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THESE

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Intitulée

Existence de Solutions Positives d'un Problème Non Linéaire via la Méthode de Quadrature et Points Fixes

Thèse Soutenue le .....

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Introduction

In the study of nonlinear phenomena in physics, engineering and many other applied sciences, some mathematical models lead to multi-point boundary value problems associated with non-linear ordinary differential equations. Started to fairly late study this kind of problem, initialed by II'in and Moiseev [19], they studied the existence of solutions for a linear multi-point boundary value problem (BVP). Gupta studied some three-point boundary value problems for nonlinear ordinary differential equations [16]. Since then, more general multi-point boundary value problems have been studied [17],[26],[27],[31]. Within the following ten years, the study on nonlocal boundary value problems for ordinary differential equations has been made great progress.

The purpose of the present thesis is to study nonlinear differential equations with nonlocal conditions. We shall obtain existence and uniqueness results based on an operator approach using fixed point theorems and the quadrature method.

This thesis consists of five chapters

In the first chapter, we introduce notations, definitions , lemmas and fixed point theorems to be used in the next chapters.

In chapter 2, we present some existence results of positive solutions for a class of nonlinear third order boundary value problem with delay given by

$$
(P_1)\begin{cases}
u'''(t)+\lambda a(t)f(t,u(t-\tau))=0, & t\in J=[0,1],\\
u(t)=\alpha u(\eta), & -\tau\leqslant t\leqslant 0,\\
u(1)=\beta u(\eta),\\
u^{'}(0)=0,
\end{cases}
$$

where α,β,η and τ are positive constants such that $\eta\in(0,1), 0<\tau\leq\frac{1}{2}$ and λ is a real positive parameter.

By the mean of Krasnoselskii's fixed point theorem, sufficient conditions are found to obtain existence of positive solutions of (P_1).

In chapter 3, we investigate the existence of positive solutions for second order nonlinear boundary value problems. By using the Leray-Schauder fixed point theorem, some sufficient conditions for the existence of positive solutions of the following nonlinear second order delay boundary value problem are obtained

$$
(P_2)\begin{cases}
u''(t)+\lambda a(t)f(t,u(t-\tau))=0, & t\in J=[0,1],\\
u(t)=\alpha u(\eta), & -\tau\leqslant t\leqslant 0,\\
u(1)=\beta u(\eta),
\end{cases}
$$

where α,β,η and τ are positive constants such that $\eta\in(0,1)$ and λ is a positive parameter.

In chapter 4, we consider the following boundary value problem involving the p-Laplacian

$$
(P_3)\begin{cases}
-(|u'(x)|^{p-2}u'(x))'=\lambda f(u(x)), & p.p.\ 0<x<1,\\
\quad u(0)=u(1)=0,
\end{cases}
$$

where $\lambda\geq 0$ and $p\in(1,2]$. We investigate the existence of positive solutions of the p-Laplacian, using the quadrature method. We prove that the number of positive solutions depends on the asymptotic growth of the nonlinearity.

The purpose of chapter 5 is to study the existence of solutions to the following nonlocal boundary value problem involving the p-Laplacian operator

$$(P_4)\begin{cases} -(|u'(x)|^{p-2}u'(x))' = \lambda f(u(x)), & p.p.\ 0 < x < 1, \\ u(0) = 0 \quad u(1) = u(\xi), \end{cases}$$

where $\lambda \geq 0$, $p \in (1, 2]$ and $0 < \xi < 1$. The existence of multiple positive solutions of the BVP $(P4)$ is proved using the quadrature method. The number of solutions is depending on the asymptotic behavior of f.

Chapter 1

Preliminaries

In this chapter, we introduce notations, definitions and preliminary results that will be used in the sequel.

We shall consider the Banach space $E = C([a,b],\mathbb{R})$ endowed with the maximum norm $\|y\|_{[a,b]} = \max\limits_{a\leq t\leq b} |y(t)|$ for $y \in E$.

Definition 1.0.1. *An operator $T : E \to E$ is completely continuous if it is continuous and maps bounded sets into relatively compact sets.*

Definition 1.0.2. *(Arzela-Ascoli Theorem). A subset A of $C([a,b],\mathbb{R})$ is relatively compact if and only if it is bounded and equicontinuous.*

Definition 1.0.3. *Let X be a real Banach space. A nonempty, closed and convex set $P \subset X$ is a cone if it satisfies the following two conditions:*

1. *If $x \in P$ and $\lambda \geq 0$ then $\lambda x \in P$,*

2. *If $x \in P$ and $-x \in P$ then $x = 0$.*

The cone P induces an ordering $\leq$ on X by

$$x \leq y \text{ if and only if } y - x \in P.$$

Now we present the well-known Krasnosel'skii fixed point Theorem on cone.

Theorem 1.0.1. *Let X be a Banach space, and let $K \subset X$ be a cone. Assume that Ω_1 and Ω_2 are open subsets of X with $0 \in \Omega_1$, $\Omega_1 \subset \Omega_2$ and let*

$$A : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$$

be a completely continuous operator such that

1. $\|Au\| \leq \|u\|$ *for* $u \in K \cap \partial\Omega_1$ *and* $\|Au\| \geq \|u\|$ *for* $u \in K \cap \partial\Omega_2$, *or*

2. $\|Au\| \geq \|u\|$ *for* $u \in K \cap \partial\Omega_1$ *and* $\|Au\| \leq \|u\|$ *for* $u \in K \cap \partial\Omega_2$.

Then A has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$

Theorem 1.0.2. *(Leray-Schauder)*
Let Ω be the convex subset of Banach space X, $0 \in \Omega$ and $T : \Omega \to \Omega$ be completely continuous operator. Then, either

1. *T has at least one fixed point in Ω; or*

2. *the set $\{x \in \Omega / x = \lambda T x, 0 < \lambda < 1\}$ is unbounded.*

Definition 1.0.4. *A function $f : [a, b] \to \mathbb{R}$ is said to be absolutely continuous on $[a, b]$ if, given $\epsilon > 0$, there exists some $\delta > 0$ such that*

$$\sum_{i=1}^{n} |f(y_i) - f(x_i)| < \epsilon.$$

whenever $\{[x_i, y_i] : i = 1, 2, ..., n\}$ is a finite collection of mutually disjoint subintervals of [a, b] with $\sum_{i=1}^{n} |y_i - x_i| < \delta$.

Proposition 1.0.1. *If f is absolutely continuous, then f' exists almost everywhere and it is integrable.*

Chapter 2

Existence of positive solutions for a third order multi-point boundary value problem with delay

In this chapter, we consider the existence of positive solutions for the following multi-point boundary value problem

$$
\begin{aligned}
&u^{'''}(t) + \lambda a(t) f(t, u(t-\tau)) = 0, \qquad t \in J = [0,1], \\
&u(t) = \alpha u(\eta), \qquad -\tau \leqslant t \leqslant 0, \\
&u(1) = \beta u(\eta), \\
&u^{'}(0) = 0,
\end{aligned}
\tag{2.0.1}
$$

where α, β, η and τ are positive constants such that $\eta \in (0,1), 0 < \tau \leq \frac{1}{2}$ and λ is a positive real parameter. We study the existence of positive solutions for a class of boundary value problems for the third order differential equations with delay by the mean of Krasnosel'skii fixed point theorem on cone.

Let the following hypotheses be satisfied

$(H1)$ $0 < \beta < \dfrac{1}{\eta^2}$, $0 < \alpha < \dfrac{1-\beta\eta^2}{1-\eta^2}$,

$(H2)$ $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous,

$(H3)$ $a : [0,1] \to [0,\infty)$ is continuous and does not vanish identically on any subinterval.

2.1 Preliminaries

In this section, we give some preliminaries needed for the rest of this chapter.

Definition 2.1.1.

A function $u \in C([-\tau,1])$ is called a solution of (2.0.1) if it satisfies the following properties

1. $u(t) \geq 0$ $\forall t \in [-\tau,1]$,
2. $u(t) = \alpha u(\eta)$ $\forall t \in [-\tau,0]$, $u(1) = \beta u(\eta)$, $u'(0) = 0$,
3. $u \in C^3([0,1])$ *and* $u'''(t) = -\lambda a(t) f(t, u(t-\tau))$ $\forall t \in [0,1]$.

Furthermore, u is a positive solution of (2.0.1) if it is a solution of (2.0.1) with $u(t) > 0\ \forall t \in (0,1)$.

Lemma 2.1.1.

For $y \in C([0,1])$ the problem

$$w^{'''}(t) + y(t) = 0, \qquad t \in (0,1), \tag{2.1.1}$$

$$w(0) = w^{'}(0) = w(1) = 0 \tag{2.1.2}$$

has a unique solution

$$w(t) = \int_0^1 g(t,s)y(s)ds, \tag{2.1.3}$$

where

$$g(t,s) = \begin{cases} \dfrac{t^2(1-s)^2 - (t-s)^2}{2}, & 0 \leqslant s \leqslant t \leqslant 1, \\ \dfrac{t^2(1-s)^2}{2}, & 0 \leqslant t < s \leqslant 1. \end{cases} \tag{2.1.4}$$

Proof.

From (2.1.1), we have $w(t) = -\frac{1}{2}\displaystyle\int_0^t (t-s)^2 y(s)ds\ + At^2 + Bt + C$.

Then $w^{'}(t) = -t\displaystyle\int_0^t y(s)ds\ + \int_0^t sy(s)ds\ + 2At + B$.

From (2.1.2), we obtain $A = \dfrac{1}{2}\displaystyle\int_0^1 (1-s)^2 y(s)ds, B = 0$ and $C = 0$.

Therefore, the boundary value problem (2.1.1),(2.1.2) has a unique solution given by

$$\begin{aligned} w(t) &= -\frac{1}{2}\int_0^t (t-s)^2 y(s)ds\ + \int_0^1 \frac{t^2}{2}(1-s)^2 y(s)ds \\ &= -\frac{1}{2}\int_0^t (t-s)^2 y(s)ds\ + \int_0^t \frac{t^2}{2}(1-s)^2 y(s)ds + \int_t^1 \frac{t^2}{2}(1-s)^2 y(s)ds \\ &= \int_0^1 g(t,s)y(s)ds\ . \end{aligned}$$

Lemma 2.1.2.

For any $y \in C([0,1])$ *the problem*

$$u'''(t) + y(t) = 0, \qquad t \in [0,1], \tag{2.1.5}$$

$$u(0) = \alpha u(\eta), \qquad u'(0) = 0, \qquad u(1) = \beta u(\eta), \tag{2.1.6}$$

has a unique solution $u(t) = \int_0^1 G(t,s)y(s)ds$, *where*

$$G(t,s) = g(t,s) + \frac{\beta t^2 + \alpha(1-t^2)}{(1-\beta\eta^2) - \alpha(1-\eta^2)} g(\eta, s). \tag{2.1.7}$$

Proof.

Suppose that the solution of (2.1.5),(2.1.6) can be expressed by

$$u(t) = w(t) + A_1 t^2 + B_1 t + C_1, \tag{2.1.8}$$

where A_1, B_1 and C_1 are constants and w is the solution of (2.1.1),(2.1.2) given by (2.1.3). From (2.1.2) and (2.1.8) we have $u(0) = C_1, u(1) = A_1 + B_1 + C_1, u(\eta) = w(\eta) + A_1\eta^2 + B_1\eta + C_1$, and $u'(t) = w'(t) + 2A_1 t + B_1$.

Then $B_1 = u'(0) - w'(0) = 0$.

From (2.1.6) we obtain $(1-\alpha)C_1 - \alpha\eta^2 A_1 = \alpha w(\eta)$ and $(1-\beta)C_1 + (1-\beta\eta^2)A_1 = \beta w(\eta)$.

From $(H1)$, we have $\alpha \neq \dfrac{1-\beta\eta^2}{1-\eta^2}$, then $A_1 = \dfrac{(\beta-\alpha)w(\eta)}{(1-\beta\eta^2) - \alpha(1-\eta^2)}$, and $C_1 = \dfrac{\alpha w(\eta)}{(1-\beta\eta^2) - \alpha(1-\eta^2)}$.

Hence

$$u(t) = w(t) + \frac{\beta t^2 + \alpha(1-t^2)}{(1-\beta\eta^2) - \alpha(1-\eta^2)} w(\eta).$$

Finally, we obtain

$$u(t) = \int_0^1 g(t,s)y(s)ds \; + \frac{\beta t^2 + \alpha(1-t^2)}{(1-\beta\eta^2) - \alpha(1-\eta^2)} \int_0^1 g(\eta,s)y(s)ds. \tag{2.1.9}$$

Thus, the Green's function $G(t,s)$ for the boundary value problem (2.1.5),(2.1.6) is given by (2.1.7).

To prove the uniqueness of the solution u, assume that v is another solution of the three-point boundary value problem (2.1.5),(2.1.6).

Let $z(t) := v(t) - u(t)$ $\forall t \in [0,1]$. Then, we get $z^{'''}(t) = v^{'''}(t) - u^{'''}(t) = 0$ $\forall t \in [0,1]$, therefore

$$z(t) = c_0 t^2 + c_1 t + c_2, \qquad z^{'}(t) = 2c_0 t + c_1 \tag{2.1.10}$$

where c_0, c_1, c_2 are constants.

