right)^{\frac{1}{2}} \right)^2 \\ &+ \frac{b}{6} \left(a (S_\mu)^3 + \left(a^2 (S_\mu)^6 + 4b (S_\mu)^3 \right)^{\frac{1}{2}} \right) \\ &= I_\lambda(u) + C_*. \end{aligned}$$

This finishes the proof of Lemma 2.5. ■

2.3 Proof of the main results

2.3.1 Existence of the first solution

First, by Lemma 2.4 we can define

$$c_1 = \inf \left\{ I_\lambda(u),\ u \in \bar{B}_{\rho_1}(0) \right\}.$$

For $t > 0$ we have

$$I_\lambda(t\Phi) = \frac{a}{4} t^4 \|\Phi\|_\mu^4 + \frac{b}{2} t^2 \|\Phi\|_\mu^2 - \frac{t^6}{6} \int_{\mathbb{R}^3} \Phi^6 \, dx - \lambda t \int_{\mathbb{R}^3} f(x) \Phi \, dx.$$

Since $f \not\equiv 0$, we can choose $\Phi \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ such that $\int_{\mathbb{R}^3} f(x) \Phi \, dx > 0$. Hence, for a fixed $\lambda \in]0,\ \lambda_1[$, there exists $t_0 > 0$ small enough such that $\|t_0 \Phi\|_\mu < \rho_1$ and

$$I_\lambda(t\Phi) < 0 \quad \text{for } t \in]0, t_0[.$$

Hence, $c_1 < I_\lambda(0) = 0$. Using the Ekeland's variational principle, for the complete metric space $\bar{B}_{\rho_1}(0)$ with respect to the norm of $H_\mu(\mathbb{R}^3)$, we obtain the result that there exists a Palais-Smale sequence $u_n \in \bar{B}_{\rho_1}(0)$ at level c_1, and from Lemma 2.3 we have $u_n \rightharpoonup u_1$ in $H_\mu(\mathbb{R}^3)$ for some u_1 with $\|u_1\|_\mu \leq \rho_1$.

Now, we shall show that $u_n \to u_1$ in $H_\mu(\mathbb{R}^3)$. Assume $u_n \nrightarrow u_1$ in $H_\mu(\mathbb{R}^3)$, then it follows from Lemma 2.5 that

$$\begin{aligned} c_1 &\geq I_\lambda(u) + C_* \\ &\geq c_1 + C_* \\ &> c_1, \end{aligned}$$

which is a contradiction. Thus u_1 is a critical point of I_λ i.e. u_1 is a solution of $(\mathcal{P}_1)$ with negative energy.

2.3.2 Existence of the Second Solution

The existence of the second solution follows immediately from the following lemma.

Lemma 2.6 *Let $\lambda_2 > 0$ such that*

$$C_* - \frac{1}{2}\left(\lambda^{\frac{3}{4}}\|f\|_*\right)^2 > 0 \quad \textit{for all } \lambda \in (0, \lambda_2).$$

Then there exists $z_\varepsilon \in H_\mu(\mathbb{R}^3)$ and $\lambda^* \in \left]0, \lambda_2\right]$ such that

$$\sup_{t \geq 0} I_\lambda(tz_\varepsilon) < C_* + c_1 \quad \textit{for all } \lambda \in (0, \lambda^*).$$

Proof. Let $z_\varepsilon(x) = \pm V_\varepsilon(x)$ such that

$$\int_{\mathbb{R}^3} f(x)\, z_\varepsilon(x)\, dx > 0.$$

We have

$$I_\lambda(tz_\varepsilon) = \frac{at^4}{4}\left\|z_\varepsilon\right\|_\mu^4 + \frac{bt^2}{2}\left\|z_\varepsilon\right\|_\mu^2 - \frac{t^6}{6}\int_{\mathbb{R}^3} z_\varepsilon^6 dx - \lambda t\int_{\mathbb{R}^3} f\left(x\right) z_\varepsilon dx.$$

We put

$$h\left(t\right) = \frac{at^4}{4}\left\|z_\varepsilon\right\|_\mu^4 + \frac{bt^2}{2}\left\|z_\varepsilon\right\|_\mu^2 - \frac{t^6}{6}\int_{\mathbb{R}^3} z_\varepsilon^6 dx$$

From the definition of S_μ, we have

$$||z_\varepsilon||_\mu^2 = \int_{\mathbb{R}^3} z_\varepsilon^6 dx = \left(S_\mu\right)^{\frac{3}{2}}.$$

Then

$$h\left(t\right) = \frac{at^4}{4}\left(S_\mu\right)^3 + \frac{bt^2}{2}\left(S_\mu\right)^{\frac{3}{2}} - \frac{t^6}{6}\left(S_\mu\right)^{\frac{3}{2}},$$

and

$$h'\left(t\right) = \left(S_\mu\right)^{\frac{3}{2}} t\left(-t^4 + a\left(S_\mu\right)^{\frac{3}{2}} t^2 + b\right).$$

Thus, the function $h\left(t\right)$ attains its maximum at

$$t_0^2 = \frac{a\left(S_\mu\right)^{\frac{3}{2}} + \left(a^2\left(S_\mu\right)^3 + 4b\right)^{1/2}}{2}.$$

By the above estimates on $h\left(t\right)$, we deduce that

$$\max_{t\geq 0} h\left(t\right) = C_*.$$

Then we have

$$\sup_{t\geq 0} I_\lambda(tz_\varepsilon) \leq C_* - \lambda t\int_{\mathbb{R}^3} f\left(x\right) z_\varepsilon dx.$$

There exists $t_1 \in (0, t_0)$ small enough such that

$$-c_1 - \lambda t_1\int_{\mathbb{R}^3} f\left(x\right) z_\varepsilon dx < 0,$$

this implies that

$$c_1 > -\lambda t_1 \int_{\mathbb{R}^3} f(x)\, z_\varepsilon dx.$$

On the other hand, using Lemma 2.4 we see that

$$c_1 \geq -\frac{1}{2}\left(\lambda^{\frac{3}{4}} \|f\|_*\right)^2 \quad \text{for all } \lambda \in (0,\ \lambda_1).$$

We choose λ_2 such that for any $\lambda \in (0,\ \lambda_2)$ we have

$$C_* + c_1 \geq C_* - \frac{1}{2}\left(\lambda^{\frac{3}{4}} \|f\|_*\right)^2 > 0,$$

then

$$C_* > \frac{1}{2}\left(\lambda^{\frac{3}{4}} \|f\|_*\right)^2,$$

this implies that

$$\lambda \leq \lambda_2 = \|f\|_*^{-\frac{4}{3}} (2C_*)^{\frac{2}{3}},$$

and we choose λ_3 such that for any $\lambda \in (0,\ \lambda_3)$ we have

$$-\lambda t_1 \int_{\mathbb{R}^3} f(x)\, z_\varepsilon dx \leq -\frac{1}{2}\left(\lambda^{\frac{3}{4}} \|f\|_*\right)^2,$$

this implies that

$$\lambda \leq \lambda_3 = 4\|f\|_*^{-4} \left(t_1 \int_{\mathbb{R}^3} f(x)\, z_\varepsilon dx\right)^2.$$

Taking

$$\lambda_* = \min(\lambda_1,\ \lambda_2,\ \lambda_3),$$

then we deduce that

$$\sup_{t \geq 0} I_\lambda(t z_\varepsilon) < c_1 + C_* \text{ for all } \lambda \in (0, \lambda_*).$$

This conclude s the proof of Lemma 2.6. ■

Note that $I_\lambda(0) = 0$ and the fact that

$$\lim_{t\to\infty} I_\lambda\left(tz_\varepsilon\right) = -\infty \ ,$$

then $I_\lambda(Tz_\varepsilon) < 0$ for T large enough, also from Lemma 2.4, we know that

$$I_\lambda\left(u\right)|_{\partial B_{\rho_1}} \geq \delta_1 > 0 \text{ for all } \lambda \in (0, \lambda_1).$$

Then, by the Mountain Pass Theorem, there exists a Palais-Smale sequence (u_n) at level c_2, such that

$$I_\lambda\left(u_n\right) \to c_2 \text{ and } I'_\lambda\left(u_n\right) \to 0 \text{ as } n \to +\infty$$

with

$$c_2 = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma\left(t\right)),$$

where

$$\Gamma = \left\{\gamma \in C\left(\left[0,1\right], H_\mu\left(\mathbb{R}^3\right)\right), \gamma\left(0\right) = 0, \gamma\left(1\right) = Tz_\varepsilon\right\}.$$

Using Lemma 2.3 we have (u_n) has a subsequence, still denoted by (u_n), such that $u_n \rightharpoonup u_2$ in $H_\mu\left(\mathbb{R}^3\right)$, for some $u_2 \in H_\mu\left(\mathbb{R}^3\right)$. Furthermore, we know by Lemma 2.6 that

$$\sup_{t\geq 0} I_\lambda(tz_\varepsilon) < C_* + c_1, \text{ for all } \lambda \in (0, \lambda_*),$$

then from Lemma 2.5 we deduce that $u_n \to u_2$ in $H_\mu\left(\mathbb{R}^3\right)$. Thus we obtain a critical point u_2 of I_λ satisfying $I_\lambda\left(u_2\right) > 0$, which achieves the proof of Theorem 2.2.

Chapter 3

Elliptic Kirchhoff problem with Caffarelli-Kohn-Nirenberg exponent

3.1 Introduction and main result

In this chapter, we study the existence of multiple solutions for the following elliptic Kirchhoff equation

$$
(\mathcal{P}_2)\begin{cases} -\left(a\left\|u\right\|_{\alpha,\mu}^{2}+b\right)\left(\operatorname{div}\left(\frac{\nabla u}{|x|^{2\alpha}}\right)+\mu\frac{u}{|x|^{2(\alpha+1)}}\right)=\frac{|u|^{p^*-2}}{|x|^{p^*\beta}}u+\lambda f\left(x\right) & \text{in } \mathbb{R}^N \\ u\in W_{\alpha,\mu}^{1,2}\left(\mathbb{R}^N\right) \end{cases}
$$

with $N\geq 3$, $a>0$, $b\geq 0$, $0\leq\alpha<(N-2)/2$, $\alpha\leq\beta<\alpha+1, -\infty<\mu<\overline{\mu}:=\left[\dfrac{(N-2(\alpha+1))}{2}\right]^2$, $f\not\equiv 0$, $\lambda>0$ is a parameter, $p^*=\dfrac{2N}{N-2(1+\alpha-\beta)}$ is the critical Caffarelli-Kohn-Nirenberg exponent and

$$\|u\|_{\alpha,\mu}^2 := \int_{\mathbb{R}^N} \left(\frac{|\nabla u|^2}{|x|^{2\alpha}} - \mu \frac{u^2}{|x|^{2(\alpha+1)}} \right) dx.$$

The problem $(\mathcal{P}_2)$ is related to the following well known Caffarelli-Kohn-Nirenberg inequality

$$\left(\int_{\mathbb{R}^N} \frac{|u|^{p^*}}{|x|^{p^*\beta}} dx \right)^{1/p^*} \leq C_{\alpha,\beta} \left(\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2\alpha}} dx \right)^{1/2} \quad \text{for all } u \in C_0^\infty \left(\mathbb{R}^N\right), \tag{3.1}$$

for some positive constant $C_{\alpha,\beta}$.

If $\beta = \alpha + 1$ in (3.1), then $p^* = 2$, $C_{\alpha,\beta} = 1/\overline{\mu}$ and we have the following weighted Hardy inequality

$$\int_{\mathbb{R}^N} \frac{u^2}{|x|^{2(\alpha+1)}} dx \leq \frac{1}{\overline{\mu}} \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2\alpha}} dx, \text{ for all } u \in C_0^\infty \left(\mathbb{R}^N\right).$$

We shall work with the space $W_{\alpha,\mu}^{1,2} := W_{\alpha,\mu}^{1,2}\left(\mathbb{R}^N\right)$ for $-\infty < \mu < \overline{\mu}$ endowed with the norm $\|.\|_{\alpha,\mu}$ which is equivalent to the norm $\|.\|_{\alpha,0}$, that is, we have the continuous embedding of $W_{\alpha,\mu}^{1,2}$ in $L^{p^*}\left(\mathbb{R}^N, |x|^{-p^*\beta}\right)$; where $L^{p^*}\left(\mathbb{R}^N, |x|^{-p^*\beta}\right)$ is the weighted $L^{p^*}\left(\mathbb{R}^N\right)$ space. We use W^* to denote the dual space of $W_{\alpha,\mu}^{1,2}$.

