N° d'ordre :

REPUBLIQUE ALGERIENNE DEMOCRATIQUE & POPULAIRE

MINISTERE DE L'ENSEIGNEMENT SUPERIEUR & DE LA RECHERCHE SCIENTIFIQUE

UNIVERSITE DJILLALI LIABES
FACULTE DES SCIENCES EXACTES
SIDI BEL ABBÈS

THESE DE DOCTORAT

Présentée par

DJILALI MEDJAHED

Spécialité : **MATHEMATIQUES**
Option : EQUATIONS AUX DERIVEES PARTIELLES

EXISTENCE GLOBALE ET ETUDE DE LA STABILISATION DES EQUATIONS D'EVOLUTION

Soutenue le 20/02/2019
Devant le jury composé de :

Président : Benaissa Abbes. Professeur Université Djilali Liabes SBA.
Examinateurs : Belghaba Kacem. Professeur université d' Oran1.
Boudaoud Fatima. Professeur université d' Oran1.
Mokeddem Soufiane. Professeur université Djilali Liabes SBA.
Abdelli Mama. MCA Université de Mascara.
Directeur de thèse : Hakem Ali. Professeur Université Djilali Liabes SBA.
Co-Directeur de thèse : ///
Année universitaire : *2018/2019*

Résumé en Francais

Titre sujet de thèse:
Existence globale et étude de la stabilisation des équations d´évolution

Resumé : Cette thèse est composé de deux éléments principaux.
Premièrement, nous avons étudié les problèmes de cauchy (3.3), (3.14), (4.1), Grâce à la méthode de la fonction test, sous certaines conditions, nous avons prouvé la non-existence des solutions globales pour ces problèmes , nous avons généralisé certains résultats obtenus par les auteurs, par exemple [28] and [53].
Deuxièmement, nous avons étudié le problème (5.1)-(5.5), et le problème(5.13)-(5.17), Grâce à la méthode de l´énergie perturbée et la technique des multiplicateurs , nous avons prouvé la stabilisation exponentielle de la solution et obtenu le taux de décroissance maximale de l´énergie pour les systèmes.

Mots clés : solution faible, exposant de Fujita, dérivé fractionnaire, Laplacian fractionnaire, onde amortie, feedback linéaire, exponentielle stabilisation.

Laboratory ACEDP, Djilali Liabes University
22000 Sidi Bel Abbes. Algeria.

Abstract in English

Title:
Global existence and study of the stabilization of evolution equations

Abstract : This thesis is composed of two main elements.
Firstly, we study the cauchy problems (3.3), (3.14), (4.1), thanks to the test function method, under some conditions, we proved the nonexistence of global solutions, we generalized some results obtained by authors, for example [28] and [53].
Secondly, we study the problem (5.1)-(5.5), and the problem (5.13)-(5.17), thanks to the perturbed energy method and the multipliers technique , we proved the exponential stabilization of solution and obtained the maximum rate decay of energy to the systems.

Keywords : weak solution, Fujita's exponent, fractional derivative, fractional Laplatian, damped wave, linear feedback, exponential stabilization.

Laboratory ACEDP, Djilali Liabes University
22000 Sidi Bel Abbes. Algeria.

Acknowledgment

First of all, I would like thank our God for his conciliation to me to achieve this point otherwise, I can not. I am deeply indebted to my supervisor, the adviser of this thesis, Prof. **Ali HAKEM**, who gave permanent support and encouraged me by his help and gold advices and encouragements about the choice of my diploma theme and for following suggestive and encouraging consultations which were practiced in general via e-mail due to the large geographical distance, and also for the help with the technical and stylistic revision of my thesis to write down the achieved results on fractional partial differential equations in general and particularly on fractional evolution equations ones with damping term.
Special thanks go to Mr. Prof. **Abbes BENAISSA** from University of Sidi Bel Abbès to have accepted to preside the jury. I would also like to thank the members of jury Mr. Prof. **Soufiane MOKEDDEM** from University of Sidi Bel Abbès, Mr. Prof. **Kacem BELGHABA** from Ahmed Ben Bella University of Oran 1, Mrs. Dr. **Mama ABDELLI** from university of Mascara and Mrs. Prof. **Fatima BOUDAOUD** from Ahmed Ben Bella University of Oran 1 to have accepted inspect the contents of my thesis, and also for there advices, remarks and orientations.
I would like to say thanks to mathematical division of university Djilali Liabes of Sidi Bel Abbès for their good treatment and for informing us of all that is new.