From (2.1.6), we have

$$z(0) = \alpha z(\eta), \qquad z(1) = \beta z(\eta), \qquad z^{'}(0) = 0. \tag{2.1.11}$$

From (2.1.10), we obtain

$$z(0) = c_2, \quad z(1) = c_0 + c_1 + c_2, \quad z^{'}(0) = c_1, \quad z(\eta) = c_0\eta^2 + c_1\eta + c_2. \tag{2.1.12}$$

From (2.1.11),(2.1.12) we have $c_1 = 0, (1-\alpha)c_2 - \alpha\eta^2 c_0 = 0$ and $(1-\beta)c_2 + (1-\beta\eta^2)c_0 = 0$.

Since $\alpha \neq \dfrac{1-\beta\eta^2}{1-\eta^2}$, we obtain $c_0 = c_1 = c_2 = 0$.

Therefore $z \equiv 0$, so $v(t) = u(t) \quad \forall t \in [0,1]$.

Lemma 2.1.3. *The function g has the following properties*

(i) $0 \leqslant g(t,s) \leqslant s(1-s)^2 \quad \forall t,s \in [0,1]$,

(ii) $g(t,s) \geqslant \Phi(t)s(1-s)^2 \quad \forall t,s \in [0,1]$, *where*

$$\Phi(t) = \begin{cases} \dfrac{t^2}{2} & t \in \left[0, \dfrac{1}{2}\right], \\[2ex] \dfrac{t(1-t)}{2} & t \in \left[\dfrac{1}{2}, 1\right]. \end{cases}$$

Proof.

It is obvious that g is positive.

Moreover, for $0 \leqslant s \leqslant t \leqslant 1$

$$\begin{aligned} g(t,s) &= \frac{1}{2}\left[t^2(1-s)^2 - (t-s)^2\right] = \frac{1}{2}s(1-t)\left[t(1-s)+(t-s)\right] \\ &\leqslant \frac{1}{2}\left[2s(1-s)^2\right] = s(1-s)^2. \end{aligned}$$

For $0 \leqslant t \leqslant s \leqslant 1$, $g(t,s) = \dfrac{1}{2}t^2(1-s)^2 \leqslant \dfrac{1}{2}s^2(1-s)^2 \leqslant s(1-s)^2$.

Thus (i) holds.

If $s=0$ or $s=1$, we easily see that (ii) holds.

If $s \in (0,1)$ and $t \in [0,\frac{1}{2}]$, we have, for $0 < s \leqslant t \leqslant \frac{1}{2}$,

$$\begin{aligned} \frac{g(t,s)}{s(1-s)^2} &= \frac{t^2(1-s)^2-(t-s)^2}{2s(1-s)^2} = \frac{s(1-t)\left[t(1-s)+(t-s)\right]}{2s(1-s)^2} \\ &\geqslant \frac{s(1-t)t(1-s)}{2s(1-s)^2} \geqslant \frac{t(1-t)}{2} \geqslant \frac{t^2}{2} \quad \forall t \in \left[0,\frac{1}{2}\right]. \end{aligned}$$

For $\dfrac{1}{2} \leqslant t \leqslant s < 1$, we have $\dfrac{g(t,s)}{s(1-s)^2} = \dfrac{t^2(1-s)^2}{2s(1-s)^2} = \dfrac{t^2}{2s} \geqslant \dfrac{t^2}{2} \geqslant \dfrac{t(1-t)}{2} \quad \forall t \in \left[\dfrac{1}{2},1\right]$.
Thus (ii) holds.

Lemma 2.1.4. *The function G has the following properties*

(i) $G(t,s) \geqslant 0 \quad \forall t,s \in [0,1]$,

(ii) $G(t,s) \leqslant M_1 s(1-s)^2 \quad \forall t,s \in [0,1]$ *and* $M_1 = \dfrac{\max\left(1+\alpha\eta^2, 1+\beta(1-\eta^2)\right)}{(1-\beta\eta^2)-\alpha(1-\eta^2)}$,

(iii) $\min\limits_{t\in\left[\sigma,\frac{1}{2}\right]} G(t,s) \geqslant M_2 s(1-s)^2 \quad \forall t,s \in [0,1]$ *where* $\sigma \in \left(0,\dfrac{1}{2}\right)$ *and* $M_2 = \dfrac{\sigma^2}{2} + \dfrac{\left(\beta\sigma^2 + \dfrac{3}{4}\alpha\right)\Phi(\eta)}{(1-\beta\eta^2)-\alpha(1-\eta^2)}$.

Proof.

It is clear that (i) holds.

Two cases will be considered for the proof of (ii).

Case(1)

For $0 \leqslant t \leqslant \eta$, by Lemma 2.1.3 (i) we have

$$G(t,s) \leqslant s(1-s)^2 + \frac{\beta\eta^2 + \alpha}{(1-\beta\eta^2) - \alpha(1-\eta^2)} s(1-s)^2 \leq M_1 s(1-s)^2.$$

Case(2)

For $\eta \leqslant t \leqslant 1$, we have $G(t,s) \leqslant s(1-s)^2 + \dfrac{\beta + \alpha(1-\eta^2)}{(1-\beta\eta^2) - \alpha(1-\eta^2)} s(1-s)^2 \leq M_1 s(1-s)^2$.

Then we have (ii).

From (ii) of Lemma 2.1.3, we have

$$\begin{aligned}
\min_{\sigma \leqslant t \leqslant \frac{1}{2}} G(t,s) &\geqslant \min_{\sigma \leqslant t \leqslant \frac{1}{2}} s(1-s)^2 \left[\Phi(t) + \frac{\beta t^2 + \alpha(1-t^2)}{(1-\beta\eta^2) - \alpha(1-\eta^2)} \Phi(\eta)\right] \\
&\geqslant s(1-s)^2 \left[\frac{\sigma^2}{2} + \frac{\beta\sigma^2 + \frac{3}{4}\alpha}{(1-\beta\eta^2) - \alpha(1-\eta^2)} \Phi(\eta)\right] = M_2 s(1-s)^2.
\end{aligned}$$

Thus (iii) holds.

Lemma 2.1.5.

If $y \in C([0,1])$ and $y \geqslant 0$, then the unique solution u of the boundary value problem (2.1.5),(2.1.6) satisfies $\min_{\sigma \leqslant t \leqslant \frac{1}{2}} u(t) \geqslant \theta \|u\|_1$ where $\|u\|_1 := \sup\{|u(t)|;\ 0 \leq t \leq 1\}$ and $\theta := \dfrac{M_2}{M_1}$.

Proof.

For any $t \in [0,1]$, by Lemma 2.1.4 we have

$$u(t) = \int_0^1 G(t,s) y(s) ds \leqslant M_1 \int_0^1 s(1-s)^2 y(s) ds,$$

thus $||u||_1 \leqslant M_1 \int_0^1 s(1-s)^2 y(s)ds$.

Moreover, from Lemma 2.1.4 for $t \in \left[\sigma, \dfrac{1}{2}\right]$ and $\sigma \in \left(0, \frac{1}{2}\right)$, we have

$$u(t) = \int_0^1 G(t,s)y(s)ds \ \geqslant M_2 \int_\sigma^{\frac{1}{2}} s(1-s)^2 y(s)ds \ \geqslant \frac{M_2}{M_1}||u||_1.$$

Lemma 2.1.6.

If u is a positive solution of (2.0.1), then $u(\eta) \geqslant \gamma||u||$, where $\gamma = \dfrac{\gamma_1}{\gamma_2}$, $\gamma_1 = \min\left\{\dfrac{\eta^2}{2}, \dfrac{\eta(1-\eta)}{2}\right\}$ and $\gamma_2 = \max\left(1, \dfrac{\gamma_1 \max(\alpha,\beta)}{\beta\eta^2 + \alpha(1-\eta^2)}\right)$.

Proof.

From (2.1.9), for every positive solution u of (2.0.1) we have

$$\begin{aligned} u(t) &= \lambda \int_0^1 g(t,s)a(s)f(s,u(s-\tau))ds \\ &\quad + \lambda \tfrac{\beta t^2 + \alpha(1-t^2)}{(1-\beta\eta^2)-\alpha(1-\eta^2)} \textstyle\int_0^1 g(\eta,s)a(s)f(s,u(s-\tau))ds. \end{aligned}$$

By Lemma 2.1.3 (i) we have

$$\begin{aligned} ||u|| &\leqslant \lambda \int_0^1 s(1-s)^2 a(s)f(s,u(s-\tau))ds \\ &\quad + \lambda \tfrac{\mu}{(1-\beta\eta^2)-\alpha(1-\eta^2)} \textstyle\int_0^1 g(\eta,s)a(s)f(s,u(s-\tau))ds \end{aligned}$$

where $\mu = \max(\alpha,\beta)$.

Then

$$\begin{aligned} \min\left\{\frac{\eta^2}{2}; \frac{\eta(1-\eta)}{2}\right\}||u|| &\leqslant \lambda \int_0^1 \min\left\{\frac{\eta^2}{2}; \frac{\eta(1-\eta)}{2}\right\} s(1-s)^2 a(s)f(s,u(s-\tau))ds \\ &\quad + \lambda \frac{\mu \min\left\{\frac{\eta^2}{2}; \frac{\eta(1-\eta)}{2}\right\}}{(1-\beta\eta^2)-\alpha(1-\eta^2)} \int_0^1 g(\eta,s)a(s)f(s,u(s-\tau))ds \end{aligned}$$

By Lemma 2.1.3 (ii) we have

$$\begin{aligned} \min\left\{\frac{\eta^2}{2}, \frac{\eta(1-\eta)}{2}\right\}||u|| &\leqslant \lambda \int_0^1 g(\eta,s)a(s)f(s,u(s-\tau))ds \\ &\quad + \lambda\nu \frac{\beta\eta^2 + \alpha(1-\eta^2)}{(1-\beta\eta^2)-\alpha(1-\eta^2)} \int_0^1 g(\eta,s)a(s)f(s,u(s-\tau))ds \end{aligned}$$

where $\nu = \dfrac{\mu\gamma_1}{\beta\eta^2 + \alpha(1-\eta^2)}$.
Then we deduce that $\gamma_1||u|| \leqslant \max(1,\nu)u(\eta) = \gamma_2 u(\eta)$.

We deduce from the results above that the boundary value problem (2.0.1) has a positive solution u if and only if u is positive and it is a fixed point of the operator T defined by

$$Tu(t) = \begin{cases} \alpha u(\eta), & -\tau \leqslant t \leqslant 0, \\ \lambda \displaystyle\int_0^1 G(t,s)a(s)f(s,u(s-\tau))ds, & 0 \leqslant t \leqslant 1. \end{cases} \tag{2.1.13}$$

Let P be given by the following set
$\{u \in C([-\tau,1]) \bigcap C^1([0,1]) : u(t) \geqslant 0 \text{ for } t \in [-\tau,1],$
$u(t) = \alpha u(\eta) \text{ for } -\tau \leqslant t \leqslant 0, u(1) = \beta u(\eta), u^{'}(0) = 0\big\}$
and K_θ be a cone in the Banach space $C([-\tau,1])$ defined by
$\mathrm{K}_\theta := \left\{ u \in \mathrm{P}, \min\limits_{\sigma \leqslant t \leqslant \frac{1}{2}} u(t) \geqslant \theta||u|| \right\}$
where $||u|| = \sup\{|u(t)| : -\tau \leqslant t \leqslant 1\}$ and $\sigma \in \left(0, \dfrac{1}{2}\right)$.

Lemma 2.1.7.

The fixed points of T are solutions of (2.0.1), furthermore $T : \mathrm{K}_\theta \to \mathrm{K}_\theta$ is completely continuous.

Proof.

From (2.1.13), we have

$$\begin{aligned} &(Tu)^{'''}(t) + \lambda a(t) f(t, u(t-\tau)) = 0, \qquad t \in J = [0,1], \\ &(Tu)(t) = \alpha(Tu)(\eta), \qquad -\tau \leqslant t \leqslant 0, \\ &(Tu)(1) = \beta(Tu)(\eta), \\ &(Tu)^{'}(0) = 0. \end{aligned}$$

Therefore, the fixed points of T are solutions of (2.0.1).

Moreover, from Lemma 2.1.5 we can see that $T : \mathrm{K}_\theta \to \mathrm{K}_\theta$ is well defined.

Thus $T(\mathrm{K}_\theta) \subset \mathrm{K}_\theta$.

Next, we shall show that T is completely continuous.

Suppose $u_n \to u$ $(n \to \infty)$ and $u_n \in \mathrm{K}_\theta$ $\forall n \in \mathbb{N}$, then there exists $M > 0$ such that $\|u_n\| \leq M$. Since f is continuous on $[0,1] \times [0, M]$, it is uniformly continuous.