In the non-degenerate case ($a > 0$) with non singular terms ($\alpha = \beta = \mu = 0$), the problem $(\mathcal{P}_2)$ is related to the stationary problem of a model introduced by Kirchhoff [31].

For the degenerate case, much interest has grown on problems involving critical exponents, starting from the celebrated paper by Brézis and Nirenberg [15]. After that many authors were dedicated to investigate all kinds of elliptic equations with critical Sobolev, Hardy-Sobolev or Caffarelli-Kohn-Nirenberg exponents in bounded or unbounded domain, (see $[1], [3], [7], [8], [11], [17]$). For $\lambda = 0$ and $b = 1$, Kang in [29] proved that the problem

$$-\mathrm{div}\left(\frac{\nabla u}{|x|^{p\alpha}}\right) - \mu\frac{u}{|x|^{2(\alpha+1)}} = \frac{|u|^{p^*-2}}{|x|^{p^*\beta}}u \text{ in } \mathbb{R}^N,$$

has radial ground state solution U_ε in $W^{1,2}_{\alpha,\mu}$, and the best constant

$$S_\mu := \inf_{W^{1,2}_{\alpha,\mu}\setminus\{0\}} \frac{\|u\|^2_{\alpha,\mu}}{\left(\int_{\mathbb{R}^N} \frac{|u|^{p^*}}{|x|^{p^*\beta}}dx\right)^{2/p^*}},$$

is achieved by a family of functions

$$V_\varepsilon(x) := \varepsilon^{-\left(\frac{N-2}{2}-\alpha\right)}U_\varepsilon\left(\frac{x}{\varepsilon}\right), \, \varepsilon > 0.$$

Moreover, it holds

$$\|V_\varepsilon\|^2_{\alpha,\mu} = \int_{\mathbb{R}^N} \frac{|V_\varepsilon|^{p^*}}{|x|^{p^*\beta}}dx = (S_\mu)^{\frac{p^*}{p^*-2}}.$$

Recently, the solvability or multiplicity of the Kirchhoff type equation with critical exponent has attracted the attention of many authors, via variational methods like the Ekeland variational principle and the Mountain Pass Theorem. See for example $([4], [13], [16], [20], [31])$.

To state our result, we make the following assumptions.

(Hf) $f \in W^*\setminus\{0\}$ and $\int_{\mathbb{R}^N} f(x)\, V_\varepsilon dx \neq 0$.

$(H1)$ $3 \leq N \leq 4$, $\beta - \alpha = 1 - \frac{N}{4}$, $0 < a < (S_\mu)^{-2}$, $b > 0$.

$(H2)$ $N = 3$, $\beta - \alpha < \frac{1}{4}$, $a > 0$, $b = 0$.

$(H3)$ $N = 3$, $\beta - \alpha = 0$, $a > 0$, $b > 0$.

The main result in this chapter is the following theorem.

Theorem 3.1 *Suppose that f satisfies (Hf) and assume that one of the hypotheses (Hi) holds for $i = \overline{1,3}$, then, there exists a constant $\lambda_* > 0$ such that the problem $(\mathcal{P}_2)$ has at least two solutions in $W^{1,2}_{\alpha,\mu}$ for any $\lambda \in \left]0, \lambda_*\right[$.*

3.2 Auxiliary results

The energy functional $I_\lambda : W^{1,2}_{\alpha,\mu} \to \mathbb{R}$, corresponding to the problem $(\mathcal{P}_2)$ is given by

$$I_\lambda(u) = \frac{a}{4}\|u\|^4_{\alpha,\mu} + \frac{b}{2}\|u\|^2_{\alpha,\mu} - \frac{1}{p^*}\int_{\mathbb{R}^N}\frac{|u|^{p^*}}{|x|^{p^*\beta}}dx - \lambda\int_{\mathbb{R}^N} f(x)\,u dx, \quad \forall u \in W^{1,2}_{\alpha,\mu}.$$

Notice that I_λ is well defined in $W^{1,2}_{\alpha,\mu}$ and belongs to $C^1\left(W^{1,2}_{\alpha,\mu},\ \mathbb{R}\right)$. We say that $u \in W^{1,2}_{\alpha,\mu}\backslash\{0\}$ is a weak solution of $(\mathcal{P}_2)$, if for any $v \in W^{1,2}_{\alpha,\mu}$, there holds

$$\left(a\|u\|^2_{\alpha,\mu} + b\right)\int_{\mathbb{R}^N}\left(\frac{\nabla u \nabla v}{|x|^{2\alpha}} - \mu\frac{uv}{|x|^{2(\alpha+1)}}\right)dx - \int_{\mathbb{R}^N}\left(\frac{|u|^{p^*-2}uv}{|x|^{p^*\beta}} - \lambda f(x)\,v\right)dx = 0.$$

To prove our main results, we need the following lemmas.

Lemma 3.2 *Let $f \in W^*\backslash\{0\}$ and suppose that one of the hypotheses (Hi) holds for $i = \overline{1,3}$. Then there exists positive numbers δ_1, ρ_1 and λ_1 such that for all $\lambda \in \left]0,\ \lambda_1\right[$ we have*

(i) $I_\lambda(u) \geq \delta_1 > 0$, *with* $\|u\|_{\alpha,\mu} = \rho_1$,

(ii) for all $u \in B_{\rho_1}(0)$ we have

$$I_\lambda(u) \geq \begin{cases} -\dfrac{1}{2}\left(\left(\dfrac{b}{2}\right)^{\frac{-1}{2}}\lambda\|f\|_*\right)^2 & \text{if } (H1) \text{ is satisfied,} \\ -\dfrac{3}{4}\left(\left(\dfrac{a}{2}\right)^{\frac{-1}{4}}\lambda\|f\|_*\right)^{\frac{4}{3}} & \text{if } (H2) \text{ or } (H3) \text{ is satisfied.} \end{cases}$$

Proof. Let $u \in W^{1,2}_{\alpha,\mu}\backslash\{0\}$ and $\rho = \|u\|_{\alpha,\mu}$. Under the hypotheses $(H2)$ or $(H3)$, we have by the definition of S_μ

$$\begin{aligned} I_\lambda(u) &\geq \frac{a}{4}\rho^4 + \frac{b}{2}\rho^2 - \frac{(S_\mu)^{-p^*/2}}{p^*}\rho^{p^*} - \lambda \|f\|_* \rho, \\ &\geq \frac{a}{4}\rho^4 - \frac{(S_\mu)^{-p^*/2}}{p^*}\rho^{p^*} - \left(\left(\frac{a}{2}\right)^{\frac{-1}{4}} \lambda \|f\|_*\right)\left(\left(\frac{a}{2}\right)^{\frac{1}{4}} \rho\right). \end{aligned}$$

By the elementary inequality

$$AB \leq \frac{A^s}{s} + \frac{B^t}{t} \quad \text{for all } A > 0,\ B > 0,\ s > 0 \text{ and } t > 0 \text{ such that } \frac{1}{s} + \frac{1}{t} = 1$$

we have

$$\begin{aligned} I_\lambda(u) &\geq \frac{a}{4}\rho^4 - \frac{(S_\mu)^{-p^*/2}}{p^*}\rho^{p^*} - \frac{3}{4}\left(\left(\frac{a}{2}\right)^{\frac{-1}{4}} \lambda \|f\|_*\right)^{\frac{4}{3}} - \frac{1}{4}\left(\left(\frac{a}{2}\right)^{\frac{1}{4}} \rho\right)^4, \\ &\geq \frac{a}{8}\rho^4 - \frac{(S_\mu)^{-p^*/2}}{p^*}\rho^{p^*} - \frac{3}{4}\left(\left(\frac{a}{2}\right)^{\frac{-1}{4}} \lambda \|f\|_*\right)^{\frac{4}{3}}. \end{aligned}$$

Now, we consider the function

$$h(\rho) = \frac{a}{8}\rho^4 - \frac{(S_\mu)^{-p^*/2}}{p^*}\rho^{p^*}$$

It is easy to see that

$$\max_{\rho \geq 0} h(\rho) = h(\rho_0) = \frac{p^* - 4}{4p^*}(S_\mu)^{\frac{2p^*}{p^*-4}}\left(\frac{a}{2}\right)^{\frac{p^*}{p^*-4}} \geq 0$$

with

$$\rho_0 = \left[\frac{a}{2}(S_\mu)^{p^*/2}\right]^{\frac{1}{p^* - 4}}.$$

So, for $\|u\|_{\alpha,\mu} = \rho_0$ and

$$\lambda \leq \left(\frac{p^* - 4}{4p^*}(S_\mu)^{\frac{2p^*}{p^*-4}}\right)^{\frac{3}{4p}}\left(\frac{a}{2}\right)^{\frac{p^*-1}{p^*-4}}\|f\|_*^{-1},$$

we have

$$
\begin{aligned}
I_\lambda(u) &\geq h\left(\rho_0\right)-\frac{3}{4}\left(\left(\frac{a}{2}\right)^{\frac{-1}{4}} \lambda\|f\|_*\right)^{\frac{4}{3}}, \\
&\geq \frac{3}{4} h\left(\rho_0\right)+\frac{1}{4} h\left(\rho_0\right)-\frac{3}{4}\left(\left(\frac{a}{2}\right)^{\frac{-1}{4}} \lambda\|f\|_*\right)^{\frac{4}{3}}, \\
&\geq \frac{p^*-4}{16 p^*}\left(S_\mu\right)^{\frac{2 p^*}{p^*-4}}\left(\frac{a}{2}\right)^{\frac{p^*}{p^*-4}} .
\end{aligned}
$$

Therefore,

$$
I_\lambda(u) \geq-\frac{3}{4}\left(\left(\frac{a}{2}\right)^{\frac{-1}{4}} \lambda\|f\|_*\right)^{\frac{4}{3}}, \text { for }\|u\|_{\alpha, \mu} \leq \rho_0 .
$$

Now, under the condition $(H1)$ and by the definition of S_μ we have

$$
\begin{aligned}
I_\lambda(u) &\geq \frac{b}{2} \rho^2+\frac{a}{4} \rho^4-\frac{\left(S_\mu\right)^{-p^* / 2}}{p^*} \rho^{p^*}-\lambda\|f\|_* \rho, \\
&\geq \frac{b}{2} \rho^2-\frac{\left(S_\mu\right)^{-p^* / 2}}{p^*} \rho^{p^*}-\left(\left(\frac{b}{2}\right)^{\frac{-1}{2}} \lambda\|f\|_*\right)\left(\frac{b}{2}\right)^{\frac{1}{2}} \rho \\
&\geq \frac{b}{2} \rho^2-\frac{\left(S_\mu\right)^{-p^* / 2}}{p^*} \rho^{p^*}-\frac{1}{2}\left(\left(\frac{b}{2}\right)^{\frac{-1}{2}} \lambda\|f\|_*\right)^2-\frac{1}{2}\left(\left(\frac{b}{2}\right)^{\frac{1}{2}} \rho\right)^2 \\
&\geq \frac{b}{4} \rho^2-\frac{\left(S_\mu\right)^{-p^* / 2}}{p^*} \rho^{p^*}-\frac{1}{2}\left(\left(\frac{b}{2}\right)^{\frac{-1}{2}} \lambda\|f\|_*\right)^2 .
\end{aligned}
$$

Consider the function

$$
\tilde{h}(\rho)=\frac{b}{4} \rho^2-\frac{\left(S_\mu\right)^{-p^* / 2}}{p^*} \rho^{p^*}
$$

and note that

$$
\max_{\rho \geq 0} \tilde{h}(\rho)=\tilde{h}\left(\tilde{\rho}_0\right)=\frac{p^*-2}{2 p^*}\left[\frac{b}{2} S_\mu\right]^{\dfrac{p^*}{p^*-2}} \geq 0
$$

with

$$
\tilde{\rho}_0=\left[\frac{b}{2}\left(S_\mu\right)^{p^* / 2}\right]^{\dfrac{1}{p^*-2}} .
$$

So, for $\|u\|_{\alpha,\mu} = \tilde{\rho}_0$ and

$$\lambda \leq \left(\frac{p^*-2}{2p^*} (S_\mu)^{\dfrac{p^*}{p^*-2}} \right)^{\frac{1}{2}} \left(\frac{b}{2}\right)^{\dfrac{p^*-1}{p^*-2}} \|f\|_*^{-1}$$

we have

$$\begin{aligned}
I_\lambda(u) &\geq \frac{p^*-2}{2p^*} \left[\frac{b}{2} S_\mu\right]^{\dfrac{p^*}{p^*-2}} - \frac{1}{2} \left(\left(\frac{b}{2}\right)^{\frac{-1}{2}} \lambda \|f\|_* \right)^2 \\
&\geq \frac{p^*-2}{4p^*} \left[\frac{b}{2} S_\mu\right]^{\frac{p^*}{p^*-2}} + \left[\frac{p^*-2}{4p^*} \left[\frac{b}{2} S_\mu\right]^{\dfrac{p^*}{p^*-2}} \right. \\
&\left. -\frac{1}{2} \left(\left(\frac{b}{2}\right)^{\frac{-1}{2}} \lambda \|f\|_* \right)^2 \right] \\
&\geq \frac{p^*-2}{4p^*} \left(\frac{b}{2} S_\mu\right)^{\frac{p^*}{p^*-2}}.
\end{aligned}$$