Therefore, $\forall \varepsilon > 0$ there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(s,x) - f(s,y)| < \epsilon$ $\forall s \in [0,1],\ x, y \in [0, M]$ and there exists N such that $\|u_n - u\| < \delta$ for $n > N$, so $|f(s, u_n(s-\tau)) - f(s, u(s-\tau))| < \varepsilon$, for $n > N$ and $s \in [0,1]$. This implies that

$$
\begin{aligned}
|Tu_n(t) - Tu(t)| &= |\lambda \int_0^1 G(t,s)a(s)(f(s,u_n(s-\tau)) - f(s,u(s-\tau)))ds| \\
&\leq \lambda \int_0^1 G(t,s)a(s)|f(s,u_n(s-\tau)) - f(s,u(s-\tau)|ds \\
&\leq \epsilon\lambda \int_0^1 G(t,s)a(s)ds.
\end{aligned}
$$

Therefore T is continuous.

Let Ω be any bounded subset of K_θ, then there exists $\gamma > 0$ such that $||u|| \leqslant \gamma$ for all $u \in \Omega$.

Since f is continuous on $[0,1] \times [0,\gamma]$ there exists $L > 0$ such that $|f(t,v)| < L$ $\forall (t,v) \in [0,1] \times [0,\gamma]$. Consequently, for all $u \in \Omega$ and $t \in [0.1]$ we have

$$
|Tu(t)| = \left|\lambda \int_0^1 G(t,s)a(s)f(s,u(s-\tau))ds\right| \leqslant \lambda M_1 L \int_0^1 s(1-s)^2 a(s)ds.
$$

Which implies the boundedness of $T\Omega$.

Since G is continuous on $[0,1] \times [0,1]$ it is uniformly continuous.

Then $\forall \epsilon > 0$ there exists $\delta > 0$ such that $|t_1 - t_2| < \delta$ implies that $|G(t_1, s) - G(t_2, s)| <$

ϵ $\forall s \in [0, 1]$.

So, if $u \in \Omega$

$$\begin{aligned} |Tu(t_1) - Tu(t_2)| &\leq \lambda \int_0^1 |G(t_1, s) - G(t_2, s)| a(s) f(s, u_n(s - \tau)) ds \\ &\leq \lambda L \epsilon \int_0^1 a(s) ds. \end{aligned}$$

From the arbitrariness of ϵ, we get the equicontinuity of $T\Omega$.

The operator T is completely continuous by the mean of the Ascoli-Arzela theorem.

The following theorem will be used to prove the existence of solutions of (2.0.1).

Theorem 2.1.1. ([29])

Let X *be a Banach space and* $\mathrm{K}(\subset \mathrm{X})$ *be a cone. Assume that* Ω_1 *and* Ω_2 *are open subsets of* X *with* $0 \in \Omega_1$ *and* $\overline{\Omega}_1 \subset \Omega_2$.

If $A : \mathrm{K} \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right) \to \mathrm{K}$ *is a completely continuous operator such that either*

(i) $||Au|| \leqslant ||u||$ *for* $u \in \mathrm{K} \cap \partial\Omega_1$, *and* $||Au|| \geqslant ||u||$ *for* $u \in \mathrm{K} \cap \partial\Omega_2$, *or*

(ii) $||Au|| \geqslant ||u||$ *for* $u \in \mathrm{K} \cap \partial\Omega_1$, *and* $||Au|| \leqslant ||u||$ *for* $u \in \mathrm{K} \cap \partial\Omega_2$.

then A *has a fixed point in* $\mathrm{K} \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$.

2.2 Main results

First, we define some important constants by the following

$$\begin{aligned} f^0 &= \limsup_{u\to 0^+} \max_{t\in J} \frac{f(t,u)}{u}, \\ f_0 &= \liminf_{u\to 0^+} \min_{t\in J} \frac{f(t,u)}{u}, \\ f^\infty &= \limsup_{u\to \infty} \max_{t\in J} \frac{f(t,u)}{u}, \\ f_\infty &= \liminf_{u\to \infty} \min_{t\in J} \frac{f(t,u)}{u}. \end{aligned}$$

Let A and B be defined by

$$A = \sup_{0\leqslant t\leqslant 1} \left(\alpha\gamma \int_0^\tau G(t,s)a(s)ds + \theta \int_{\sigma+\tau}^{\frac{1}{2}+\tau} G(t,s)a(s)ds \right)$$

and

$$B = M_1 \left(\alpha \int_0^\tau s(1-s)^2 a(s)ds \ + \int_\tau^1 s(1-s)^2 a(s)ds \right).$$

Theorem 2.2.1.

Suppose that $Af_\infty > Bf^0$.

Then for each

$$\lambda \in ((Af_\infty)^{-1}, (Bf^0)^{-1}) \tag{2.2.1}$$

the problem (2.0.1) has at least one positive solution.

Proof.

Let $\lambda \in ((Af_\infty)^{-1}, (Bf^0)^{-1})$, then there exists $\epsilon > 0$ such that

$$0 < \frac{1}{A(f_\infty - \epsilon)} \leq \lambda \leq \frac{1}{B(f^0 + \epsilon)}. \tag{2.2.2}$$

Let ϵ be fixed. By the definition of f^0, there exists $r > 0$ such that

$$f(s,u) \leqslant (f^0 + \epsilon)u \ \text{ for } \ 0 < u \leqslant r. \tag{2.2.3}$$

Let $\Omega_1 = \{u \in C([-\tau, 1]) : ||u|| < r\}$, then for $u \in \mathrm{K}_\theta \cap \partial\Omega_1$ we have by (2.2.3)

$$\begin{aligned}
Tu(t) &= \lambda \int_0^1 G(t,s)a(s)f(s,u(s-\tau))ds \\
&\leqslant \lambda M_1(f^0+\epsilon) \int_0^1 s(1-s)^2 a(s)u(s-\tau)ds \\
&= \lambda M_1(f^0+\epsilon) \left(\int_0^\tau s(1-s)^2 a(s)\alpha u(\eta)ds \;+ \int_\tau^1 s(1-s)^2 a(s)u(s-\tau)ds \right) \\
&\leqslant \lambda M_1(f^0+\epsilon) \left(\alpha \int_0^\tau s(1-s)^2 a(s)ds \;+ \int_\tau^1 s(1-s)^2 a(s)ds \right) ||u|| \\
&= \lambda B(f^0+\epsilon)||u||.
\end{aligned}$$

Then $Tu(t) \leq ||u||$. Therefore $||Tu|| \leq ||u||$.

Moreover, there exists $R > r$ such that $f(s,u) \geqslant (f_\infty - \epsilon)u$ for $u \geqslant R$.

Let $\Omega_2 = \{u \in C[-\tau, 1] : ||u|| < R\}$, then for $u \in \mathrm{K}_\theta \cap \partial\Omega_2$ we have

$$\begin{aligned}
||Tu|| &\geqslant \lambda(f_\infty-\epsilon) \sup_{0\leqslant t\leqslant 1} \int_0^1 G(t,s)a(s)u(s-\tau)ds \\
&= \lambda(f_\infty-\epsilon) \sup_{0\leqslant t\leqslant 1} \left(\int_0^\tau G(t,s)a(s)\alpha u(\eta)ds \;+ \int_\tau^1 G(t,s)a(s)u(s-\tau)ds \right) \\
&= \lambda(f_\infty-\epsilon) \sup_{0\leqslant t\leqslant 1} \left(\int_0^\tau G(t,s)a(s)\alpha u(\eta)ds \;+ \int_0^{1-\tau} G(t,s+\tau)a(s+\tau)u(s)ds \right) \\
&\geqslant \lambda(f_\infty-\epsilon) \sup_{0\leqslant t\leqslant 1} \left(\int_0^\tau G(t,s)a(s)\alpha u(\eta)ds \;+ \int_\sigma^{\frac{1}{2}} G(t,s+\tau)a(s+\tau)u(s)ds \right) \\
&\geqslant \lambda(f_\infty-\epsilon) \sup_{0\leqslant t\leqslant 1} \left(\int_0^\tau G(t,s)a(s)\alpha\gamma||u||ds \;+ \int_\sigma^{\frac{1}{2}} G(t,s+\tau)a(s+\tau)\theta||u||ds \right) \\
&\geqslant \lambda(f_\infty-\epsilon) \sup_{0\leqslant t\leqslant 1} \left(\alpha\gamma \int_0^\tau G(t,s)a(s)ds \;+ \theta \int_{\sigma+\tau}^{\frac{1}{2}+\tau} G(t,s)a(s)ds \right) ||u|| \\
&= \lambda A(f_\infty-\epsilon)||u||.
\end{aligned}$$

Then $||Tu|| \geq ||u||$.

Therefore, by (i) of Theorem 2.1.1, T has a fixed point $u \in \mathrm{K}_\theta \cap (\overline{\Omega}_2 \setminus \Omega_1)$ and $||u|| \geqslant r$.

From Lemma 2.1.7, u is a positive solution of (2.0.1).

Theorem 2.2.2.

Suppose that $Af_0 > Bf^\infty$, then for each

$$\lambda \in ((Af_0)^{-1}, (Bf^\infty)^{-1}) \tag{2.2.4}$$

the problem (2.0.1) has at least one positive solution.

Proof.

From (2.2.4) there exists $\epsilon > 0$ such that

$$0 < \frac{1}{A(f_0 - \epsilon)} \leq \lambda \leq \frac{1}{B(f^\infty + \epsilon)}. \tag{2.2.5}$$

Then there exists $r^* > 0$ such that

$$f(s,u) \geqslant (f_0 - \epsilon)u \quad \text{for} \quad 0 < u \leqslant r^*. \tag{2.2.6}$$

Let $\Omega_1 = \{u \in C[-\tau, 1] : ||u|| < r^*\}$, then for $u \in \mathrm{K}_\theta \cap \partial\Omega_1$ we have by (2.2.6)

$$\begin{aligned}
||Tu|| &\geqslant \lambda \sup_{0\leqslant t\leqslant 1} \int_0^1 G(t,s)a(s)(f_0-\epsilon)u(s-\tau)ds \\
&= \lambda(f_0-\epsilon) \sup_{0\leqslant t\leqslant 1} \left(\int_0^\tau G(t,s)a(s)\alpha u(\eta)ds \ + \int_\tau^1 G(t,s)a(s)u(s-\tau)ds \right) \\
&= \lambda(f_0-\epsilon) \sup_{0\leqslant t\leqslant 1} \left(\int_0^\tau G(t,s)a(s)\alpha u(\eta)ds \ + \int_0^{1-\tau} G(t,s+\tau)a(s+\tau)u(s)ds \right) \\
&\geqslant \lambda(f_0-\epsilon) \sup_{0\leqslant t\leqslant 1} \left(\int_0^\tau G(t,s)a(s)\alpha u(\eta)ds \ + \int_\sigma^{\frac{1}{2}} G(t,s+\tau)a(s+\tau)u(s)ds \right) \\
&\geqslant \lambda(f_0-\epsilon) \sup_{0\leqslant t\leqslant 1} \left(\int_0^\tau G(t,s)a(s)\alpha\gamma||u||ds \ + \int_\sigma^{\frac{1}{2}} G(t,s+\tau)a(s+\tau)\theta||u||ds \right) \\
&\geqslant \lambda(f_0-\epsilon) \sup_{0\leqslant t\leqslant 1} \left(\alpha\gamma \int_0^\tau G(t,s)a(s)ds \ + \theta \int_{\sigma+\tau}^{\frac{1}{2}+\tau} G(t,s)a(s)ds \right) ||u||. \\
&= \lambda A(f_0-\epsilon)||u||.
\end{aligned}$$

Then $||Tu|| \geq ||u||$.

By definition of f^{∞} we can choose $R_* > r^*$ such that for $u \geqslant R_*$, $f(s,u) \leqslant (f^{\infty}+\epsilon)u$.

Then

$$\begin{aligned}
Tu(t) &\leqslant \lambda \int_0^1 M_1 s(1-s)^2 a(s) f(s,u(s-\tau))ds \\
&\leqslant \lambda M_1 \int_0^1 s(1-s)^2 a(s) f(s,R^*)ds \\
&= \lambda M_1 (f^{\infty}+\epsilon)R_* \left(\alpha \int_0^{\tau} s(1-s)^2 a(s)ds \ + \int_{\tau}^1 s(1-s)^2 a(s)ds \ \right) \\
&= \lambda B (f^{\infty}+\epsilon) R_* \leq R_* = ||u||.
\end{aligned}$$

Then $||Tu|| \leq ||u||$.

Therefore, by (ii) of Theorem 2.1.1, T has a fixed point $u \in \mathrm{K}_{\theta} \cap (\overline{\Omega}_2 \setminus \Omega_1)$ and $||u|| \geqslant r^*$.

From Lemma 2.1.7, u is a positive solution of (2.0.1).

Chapter 3

Positive solutions for a second order three-point boundary value problem with delay

In this chapter, we investigate the existence and multiplicity of positive solutions to the following nonlinear second order boundary value problem with delay

$$
\begin{aligned}
& u''(t) + a(t)f(t, u(t-\tau)) = 0, \qquad t \in [0,1], \\
& u(t) = \beta u(\eta), \qquad -\tau \leqslant t \leqslant 0, \\
& u(1) = \alpha u(\eta),
\end{aligned}
\tag{3.0.1}
$$

where $0 < \eta < 1$, $0 < \alpha < \frac{1}{\eta}$ and $0 < \beta < \frac{1-\alpha\eta}{1-\eta}$ are given constants.

3.1 Preliminaries

Lemma 3.1.1.

Let $\beta \neq \frac{1-\alpha\eta}{1-\eta}$*. Then for* $y \in C([0,T],\ \mathbb{R})$*, the problem*

$$u''(t) + y(t) = 0, \quad t \in [0,T], \tag{3.1.1}$$

$$u(0) = \beta u(\eta), \quad u(1) = \alpha u(\eta) \tag{3.1.2}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)y(s)ds + \frac{\beta + (\alpha - \beta)t}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^1 G(\eta,s)y(s)ds \tag{3.1.3}$$

where

$$G(t,s) = \begin{cases} s(1-t), & 0 \leq s \leq t \leq 1, \\ t(1-s), & 0 \leq t \leq s \leq 1. \end{cases}$$

Proof.

From (3.1.1), we have

$$u(t) = u(0) + u'(0)t - \int_0^t (t-s)y(s)ds \ := A + Bt - \int_0^t (t-s)y(s)ds.$$

With

$$\begin{aligned} u(0) &= A, \\ u(\eta) &= A + B\eta - \textstyle\int_0^\eta (\eta - s)y(s)ds, \\ u(1) &= A + B - \textstyle\int_0^1 (1-s)y(s)ds. \end{aligned}$$

By (3.1.2) and from $u(0) = \beta u(\eta)$, we have

$$(1-\beta)A - B\beta\eta = -\beta \int_0^\eta (\eta - s)y(s)ds.$$

From $u(1) = \alpha u(\eta)$, we have

$$(1-\alpha)A + B(1-\alpha\eta) = \int_0^1 (1-s)y(s)ds - \alpha \int_0^\eta (\eta - s)y(s)ds.$$

Therefore,

$$\begin{aligned}A = &\frac{\beta\eta}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^1 (1-s)y(s)ds\\ &-\frac{\beta 1}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^\eta (\eta-s)y(s)ds\end{aligned}$$

and

$$\begin{aligned}B = &\frac{1-\beta}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^1 (1-s)y(s)ds\\ &-\frac{\alpha-\beta}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^\eta (\eta-s)y(s)ds\,,\end{aligned}$$

from which it follows that

$$\begin{aligned}u(t) =&\frac{\beta\eta}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^1 (1-s)y(s)ds\\ &-\frac{\beta}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^\eta (\eta-s)y(s)ds\\ &+\frac{(1-\beta)t}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^1 (1-s)y(s)ds\\ &-\frac{(\alpha-\beta)t}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^\eta (\eta-s)y(s)ds-\int_0^t (t-s)y(s)ds\\ =&-\int_0^t (t-s)y(s)ds+\frac{(\beta-\alpha)t-\beta}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^\eta (\eta-s)y(s)ds\\ &+\frac{(1-\beta)t+\beta\eta}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^1 (1-s)y(s)ds\\ =&\int_0^1 G(t,s)y(s)ds+\frac{\beta+(\alpha-\beta)t}{(1-\alpha\eta)-\beta(1-\eta)}\int_0^1 G(\eta,s)y(s)ds.\end{aligned}$$

The function u presented above is a solution to the problem (3.1.1)-(3.1.2), and the uniqueness of u is obvious.

Lemma 3.1.2.

Let $0<\alpha<\frac{1}{\eta}$, $0\le\beta<\frac{1-\alpha\eta}{1-\eta}$. *If* $y\in C([0,1],\ [0,\infty))$, *then the unique solution* u *of the problem* (3.1.1)-(3.1.2) *satisfies*

$$u(t)\ge 0,\quad t\in[0,1].$$

Proof.

It is known that the graph of u is concave down on $[0,1]$ from $u''(t) = -y(t) \le 0$, so

$$\frac{u(\eta)-u(0)}{\eta} \ge \frac{u(1)-u(0)}{1}.$$

Combining this with (3.1.2), we have

$$\frac{1-\beta}{\eta}u(\eta) \ge \frac{\alpha-\beta}{1}u(\eta).$$

If $u(0) < 0$, then $u(\eta) < 0$. This implies that $\beta \ge \frac{1-\alpha\eta}{1-\eta}$, which is a contradiction with $\beta < \frac{1-\alpha\eta}{1-\eta}$.

If $u(1) < 0$, then $u(\eta) < 0$, and the same contradiction emerges.

Thus, it is true that $u(0) \ge 0$, $u(1) \ge 0$, together with the concavity of u, we have

$$u(t) \ge 0, \quad t \in [0,1]$$

as required.

Lemma 3.1.3.

Let $\alpha\eta \ne 1$ and $\beta > \max\{\frac{1-\alpha\eta}{1-\eta}, 0\}$. If $y \in C([0,1],[0,\infty))$, then problem (3.1.1)-(3.1.2) *has no nonnegative solutions.*

Proof.

Suppose that problem (3.1.1)-(3.1.2) has a nonnegative solution u satisfying $u(t) \ge 0, t \in [0,1]$ and there is a $t_0 \in (0,1)$ such that $u(t_0) > 0$.

If $u(1) > 0$, then $u(\eta) > 0$.

This implies

$$u(0) = \beta u(\eta) > \frac{1-\alpha\eta}{1-\eta}u(\eta) = \frac{u(\eta)-\eta u(1)}{1-\eta},$$

that is

$$\frac{u(1)-u(0)}{1} > \frac{u(\eta)-u(0)}{\eta},$$

which is a contradiction with the concavity of u.

If $u(1) = 0$, then $u(\eta) = 0$.

When $t_0 \in (0, \eta)$, we get $u(t_0) > u(\eta) = u(1)$, a violation of the concavity of u.

When $t_0 \in (\eta, 1)$, we get $u(0) = \beta u(\eta) = 0 = u(\eta) < u(t_0)$, another violation of the concavity of u.

Therefore, no nonnegative solutions exist.

Lemma 3.1.4.

Let $0 < \alpha < \frac{1}{\eta}$ and $0 < \beta < \frac{1-\alpha\eta}{1-\eta}$. If $y \in C([0,1], [0, \infty))$, then the unique solution to the problem (3.1.1)-(3.1.2) satisfies

$$\min_{t\in[0,1]} u(t) \geq \gamma\|u\|, \tag{3.1.4}$$

where

$$\gamma := \min\left\{\frac{\alpha(1-\eta)}{1-\alpha\eta}, \frac{\alpha\eta}{1}, \frac{\beta(1-\eta)}{1}, \frac{\beta\eta}{1}\right\}. \tag{3.1.5}$$

Proof.

It is known that the graph of u is concave down on $[0, 1]$ from $u''(t) = -y(t) \leq 0$.

We divide the proof into two cases. Case 1.

$0 < \alpha < 1$, then $\frac{1-\alpha\eta}{1-\eta} > \alpha$.

For $u(0) = \beta u(\eta) = \frac{\beta}{\alpha}u(1)$, it may develop in the following two possible directions.

(i) If $0 < \alpha \leq \beta$, then $u(0) \geq u(1)$, so

$$\min_{t\in[0,1]} u(t) = u(1).$$

Assume $\|u\| = u(t_1)$ for $t_1 \in [0, 1)$, then either $0 \leq t_1 \leq \eta < \rho(1)$, or $0 < \eta < t_1 < 1$.

If $0 \leq t_1 \leq \eta < \rho(1)$, then

$$\begin{aligned} u(t_1) &\leq u(1) + \frac{u(1) - u(\eta)}{1-\eta}(t_1 - 1) \\ &\leq u(1) + \frac{u(1) - u(\eta)}{1-\eta}(0 - 1) \\ &= \frac{u(\eta) - \eta u(1)}{1-\eta} \\ &= \frac{1-\alpha\eta}{\alpha(1-\eta)} u(1), \end{aligned}$$

from which it follows that $\min_{t\in[0,1]} u(t) \geq \frac{\alpha(1-\eta)}{1-\alpha\eta}\|u\|$.
If $0 < \eta < t_1 < 1$, from

$$\frac{u(\eta)}{\eta} \geq \frac{u(t_1)}{t_1} \geq \frac{u(t_1)}{1},$$

together with $u(1) = \alpha u(\eta)$, we have

$$u(1) > \frac{\alpha\eta}{1} u(t_1).$$

So $\min_{t\in[0,1]} u(t) \geq \frac{\alpha\eta}{1}\|u\|$. (ii) If $0 < \beta < \alpha$, then $u(0) \leq u(1)$, so

$$\min_{t\in[0,1]} u(t) = u(0).$$

Assume $\|u\| = u(t_2)$ for $t_2 \in (0,1]$, then either $0 < t_2 < \eta < \rho(1)$, or $0 < \eta \leq t_2 \leq 1$.

If $0 < t_2 < \eta < \rho(1)$, from

$$\frac{u(\eta)}{1-\eta} \geq \frac{u(t_2)}{1-t_2} \geq \frac{u(t_2)}{1},$$

together with $u(0) = \beta u(\eta)$, we have

$$u(0) \geq \frac{\beta(1-\eta)}{1} u(t_2).$$

Hence, $\min_{t\in[0,1]} u(t) \geq \frac{\beta(1-\eta)}{1}\|u\|$.
If $0 < \eta \leq t_2 \leq 1$, from

$$\frac{u(t_2)}{1} \leq \frac{u(t_2)}{t_2} \leq \frac{u(\eta)}{\eta},$$

together with $u(0) = \beta u(\eta)$, we have

$$u(0) \geq \frac{\beta\eta}{1} u(t_2).$$

So $\min\limits_{t\in[0,1]} u(t) \geq \dfrac{\beta\eta}{1}\|u\|$.

Case 2.

$\frac{1}{\eta} > \alpha \geq 1$, then $\frac{1-\alpha\eta}{1-\eta} \leq \alpha$.

In this case, $\beta < \alpha$ is true. This implies that $u(0) \leq u(1)$. So,

$$\min_{t\in[0,1]} u(t) = u(0).$$

Assume $\|u\| = u(t_2)$ for $t_2 \in (0,1]$. Since $\alpha \geq 1$, it is known that $u(\eta) \leq u(1)$, together with the concavity of u, we have $0 < \eta \leq t_2 \leq 1$. Similar to the above discussion,

$$\min_{t\in[0,1]} u(t) \geq \frac{\beta\eta}{1}\|u\|.$$

Summing up, we have

$$\min_{t\in[0,1]} u(t) \geq \gamma\|u\|,$$

where

$$0 < \gamma = \min\left\{\frac{\alpha(1-\eta)}{1-\alpha\eta},\ \frac{\alpha\eta}{1},\ \frac{\beta(1-\eta)}{1},\ \frac{\beta\eta}{1}\right\} < 1.$$

This completes the proof.

By Lemma (3.1.1), it is easy to see that the BVP (3.1.1)-(3.1.2) has a solution $u = u(t)$ if and only if u is a solution of the operator equation $u = Tu$, where

$$Tu(t) = \begin{cases} \beta u(\eta), \qquad -\tau \leq t \leq 0, \\ -\displaystyle\int_0^t (t-s)a(s)f(s,u(s-\tau))ds \\ -\dfrac{(-\beta+\alpha)t+\beta}{(1-\alpha\eta)-\beta(1-\eta)}\displaystyle\int_0^\eta (\eta-s)a(s)f(s,u(s-\tau))ds \\ +\dfrac{(1-\beta)t+\beta\eta}{(1-\alpha\eta)-\beta(1-\eta)}\displaystyle\int_0^1 (1-s)a(s)f(s,u(s-\tau))ds, \qquad 0 \leq t < s \leq 1. \end{cases}$$

We assume the following hypotheses:

(A_1) $f \in C([0,\infty),[0,\infty))$,

(A_2) $a \in C([0,1],[0,\infty))$ and there exists $t_0 \in (0,1)$, such that $a(t_0) > 0$.

Define

$$f_0 = \lim_{u\to 0^+} \frac{f(u)}{u}, \qquad f_\infty = \lim_{u\to\infty} \frac{f(u)}{u}.$$

And

$$M_1 = \int_0^1 (1-s)a(s)ds, \qquad M_2 = \beta \int_0^\tau (1-s)a(s)ds + \int_\tau^1 (1-s)a(s)ds.$$

Theorem 3.1.1.

Let Ω be the convex subset of Banach space X, $0 \in \Omega$ and $\Phi : \Omega \to \Omega$ be a completely continuous operator.

Then either

1. *Φ has at least one fixed point in Ω; or*

2. *the set $\{x \in \Omega / x = \lambda\Phi x,\ 0 < \lambda < 1\}$, is unbounded.*

3.2 Main results

Theorem 3.2.1.

Assume (A_1) and (A_2) hold. If $f_0 = 0$, then the boundary value problem (3.0.1) has at least one positive solution.

Proof.

Choose $\epsilon > 0$ such that $\epsilon \leq \dfrac{(1-\alpha\eta)-\beta(1-\eta)}{(1+\beta(1+\eta))M_2}$.