Therefore,

$$I_\lambda(u) \geq -\frac{1}{2} \left(\left(\frac{b}{2}\right)^{\frac{-1}{2}} \lambda \|f\|_* \right)^2, \text{ for } \|u\|_{\alpha,\mu} \leq \tilde{\rho}_0.$$

Then we can choose δ_1, $\rho_1$and λ_1 such that

$$\delta_1 = \begin{cases} \dfrac{p^*-2}{4p^*} \left(\dfrac{b}{2} S_\mu\right)^{\frac{p^*}{p^*-2}} & \text{if } (H1) \text{ is satisfied,} \\ \dfrac{p^*-4}{16p^*} (S_\mu)^{\frac{2p^*}{p^*-4}} \left(\dfrac{a}{2}\right)^{\frac{p^*}{p^*-4}} & \text{if } (H2) \text{ or } (H3) \text{ is satisfied,} \end{cases}$$

$$\rho_1 = \begin{cases} \left[\dfrac{b}{2} (S_\mu)^{p^*/2}\right]^{\frac{1}{p^*-2}} & \text{if } (H1) \text{ is satisfied,} \\ \left[\dfrac{a}{2} (S_\mu)^{p^*/2}\right]^{\frac{1}{p^*-4}} & \text{if } (H2) \text{ or } (H3) \text{ is satisfied,} \end{cases}$$

$$\lambda_1 = \begin{cases} \left(\dfrac{p^*-2}{2p^*} (S_\mu)^{\frac{p^*}{p^*-2}}\right)^{\frac{1}{2}} \left(\dfrac{b}{2}\right)^{\frac{p^*-1}{p^*-2}} \|f\|_*^{-1} & \text{if } (H1) \text{ is satisfied,} \\ \left(\dfrac{p^*-4}{4p^*} (S_\mu)^{\frac{2p^*}{p^*-4}}\right)^{\frac{3}{4}} \left(\dfrac{a}{2}\right)^{\frac{p^*-1}{p^*-4}} \|f\|_*^{-1} & \text{if } (H2) \text{ or } (H3) \text{ is satisfied.} \end{cases}$$

■

Lemma 3.3 *Let $f \in W^{*} \backslash \{0\}$ and $(u_n) \subset W^{1,2}_{\alpha,\mu}$ be a Palais-Smale sequence for I_λ at level c, then*

$$u_n \rightharpoonup u \text{ in } W^{1,2}_{\alpha,\mu}$$

for some $u \in W^{1,2}_{\alpha,\mu}$ with $I'_\lambda(u) = 0$.

Proof. Let $(u_n) \subset W^{1,2}_{\alpha,\mu}$ be a Palais-Smale sequence for I_λ such that

$$I_\lambda(u_n) \to c \text{ and } I'_\lambda(u_n) \to 0.$$

We have

$$c + o_n(1) = I_\lambda(u_n) \text{ and } o_n(1) = \langle I'_\lambda(u_n), u_n \rangle,$$

that is

$$\begin{aligned} c + o_n(1) &= I_\lambda(u_n) - \frac{1}{p^*} \langle I'_\lambda(u_n), u_n \rangle \\ &= a\frac{p^*-4}{4p^*} \|u\|^4_{\alpha,\mu} + b\frac{p^*-2}{2p^*} \|u\|^2_{\alpha,\mu} - \lambda\frac{p^*-1}{p^*} \int_{\mathbb{R}^N} f(x) u_n dx, \\ &\geq a\frac{p^*-4}{4p^*} \|u\|^4_{\alpha,\mu} + b\frac{p^*-2}{2p^*} \|u\|^2_{\alpha,\mu} - \lambda\frac{p^*-1}{p^*} \|f\|_* \|u\|_{\alpha,\mu}. \end{aligned}$$

Then (u_n) is bounded in $W^{1,2}_{\alpha,\mu}$. Up to a subsequence if necessary, we obtain

$$u_n \rightharpoonup u \text{ in } W^{1,2}_{\alpha,\mu} \text{ and in } L^{p^*}\left(\mathbb{R}^N, |x|^{-p^*\beta}\right), \; u_n \to u \text{ a. e. in } \mathbb{R}^N$$

and

$$\int_{\mathbb{R}^N} f(x) u_n dx \to \int_{\mathbb{R}^N} f(x) u dx.$$

Then

$$\langle I'_\lambda(u_n), \Phi \rangle = 0 \text{ for all } \Phi \in C_0^\infty\left(\mathbb{R}^N\right),$$

thus $I'_\lambda(u) = 0$. This completes the proof of Lemma 3.3. ■

Now, we prove an important lemma which ensures the local compactness of the Palais-Smale sequence for I_λ.

Let

$$C_* = \begin{cases} \dfrac{b^2}{4\left[(S_\mu)^{-2} - a\right]} & \text{if } (H1) \text{ is satisfied,} \\ \dfrac{4(1+\alpha-\beta)-N}{4N}\left(a\,(S_\mu)^2\right)^{\frac{N}{4(1+\alpha-\beta)-N}} & \text{if } (H2) \text{ is satisfied,} \\ \dfrac{a}{48} z_1^2 + \dfrac{b}{6} z_1 & \text{if } (H3) \text{ is satisfied,} \end{cases}$$

with

$$z_1 = a\,(S_\mu)^3 + \sqrt{a^2\,(S_\mu)^6 + 4b\,(S_\mu)^3}$$

Lemma 3.4 *Let $f \in W^*\backslash\{0\}$ and $(u_n) \subset W^{1,2}_{\alpha,\mu}$ be a Palais-Smale sequence for I_λ for some $c \in \mathbb{R}^+$ such that $u_n \rightharpoonup u$ in $W^{1,2}_{\alpha,\mu}$. If one of the hypothesis (Hi), $i = \overline{1,3}$ occurs, then*

$$\textit{either } u_n \to u \quad \textit{or } c \geq I_\lambda(u) + C_*.$$

Proof. By the proof of Lemma 3.3, the sequence (u_n) is bounded in $W^{1,2}_{\alpha,\mu}$ and as $f \in W^*\backslash\{0\}$ we have

$$\int_{\mathbb{R}^N} f(x)\,u_n dx \to \int_{\mathbb{R}^N} f(x)\,u dx. \tag{3.2}$$

Furthermore, if we write $v_n = u_n - u$, we derive that $v_n \rightharpoonup 0$ in $W^{1,2}_{\alpha,\mu}$. Then by using the Brezis-Lieb result [14] we have

$$\begin{cases} \|u_n\|^2_{\alpha,\mu} = \|v_n\|^2_{\alpha,\mu} + \|u\|^2_{\alpha,\mu} + o_n(1), \\ \displaystyle\int_{\mathbb{R}^N} \frac{|u_n|^{p^*}}{|x|^{p^*\beta}} dx = \int_{\mathbb{R}^N} \frac{|v_n|^{p^*}}{|x|^{p^*\beta}} dx + \int_{\mathbb{R}^N} \frac{|u|^{p^*}}{|x|^{p^*\beta}} dx + o_n(1). \end{cases} \tag{3.3}$$

Combining (3.2) and (3.3), we get

$$c + o_n(1) = I_\lambda(u) + a\frac{p^*-4}{4p^*}\|v_n\|^4_{\alpha,\mu} + b\frac{p^*-2}{2p^*}\|v_n\|^2_{\alpha,\mu} + a\frac{p^*-4}{2p^*}\|v_n\|^2_{\alpha,\mu}\|u\|^2_{\alpha,\mu} \tag{3.4}$$

and

$$o_n(1) = \langle I'_\lambda(u), u\rangle + a\|v_n\|^4_{\alpha,\mu} + b\|v_n\|^2_{\alpha,\mu} + 2a\|v_n\|^2_{\alpha,\mu}\|u\|^2_{\alpha,\mu} - \int_{\mathbb{R}^N}\frac{|v_n|^{p^*}}{|x|^{p^*\beta}}dx. \quad (3.5)$$

Assume that $\|v_n\|_{\alpha,\mu} \to l > 0$, then by (3.5) and the Caffarelli-Kohn-Nirenberg inequality we obtain

$$l^2 \geq S_\mu\left(bl^2 + al^4\right)^{2/p^*},$$

this implies that

$$(S_\mu)^{-\frac{p^*}{2}} l^{p^*} - al^4 - bl^2 \geq 0.$$

Therefore,

$$(S_\mu)^{-\frac{p^*}{2}} l^{p^*-2} - al^2 - b \geq 0.$$

Now, we consider the functions $g : \mathbb{R}^+ \to \mathbb{R}$, given by

$$g(y) = (S_\mu)^{-\frac{p^*}{2}} y^{p^*-2} - ay^2 - b.$$

If $(H1)$ is satisfied, we get

$$g(y) = \left((S_\mu)^{-2} - a\right)y^2 - b,$$

that is,

$$g(y) \geq 0 \text{ if } y^2 \geq \frac{b}{(S_\mu)^{-2} - a}.$$

From the above inequality and (3.4), we conclude that

$$c \geq I_\lambda(u) + \frac{1}{4}\left(\frac{b^2}{(S_\mu)^{-2} - a}\right)$$

$$= I_\lambda(u) + C_*.$$

If $(H2)$ is satisfied, we get

$$g(y) = y^2 \left((S_\mu)^{-\frac{3}{1-2(\alpha-\beta)}} y^{\frac{2(1+4(\alpha-\beta))}{1-2(\alpha-\beta)}} - a \right),$$

that is,

$$g(y) \geq 0 \text{ if } y^2 \geq \left(a (S_\mu)^{\frac{3}{1-2(\alpha-\beta)}} \right)^{\frac{1-2(\alpha-\beta)}{1+4(\alpha-\beta)}},$$

and then

$$\begin{aligned} c &\geq I_\lambda(u) + \frac{4(1+\alpha-\beta)-3}{12} \left(a (S_\mu)^2 \right)^{\frac{3}{4(1+\alpha-\beta)-3}} \\ &= I_\lambda(u) + C_*. \end{aligned}$$

If $(H3)$ is satisfied, we get

$$g(y) = (S_\mu)^{-3} y^4 - a y^2 - b,$$

that is,

$$g(y) \geq 0 \text{ if } y^2 \geq \frac{a (S_\mu)^3 + \sqrt{a^2 (S_\mu)^6 + 4b (S_\mu)^3}}{2}.$$

Thus

$$\begin{aligned} c &\geq I_\lambda(u) + \frac{a}{48} z_1^2 + \frac{b}{6} z_1 \\ &= I_\lambda(u) + C_*. \end{aligned}$$

This finishes the proof of Lemma 3.4. ■

3.3 Proof of the main result

3.3.1 Existence of the first solution

Proposition 3.5 *Let $f \in W^* \setminus \{0\}$ and assume that one of the hypotheses (Hi) holds for $i = \overline{1,3}$. For all $\lambda \in]0, \ \lambda_1[$, there exists a solution u_1 of $(\mathcal{P}_2)$ with negative energy.*

Proof. By Lemma 3.2, we can define

$$c_1 = \inf \left\{ I_\lambda (u) \,, \; u \in \overline{B}_{\rho_1} (0) \right\}.$$

For $t > 0$ we have

$$I_\lambda(t\varphi) = \frac{a}{4} t^4 \left\| \varphi \right\|_{\alpha,\mu}^4 + \frac{b}{2} t^2 \left\| \varphi \right\|_{\alpha,\mu}^2 - \frac{1}{p^*} t^{p^*} \int_{\mathbb{R}^N} \frac{|\varphi|^{p^*}}{|x|^{p^* \beta}} dx - \lambda t \int_{\mathbb{R}^N} f(x) \varphi dx.$$

By (Hf) we can choose $\varphi \in W_{\alpha,\mu}^{1,2}$ such that $\displaystyle\int_{\mathbb{R}^N} f(x) \varphi dx > 0$. Then, for a fixed $\lambda \in \left]0, \ \lambda_1\right[$, there exists $t_0 > 0$ such that $||t_0 \varphi||_{\alpha,\mu} < \rho_1$ and

$$I_\lambda(t\varphi) < 0 \quad \text{for } t \in \left]0, t_0\right[.$$

Hence, $c_1 < I_\lambda(0) = 0$. Using the Ekeland's variational principle, for the complete metric space $\overline{B}_{\rho_1} (0)$ with respect to the norm of $W_{\alpha,\mu}^{1,2}$, we obtain the result that there exists a Palais-Smale sequence $(u_n) \in \overline{B}_{\rho_1} (0)$ at level c_1, and from Lemma 3.3 we have $u_n \rightharpoonup u_1$ in $W_{\alpha,\mu}^{1,2}$ for some u_1 with $\left\| u_1 \right\|_{\alpha,\mu} < \rho_1$.

Now, we shall show that $u_n \rightarrow u_1$ in $W_{\alpha,\mu}^{1,2}$. Assume that $u_n \nrightarrow u_1$ in $W_{\alpha,\mu}^{1,2}$, then, it follows from Lemma 3.4 that

$$\begin{aligned} c_1 &\geq I_\lambda (u_1) + C_* \\ &\geq c_1 + C_* \\ &> c_1, \end{aligned}$$

which is a contradiction. Thus u_1 is a critical point of I_λ i.e. u_1 is a solution of $(\mathcal{P}_2)$. As $I_\lambda (0) = 0$ and $I_\lambda (u_1) < 0$ then, $u_1 \neq 0$. Thus u_1 is a solution of $(\mathcal{P}_2)$ with negative energy. ■

3.3.2 Existence of a Mountain Pass type solution

Now, we proof the existence of the second solution.

Proposition 3.6 *Suppose that f satisfies (Hf) and assume that one of the hypotheses (Hi) holds for $i = \overline{1,3}$. Then there exists $\lambda_* > 0$ such that the problem $(\mathcal{P}_2)$ has a solution u_2 with positive energy.*

We need the following lemma.

Lemma 3.7 *Suppose that f satisfies (Hf) and assume that one of the hypotheses (Hi) holds for $i = \overline{1,3}$. Then there exists $\lambda_* > 0$ such that*

$$\sup_{t\geq 0} I_\lambda(tV_\varepsilon) < C_* + c_1, \quad \text{for all } \lambda \in \left]0, \lambda_*\right[.$$

Proof. We have

$$I_\lambda(tV_\varepsilon) = \frac{a}{4}t^4 \left\|V_\varepsilon\right\|_{\alpha,\mu}^4 + \frac{b}{2}t^2 \left\|V_\varepsilon\right\|_{\alpha,\mu}^2 - \frac{1}{p^*}t^{p^*}\int_{\mathbb{R}^N} \frac{|V_\varepsilon|^{p^*}}{|x|^{p^*\beta}}dx - \lambda t\int_{\mathbb{R}^N} f\left(x\right) V_\varepsilon dx.$$

put

$$h\left(t\right) = \frac{a}{4}t^4 \left\|V_\varepsilon\right\|_{\alpha,\mu}^4 + \frac{b}{2}t^2 \left\|V_\varepsilon\right\|_{\alpha,\mu}^2 - \frac{1}{p^*}t^{p^*}\int_{\mathbb{R}^N} \frac{|V_\varepsilon|^{p^*}}{|x|^{p^*\beta}}dx$$

From the definition of S_μ, we have

$$\left\|V_\varepsilon\right\|_{\alpha,\mu}^2 = \int_{\mathbb{R}^N} \frac{|V_\varepsilon|^{p^*}}{|x|^{p^*\beta}}dx = \left(S_\mu\right)^{\frac{N}{2(1+\alpha-\beta)}}.$$

Then

$$h\left(t\right) = \frac{a}{4}t^4 \left(S_\mu\right)^{\frac{N}{1+\alpha-\beta}} + \frac{b}{2}t^2 \left(S_\mu\right)^{\frac{N}{2(1+\alpha-\beta)}} - \frac{1}{p^*}t^{p^*} \left(S_\mu\right)^{\frac{N}{2(1+\alpha-\beta)}},$$

and

$$h'\left(t\right) = t\left(S_\mu\right)^{\frac{N}{2(1+\alpha-\beta)}}\left(-t^{p^*-2} + at^2 \left(S_\mu\right)^{\frac{N}{2(1+\alpha-\beta)}} + b\right).$$

Thus, the function $h(t)$ attains its maximum at

$$t_0 = \begin{cases} \left[\dfrac{b}{1-a\left(S_\mu\right)^2}\right]^{\frac{1}{2}} & \text{if } (H1) \text{ is satisfied,} \\ \left[a\left(S_\mu\right)^{\frac{3}{2(1+\alpha-\beta)}}\right]^{\frac{1-2(\alpha-\beta)}{8(\alpha-\beta)+5}} & \text{if } (H2) \text{ is satisfied,} \\ \dfrac{1}{2}\left(a\left(S_\mu\right)^{\frac{3}{2}} + \sqrt{a^2\left(S_\mu\right)^3 + 4b}\right) & \text{if } (H3) \text{ is satisfied.} \end{cases}$$

The above estimate on $h(t)$ yields that

$$\max_{t\geq 0} h(t) = C_*.$$

Then we have

$$\sup_{t\geq 0} I_\lambda(tV_\varepsilon) \leq C_* - \lambda t \int_{\mathbb{R}^N} f(x)\, V_\varepsilon dx.$$

Using Lemma 3.2 we see that

$$c_1 \geq \begin{cases} -\dfrac{1}{2}\left(\left(\dfrac{b}{2}\right)^{\frac{-1}{2}} \lambda \left\|f\right\|_*\right)^2 & \text{if } (H1) \text{ is satisfied,} \\ -\dfrac{3}{4}\left(\left(\dfrac{a}{2}\right)^{\frac{-1}{4}} \lambda \left\|f\right\|_*\right)^{\frac{4}{3}} & \text{if } (H2) \text{ or } (H3) \text{ is satisfied,} \end{cases}$$

for all $\lambda \in \left]0, \lambda_1\right[$.

Let $\lambda_2 > 0$ such that

$$C_* > \begin{cases} \dfrac{p-1}{p}\left(\left(\dfrac{b}{2}\right)^{\frac{-1}{p}} \lambda \left\|f\right\|_*\right)^{\frac{p}{p-1}} & \text{if } (H1) \text{ is satisfied,} \\ \dfrac{2p-1}{2p}\left(\left(\dfrac{a}{2}\right)^{\frac{-1}{2p}} \lambda \left\|f\right\|_*\right)^{\frac{2p}{2p-1}} & \text{if } (H2) \text{ or } (H3) \text{ is satisfied,} \end{cases}$$

for any $\lambda \in \left]0, \lambda_2\right[$, we choose λ_3 such that for any $\lambda \in \left]0, \lambda_3\right[$ we have

$$-\lambda t_1 \int_{\mathbb{R}^N} f(x)\, V_\varepsilon dx \leq \begin{cases} -\dfrac{1}{2}\left(\left(\dfrac{b}{2}\right)^{\frac{-1}{2}} \lambda \left\|f\right\|_*\right)^2 & \text{if } (H1) \text{ is satisfied} \\ -\dfrac{3}{4}\left(\left(\dfrac{a}{2}\right)^{\frac{-1}{4}} \lambda \left\|f\right\|_*\right)^{\frac{4}{3}} & \text{if } (H2) \text{ or } (H3) \text{ is satisfied,} \end{cases}$$

this implies that

$$\lambda \leq \lambda_3 = \begin{cases} t_1 b \|f\|_*^{-2} \displaystyle\int_{\mathbb{R}^N} f(x) V_\varepsilon dx & \text{if } (H1) \text{ is satisfied,} \\ \dfrac{a}{2} \|f\|_*^{-4} \left(\frac{4}{3} t_1 \displaystyle\int_{\mathbb{R}^N} f(x) V_\varepsilon dx \right)^3 & \text{if } (H2) \text{ or } (H3) \text{ is satisfied.} \end{cases}$$

Taking

$$\lambda_* = \min(\lambda_1, \ \lambda_2, \ \lambda_3).$$

Thus for any $\lambda \in]0, \ \lambda_*[$ we obtain

$$\sup_{t \geq 0} I_\lambda(tV_\varepsilon) < C_* + c_1.$$

This conclude the proof of Lemma 3.7. ■

Proof of Proposition 3.6. We know by Lemma 3.2 and the fact that

$$\lim_{t \to \infty} I_\lambda(tV_\varepsilon) = -\infty,$$

that I_λ satisfies the geometrical conditions of the Mountain Pass Theorem. Then, there exists a Palais-Smale sequence (u_n) at level c_2, such that

$$I_\lambda(u_n) \to c_2 \text{ and } I'_\lambda(u_n) \to 0 \text{ as } n \to +\infty$$

with

$$c_2 = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)),$$

where for T large enough

$$\Gamma = \left\{ \gamma \in C\left([0,1], W^{1,2}_{\alpha,\mu}\right), \gamma(0) = 0, \gamma(1) = TV_\varepsilon \right\}.$$

Using Lemma 3.4 we have (u_n) has a subsequence, still denoted by (u_n), such that $u_n \to u_2$ in $W^{1,2}_{\alpha,\mu}$, for some $u_2 \in W^{1,2}_{\alpha,\mu}$, as $c_2 > 0$ then $u_2 \neq 0$. Thus the existence of the solution with energy positive follows immediately from the preceding lemma which achieves the proof of Proposition 3.6. ■

Proof of Theorem 3.1. It follows immediately from the combination of Proposition 3.5 and Proposition 3.6. ■

Chapter 4

Quasilinear elliptic Kirchhoff problem with Caffarelli-Kohn-Nirenberg exponent

4.1 Introduction and main results

In this chapter, we explore the existence and the multiplicity of nontrivial solutions for the following quasilinear nonlocal elliptic equation

$$(\mathcal{P}_3)\begin{cases} -\left(a\left\|u\right\|_{\alpha,\mu}^{p}+b\right)\left(\operatorname{div}\left(\frac{|\nabla u|^{p-2}}{|x|^{p\alpha}}\nabla u\right)+\mu\frac{|u|^{p-2}}{|x|^{p(\alpha+1)}}u\right)=\frac{|u|^{p_*-2}}{|x|^{p_*\beta}}u+\lambda f\left(x\right) & \text{in } \mathbb{R}^N \\ u\in W_{\alpha,\mu}^{1,p} & \end{cases}$$

with $N\geq 3$, $1<p<N$, $a>0$, $b\geq 0$, $0\leq\alpha<\left(N-p\right)/p$, $\alpha\leq\beta<\alpha+1, -\infty<\mu<\overline{\mu}:=\left[\left(N-\left(\alpha+1\right)p\right)/p\right]^{p}$, $f\not\equiv 0$, $\lambda>0$ is a parameter and $p_*=$

$pN/\left[N-p\left(1+\alpha-\beta\right)\right]$ is the critical Caffarelli-Kohn-Nirenberg exponent and

$$\|u\|_{\alpha,\mu}^{p}:=\int_{\mathbb{R}^N}\left(\frac{|\nabla u|^p}{|x|^{p\alpha}}-\mu\frac{|u|^p}{|x|^{p(\alpha+1)}}\right)dx.$$

Also, we use W^* to denote the dual space of $W^{1,p}_{\alpha,\mu}$.

For the degenerate case, much interest has grown on problems involving critical exponents, many authors investigated all kinds of elliptic equations with critical Sobolev or Caffarelli-Kohn-Nirenberg exponents in bounded or unbounded domain. For $a=\lambda=0$ and $b=1$, Kang in [29] proved that the problem

$$-\mathrm{div}\left(\frac{|\nabla u|^{p-2}}{|x|^{p\alpha}}\nabla u\right)-\mu\frac{|u|^{p-2}}{|x|^{p(\alpha+1)}}u=\frac{|u|^{p_*-2}}{|x|^{p_*\beta}}u \text{ in } \mathbb{R}^N,$$

has radial ground state solution U_ε in $W^{1,p}_{\alpha,\mu}$, and the best constant

$$S_\mu:=\inf_{W^{1,p}_{\alpha,\mu}\setminus\{0\}}\frac{\|u\|_{\alpha,\mu}^{p}}{\left(\int_{\mathbb{R}^N}\frac{|u|^{p_*}}{|x|^{p_*\beta}}dx\right)^{p/p_*}},$$

is achieved by a family of functions

$$V_\varepsilon(x):=\varepsilon^{-\left(\frac{N-p}{p}-\alpha\right)}U_\varepsilon\left(\frac{x}{\varepsilon}\right),\ \varepsilon>0.$$

Moreover, it holds

$$\|V_\varepsilon\|_{\alpha,\mu}^{p}=\int_{\mathbb{R}^N}\frac{|V_\varepsilon|^{p_*}}{|x|^{p_*\beta}}dx=\left(S_\mu\right)^{\frac{p_*}{p_*-p}}.$$

It is worth to point out that Kang in [29] established an existence result however in this work, we also prove a nonexistence results under some conditions on certain parameters.