By $f_0 = 0$, we know that there exists constant $B > 0$, such that $f(u) < \epsilon u$ for $0 < u \leq B$.
Let

$$\Omega = \left\{ u \ /u \in C([-\tau, 1]), u \geq 0, \|u\| \leq B, \min_{0 \leq t \leq 1} u(t) \geq \gamma \|u\| \right\}.$$

Then Ω is the convex subset of X.

For $u \in \Omega$, by Lemmas 3.1.2 and 3.1.4, we know that $Tu(t) \geq 0$ and $\min_{0 \leq t \leq 1} Tu(t) \geq \gamma \|Tu\||$.
Moreover,

$$\begin{aligned}
Tu(t) &\leq \frac{(\beta - \alpha)t - \beta}{(1 - \alpha\eta) - \beta(1 - \eta)} \int_0^\eta (\eta - s)a(s)f(s, u(s - \tau))ds \\
&\quad + \frac{(1 - \beta)t + \beta\eta}{(1 - \alpha\eta) - \beta(1 - \eta)} \int_0^1 (1 - s)a(s)f(s, u(s - \tau))ds \\
&\leq \frac{\beta t}{(1 - \alpha\eta) - \beta(1 - \eta)} \int_0^\eta (\eta - s)a(s)f(s, u(s - \tau))ds \\
&\quad + \frac{t + \beta\eta}{(1 - \alpha\eta) - \beta(1 - \eta)} \int_0^1 (1 - s)a(s)f(s, u(s - \tau))ds
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\beta}{(1 - \alpha\eta) - \beta(1 - \eta)} \int_0^\eta (\eta - s)a(s)f(s, u(s - \tau))ds \\
&\quad + \frac{1 + \beta\eta}{(1 - \alpha\eta) - \beta(1 - \eta)} \int_0^1 (1 - s)a(s)f(s, u(s - \tau))ds \\
&\leq \frac{1 + \beta(1 + \eta)}{(1 - \alpha\eta) - \beta(1 - \eta)} \int_0^1 (1 - s)a(s)f(s, u(s - \tau))ds \\
&\leq \epsilon \frac{1 + \beta(1 + \eta)}{(1 - \alpha\eta) - \beta(1 - \eta)} \int_0^1 (1 - s)a(s)u(s - \tau)ds \\
&= \epsilon \frac{1 + \beta(1 + \eta)}{(1 - \alpha\eta) - \beta(1 - \eta)} \left(\int_0^\tau (1 - s)a(s)\beta u(\eta)ds + \int_\tau^1 (1 - s)a(s)u(s - \tau)ds \right) \\
&\leq \epsilon \|u\| \frac{1 + \beta(1 + \eta)}{(1 - \alpha\eta) - \beta(1 - \eta)} \left(\beta \int_0^\tau (1 - s)a(s)ds \ + \int_\tau^1 (1 - s)a(s)ds \ \right) \leq \|u\| \leq B.
\end{aligned}$$

Thus, $\|Tu\| \leq B$. Hence, $T\Omega \subset \Omega$.

We shall show that T is completely continuous.

Suppose $u_n \to u$ $(n \to \infty)$ and $u_n \in \Omega$ $\forall n \in \mathbb{N}$, then there exists $M > 0$ such that $\|u_n\| \leq M$.

Since f is continuous on $[0,1] \times [0,M]$, it is uniformly continuous.

Therefore, $\forall \varepsilon > 0$ there exists $\delta > 0$ such that $|x-y| < \delta$ implies $|f(s,x) - f(s,y)| < \epsilon$ $\forall s \in [0,1]$, $x, y \in [0,M]$ and there exists N such that $\|u_n - u\| < \delta$ for $n > N$, so $|f(s,u_n(s-\tau)) - f(s,u(s-\tau))| < \varepsilon$, for $n > N$ and $s \in [0,1]$.

This implies

$$
\begin{aligned}
|Tu_n(t) - Tu(t)| = |\int_0^1 G(t,s)a(s)(f(s,u_n(s-\tau)) - f(s,u(s-\tau))) \\
+\frac{\beta + (\alpha-\beta)t}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^1 G(\eta,s)(f(s,u_n(s-\tau)) - f(s,u(s-\tau)))ds| \\
\leq [1 + \frac{\beta + (\alpha-\beta)}{(1-\alpha\eta) - \beta(1-\eta)}] \int_0^1 G(s,s)a(s)|f(s,u_n(s-\tau)) - f(s,u(s-\tau)|ds \\
\leq [1 + \frac{\beta + (\alpha-\beta)}{(1-\alpha\eta) - \beta(1-\eta)}]\epsilon \int_0^1 G(s,s)a(s)ds.
\end{aligned}
$$

Therefore T is continuous.

Let D be any bounded subset of Ω, then there exists $\gamma > 0$ such that $||u|| \leq \gamma$ for all $u \in D$.

Since f is continuous on $[0,1] \times [0,\gamma]$ there exists $L > 0$ such that $|f(t,v)| < L$ $\forall (t,v) \in [0,1] \times [0,\gamma]$.

Consequently, for all $u \in D$ and $t \in [0.1]$ we have

$$
\begin{aligned}
|Tu(t)| \leq \left| [1 + \frac{\beta + (\alpha-\beta)}{(1-\alpha\eta) - \beta(1-\eta)}] \int_0^1 G(s,s)a(s)f(s,u(s-\tau))ds \right| \\
\leq [1 + \frac{\beta + (\alpha-\beta)}{(1-\alpha\eta) - \beta(1-\eta)}]L \int_0^1 s(1-s)a(s)ds.
\end{aligned}
$$

Which implies the boundedness of TD.

Since G is continuous on $[0,1] \times [0,1]$ it is uniformly continuous.

Then $\forall\epsilon > 0$ there exists $\delta > 0$ such that $|t_1 - t_2| < \delta$ implies that $|G(t_1, s) - G(t_2, s)| < \epsilon\ \forall s \in [0, 1]$. So, if $u \in D$ $|Tu(t_1) - Tu(t_2)| \leq \int_0^1 |G(t_1, s) - G(t_2, s)| a(s) f(s, u_n(s - \tau)) ds \leq L\epsilon \int_0^1 a(s) ds$.

From the arbitrariness of ϵ, we get the equicontinuity of TD.

The operator T is completely continuous by the mean of the Ascoli-Arzela theorem.

For $u \in \Omega$ and $u = \lambda Tu$, $0 < \lambda < 1$, we have $u(t) = \lambda Tu(t) < Tu(t) < B$, which implies $\|u\| \leq B$. So $\{x \in \Omega/x = \lambda\Phi x,\ 0 < \lambda < 1\}$, is unbounded.

By theorem 3.1.1, we know the operator T has at least one fixed point in Ω.

Thus the boundary value problem (3.0.1) has at least one positive solution. The proof is complete.

Theorem 3.2.2.

Assume $(H1) - (H4)$ *hold. If* $f_\infty = 0$, *then the boundary value problem (3.0.1) has at least one positive solution.*

Proof. Choose $\epsilon > 0$ and $\epsilon \leq \frac{(1 - \alpha\eta) - \beta(1 - \eta)}{2(1 - \beta + \beta\eta)M_1}$. By $f_\infty = 0$ we know that there exists constant $N > 0$, such that $f(t, u) < \epsilon u$ for $u > N$.

Select

$$B \geq N + 1 + \frac{2(1 - \beta + \beta\eta)}{(1 - \alpha\eta) - \beta(1 - \eta)} M_2 \max_{\substack{0 \leq s \leq 1 \\ 0 \leq u \leq N}} f(u).$$

Let $\Omega = \{u/u \in C[-\tau, 1], \|u\| \le B, \min\limits_{0<t<1} u(t) \le \gamma \|u\|\}$. Then for $u \in \Omega$, we have

$$\begin{aligned}
Tu(t) &\le \frac{(1-\beta)t + \beta\eta}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^1 (1-s)a(s)f(s, u(s-\tau))ds \\
&\le \frac{1-\beta+\beta\eta}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^1 (1-s)a(s)\ f(s, u(s-\tau))ds \\
&= \frac{1-\beta+\beta\eta}{(1-\alpha\eta) - \beta(1-\eta)} \int_{J_1=\{s\in[0,1]/u>N\}} (1-s)a(s)f(s, u(s-\tau))ds \\
&\quad + \frac{1-\beta+\beta\eta}{(1-\alpha\eta) - \beta(1-\eta)} \int_{J_2=\{s\in[0,1]/u\le N\}} (1-s)a(s)f(s, u(s-\tau))ds \\
&\le \frac{1-\beta+\beta\eta}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^1 (1-s)a(s)\epsilon u(s-\tau)ds \\
&\quad + \frac{1-\beta+\beta\eta}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^1 (1-s)a(s) \max_{\substack{0\le s\le 1\\ 0\le u\le N}} f(s, u(s-\tau))ds \\
&\le \epsilon \frac{1-\beta+\beta\eta}{(1-\alpha\eta) - \beta(1-\eta)} \left(\beta \int_0^\tau (1-s)a(s)ds + \int_\tau^1 (1-s)a(s)ds\right) \|u\| \\
&\quad + \frac{1-\beta+\beta\eta}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^1 (1-s)a(s) \max_{\substack{0\le s\le 1\\ 0\le u\le N}} f(s, u(s-\tau))ds
\end{aligned}$$

$$\begin{aligned}
&\le \epsilon \frac{1-\beta+\beta\eta}{(1-\alpha\eta) - \beta(1-\eta)} M_1 B + \frac{1-\beta+\beta\eta}{(1-\alpha\eta) - \beta(1-\eta)} M_2 \max_{\substack{0\le s\le 1\\ 0\le u\le N}} f(s, u(s-\tau)) \\
&= \frac{1}{2}B + \frac{1-\beta+\beta\eta}{(1-\alpha\eta) - \beta(1-\eta)} M_2 \max_{\substack{0\le s\le 1\\ 0\le u\le N}} f(s, u(s-\tau)) \\
&\le \frac{1}{2}B + \frac{1}{2}B \quad = B.
\end{aligned}$$

Thus, $\|Tu\| \leq B$. Hence, $T\Omega \subset \Omega$.

It is easy that $T : \Omega \to \Omega$ is completely continuous.

For $u \in \Omega$ and $u = \lambda Tu$, $0 < \lambda < 1$, we have $u(t) = \lambda Tu(t) < Tu(t) < B$, which implies $\|u\| \leq B$. So $\{x \in \Omega/x = \lambda \Phi x,\ 0 < \lambda < 1\}$, is unbounded.

By theorem 3.1.1, we know the operator T has at least one fixed point in Ω. Thus the boundary value problem (3.0.1) has at least one positive solution. The proof is complete.

Chapter 4

Multiple positive solutions of the p-Laplacian

In this chapter, we consider the following boundary value problem involving the p-laplacian

$$-(|u'(x)|^{p-2}u'(x))' = \lambda f(u(x)), \quad a.e.\ 0 < x < 1 \tag{4.0.1}$$

$$u(0) = u(1) = 0 \tag{4.0.2}$$

where $\lambda \geq 0$, $p \in (1, 2]$ and $f : \mathbb{R}_+ \to \mathbb{R}_+^*$ smooth enough.

We investigate the existence of positive solutions of the p-Laplacian using the quadrature method. We prove the existence of multiple positive solutions of (4.0.1), (4.0.2), in both case $\lim\limits_{s\to+\infty} \dfrac{f(s)}{s^{p-1}} = A$ with $0 < A \leq +\infty$ and $\lim\limits_{s\to+\infty} \dfrac{f(s)}{s^{p-1}} = 0$.

4.1 Preliminaries

In this section, we give some definitions and preliminaries.

Definition 4.1.1.

A pair $(u, \lambda) \in \mathrm{C}^1([0,1];\mathbb{R}_+) \times [0, +\infty[$ *is called a solution of* (4.0.1), (4.0.2), *if*

1. $(|u'|^{p-2}u')$ *is absolutely continuous, and*

2. $-(|u'|^{p-2}u')' = \lambda f(u)$ *a.e. in* $(0,1)$, *and* $u(0) = u(1) = 0$.

Remarque 4.1.1.

The pair $(0,0)$ *is a solution of* (4.0.1), (4.0.2).

Definition 4.1.2.

The function f *be called p-sublinear if* $\lim_{s \to +\infty} \frac{f(s)}{s^{p-1}} = 0$, *and it is called p-superlinear if* $\lim_{s \to +\infty} \frac{f(s)}{s^{p-1}} = A$, $(0 < A \leq \infty)$.

Let $F : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $F(u) = \int_0^\rho f(s)ds$, and $g : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by

$$g(\rho) := 2\left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_0^\rho \frac{ds}{[F(\rho) - F(s)]^{\frac{1}{p}}}$$

for $\rho > 0$, and $g(0) = 0$.

Then we have

Lemma 4.1.1.

The function g *is continuous.*

Proof.

Let $r > 0$ fixed and $\rho \in [0, r]$, we have

$$(\lambda(\rho))^{\frac{1}{p}} = 2\left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_0^\rho \frac{ds}{(F(\rho) - F(s))^{\frac{1}{p}}} = 2\left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_0^1 \frac{\rho dv}{(F(\rho) - F(\rho v))^{\frac{1}{p}}} \text{ .(4.1.1)}$$

Let $m(r)=\inf\limits_{t\in[0,r]}\{f(t)\}$. on a $m(r)>0$, $\forall r\geq 0$.
by using mean value theorem, for all $v\in[0,1]$

$$\exists c_0 \in]\rho v, \rho[\subset]0,r[, \text{ tel que } F(\rho)-F(\rho v)=\rho(1-v)f(c_0)\geq \rho(1-v)m(r). \quad (4.1.2)$$

from where

$$F(\rho)-F(\rho v)\geq \rho(1-v)m(r). \quad (4.1.3)$$

so

$$0\leq \frac{\rho}{(F(\rho)-F(\rho v))^{\frac{1}{p}}}\leq \frac{\rho}{(m(r)\rho(1-v))^{\frac{1}{p}}}=\frac{\rho^{1-\frac{1}{p}}}{(m(r)(1-v))^{\frac{1}{p}}}. \quad (4.1.4)$$

Therefore

$$0\leq(\lambda(\rho))^{\frac{1}{p}}\leq 2\rho^{1-\frac{1}{p}}\left(\frac{p-1}{p\ m(r)}\right)^{\frac{1}{p}}\int_0^1\frac{dv}{(1-v)^{\frac{1}{p}}}\leq 2r^{1-\frac{1}{p}}\left(\frac{p-1}{p\ m(r)}\right)^{\frac{1}{p}}\int_0^1\frac{dv}{(1-v)^{\frac{1}{p}}}<\infty \quad (4.1.5)$$

The convergence of the improper integral $\displaystyle\int_0^1\frac{dv}{(1-v)^{\frac{1}{p}}}$ implies that the improper integral $\displaystyle\int_0^\rho\frac{ds}{(F(\rho)-F(s))^{\frac{1}{p}}}$ converges uniformly in $[0,r]$.
So $\lambda(\rho)$ is contained in $[0,r]$, r being arbitrary $\mathbb{R}_+$ so $\lambda(\rho)$ is contained in $\mathbb{R}_+$.

Lemma 4.1.2.

If f is class C^1, then g is differentiable and

$$g'(\rho)=2\left(\frac{p-1}{p}\right)^{\frac{1}{p}}\int_0^1\frac{H(\rho)-H(\rho v)}{[F(\rho)-F(\rho v)]^{1+\frac{1}{p}}}dv, \quad (4.1.6)$$

where $H(s)=F(s)-\frac{s}{p}f(s)$.

Proof.

Let $r>0$ fixed and $\rho\in[0,r]$, we have

$$g'(\rho)=2\left(\frac{p-1}{p}\right)^{\frac{1}{p}}\int_0^1\frac{H(\rho)-H(\rho v)}{[F(\rho)-F(\rho v)]^{1+\frac{1}{p}}}dv$$

$$= 2\left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_0^1 \frac{[F(\rho)-\frac{\rho}{p}f(\rho)]-[F(\rho v)-\frac{\rho v}{p}f(\rho v)]}{[F(\rho)-F(\rho v)]^{1+\frac{1}{p}}}dv,$$
$$= \frac{2}{p}\left(\frac{p-1}{p}\right)^{\frac{1}{p}}\left[\int_0^1 \frac{(p-1)dv}{[F(\rho)-F(\rho v)]^{\frac{1}{p}}} - \int_0^1 \frac{\rho^2}{[F(\rho)-F(\rho v)]^{1+\frac{1}{p}}}\int_v^1 tf'(t\rho)dtdv\right]$$

by using mean value theorem, for all $v \in [0,1]$

$$\exists c_0 \in]\rho v, \rho[\subset]0, r[, \text{ tel que } F(\rho)-F(\rho v) = \rho(1-v)f(c_0) \geq \rho(1-v)m(r).$$

D'où
$$0 \leq \int_0^1 \frac{dv}{[F(\rho)-F(\rho v)]^{\frac{1}{p}}} \leq \left(\frac{1}{\rho m(r)}\right)^{\frac{1}{p}} \int_0^1 \frac{dv}{(1-v)^{\frac{1}{p}}} < \infty.$$

On the other hand we have

$|\int_v^1 tf'(t\rho)dt| \leq \max\limits_{0\leq\rho t\leq\rho} |f'(\rho t)|(1-v) = \max\limits_{0\leq s\leq\rho} |f'(s)|(1-v) = M(\rho)(1-v)$, where $M(\rho) = \max\limits_{0\leq s\leq\rho} |f'(s)|$

therefore

$$\int_0^1 \frac{1}{[F(\rho)-F(\rho v)]^{1+\frac{1}{p}}}|\int_v^1 tf'(t\rho)dt|dv \leq \frac{M(\rho)}{(\rho m(r))^{1+\frac{1}{p}}}\int_0^1 (1-v)^{-(1+\frac{1}{p})}(1-v)dv$$
$$= \frac{M(\rho)}{(\rho m(r))^{1+\frac{1}{p}}}\int_0^1 \frac{1}{(1-v)^{\frac{1}{p}}}dv < \infty.$$

It follows that f is continuously differentiable.

Lemma 4.1.3. *1. If* $\lim\limits_{s\to+\infty} \frac{f(s)}{s^{p-1}} = 0$, *then* $\lim\limits_{\rho\to+\infty} g(\rho) = +\infty$.

2. If $\lim\limits_{s\to+\infty} \frac{f(s)}{s^{p-1}} = +\infty$, *then* $\lim\limits_{\rho\to+\infty} g(\rho) = 0$.

Proof.

1. Let $\lim\limits_{s\to+\infty} \frac{f(s)}{s^{p-1}} = 0$. We have $g(\rho) \geq 2\left(\frac{p-1}{p}\right)^{\frac{1}{p}}\left(\frac{\rho^p}{F(\rho)}\right)^{\frac{1}{p}}$.

   Then $\lim\limits_{\rho\to+\infty} \frac{\rho^p}{F(\rho)} = \lim\limits_{\rho\to+\infty} p\frac{\rho^{p-1}}{f(\rho)} = +\infty$.

   Hence $\lim\limits_{\rho\to+\infty} G(\rho) = +\infty$.

2. Without loss of generality, assume that $0 < A < +\infty$ and $M = \frac{A}{2}$. Then, there exists a positive number R such that $f(s) > Ms^{p-1}$ for $s \geq R$.

   Thus, for $R \leq s \leq \rho$, we have

$$F(\rho) - F(s) = \int_s^\rho f(t)dt \geq \frac{M}{p}(\rho^p - s^p). \tag{4.1.7}$$

   This implies

$$\int_R^\rho (F(\rho) - F(s))^{-\frac{1}{p}} ds \leq \left(\frac{M}{p}\right)^{-\frac{1}{p}} \rho^{-1} \int_{\frac{R}{\rho}}^1 (1-w)^{-\frac{1}{p}} dw. \tag{4.1.8}$$

   That is

$$\int_R^\rho (F(\rho) - F(s))^{-\frac{1}{p}} ds \leq \left(\frac{M}{p}\right)^{-\frac{1}{p}} \frac{p-1}{p} \rho^{-1} \left(1 - \frac{R}{\rho}\right)^{\frac{p-1}{p}}. \tag{4.1.9}$$

   Then

$$\lim_{\rho \to +\infty} \int_R^\rho (F(\rho) - F(s))^{-\frac{1}{p}} ds = 0. \tag{4.1.10}$$

   Moreover, for $0 \leq s \leq R \leq \rho$ we have $F(\rho) - F(s) \geq F(\rho) - F(R)$, using (4.1.7) we obtain

$$F(\rho) - F(s) \geq \frac{M}{p}(\rho^p - R^p). \tag{4.1.11}$$

   Which gives

$$\int_0^R (F(\rho) - F(s))^{-\frac{1}{P}} ds \leq \int_0^R (\frac{M}{p})^{-\frac{1}{P}} (\rho^p - R^p)^{-\frac{1}{P}} ds. \tag{4.1.12}$$

   That is

$$\int_0^R (F(\rho) - F(s))^{-\frac{1}{P}} ds \leq (\frac{M}{p})^{-\frac{1}{P}} (\rho^p - R^p)^{-\frac{1}{P}} R. \tag{4.1.13}$$

   Then

$$\lim_{\rho \to +\infty} \int_0^R (F(\rho) - F(s))^{-\frac{1}{p}} ds = 0. \tag{4.1.14}$$

   From (4.1.10) and (4.1.14), we deduce that $\lim\limits_{\rho \to +\infty} g(\rho) = 0$.

The case $A = +\infty$ can be handled by similar arguments.

For $u \in \mathrm{C}^1([0,1];\mathbb{R}_+)$, we define $||u|| := \sup\{u(s); s \in (0,1)\}$.

Lemma 4.1.4.

If (u, λ) is a solution of (4.0.1), (4.0.2) with $\lambda > 0$, then $\lambda^{\frac{1}{p}} = g(||u||)$.

Proof.

Let (u, λ) be a positive solution of (1), (2) with $\lambda > 0$, and $u > 0$ in $(0, 1)$.

By the maximum principle and symmetry. We have $u'(\frac{1}{2}) = 0$, $u(\frac{1}{2}) = ||u||$.

Moreover, $u'(x) > 0$ for $x \in (0, \frac{1}{2})$ and $u'(x) < 0$ for $x \in (\frac{1}{2}, 1)$.

Let $\rho = ||u||$. Multiplying (4.0.1) by $u'(x)$, and integrate it for $x \in [0, \frac{1}{2}]$, we obtain

$$-\int_x^{\frac{1}{2}} (|u'(t)|^{p-2}u'(t))'u'(t)dt = \int_x^{\frac{1}{2}} \lambda f(u(t))u'(t)dt. \tag{4.1.15}$$

We have in one hand

$$\int_x^{\frac{1}{2}} \lambda f(u(t))u'(t)dt = \lambda \int_{u(x)}^{u(\frac{1}{2})} f(y)dy = \lambda(F(\rho) - F(u(x))), \tag{4.1.16}$$

and in the other hand

$$-\int_x^{\frac{1}{2}} (|u'(t)|^{p-2}u'(t))'u'(t)dt = \frac{(p-1)}{p}(u'(x))^p. \tag{4.1.17}$$

From (4.1.15),(4.1.16) and (4.1.17), we have

$$\frac{(p-1)}{p}(u'(x))^p = \lambda(F(\rho) - F(u(x))). \tag{4.1.18}$$

Then for all $x \in (0, \frac{1}{2})$, we have

$$(u'(x))^p = \left(\frac{p}{p-1}\right)\lambda(F(\rho) - F(u(x))), \tag{4.1.19}$$

which implies

$$u'(x) = \left(\frac{p}{p-1}\right)^{\frac{1}{p}} [\lambda(F(\rho) - F(u(x)))]^{\frac{1}{p}} \quad \text{for } x \in [0, \frac{1}{2}], \tag{4.1.20}$$

and by symmetry

$$u'(x) = -\left(\frac{p}{p-1}\right)^{\frac{1}{p}} [\lambda(F(\rho) - F(u(x)))]^{\frac{1}{p}} \quad \text{for } x \in [\frac{1}{2}, 1]. \tag{4.1.21}$$

From (4.1.16), we obtain

$$\frac{1}{2}\lambda^{\frac{1}{p}} = \left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_0^\rho \frac{ds}{[F(\rho) - F(s)]^{\frac{1}{p}}}. \tag{4.1.22}$$

Then $\lambda^{\frac{1}{2}} = g(\rho) = g(||u||)$.

Moreover, we have

$$g'(\rho) = 2\left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_0^1 \frac{H(\rho) - H(\rho v)}{[F(\rho) - F(\rho v)]^{1+\frac{1}{p}}} dv. \tag{4.1.23}$$

Where $H(s) = F(s) - \frac{s}{p} f(s)$.

4.2 Main results

In the section we give our main results.

Theorem 4.2.1.
If $\lim_{s \to +\infty} \frac{f(s)}{s^{p-1}} = A$, *with* $0 < A \leq +\infty$, *then there exists* $\lambda^* > 0$ *such that the problem* (4.0.1), (4.0.2) *has at least two positive solutions for* $\lambda \in (0, \lambda^*)$, *and at least one positive solution for* $\lambda = \lambda^*$, *and zero positive solution for* $\lambda > \lambda^*$.

Proof.

From lemma 4.1.3, $\lim_{s\to+\infty} g(s) = g(0) = 0$. Then g is bounded and reaches its maximum at some point $\rho_0 > 0$. Further $\lambda^* = (g(\rho_0))^p$.

Theorem 4.2.2.
If $\lim_{s\to+\infty} \frac{f(s)}{s^{p-1}} = 0$, *then the problem* (4.0.1), (4.0.2) *has at least one positive solutions for all* $\lambda > 0$.

Proof.