The mathematical interest in equation $(\mathcal{P}_3)$ lies in the fact that it involves non-local singular and quasilinear operator and also critical singular nonlinearities so, it becomes difficult to apply variational methods directly moreover we never have compact embedding in the unbounded domain $\mathbb{R}^N$. To deal with all these difficulties, we attempt to use Mountain Pass Theorem as well as the Ekeland's variational principle to explore the existence of two distinct solutions.

To state our results, we make the following assumptions.

$(H0)$ $f \in W^{*} \backslash \{0\}$ and $\int_{\mathbb{R}^N} f(x) V_{\varepsilon} dx \neq 0$.

$(H1)$ $3 \leq N \leq 2p$, $\beta - \alpha = 1 - \dfrac{N}{2p}$, $b = 0$, $a > (S_\mu)^{-2}$.

$(H2)$ $3 \leq N \leq 2p$, $\beta - \alpha = 1 - \dfrac{N}{2p}$, $a \geq (S_\mu)^{-2}$, $b > 0$.

$(H3)$ $N \geq 3$, $\beta - \alpha \in [0, 1[\cap \left]1 - \dfrac{N}{2p}, 1\right[$, $a > \dfrac{p^{*} - p}{p} (S_\mu)^{-\frac{p^{*}}{p^{*}-p}}$, $b > \dfrac{2p - p^{*}}{p}$.

$(H4)$ $3 \leq N \leq 2p$, $\beta - \alpha = 1 - \dfrac{N}{2p}$, $0 < a < (S_\mu)^{-2}$, $b > 0$.

$(H5)$ $3 \leq N < 2p$, $\beta - \alpha < 1 - \dfrac{N}{2p}$, $a > 0$, $b = 0$.

$(H6)$ $3 \leq N \leq \dfrac{3}{2}p$, $\beta - \alpha = 1 - \dfrac{2N}{3p}$, $a > 0$, $b > 0$.

$(H7)$ $3 \leq N \leq \dfrac{4}{3}p$, $\beta - \alpha = 1 - \dfrac{3N}{4p}$, $a > 0$, $b = 2\left(\dfrac{a}{3}\right)^{\frac{3}{2}} (S_\mu)^2$.

$(H8)$ $3 \leq N \leq \dfrac{4}{3}p$, $\beta - \alpha = 1 - \dfrac{3N}{4p}$, $a > 0$, $b > 2\left(\dfrac{a}{3}\right)^{\frac{3}{2}} (S_\mu)^2$.

$(H9)$ $3 \leq N \leq \dfrac{4}{3}p$, $\beta - \alpha = 1 - \dfrac{3N}{4p}$, $a > 0$, $0 < b < 2\left(\dfrac{a}{3}\right)^{\frac{3}{2}} (S_\mu)^2$.

$(H10)$ $3 \leq N \leq \dfrac{5}{4}p$, $\beta - \alpha = 1 - \dfrac{4N}{5p}$, $a > 0$, $b > 0$.

The main results in this chapter are the following theorems.

Theorem 4.1 *Assume that one of the hypotheses* (Hi) *holds for* $1 \leq i \leq 3$. *Then problem* $(\mathcal{P}_3)$ *has no solutions for* $\lambda = 0$.

Theorem 4.2 *Suppose that f satisfies $(H0)$ and assume that one of the hypotheses (Hi) holds for $4 \leq i \leq 10$, then, there exists a constant $\lambda_* > 0$ such that the problem $(\mathcal{P}_3)$ has at least two solutions in $W^{1,p}_{\alpha,\mu}$ for any $\lambda \in]0, \lambda_*[$.*

4.2 Preliminaries

The energy functional $I_\lambda : W^{1,p}_{\alpha,\mu} \to \mathbb{R}$, corresponding to the problem $(\mathcal{P}_3)$ is given by

$$I_\lambda(u) = \frac{a}{2p} \|u\|^{2p}_{\alpha,\mu} + \frac{b}{p} \|u\|^{p}_{\alpha,\mu} - \frac{1}{p^*} \int_{\mathbb{R}^N} \frac{|u|^{p^*}}{|x|^{p^*\beta}} dx - \lambda \int_{\mathbb{R}^N} f(x)\, u dx, \quad \forall u \in W^{1,p}_{\alpha,\mu}.$$

Notice that I_λ is well defined in $W^{1,p}_{\alpha,\mu}$ and belongs to $C^1\left(W^{1,p}_{\alpha,\mu},\ \mathbb{R}\right)$. We say that $u \in W^{1,p}_{\alpha,\mu} \backslash \{0\}$ is a weak solution of $(\mathcal{P}_3)$, if for any $v \in W^{1,p}_{\alpha,\mu}$ there holds

$$\left(a \|u\|^{p}_{\alpha,\mu} + b\right) \int_{\mathbb{R}^N} \left(\frac{|\nabla u|^{p-2} \nabla u \nabla v}{|x|^{p\alpha}} - \mu \frac{|u|^{p-2} uv}{|x|^{p(\alpha+1)}} \right) dx - \int_{\mathbb{R}^N} \left(\frac{|u|^{p^*-2} uv}{|x|^{p^*\beta}} - \lambda f(x)\, v \right) dx = 0.$$

To prove our main results, we need the following lemmas.

Lemma 4.3 *Let $f \in W^* \backslash \{0\}$ and suppose that one of the hypotheses (Hi) holds for $4 \leq i \leq 10$. Then there exists positive numbers δ_1, ρ_1 and λ_1 such that for all $\lambda \in]0,\ \lambda_1[$ we have*

(i) $I_\lambda(u) \geq \delta_1 > 0$, *with* $\|u\|_{\alpha,\mu} = \rho_1$,

(ii) For all $u \in B_{\rho_1}(0)$ *we have*

$$I_\lambda(u) \geq \begin{cases} -\dfrac{p-1}{p} \left(\left(\dfrac{b}{2}\right)^{\frac{-1}{p}} \lambda \|f\|_* \right)^{\frac{p}{p-1}} & \text{if } (H4) \text{ is satisfied,} \\ -\dfrac{2p-1}{2p} \left(\left(\dfrac{a}{2}\right)^{\frac{-1}{2p}} \lambda \|f\|_* \right)^{\frac{2p}{2p-1}} & \text{if one of } (Hi) \text{ is satisfied} \\ & \text{with } 5 \leq i \leq 10. \end{cases}$$

Proof. Let $u \in W^{1,p}_{\alpha,\mu} \backslash \{0\}$ and $\rho = \|u\|_{\alpha,\mu}$. Under one of hypotheses (Hi) with $5 \leq i \leq 10$, we have by the definition of S_μ

$$\begin{aligned} I_\lambda(u) &\geq \frac{a}{2p}\rho^{2p} + \frac{b}{p}\rho^p - \frac{(S_\mu)^{-p^*/p}}{p^*}\rho^{p^*} - \lambda \|f\|_* \rho, \\ &\geq \frac{a}{2p}\rho^{2p} - \frac{(S_\mu)^{-p^*/p}}{p^*}\rho^{p^*} - \left(\left(\frac{a}{2}\right)^{\frac{-1}{2p}} \lambda \|f\|_*\right)\left(\left(\frac{a}{2}\right)^{\frac{1}{2p}} \rho\right). \end{aligned}$$

By the elementary inequality

$$AB \leq \frac{A^s}{s} + \frac{B^t}{t} \text{ for all } A > 0,\ B > 0,\ s > 0 \text{ and } t > 0 \text{ such that } \frac{1}{s} + \frac{1}{t} = 1$$

we have that

$$\begin{aligned} I_\lambda(u) &\geq \frac{a}{2p}\rho^{2p} - \frac{(S_\mu)^{-p^*/p}}{p^*}\rho^{p^*} - \frac{2p-1}{2p}\left(\left(\frac{a}{2}\right)^{\frac{-1}{2p}} \lambda \|f\|_*\right)^{\frac{2p}{2p-1}} - \frac{1}{2p}\left(\left(\frac{a}{2}\right)^{\frac{1}{2p}} \rho\right)^{2p}, \\ &\geq \frac{a}{4p}\rho^{2p} - \frac{(S_\mu)^{-p^*/p}}{p^*}\rho^{p^*} - \frac{2p-1}{2p}\left(\left(\frac{a}{2}\right)^{\frac{-1}{2p}} \lambda \|f\|_*\right)^{\frac{2p}{2p-1}}. \end{aligned}$$

Now, we consider the function

$$h(\rho) = \frac{a}{4p}\rho^{2p} - \frac{(S_\mu)^{-p^*/p}}{p^*}\rho^{p^*}$$

It is easy to see that

$$\max_{\rho \geq 0} h(\rho) = h(\rho_0) = \frac{p^* - 2p}{2p.p^*}(S_\mu)^{\frac{2p^*}{p^*-2p}}\left(\frac{a}{2}\right)^{\frac{p^*}{p^*-2p}} \geq 0$$

with

$$\rho_0 = \left[\frac{a}{2}(S_\mu)^{p^*/p}\right]^{\frac{1}{p^*-2p}}.$$

So, for $\|u\|_{\alpha,\mu} = \rho_0$ and

$$\lambda \leq \left(\frac{p^* - 2p}{2pp^*}(S_\mu)^{\frac{2p^*}{p^*-2p}}\right)^{\frac{2p-1}{2p}}\left(\frac{a}{2}\right)^{\frac{p^*-1}{p^*-2p}}\|f\|_*^{-1},$$

we have

$$
\begin{aligned}
I_\lambda(u) &\geq h(\rho_0) - \frac{2p-1}{2p}\left(\left(\frac{a}{2}\right)^{\frac{-1}{2p}} \lambda \|f\|_*\right)^{\frac{2p}{2p-1}}, \\
&\geq \frac{2p-1}{2p} h(\rho_0) + \frac{1}{2p} h(\rho_0) - \frac{2p-1}{2p}\left(\left(\frac{a}{2}\right)^{\frac{-1}{2p}} \lambda \|f\|_*\right)^{\frac{2p}{2p-1}}, \\
&\geq \frac{p^*-2p}{4p^2p^*} (S_\mu)^{\frac{2p^*}{p^*-2p}} \left(\frac{a}{2}\right)^{\frac{p^*}{p^*-2p}}.
\end{aligned}
$$

Therefore,

$$
I_\lambda(u) \geq -\frac{2p-1}{2p}\left(\left(\frac{a}{2}\right)^{\frac{-1}{2p}} \lambda \|f\|_*\right)^{\frac{2p}{2p-1}}, \text{ for } \|u\|_{\alpha,\mu} \leq \rho_0.
$$

Now, under the condition $(H4)$ and the definition of S_μ we have

$$
\begin{aligned}
I_\lambda(u) &\geq \frac{b}{p}\rho^p + \frac{a}{2p}\rho^{2p} - \frac{(S_\mu)^{-p^*/p}}{p^*}\rho^{p^*} - \lambda \|f\|_* \rho, \\
&\geq \frac{b}{p}\rho^p - \frac{(S_\mu)^{-p^*/p}}{p^*}\rho^{p^*} - \left(\left(\frac{b}{2}\right)^{\frac{-1}{p}} \lambda \|f\|_*\right)\left(\frac{b}{2}\right)^{\frac{1}{p}} \rho \\
&\geq \frac{b}{p}\rho^p - \frac{(S_\mu)^{-p^*/p}}{p^*}\rho^{p^*} - \frac{p-1}{p}\left(\left(\frac{b}{2}\right)^{\frac{-1}{p}} \lambda \|f\|_*\right)^{\frac{p}{p-1}} - \frac{1}{p}\left(\left(\frac{b}{2}\right)^{\frac{1}{p}} \rho\right)^p \\
&\geq \frac{b}{2p}\rho^p - \frac{(S_\mu)^{-p^*/p}}{p^*}\rho^{p^*} - \frac{p-1}{p}\left(\left(\frac{b}{2}\right)^{\frac{-1}{p}} \lambda \|f\|_*\right)^{\frac{p}{p-1}}.
\end{aligned}
$$

We consider the function

$$
\tilde{h}(\rho) = \frac{b}{2p}\rho^p - \frac{(S_\mu)^{-p^*/p}}{p^*}\rho^{p^*}
$$

notice that

$$
\max_{\rho \geq 0} \tilde{h}(\rho) = \tilde{h}(\tilde{\rho}_0) = \frac{p^*-p}{pp^*}\left[\frac{b}{2}S_\mu\right]^{\dfrac{p^*}{p^*-p}} \geq 0
$$

with

$$
\tilde{\rho}_0 = \left[\frac{b}{2}(S_\mu)^{p^*/p}\right]^{\dfrac{1}{p^*-p}}.
$$