Let $\lim_{s\to+\infty} \frac{f(s)}{s^{p-1}} = 0$. Then from lemma 5.1.10, we have $\lim_{s\to+\infty} g(s) = +\infty$ and $g(0) = 0$, then (5.0.1), (5.0.2) has at least one positive solution for all $\lambda > 0$.

Theorem 4.2.3.
If $\lim_{s\to+\infty} \frac{f(s)}{s^{p-1}} = 0$ *and* H *is nondecreasing, then* (4.0.1), (4.0.2) *has a unique positive solution for each* $\lambda > 0$.

Proof.

If H is nondecreasing $H(\rho) - H(\rho v) \geq 0$ for all $v \in (0,1) and\ \rho > 0$. From (4.1.23) we have $g'(\rho) > 0$ for all $\rho > 0$. That is g increasing from 0 to $+\infty$. hence (4.0.1), (4.0.2) has one and only one positive solution for each $\lambda > 0$.

Corollaire 4.2.1.
If $\lim_{s\to+\infty} \frac{f(s)}{s^{p-1}} = 0$, *and either* $(p-2)f'(s) > sf''(s)$ *for all* $s > 0$, *or* $(p-1)f(s) > sf'(s)$ *for all* $s > 0$, *then* (4.0.1), (4.0.2) *has a unique positive solution for each* $\lambda > 0$.

Proof.

If $(p-2)f'(s) > sf''(s)$ for all $s > 0$, then $H''(s) = \frac{1}{p}[(p-2)f'(s) - sf''(s)] > 0$ for $s > 0$.

That is H' is increasing. Further, we have $H'(0) > 0$, then $H'(s) > 0$ for $s > 0$, which implies that H is nondecreasing.

If $(p-1)f(s) > sf'(s)$ for $s > 0$, then $H'(s) > 0$ for $s > 0$, which implies that H is nondecreasing.

Then, in both cases we have H nondecreasing, from theorem 4.2.3 we deduce the result.

Theorem 4.2.4.
Assume that $\lim_{s\to+\infty} \frac{f(s)}{s^{p-1}} = 0$ *and* $\lim_{s\to+\infty} H(s) = +\infty$. *If there exist* M *and* $\sigma > 0$ *such that* $M > \sigma > 0$, H *is nondecreasing for* $s > M$ *and* $pF(\sigma) < \sigma f(\sigma)$, *then there exist* λ_1, λ_2 *with* $0 < \lambda_1 < \lambda_2$, *such that the problem* (4.0.1), (4.0.2) *has at least three positives solutions for* $\lambda \in (\lambda_1, \lambda_2)$, *at least two positives solutions for* $\lambda = \lambda_1$ *and* $\lambda = \lambda_2$, *a unique positive solution for* $\lambda \in (0, \lambda_1) \cup (\lambda_2, +\infty)$.

Proof.
We have in one hand $H(0) = 0$, $H'(0) > 0$ and $H(\sigma) < 0$, then there exists $s_1 \in (0, \sigma)$ such that $H'(s_1) = 0$, $H(s_1) > 0$ and $H(s_1) > H(s)$ for $s \in (0, s_1)$. That is $g'(s) > 0$.

And in the other hand, $H(s)$ is increasing in $(M, +\infty)$, then there exists $M_0 \geq M$ such that $H(M_0) > H(s)$ for $s \in (0, M_0)$, we have $H(s) \geq H(M_0) > H(s_1) > 0$ for $s \in [M_0, +\infty)$. for $s \in [M_0, +\infty)$. That is $g'(s) > 0$ for $s \in [M_0, +\infty)$.

Moreover, there exists $s_2 \in (s_1, \sigma)$ such that $H(s_2) = 0$, then $g'(s_2) < 0$,

From the precedent arguments, we deduce the existence of nonnegative numbers λ_1 and λ_2, such that (4.0.1), (4.0.2) has exactly one positive solution for each $\lambda \in (0, \lambda_1) \cup (\lambda_2, +\infty)$, at least two positive solutions for $\lambda = \lambda_1$ and $\lambda = \lambda_2$, and at least three positive solutions for $\lambda \in (\lambda_1, \lambda_2)$.

Corollaire 4.2.2.
Assume that $\lim_{s\to+\infty} \frac{f(s)}{s^{p-1}} = 0$ *and* $\lim_{s\to+\infty} ((p-1)f(s) - sf'(s)) > 0$.

If there exists $\sigma > 0$ such that $pF(\sigma) > \sigma f(\sigma)$, then there exist λ_1, λ_2 with $0 < \lambda_1 < \lambda_2$, such that the problem (4.0.1), (4.0.2) *has at least three positives solutions for $\lambda \in (\lambda_1, \lambda_2)$, at least two positives solutions for $\lambda = \lambda_1$ and $\lambda = \lambda_2$, a unique positive solution for $\lambda \in (0, \lambda_1) \cup (\lambda_2, +\infty)$.*

Proof.

From $\lim\limits_{s \to +\infty} ((p-1)f(s) - sf'(s)) > 0$, we have $\lim\limits_{s \to +\infty} H'(s) > 0$, then there exists $M > \sigma$ such that $H(s)$ is increasing in $(M, +\infty)$ and $\lim\limits_{s \to +\infty} H'(s) = +\infty$. From the precedent theorem we deduce the result.

Concluding remarks

As application, we can cite the example in Anuradha et al. **??** $f_c(u) = e^{\frac{cu}{c+u}}$. Conditions of theorem 4.2.2 and 4.2.3 are satisfied for $0 < c \leq 4(p-1)$ for all $p \in (1, 2]$, and conditions of theorem 4.2.4 are satisfied for $c > 4(p-1)$ and $p = 2$, by continuation we can say that there exists a subinterval $I_q := (q, 2]$ whit $1 < q < 2$, for which conditions of theorem 4.2.4 are satisfied, but we could not prove that $q = 1$. So, we construct an example satisfying conditions of theorem 4.2.4 for $p \in (1, 2)$, it is given by the following equations:

$$f_{a,b}(u) = \begin{cases} u + a, & for \quad 0 \leq u \leq b, \\ b + a, & for \quad u > b \end{cases}$$

With $\sigma = b$ such that $\sigma > 2a\dfrac{p-1}{2-p}$ and $a, b > 0$.

We have disussed the number of solutions only in the case where f is p-sublinear, because of the approach adopted in this work we could not discuss the p-superlinear case, the approach in Lakmeche and Hammoudi is more adapted to the last case.

Chapter 5

Double solutions of three-point boundary value problems for p-Laplacian

In this chapter, we consider the following nonlocal boundary value problem

$$-(|u'(x)|^{p-2}u'(x))' = \lambda f(u(x)), \quad a.e.\ 0 < x < 1 \tag{5.0.1}$$

$$u(0) = 0, \quad u(\xi) - u(1) = 0 \tag{5.0.2}$$

where $\lambda \geq 0$, $p \in (1, 2]$, $0 < \xi < 1$, and $f : \mathbb{R}_+ \to \mathbb{R}_+^*$ smooth enough.

When $p = 2$ the problem (5.0.1), (5.0.2) becomes similar to the problem studied by J. Henderson [18]. He proved the following theorem

Theorem 5.0.1. *([18], th. 3.1)*

Let $r > 0$ and $0 < a < b < c$, such that $0 < a < \frac{r[r(1-r)+\xi(1-\xi)]}{\xi(1-\xi)} b < \frac{r[r(1-r)+\xi(r-\xi)]}{(1-\xi)} c$.

And suppose that f satisfies the following conditions :

(A) $f(u) > \frac{2c}{\xi(1-\xi)}$*, if* $c \le u \le \frac{c}{\xi}$*,*

(B) $f(u) < \frac{2b}{\xi}$*, if* $0 \le u \le \frac{b}{\xi}$*,*

(C) $f(u) > \frac{2(1-\xi)a}{r[r(1-r)+\xi(1-\xi)]}$*, if* $0 \le u \le a$*.*

Then (5.0.1), (5.0.2) (for $p = 2, \lambda = 1$*) has at least two positive solutions,* u_1 *and* u_2 *such that*

$a < \max\limits_{0 \le t \le r} u_1(t)$*, with* $\max\limits_{0 \le t \le \xi} u_1(t) < b$*,*

and

$b < \max\limits_{0 \le t \le \xi} u_2(t)$*, with* $\min\limits_{\xi \le t \le r} u_2(t) < c$*.*

The main aim of this work is to prove the existence of multiple positive solutions for (5.0.1), (5.0.2). To reach our aim we use the quadrature method which is constructive and simple.

5.1 Preliminaries

In this section we give some definitions and preliminaries.

Definition 5.1.1.

A pair $(u, \lambda) \in \mathrm{C}^1([0, 1]; \mathbb{R}) \times [0, +\infty)$ *is called a solution of (5.0.1), (5.0.2), if*

1. $(|u'|^{p-2}u')$ *is absolutely continuous, and*

2. $-(|u'|^{p-2}u')' = \lambda f(u)$ *a.e. in* $(0, 1)$*, with* $u(0) = u(\xi) - u(1) = 0$*.*

Remarque 5.1.1.

The pair $(0, \lambda)$ *is a solution of (5.0.1), (5.0.2) if and only if* $\lambda = 0$*.*

Lemma 5.1.1.

If (u, λ) *is a solution of the problem (5.0.1), (5.0.2) with* $\lambda \neq 0$*, then* $u(x) \ge 0$ $\forall x \in [0, 1]$*.*

Proof.

Assume that there exists $x' \in]0,1[$ such that $u(x') < 0$, where $u(x') = \min_{x\in[0,1]} u(x)$. Since u is continuous there exists an interval $[\alpha, \beta] \subset [0, 1]$, such that $x' \in (\alpha, \beta)$ and $u(x) < 0$ for $x \in (\alpha, \beta)$. Moreover, we have $u(\alpha) = 0 = u(\beta)$ and $u'(\alpha) < 0 < u'(\beta)$.

Then

$$|u'(\alpha)|^{p-2}u'(\alpha) < 0 < |u'(\beta)|^{p-2}u'(\beta).$$

Integrating (5.0.1) on $[\alpha, \beta]$, we obtain

$$|u'(\beta)|^{p-2}u'(\beta) - |u'(\alpha)|^{p-2}u'(\alpha) = -\lambda \int_{\alpha}^{\beta} f(s)ds < 0,$$

since $f(u(x)) > 0$ for $x \in [\alpha, \beta]$, hence $u'(\beta) \leq 0 \leq u'(\alpha)$, which is impossible.

Then $u \geq 0$ $\forall x \in [0, 1]$.

Lemma 5.1.2.

Let (u, λ) be a solution of (5.0.1), (5.0.2) with $\lambda \neq 0$, then $u \neq 0$ and admits a unique maximum at $\frac{\xi+1}{2}$. Moreover $u'(0) > 0 > u'(1) = -u'(\xi)$, u' is increasing on $\left[0, \frac{\xi+1}{2}\right)$ and decreasing on $\left(\frac{\xi+1}{2}, 1\right]$, and $u(\xi) = u(1) > 0$.

Proof.

For $\lambda \neq 0$ we have $u \neq 0$ since f is positive.

The function u is continuous on the compact set $[0, 1]$, then it reaches its maximum at some point of $[0, 1]$. Since $u(\xi) = u(1)$, then there exists $x_0 \in (\xi, 1)$ such that $u'(x_0) = 0$. We have $|u'(x)|^{p-2}u'(x) = -\lambda \int_{x_0}^{x} f(s)ds < 0$, for $x > x_0$ and $|u'(x)|^{p-2}u'(x) = \lambda \int_{x}^{x_0} f(s)ds > 0$, for $x < x_0$. Thus $u'(x) > 0, \forall x \in [0, x_0)$ and $u'(x) < 0, \forall x \in (x_0, 1]$.

By symmetry of the solution in the interval $[\xi, 1]$ with respect to $\frac{\xi+1}{2}$ (see [11]), we find $x_0 = \frac{\xi+1}{2}$ and $0 = u(0) < u(\xi) = u(1)$.

Let $F : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by

$$F(u) = \int_0^\rho f(s)ds,$$

and $g : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by

$$g(\rho) = \begin{cases} 2\left(\frac{p-1}{p}\right)^{1/p} \displaystyle\int_0^\rho \frac{ds}{[F(\rho) - F(s)]^{1/p}}, & \text{for } \rho > 0, \\ 0 & \text{for } \rho = 0. \end{cases}$$

Let $\eta \geq 0$, and define $h_\eta : [\eta, +\infty) \to \mathbb{R}_+^*$, by

$$h_\eta(\rho) := \left(\frac{p-1}{p}\right)^{\frac{1}{p}} \left[\int_0^\rho \frac{ds}{[F(\rho) - F(s)]^{\frac{1}{p}}} + \int_\eta^\rho \frac{ds}{[F(\rho) - F(s)]^{\frac{1}{p}}}\right]$$

.

Remarque 5.1.2.

For $\eta \geq 0$, the functions g and h_η are continuous. Moreover $g(\rho) \leq 2h_\eta(\rho) \leq 2g(\rho)$, $h_0(\rho) = g(\rho), \forall \rho \geq \eta$ and $h_\eta(\eta) = 2g(\eta)$.