So, for $\|u\|_{\alpha,\mu} = \tilde{\rho}_0$ and

$$\lambda \leq \left(\frac{p^* - p}{pp^*} (S_\mu)^{\dfrac{p^*}{p^* - p}} \right)^{\frac{p-1}{p}} \left(\frac{b}{2}\right)^{\dfrac{p^* - 1}{p^* - p}} \|f\|_*^{-1}$$

we have

$$\begin{aligned} I_\lambda(u) \quad &\geq \quad \frac{p^* - p}{pp^*} \left[\frac{b}{2} S_\mu\right]^{\dfrac{p^*}{p^* - p}} - \frac{p-1}{p} \left(\left(\frac{b}{2}\right)^{\frac{-1}{p}} \lambda \|f\|_* \right)^{\frac{p}{p-1}} \\ &\geq \quad \frac{1}{p} \frac{p^* - p}{pp^*} \left[\frac{b}{2} S_\mu\right]^{\frac{p^*}{p^* - p}} + \left[\frac{p-1}{p} \frac{p^* - p}{pp^*} \left[\frac{b}{2} S_\mu\right]^{\dfrac{p^*}{p^* - p}} \right. \\ &\quad \left. - \frac{p-1}{p} \left(\left(\frac{b}{2}\right)^{\frac{-1}{p}} \lambda \|f\|_* \right)^{\frac{p}{p-1}} \right] \\ &\geq \quad \frac{p^* - p}{p^2 p^*} \left(\frac{b}{2} S_\mu\right)^{\frac{p^*}{p^* - p}}. \end{aligned}$$

Therefore,

$$I_\lambda(u) \geq -\frac{p-1}{p} \left(\left(\frac{b}{2}\right)^{\frac{-1}{p}} \lambda \|f\|_* \right)^{\frac{p}{p-1}}, \text{ for } \|u\|_{\alpha,\mu} \leq \tilde{\rho}_0.$$

then we can choose δ_1, $\rho_1$and λ_1 such that

$$\delta_1 = \begin{cases} \dfrac{p^*-p}{p^2p^*}\left(\dfrac{b}{2}S_\mu\right)^{\frac{p^*}{p^*-p}} & \text{if } (H4) \text{ is satisfied,} \\ \dfrac{p^*-2p}{4p^2p^*}\left(S_\mu\right)^{\frac{2p^*}{p^*-2p}}\left(\dfrac{a}{2}\right)^{\frac{p^*}{p^*-2p}} & \text{if one of } (Hi) \text{ is satisfied} \\ & \text{with } 5 \leq i \leq 10, \end{cases}$$

$$\rho_1 = \begin{cases} \left[\dfrac{b}{2}\left(S_\mu\right)^{p^*/p}\right]^{\frac{1}{p^*-p}} & \text{if } (H4) \text{ is satisfied,} \\ \left[\dfrac{a}{2}\left(S_\mu\right)^{p^*/p}\right]^{\frac{1}{p^*-2p}} & \text{if one of } (Hi) \text{ is satisfied} \\ & \text{with } 5 \leq i \leq 10, \end{cases}$$

$$\lambda_1 = \begin{cases} \left(\dfrac{p^*-p}{pp^*}\left(S_\mu\right)^{\frac{p^*}{p^*-p}}\right)^{\frac{p-1}{p}}\left(\dfrac{b}{2}\right)^{\frac{p^*-1}{p^*-p}}\|f\|_*^{-1} & \text{if } (H4) \text{ is satisfied,} \\ \left(\dfrac{p^*-2p}{2pp^*}\left(S_\mu\right)^{\frac{2p^*}{p^*-2p}}\right)^{\frac{2p-1}{2p}}\left(\dfrac{a}{2}\right)^{\frac{p^*-1}{p^*-2p}}\|f\|_*^{-1} & \text{if one of } (Hi) \text{ is satisfied} \\ & \text{for } 5 \leq i \leq 10. \end{cases}$$

■

Lemma 4.4 *Let $f \in W^*\backslash\{0\}$ and $(u_n) \subset W^{1,p}_{\alpha,\mu}$ be a Palais-Smale sequence for I_λ at level c, then*

$$u_n \rightharpoonup u \ in\ W^{1,p}_{\alpha,\mu}$$

for some $u \in W^{1,p}_{\alpha,\mu}$ with $I'_\lambda(u) = 0$.

Proof. Let $(u_n) \subset W^{1,p}_{\alpha,\mu}$ be a Palais-Smale sequence for I_λ such that

$$I_\lambda(u_n) \to c \text{ and } I'_\lambda(u_n) \to 0.$$

We have

$$c + o_n(1) = I_\lambda(u_n) \text{ and } o_n(1) = \langle I'_\lambda(u_n), u_n\rangle,$$

that is

$$\begin{aligned}
c+o_n(1) &= I_\lambda(u_n)-\frac{1}{p^*}\left\langle I'_\lambda(u_n),u_n\right\rangle \\
&= a\frac{p^*-2p}{2pp^*}\|u\|^{2p}_{\alpha,\mu}+b\frac{p^*-p}{pp^*}\|u\|^{p}_{\alpha,\mu}-\lambda\frac{p^*-1}{p^*}\int_{\mathbb{R}^N} f(x)\,u_n dx, \\
&\geq a\frac{p^*-2p}{2pp^*}\|u\|^{2p}_{\alpha,\mu}+b\frac{p^*-p}{pp^*}\|u\|^{p}_{\alpha,\mu}-\lambda\frac{p^*-1}{p^*}\|f\|_*\|u\|_{\alpha,\mu}.
\end{aligned}$$

Then (u_n) is bounded in $W^{1,p}_{\alpha,\mu}$. Up to a subsequence if necessary, we obtain

$$u_n \rightharpoonup u \text{ in } W^{1,p}_{\alpha,\mu} \text{ and in } L^{p^*}\left(\mathbb{R}^N,|x|^{-p^*\beta}\right),\ u_n \to u \text{ a. e. in } \mathbb{R}^N$$

and

$$\int_{\mathbb{R}^N} f(x)\,u_n dx \to \int_{\mathbb{R}^N} f(x)\,u dx.$$

Then

$$\left\langle I'_\lambda(u_n),\Phi\right\rangle = 0 \text{ for all } \Phi\in C_0^\infty\left(\mathbb{R}^N\right),$$

thus $I'_\lambda(u)=0$. This completes the proof of Lemma 4.4. ■

Now, we prove an important lemma which ensures the local compactness of the Palais-Smale sequence for I_λ.

Let

$$C_* = \begin{cases} \dfrac{1}{2p}\dfrac{b^2}{\left(S_\mu\right)^{-2} - a} & \text{if } (H4) \text{ is satisfied,} \\[2ex] \dfrac{2p\left(1+\alpha-\beta\right)-N}{2pN}\left(a\left(S_\mu\right)^2\right)^{\frac{N}{2p(1+\alpha-\beta)-N}} & \text{if } (H5) \text{ is satisfied,} \\[2ex] \dfrac{a}{24p}z_1^2 + \dfrac{b}{3p}z_1 & \text{if } (H6) \text{ is satisfied,} \\[2ex] \dfrac{a}{4p}z_2^2 + \dfrac{3b}{4p}z_2 & \text{if } (H7) \text{ is satisfied,} \\[2ex] \dfrac{a}{4p}z_3^2 + \dfrac{3b}{4p}z_3 & \text{if } (H8) \text{ is satisfied,} \\[2ex] \dfrac{a}{4p}z_{k+4}^2 + \dfrac{3b}{4p}z_{k+4}, \ \ k=0 \text{ or } k=1 \text{ or } k=2 & \text{if } (H9) \text{ is satisfied,} \\[2ex] \dfrac{3a}{10p}z_7^2 + \dfrac{4b}{5p}z_7 & \text{if } (H10) \text{ is satisfied,} \end{cases}$$

with

$$z_1 = a\left(S_\mu\right)^3 + \sqrt{a^2\left(S_\mu\right)^6 + 4b\left(S_\mu\right)^3}$$

$$z_2 = 2\sqrt{\frac{a}{3}}\left(S_\mu\right)^2$$

$$z_3 = \left(\tfrac{\left(S_\mu\right)^4}{2}\right)^{\frac{1}{3}}\left[\left(b+\delta\right)^{\frac{1}{3}} + \left(b-\delta\right)^{\frac{1}{3}}\right]$$

$$\delta = \sqrt{b^2 - 4\left(\tfrac{a}{3}\right)^3\left(S_\mu\right)^4}$$

$$z_{k+4} = 2\sqrt{\frac{a}{3}}\left(S_\mu\right)^2\cos\tfrac{\theta+2k\pi}{3} \text{ with } k\in\{0,1,2\},$$

$$\cos\theta = \frac{b}{2\left(\frac{a}{3}\right)^{\frac{3}{2}}\left(S_\mu\right)^2} \quad \text{and} \quad \sin\theta = \frac{\sqrt{4\left(\dfrac{a}{3}\right)^3\left(S_\mu\right)^4 - b^2}}{2\left(\frac{a}{3}\right)^{\frac{3}{2}}\left(S_\mu\right)^2}$$

$$z_7 = \tfrac{1}{2}\left[2^{\frac{1}{3}}\sqrt{\omega} + \sqrt{-2^{\frac{2}{3}}\omega + 2^{\frac{2}{3}}\left(S_\mu\right)^5 a\omega^{-\frac{1}{2}}}\right]$$

$$\omega = \omega_1 + \omega_2$$

$$\omega_1 = \left[\frac{a^2}{2^3}\left(S_\mu\right)^{10} - \sqrt{\frac{a^4}{2^6}\left(S_\mu\right)^{20} + \frac{4}{3^3}b^3\left(S_\mu\right)^{15}}\right]^{\frac{1}{3}}$$

$$\omega_2 = \left[\frac{a^2}{2^3}\left(S_\mu\right)^{10} + \sqrt{\frac{a^4}{2^6}\left(S_\mu\right)^{20} + \frac{4}{3^3}b^3\left(S_\mu\right)^{15}}\right]^{\frac{1}{3}}.$$

Lemma 4.5 *Let $f \in W^{*}\backslash\{0\}$ and $(u_n) \subset W^{1,p}_{\alpha,\mu}$ be a Palais-Smale sequence for I_λ for some $c \in \mathbb{R}^+$ such that $u_n \rightharpoonup u$ in $W^{1,p}_{\alpha,\mu}$. If one of the hypotheses (Hi), $4 \leq i \leq 10$ occurs, then*

$$\textit{either } u_n \to u \quad \textit{or } c \geq I_\lambda(u) + C_*.$$

Proof. By the proof of Lemma 4.4 the sequence (u_n) is bounded in $W^{1,p}_{\alpha,\mu}$ and as $f \in W^{*}\backslash\{0\}$ we have

$$\int_{\mathbb{R}^N} f(x)\, u_n dx \to \int_{\mathbb{R}^N} f(x)\, u dx. \tag{4.2}$$

Furthermore, if we write $v_n = u_n - u$; we derive that $v_n \rightharpoonup 0$ in $W^{1,p}_{\alpha,\mu}$. Then by using the Brezis-Lieb result [14] we have

$$\begin{cases} \|u_n\|^p_{\alpha,\mu} = \|v_n\|^p_{\alpha,\mu} + \|u\|^p_{\alpha,\mu} + o_n(1), \\ \displaystyle\int_{\mathbb{R}^N} \frac{|u_n|^{p^*}}{|x|^{p^*\beta}} dx = \int_{\mathbb{R}^N} \frac{|v_n|^{p^*}}{|x|^{p^*\beta}} dx + \int_{\mathbb{R}^N} \frac{|u|^{p^*}}{|x|^{p^*\beta}} dx + o_n(1). \end{cases} \tag{4.3}$$

Combining (4.2) and (4.3), we get

$$\begin{aligned} c + o_n(1) &= I_\lambda(u) + a\frac{p^*-2p}{2pp^*}\|v_n\|^{2p}_{\alpha,\mu} + b\frac{p^*-p}{pp^*}\|v_n\|^p_{\alpha,\mu} \\ &\quad + a\frac{p^*-2p}{pp^*}\|v_n\|^p_{\alpha,\mu}\|u\|^p_{\alpha,\mu} \end{aligned} \tag{4.4}$$

and

$$o_n(1) = \langle I'_\lambda(u), u\rangle + a\|v_n\|^{2p}_{\alpha,\mu} + b\|v_n\|^p_{\alpha,\mu} + 2a\|v_n\|^p_{\alpha,\mu}\|u\|^p_{\alpha,\mu} - \int_{\mathbb{R}^N} \frac{|v_n|^{p^*}}{|x|^{p^*\beta}} dx. \tag{4.5}$$