For $u \in \mathrm{C}^1([0, 1]; \mathbb{R}_+)$, we define $||u|| := \max\limits_{0 \leq x \leq 1} u(x) = \rho$.

Lemma 5.1.3.

If (u, λ) is a solution of (5.0.1), (5.0.2) with $\lambda > 0$, then $||u|| > u(1) > 0$ and $\lambda^{\frac{1}{p}} = h_\eta(||u||)$ where $\eta = u(1)$.

Proof.

Let (u, λ) be a positive solution of (5.0.1), (5.0.2) with $\lambda > 0$, then $u \neq 0$. From lemmas

5.1.1 and 5.1.2, we have $||u|| = u\left(\frac{\xi+1}{2}\right) > u(1) = u(\xi) = \eta > 0$, $u'\left(\frac{\xi+1}{2}\right) = 0$, $u'(x) > 0$ for $x \in \left(0, \frac{\xi+1}{2}\right)$, and $u'(x) < 0$ for $x \in \left(\frac{\xi+1}{2}, 1\right)$.

Multiplying (5.0.1) by $u'(x)$, and integrating it for $x \in \left[0, \frac{\xi+1}{2}\right]$. We obtain

$$-\int_x^{\frac{\xi+1}{2}} (|u'(t)|^{p-2}u'(t))'u'(t)dt = \int_x^{\frac{\xi+1}{2}} \lambda f(u(t))u'(t)dt. \tag{5.1.1}$$

We have in one hand

$$\int_x^{\frac{\xi+1}{2}} \lambda f(u(t))u'(t)dt = \lambda \int_{u(x)}^{u(\frac{\xi+1}{2})} f(y)dy = \lambda(F(\rho) - F(u(x))), \tag{5.1.2}$$

and in the other hand

$$-\int_x^{\frac{\xi+1}{2}} (|u'(t)|^{p-2}u'(t))'u'(t)dt = \frac{(p-1)}{p}(u'(x))^p. \tag{5.1.3}$$

From (5.1.1), (5.1.2) and (5.1.3), we have

$$\frac{(p-1)}{p}(u'(x))^p = \lambda(F(\rho) - F(u(x))). \tag{5.1.4}$$

Then for all $x \in \left(0, \frac{\xi+1}{2}\right)$, we have

$$(u'(x))^p = \left(\frac{p}{p-1}\right)\lambda(F(\rho) - F(u(x))), \tag{5.1.5}$$

which implies

$$u'(x) = \left(\frac{p}{p-1}\right)^{\frac{1}{p}} [\lambda(F(\rho) - F(u(x)))]^{\frac{1}{p}} \quad \text{for } x \in \left[0, \frac{\xi+1}{2}\right], \tag{5.1.6}$$

and by symmetry

$$u'(x) = -\left(\frac{p}{p-1}\right)^{\frac{1}{p}} [\lambda(F(\rho) - F(u(x)))]^{\frac{1}{p}} \quad \text{for } x \in \left[\frac{\xi+1}{2}, 1\right]. \tag{5.1.7}$$

Integrate (5.1.6) between 0 and $\frac{\xi+1}{2}$, we obtain

$$\lambda^{\frac{1}{p}}\left(\frac{\xi+1}{2}\right) = \left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_0^{\rho} \frac{ds}{[F(\rho) - F(s)]^{\frac{1}{p}}}. \tag{5.1.8}$$

Similarly, by integration of (5.1.7) between $\frac{\xi+1}{2}$ and 1, we obtain

$$\lambda^{\frac{1}{p}}\left(1-\frac{\xi+1}{2}\right)=\left(\frac{p-1}{p}\right)^{\frac{1}{p}}\int_{\eta}^{\rho}\frac{ds}{[F(\rho)-F(s)]^{\frac{1}{p}}}. \tag{5.1.9}$$

From (5.1.8) and (5.1.9) we deduce that

$$\lambda^{\frac{1}{p}}=\left(\frac{p-1}{p}\right)^{\frac{1}{p}}\left[\int_{0}^{\rho}\frac{ds}{[F(\rho)-F(s)]^{\frac{1}{p}}}+\int_{\eta}^{\rho}\frac{ds}{[F(\rho)-F(s)]^{\frac{1}{p}}}\right]. \tag{5.1.10}$$

From equation (5.1.10) , we deduce the results of lemma (5.1.3) .

Consider the Dirichlet boundary value problem constituted by (5.0.1) and the Dirichlet boundary conditions

$$u(0)=u(1)=0 \tag{5.1.11}$$

The problem (5.0.1), (5.1.11) has been studied in [22],[23] where the authors proved the following results.

Lemma 5.1.4. *([22])*

1. *If* $\lim\limits_{s\to+\infty}\dfrac{f(s)}{s^{p-1}}=0$*, then* $\lim\limits_{s\to+\infty}g(s)=+\infty$
2. *If* $\lim\limits_{s\to+\infty}\dfrac{f(s)}{s^{p-1}}=+\infty$*, then* $\lim\limits_{s\to+\infty}g(s)=0$

Theorem 5.1.1. *([23])*
If $\lim\limits_{s\to+\infty}\dfrac{f(s)}{s^{p-1}}=0$*, then the problem (5.0.1), (5.1.11) has at least one positive solution for all* $\lambda>0$.

Proof.
From lemma (5.1.4), $g(0)=0$ and $\lim\limits_{s\to+\infty}g(s)=+\infty$, then for all $\lambda>0$ there exist at least one positive solution.

Theorem 5.1.2. *([23])*

If $\lim_{s\to+\infty} \frac{f(s)}{s^{p-1}} = +\infty$, *then there exist* $\lambda^* > 0$ *such that the problem (*$5.0.1$*), (*$5.1.11$*) has at least two positive solutions for* $\lambda \in (0, \lambda^*)$, *and no positive solution for* $\lambda > \lambda^*$.

Proof.

From lemma (5.1.4), $g(0) = \lim_{s\to+\infty} g(s) = 0$, then g is bounded and admits a maximum at some point $\rho^* > 0$, we have then $\lambda^* = g(\rho^*)^p > 0$.

5.2 Main results

Let (u, λ) be a solution of (5.0.1), (5.1.11), $u(1) = \eta$ with $0 < \eta < \rho$ and $u(\frac{\xi+1}{2}) = \max_{x\in[0,1]} |u(x)| = \rho$.

From equation (5.1.7) we obtain

$$\lambda^{\frac{1}{p}}\xi = \left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_0^\eta \frac{ds}{[F(\rho) - F(s)]^{\frac{1}{p}}}. \tag{5.2.1}$$

Hence

$$\lambda^{\frac{1}{p}} = \frac{1}{\xi}\left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_0^\eta \frac{ds}{[F(\rho) - F(s)]^{\frac{1}{p}}}. \tag{5.2.2}$$

From (5.1.10) and (5.2.2), we have

$$\int_0^\rho \frac{ds}{[F(\rho) - F(s)]^{\frac{1}{p}}} + \int_\eta^\rho \frac{ds}{[F(\rho) - F(s)]^{\frac{1}{p}}} = \frac{1}{\xi}\int_0^\eta \frac{ds}{[F(\rho) - F(s)]^{\frac{1}{p}}}. \tag{5.2.3}$$

Theorem 5.2.1. *Let* $\rho > 0$, *then*

1. *there exist a unique* $\eta^*(\rho) \in (0, \rho)$, *such that (*$5.2.3$*) is satisfied for* $\eta = \eta^*(\rho)$, *further* η^* *is continuously differentiable,*

2. *for each* η^* *satisfying (*$5.2.3$*), there is a unique* $\lambda = \lambda(\eta^*(\rho))$ *given by* (5.1.10) *or* (5.2.2) *such that* (5.0.1), (5.0.2) *has exactly one solution* (u, λ), *with* $||u|| = u\left(\frac{\xi+1}{2}\right) = \rho$, $u(1) = u(\xi) = \eta^*(\rho)$ *and* $u'\left(\frac{\xi+1}{2}\right) = 0$,

Proof.

Let $G : [0, \rho] \to \mathbb{R}_+$ be defined by

$$G(\eta) = \int_0^\rho \frac{ds}{[F(\rho) - F(s)]^{\frac{1}{p}}} + \int_\eta^\rho \frac{ds}{[F(\rho) - F(s)]^{\frac{1}{p}}}. \tag{5.2.4}$$

We have $G(0) = 2G(\rho) = 2\displaystyle\int_0^\rho \frac{ds}{[F(\rho) - F(s)]^{\frac{1}{p}}}$ and G is differentiable on $(0, \rho)$, with

$$G'(n) = \frac{-1}{[F(\rho) - F(\eta)]^{\frac{1}{p}}} < 0, \quad \forall \eta \in (0, \rho) \tag{5.2.5}$$

Hence $G(\eta)$ is a decreasing function of η.

Let $H : [0, \rho] \to \mathbb{R}_+$ be defined by

$$H(\eta) = \frac{1}{\xi}\int_0^\eta \frac{ds}{[F(\rho) - F(s)]^{\frac{1}{p}}}. \tag{5.2.6}$$

Then $H(0) = 0$, $H(\rho) = \dfrac{1}{\xi}\displaystyle\int_0^\rho \frac{ds}{[F(\rho) - F(s)]^{\frac{1}{p}}} > g(\rho)$ and H is differentiable on $(0, \rho)$, with

$$H'(n) = \frac{1}{\xi}\frac{1}{[F(\rho) - F(\eta)]^{\frac{1}{p}}} > 0, \quad \forall \eta \in (0, \rho). \tag{5.2.7}$$

Hence, $H(\eta)$ is an increasing function of η.

Thus there exist a unique $\eta = \eta^*(\rho) \in (0, \rho)$ such that $G(\eta^*) = H(\eta^*)$. From the implicit function theorem $\eta^*(\rho)$ is continuously differentiable.

There exist unique $\lambda = \lambda(\eta^*(\rho))$ given by (5.2.2) for $\eta = \eta^*$, hence the problem (5.0.1), (5.0.2) has unique positive solution.

Corollaire 5.2.1.

Let $\rho > 0$, then the bifurcation diagram (λ, ρ) of the positive solutions of (5.0.1), (5.0.2) *is given by*
$\lambda^{\frac{1}{p}}(\rho) = \left(\dfrac{p-1}{p}\right)^{\frac{1}{p}} \left[\displaystyle\int_0^\rho \frac{ds}{[F(\rho) - F(s)]^{\frac{1}{p}}} + \int_{\eta^*(\rho)}^\rho \frac{ds}{[F(\rho) - F(s)]^{\frac{1}{p}}}\right]$, *where η^* is the solution of (*5.2.3*).*

Theorem 5.2.2.

1) If $\lim\limits_{s\to+\infty}\dfrac{f(s)}{s^{p-1}}=0$*, then* (5.0.1), (5.0.2) *has at least one positive solution for all* $\lambda>0$.

2) If $\lim\limits_{s\to+\infty}\dfrac{f(s)}{s^{p-1}}=+\infty$*, then there exist* $\lambda_0^*=\left(sup\left\{h_{\eta^*(s)}\left(s\right);s\in(0,+\infty)\right\}\right)^p$ *such that* (5.0.1), (5.0.2) *has at least two positive solutions for* $\lambda\in(0,\lambda_0^*)$*, and zero positive solution for* $\lambda>\lambda_0^*$.

Proof.

We have $g(\rho)\le 2h_{\eta^*(\rho)}(\rho)\le 2g(\rho)$, for all $\rho>0$. From theorems 5.1.1 and 5.1.2, we deduce the results above.

5.3 Concluding remarks

In this work we have studied a nonlocal boundary value problem involving the p-Laplacian, this problem has been studied by J. Henderson [[18].], who proved the existence of two positive solutions, for specific function f using the Avery-Henderson double fixed points theorem [4]. In our work we proved that the generalized problem, considered in this paper, has at least one positive solution for all $\lambda>0$ when $\lim\limits_{s\to+\infty}\dfrac{f(s)}{s^{p-1}}=0$, and at least two positive solutions when $\lim\limits_{s\to+\infty}\dfrac{f(s)}{s^{p-1}}=+\infty$, for $\lambda\in(0,\lambda^*)$. These results extend those obtained in [[18].], for $p\neq 0$. We have used the quadrature method which gives more information than the method used in [[18].], in fact, in lemma 2 we have information on the maximum of the solutions and on the signs of its derivative on $[0,1]$, and the description of the bifurcation diagram of the solutions in corollary 1.

It will be very interesting to study the exact number of solutions of (5.0.1), (5.0.2) when $\lim\limits_{s\to+\infty}\dfrac{f(s)}{s^{p-1}}=A$ where $0\le A\le\infty$, we might obtain three or more positive solutions for $A=0$, if we combine the conditions assumed by J. Henderson (theorem 1) and the conditions in 1) of theorem 5. Some recent works related to our problem (see [17]) consider a multi-point boundary value problem with $u(1) = \sum_{i=1}^{m} \alpha_i u(\xi_i), \xi_i \in (0,1), \alpha_i > 0$, in our knowledge there is no works on this problem using the quadrature method, it will be very interesting to see if this method is useful in this case.

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