Assume that $\|v_n\|_{\alpha,\mu} \to l > 0$, then by (4.5) and the Caffarelli-Kohn-Nirenberg inequality we obtain

$$l^p \geq S_\mu \left(bl^p + al^{2p}\right)^{p/p^*},$$

this implies that

$$(S_\mu)^{-\frac{p^*}{p}} l^{p^*} - al^{2p} - bl^p \geq 0.$$

Therefore,

$$(S_\mu)^{-\frac{p^*}{p}} l^{p^*-p} - al^p - b \geq 0.$$

Now, we consider the functions $g : \mathbb{R}^+ \to \mathbb{R}$, given by

$$g(y) = (S_\mu)^{-\frac{p^*}{p}} y^{p^*-p} - ay^p - b,$$

If $(H4)$ is satisfied, we get

$$g(y) = \left((S_\mu)^{-2} - a\right) y^p - b,$$

that is,

$$g(y) \geq 0 \text{ if } y^p \geq \frac{b}{(S_\mu)^{-2} - a}.$$

From the above inequality and (4.4), we conclude that

$$\begin{aligned} c &\geq I_\lambda(u) + \frac{1}{2p}\left(\frac{b^2}{(S_\mu)^{-2} - a}\right) \\ &= I_\lambda(u) + C_*. \end{aligned}$$

If $(H5)$ is satisfied, we get

$$g(y) = y^p \left((S_\mu)^{-\frac{N}{N-p(1+\alpha-\beta)}} y^{\frac{p(2p(1+\alpha-\beta)-N)}{N-p(1+\alpha-\beta)}} - a\right),$$

that is,

$$g(y) \geq 0 \text{ if } y^p \geq \left(a(S_\mu)^{\frac{N}{N-p(1+\alpha-\beta)}}\right)^{\frac{N-p(1+\alpha-\beta)}{2p(1+\alpha-\beta)-N}}.$$

and then

$$\begin{aligned} c &\geq I_\lambda(u) + \frac{2p(1+\alpha-\beta) - N}{2pN}\left(a(S_\mu)^2\right)^{\frac{N}{2p(1+\alpha-\beta)-N}} \\ &= I_\lambda(u) + C_*. \end{aligned}$$

If $(H6)$ is satisfied, we get

$$g(y) = (S_\mu)^{-3} y^{2p} - ay^p - b,$$

that is,

$$g(y) \geq 0 \text{ if } y^p \geq \frac{a(S_\mu)^3 + \sqrt{a^2(S_\mu)^6 + 4b(S_\mu)^3}}{2}.$$

Thus

$$\begin{aligned} c &\geq I_\lambda(u) + \frac{a}{24p} z_1^2 + \frac{b}{3p} z_1 \\ &= I_\lambda(u) + C_*. \end{aligned}$$

If $(H7)$ or $(H8)$ or $(H9)$ is satisfied, we get

$$g(y) = (S_\mu)^{-4} y^{3p} - ay^p - b,$$

and we have to distinguish three cases:

Case 1: $b = 2\left(\frac{a}{3}\right)^{\frac{3}{2}} (S_\mu)^2$.

If $y^p \geq 2\left(\frac{a}{3}\right)^{\frac{1}{2}} (S_\mu)^2$, then $g(y) \geq 0$. So, by (4.4) we conclude

$$\begin{aligned} c &\geq I_\lambda(u) + \frac{a}{4p} z_2^2 + \frac{3b}{4p} z_2 \\ &= I_\lambda(u) + C_*, \end{aligned}$$

Case 2: $b > 2\left(\frac{a}{3}\right)^{\frac{3}{2}} (S_\mu)^2$.

Since $g(y) \geq 0$ for $y^p \geq z_3$, we get by (4.4)

$$\begin{aligned} c &\geq I_\lambda(u) + \frac{a}{4p} z_3^2 + \frac{3b}{4p} z_3 \\ &= I_\lambda(u) + C_*. \end{aligned}$$

Case 3: $b < 2\left(\frac{a}{3}\right)^{\frac{3}{2}} (S_\mu)^2$.

In this case, $g(y) \geq 0$ for $y^p \geq z_{k+4}$, $k \in \{0,1,2\}$.

Since $\cos\theta$ and $\sin\theta$ are positive, we have $\theta \in \left[2K\pi, \dfrac{\pi}{2} + 2K\pi\right]$, $K \in \mathbb{Z}$.

Note that for $\theta \in \left[6K\pi, \frac{\pi}{2} + 6K\pi\right]$, $K \in \mathbb{Z}$, we have

$$\cos\frac{\theta}{3} > 0, \quad \cos\frac{\theta+2\pi}{3} < 0 \text{ and } \quad \cos\frac{\theta+4\pi}{3} < 0,$$

for $\theta \in \left[2(1+3K)\pi, \dfrac{\pi}{2} + 2(1+3K)\pi\right]$, $K \in \mathbb{Z}$, we have

$$\cos\frac{\theta+4\pi}{3} > 0, \quad \cos\frac{\theta}{3} < 0 \text{ and } \quad \cos\frac{\theta+2\pi}{3} < 0,$$

and for $\theta \in \left[2(2+3K)\pi, \dfrac{\pi}{2} + 2(2+3K)\pi\right]$, $K \in \mathbb{Z}$, we have

$$\cos\frac{\theta+2\pi}{3} > 0, \quad \cos\frac{\theta+4\pi}{3} < 0 \text{ and } \quad \cos\frac{\theta+4\pi}{3} < 0.$$

So $g(y) \geq 0$ for

$$y^p \geq \begin{cases} 2\sqrt{\dfrac{a}{3}}(S_\mu)^2 \cos\dfrac{\theta}{3} & \text{if } \theta \in \left[6K\pi, \frac{\pi}{2} + 6K\pi\right], \\ 2\sqrt{\dfrac{a}{3}}(S_\mu)^2 \cos\dfrac{\theta+2\pi}{3} & \text{if } \theta \in \left[2(2+3K)\pi, \frac{\pi}{2} + 2(2+3K)\pi\right], \\ 2\sqrt{\dfrac{a}{3}}(S_\mu)^2 \cos\dfrac{\theta+4\pi}{3} & \text{if } \theta \in \left[2(1+3K)\pi, \frac{\pi}{2} + 2(1+3K)\pi\right], \end{cases}$$

for $K \in \mathbb{Z}$.

From the above inequality and (4.4), we conclude

$$\begin{aligned} c &\geq I_\lambda(u) + \frac{a}{4p} z_{k+4}^2 + \frac{3b}{4p} z_{k+4}, \; k \in \{0,1,2\} \\ &= I_\lambda(u) + C_*. \end{aligned}$$

If $(H10)$ is satisfied, we have

$$g(y) = (S_\mu)^{-5} y^{4p} - ay^p - b.$$

Then $g(y) \geq 0$ for $y^p \geq y_7$. From the above inequality and (4.4), we conclude

$$\begin{aligned} c &\geq I_\lambda(u) + \frac{3a}{10p} y_7^2 + \frac{4b}{5p} y_7 \\ &= I_\lambda(u) + C_*. \end{aligned}$$

This finishes the proof of Lemma 4.5. ■

4.3 Nonexistence Result

Proof of Theorem 4.1. Suppose that $(H1)$ is satisfied and that $u \in W^{1,p}_{\alpha,\mu} \backslash \{0\}$ is a solution of the problem $(\mathcal{P}_3)$. Then

$$a \|u\|^{2p}_{\alpha,\mu} = \int_{\mathbb{R}^N} \frac{|u|^{p^*}}{|x|^{p^*\beta}} dy. \tag{4.6}$$

As $a > (S_\mu)^{-2}$ and $\int_{\mathbb{R}^N} \frac{|u|^{p^*}}{|x|^{p^*\beta}} dy \leq (S_\mu)^{-2} \|u\|^{p^*}_{\alpha,\mu}$, we have by (4.6)

$$\begin{aligned} (S_\mu)^{-2} \|u\|^{2p}_{\alpha,\mu} &< a \|u\|^{2p}_{\alpha,\mu} \\ &= \int_{\mathbb{R}^N} \frac{|u|^{p^*}}{|x|^{p^*\beta}} dy \\ &\leq (S_\mu)^{-2} \|u\|^{2p}_{\alpha,\mu}, \end{aligned}$$

which leads to a contradiction.

Suppose now that $(H2)$ is satisfied and that $u \in W^{1,p}_{\alpha,\mu} \backslash \{0\}$ is a solution of $(\mathcal{P}_3)$. Then

$$a \|u\|^{2p}_{\alpha,\mu} + b \|u\|^{p}_{\alpha,\mu} = \int_{\mathbb{R}^N} \frac{|u|^{p^*}}{|x|^{p^*\beta}} dy.$$

From this last equality and because $a \geq (S_\mu)^{-2}$, $b > 0$ and the fact that

$$\int_{\mathbb{R}^N} \frac{|u|^{p^*}}{|x|^{p^*\beta}} dy \leq (S_\mu)^{-2} \|u\|^{p^*}_{\alpha,\mu}$$

we get

$$
\begin{aligned}
(S_\mu)^{-2} \|u\|_{\alpha,\mu}^{2p} &< a\|u\|_{\alpha,\mu}^{2p} + b\|u\|_{\alpha,\mu}^{p} \\
&= \int_{\mathbb{R}^N} \frac{|u|^{p^*}}{|x|^{p^*\beta}} dy \\
&\leq (S_\mu)^{-2} \|u\|_{\alpha,\mu}^{2p},
\end{aligned}
$$

which is a contradiction.

In the same way as above, we suppose that under the condition $(H3)$ we have the existence of a solution $u \in W_{\alpha,\mu}^{1,p} \backslash \{0\}$, that is,

$$
a\|u\|_{\alpha,\mu}^{2p} + b\|u\|_{\alpha,\mu}^{p} = \int_{\mathbb{R}^N} \frac{|u|^{p^*}}{|x|^{p^*\beta}} dy,
$$

and then we get

$$
\begin{aligned}
\int_{\mathbb{R}^N} \frac{|u|^{p^*}}{|x|^{p^*\beta}} dy &\leq (S_\mu)^{-\frac{p^*}{p}} \|u\|_{\alpha,\mu}^{p^*} \\
&\leq (S_\mu)^{-\frac{p^*}{p}} \|u\|_{\alpha,\mu}^{2p^*-2p} \|u\|_{\alpha,\mu}^{2p-p^*} \\
&\leq \frac{p^*-p}{p} \left((S_\mu)^{-\frac{p^*}{p}} \|u\|_{\alpha,\mu}^{2p^*-2p} \right)^{\frac{p}{p^*-p}} + \frac{2p-p^*}{p} \left(\|u\|_{\alpha,\mu}^{2p-p^*} \right)^{\frac{p}{2p-p^*}} \\
&\leq \frac{p^*-p}{p} (S_\mu)^{-\frac{p^*}{p^*-p}} \|u\|_{\alpha,\mu}^{2p} + \frac{2p-p^*}{p} \|u\|_{\alpha,\mu}^{p} \\
&< a\|u\|_{\alpha,\mu}^{2p} + b\|u\|_{\alpha,\mu}^{p} \\
&= \int_{\mathbb{R}^N} \frac{|u|^{p^*}}{|x|^{p^*\beta}} dy,
\end{aligned}
$$

which leads to a contradiction. ∎

4.4 Existence Result

4.4.1 Existence of a first solution

Proposition 4.6 *Let $f \in W^{*}\setminus\{0\}$ and assume that one of the hypotheses (Hi) holds for $4 \leq i \leq 10$. For all $\lambda \in \left]0,\ \lambda_1\right[$, there exists a solution u_1 of $(\mathcal{P}_3)$ with negative energy.*

Proof. By Lemma 4.3, we can define

$$c_1 = \inf\left\{I_\lambda(u),\ u \in \overline{B}_{\rho_1}(0)\right\}.$$

For $t > 0$ we have

$$I_\lambda(t\varphi) = \frac{a}{2p}t^{2p}||\varphi||^{2p}_{\alpha,\mu} + \frac{b}{p}t^{p}||\varphi||^{p}_{\alpha,\mu} - \frac{1}{p^*}t^{p^*}\int_{\mathbb{R}^N}\frac{|\varphi|^{p^*}}{|x|^{p^*\beta}}dy - \lambda t\int_{\mathbb{R}^N} f(x)\,\varphi dx.$$

By $(H0)$ we can choose $\varphi \in W^{1,p}_{\alpha,\mu}$ such that $\displaystyle\int_{\mathbb{R}^N} f(x)\,\varphi dx > 0$. Then, for a fixed $\lambda \in \left]0,\ \lambda_1\right[$, there exists $t_0 > 0$ such that $||t_0\varphi||_{\alpha,\mu} < \rho_1$ and

$$I_\lambda(t\varphi) < 0 \quad \text{for } t \in \left]0, t_0\right[.$$

Hence, $c_1 < I_\lambda(0) = 0$. Using the Ekeland's variational principle, for the complete metric space $\overline{B}_{\rho_1}(0)$ with respect to the norm of $W^{1,p}_{\alpha,\mu}$, we obtain the result that there exists a Palais-Smale sequence $u_n \in \overline{B}_{\rho_1}(0)$ at level c_1, and from Lemma 4.4 we have $u_n \rightharpoonup u_1$ in $W^{1,p}_{\alpha,\mu}$ for some u_1 with $||u_1||_{\alpha,\mu} < \rho_1$.

Now, we shall show that $u_n \to u_1$ in $W^{1,p}_{\alpha,\mu}$. Assume that $u_n \nrightarrow u_1$ in $W^{1,p}_{\alpha,\mu}$, then, it follows from Lemma 4.5 that

$$\begin{aligned} c_1 &\geq I_\lambda(u_1) + C_* \\ &\geq c_1 + C_* \\ &> c_1, \end{aligned}$$

which is a contradiction. Thus u_1 is a critical point of I_λ i.e. u_1 is a solution of $(\mathcal{P}_3)$. As $I_\lambda(0) = 0$ and $I_\lambda(u_1) < 0$ then, $u_1 \neq 0$. Thus u_1 is a solution of $(\mathcal{P}_3)$ with negative energy. ■

4.4.2 Existence of a second solution

Now, we proof the existence of a Mountain Pass type solution.

Proposition 4.7 *Suppose that f satisfies $(H0)$ and assume that one of the hypotheses (Hi) holds for $4 \leq i \leq 10$. Then there exists $\lambda_* \in \left]0,\ \lambda_1\right]$ such that the problem $(\mathcal{P}_3)$ has a solution u_2 with positive energy.*

We need the following lemma.

Lemma 4.8 *Suppose that f satisfies $(H0)$ and assume that one of the hypotheses (Hi) holds for $4 \leq i \leq 10$. Then there exists $\lambda_* > 0$ such that*

$$\sup_{t\geq 0} I_\lambda(tV_\varepsilon) < C_* + c_1, \quad \textit{for all } \lambda \in \left]0, \lambda_*\right[.$$

Proof. We have

$$I_\lambda(tV_\varepsilon) = \frac{a}{2p}t^{2p}\left\|V_\varepsilon\right\|^{2p}_{\alpha,\mu} + \frac{b}{p}t^p\left\|V_\varepsilon\right\|^p_{\alpha,\mu} - \frac{1}{p^*}t^{p^*}\int_{\mathbb{R}^N}\frac{|V_\varepsilon|^{p^*}}{|x|^{p^*\beta}}dy - \lambda t\int_{\mathbb{R}^N} f(x)\, V_\varepsilon dx,$$

put

$$h(t) = \frac{a}{2p} t^{2p} \|V_\varepsilon\|_{\alpha,\mu}^{2p} + \frac{b}{p} t^p \|V_\varepsilon\|_{\alpha,\mu}^{p} - \frac{1}{p^*} t^{p^*} \int_{\mathbb{R}^N} \frac{|V_\varepsilon|^{p^*}}{|x|^{p^*\beta}} dy.$$

From the definition of S_μ, we have

$$\|V_\varepsilon\|_{\alpha,\mu}^{p} = \int_{\mathbb{R}^N} \frac{|V_\varepsilon|^{p^*}}{|x|^{p^*\beta}} dy = (S_\mu)^{\frac{N}{p(1+\alpha-\beta)}}.$$

Then

$$h(t) = \frac{a}{2p} t^{2p} (S_\mu)^{\frac{2N}{p(1+\alpha-\beta)}} + \frac{b}{p} t^p (S_\mu)^{\frac{N}{p(1+\alpha-\beta)}} - \frac{1}{p^*} t^{p^*} (S_\mu)^{\frac{N}{p(1+\alpha-\beta)}},$$

and

$$h'(t) = t^{p-1} (S_\mu)^{\frac{N}{p(1+\alpha-\beta)}} \left(-t^{p^*-p} + a t^p (S_\mu)^{\frac{N}{p(1+\alpha-\beta)}} + b \right).$$

Thus, the function $h(t)$ attains its maximum at

$$t_0 = \begin{cases} \left[\dfrac{b}{1 - a (S_\mu)^2} \right]^{\frac{1}{p}} & \text{if } (H4) \text{ is satisfied,} \\ \left[a (S_\mu)^{\frac{N}{p(1+\alpha-\beta)}} \right]^{\frac{N-p(1+\alpha-\beta)}{2p^2(1+\alpha-\beta)-N}} & \text{if } (H5) \text{ is satisfied,} \\ \frac{1}{2} \left(a (S_\mu)^{\frac{3}{2}} + \sqrt{a^2 (S_\mu)^3 + 4b} \right) & \text{if } (H6) \text{ is satisfied,} \\ \left[\dfrac{1}{2 (S_\mu)^{\frac{5}{4}}} \left(2^{\frac{1}{3}} \sqrt{\omega} + \sqrt{-2^{\frac{2}{3}} \omega + 2^{\frac{2}{3}} a (S_\mu)^5 \omega^{-\frac{1}{2}}} \right) \right]^{\frac{1}{p}} & \text{if } (H10) \text{ is satisfied.} \end{cases}$$

In the other hand, if $(H7)$ or $(H8)$ or $(H9)$ is satisfied, the function $h(t)$ attains its maximum at

$$t_0 = \begin{cases} \left[2\sqrt{\tfrac{a}{3}} (S_\mu)^{\frac{2}{3}} \right]^{\frac{1}{p}} & \text{if } b = 2 \left(\dfrac{a}{3}\right)^{\frac{3}{2}} (S_\mu)^2 \\ 2^{\frac{-1}{3p}} \left[(b+\delta)^{\frac{1}{3}} + (b-\delta)^{\frac{1}{3}} \right]^{\frac{1}{p}} & \text{if } b > 2 \left(\dfrac{a}{3}\right)^{\frac{3}{2}} (S_\mu)^2, \end{cases}$$

with $\delta=\sqrt{b^2-4\left(\frac{a}{3}\right)^3(S_\mu)^4}$, and if $b<2\left(\dfrac{a}{3}\right)^{\frac{3}{2}}(S_\mu)^2$, the function $h(t)$ attains its maximum at

$$t_0=\begin{cases}\left(2\left(\frac{a}{3}\right)^{\frac{1}{2}}(S_\mu)^{\frac{2}{3}}\cos\frac{\theta}{3}\right)^{\frac{1}{p}} & \text{if } \theta\in\left[6K\pi,\dfrac{\pi}{2}+6K\pi\right],\\ \left(2\left(\frac{a}{3}\right)^{\frac{1}{2}}(S_\mu)^{\frac{2}{3}}\cos\frac{\theta+2\pi}{3}\right)^{\frac{1}{p}} & \text{if } \theta\in\left[(4+6K)\pi,\dfrac{\pi}{2}+(4+6K)\pi\right],\\ \left(2\left(\frac{a}{3}\right)^{\frac{1}{2}}(S_\mu)^{\frac{2}{3}}\cos\frac{\theta+4\pi}{3}\right)^{\frac{1}{p}} & \text{if } \theta\in\left[(2+6K)\pi,\dfrac{\pi}{2}+(2+6K)\pi\right],\end{cases}$$

for $K\in\mathbb{Z}$.

The above estimate on $h(t)$ yields

$$\max_{t\geq 0}h(t)=C_*.$$

Then we have

$$\sup_{t\geq 0}I_\lambda(tV_\varepsilon)\leq C_*-\lambda t\int_{\mathbb{R}^N}f(x)V_\varepsilon dx,$$

On the other hand, using Lemma 4.3 we see that for all $\lambda\in\,]0,\ \lambda_1[$

$$c_1\geq\begin{cases}-\dfrac{p-1}{p}\left(\left(\dfrac{b}{2}\right)^{\frac{-1}{p}}\lambda\|f\|_*\right)^{\frac{p}{p-1}} & \text{if } (H4) \text{ is satisfied}\\ -\dfrac{2p-1}{2p}\left(\left(\dfrac{a}{2}\right)^{\frac{-1}{2p}}\lambda\|f\|_*\right)^{\frac{2p}{2p-1}} & \text{if one of } (Hi) \text{ is satisfied}\\ & \text{with } i=\overline{5,10}.\end{cases}.$$

Let $\lambda_2>0$ such that

$$C_*>\begin{cases}\dfrac{p-1}{p}\left(\left(\dfrac{b}{2}\right)^{\frac{-1}{p}}\lambda\|f\|_*\right)^{\frac{p}{p-1}} & \text{if } (H4) \text{ is satisfied}\\ \dfrac{2p-1}{2p}\left(\left(\dfrac{a}{2}\right)^{\frac{-1}{2p}}\lambda\|f\|_*\right)^{\frac{2p}{2p-1}} & \text{if one of } (Hi) \text{ is satisfied}\\ & \text{with } i=\overline{5,10},\end{cases}$$

for any $\lambda \in]0,\ \lambda_2[$, we choose λ_3 such that for any $\lambda \in]0,\ \lambda_3[$ we have

$$-\lambda t_1 \int_{\mathbb{R}^N} f(x) V_\varepsilon dx \leq \begin{cases} -\dfrac{p-1}{p}\left(\left(\dfrac{b}{2}\right)^{\frac{-1}{p}} \lambda \|f\|_*\right)^{\frac{p}{p-1}} & \text{if } (H4) \text{ is satisfied} \\ -\dfrac{2p-1}{2p}\left(\left(\dfrac{a}{2}\right)^{\frac{-1}{2p}} \lambda \|f\|_*\right)^{\frac{2p}{2p-1}} & \text{if } (Hi) \text{ is satisfied} \\ & \text{with } i=\overline{5,10}, \end{cases}$$

this implies that

$$\lambda \leq \lambda_3 = \begin{cases} \dfrac{b}{2}\|f\|_*^{-p}\left(\frac{pt_1}{p-1}\displaystyle\int_{\mathbb{R}^N} f(x) V_\varepsilon dx\right)^{p-1} & \text{if } (H4) \text{ is satisfied} \\ \dfrac{a}{2}\|f\|_*^{-2p}\left(\frac{2pt_1}{2p-1}\displaystyle\int_{\mathbb{R}^N} f(x) V_\varepsilon dx\right)^{2p-1} & \text{if one of } (Hi) \text{ is satisfied} \\ & \text{with } i=\overline{5,10}. \end{cases}$$

Taking

$$\lambda_* = \min(\lambda_1,\ \lambda_2,\ \lambda_3).$$

Thus for any $\lambda \in]0,\ \lambda_*[$ we obtain

$$\sup_{t\geq 0} I_\lambda(tV_\varepsilon) < C_* + c_1.$$

This concludes the proof of Lemma 4.8. ∎

Proof of Proposition 4.7. We know by Lemma 4.3 and the fact that

$$\lim_{t\to\infty} I_\lambda(tV_\varepsilon) = -\infty,$$

that I_λ satisfies the geometrical conditions of the Mountain Pass Theorem. Then, there exists a Palais-Smale sequence (u_n) at level c_2, such that

$$I_\lambda(u_n) \to c_2 \text{ and } I'_\lambda(u_n) \to 0 \text{ as } n \to +\infty$$

with

$$c_2 = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)),$$

where for T large enough

$$\Gamma = \left\{ \gamma \in C\left([0,1], W^{1,p}_{\alpha,\mu} \right), \gamma(0) = 0, \gamma(1) = TV_\varepsilon \right\}.$$

Using Lemma 4.5 we have that (u_n) has a subsequence, still denoted by (u_n), such that $u_n \rightarrow u_2$ in $W^{1,p}_{\alpha,\mu}$, for some $u_2 \in W^{1,p}_{\alpha,\mu}$, as $c_2 > 0$ then $u_2 \neq 0$.

Thus the existence of the solution with energy positive follows immediately from the preceding lemma which achieves the proof of Proposition 4.7. ■

Proof of Theorem 4.2. It follows immediately from the combination of Proposition 4.6 and Proposition 4.7. ■

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