Sidi Bel Abbes Medjahed Djilali

Dedicate

This modest work is dedicated to my family,

My supervisor Professeur Hakem Ali

My Colleagues of University of Sidi Belabbes

My Colleagues of High School of Economic of Oran.

MEDJAHED DJILALI

Remerciements

Je sais tout particulièrement gré à mon épouse **Meriem**, pour tes conseils toujours trés pertinents.
On s'est toujours partagé les soucis de la thèse.
Ta présence à mes côtés et tes encouragements sont pour moi les piliers fondateurs de ce que je suis et de ce que je fais...

MEDJAHED

Notations

Ω: Bounded domain in $\mathbb{R}^N$.

Γ: Topological boundary of Ω.

$x = (x_1, x_1, ..., x_N)$:Generic point of $\mathbb{R}^N$.

$dx = dx_1 dx_1 ...dx_N$: Lebesgue measuring on Ω.

∇u: Gradient of u.

Δu: Laplacien of u.

f^+, f^- : $\max(f, 0)$, $\quad \max(-f, 0)$.

a.e: Almost everywhere.

$p^{'}$: Conjugate of p, i.e $\frac{1}{p} + \frac{1}{p'} = 1$.

$D(\Omega)$: Space of differentiable functions with compact support in Ω.

$D'(\Omega)$: Distribution space.

$C^k(\Omega)$: Space of functions k-times continuously differentiable in Ω.

$C_0(\Omega)$: Space of continuous functions null board in Ω.

$L^p(\Omega)$: Space of functions p-th power integrated on Ω with measure of dx.

$||f||_p = \left(\int_\Omega |f(x)|^p \right)^{\frac{1}{p}}$.

$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega), \nabla u \in (L^p(\Omega))^N \right\}$.

$W^{1,p}(\Omega)$: The closure of $D(\Omega)$ in $W^{1,p}(\Omega)$.

$||u||_{1,p} = \left(||u||_p^p + ||\nabla u||_p^p \right)^{\frac{1}{p}}$.

$W_0^{1,p}(\Omega)$: The closure of $D(\Omega)$ in $W^{1,p}(\Omega)$.

$W_0^{-1,p'}(\Omega)$: The dual space of $W_0^{1,p}(\Omega)$.

H: Hilbert space.

$H_0^1 = W_0^{1,2}(\Omega)$.

If X is a Banach space, we denote

$L^p(0, T; X) = \left\{ f : (0, T) \to X \text{is measurable; } \int_0^T |f(t)|_X^p dt < \infty \right\}$.

$L^\infty(0, T; X) = \left\{ f : (0, T) \to X \text{is measurable; } \sup_{t \in (0,T)} ess|f(t)|_X^p \right\}$.

$C^k([0, T]; X)$: Space of functions k-times continuously differentiable for $[0, T] \to X$.

$D([0, T]; X)$: Space of functions continuously differentiable with compact support in $[0, T]$.

$B_X = \{x \in X; ||x|| \leq 1\}$: unit ball.

Contents

Chapter 1

Intoduction

1.1 Brief History

1.1.1 History and Background of fractional derivation and its applications

The Fractional Calculus (**FC**) is a generalization of classical calculus concerned with operations of integration and differentiation of non-integer (fractional) order. The concept of fractional operators has been introduced almost simultaneously with the development of the classical ones. The first known reference can be found in the correspondence of G. W. Leibniz and Marquis de l'Hospital in 1695 where the question of meaning of the semi-derivative has been raised. This question consequently attracted the interest of many well- known mathematicians, including Euler, Liouville, Laplace, Riemann, Grünwald, Letnikov and many others. Since the 19th century, the theory of fractional calculus developed rapidly, mostly as a foundation for a number of applied disciplines, including fractional geometry, fractional differential equations (**FDE**) and fractional dynamics. The applications of FC are very wide nowadays. It is safe to say that almost no discipline of modern engineering and science in general, remains untouched by the tools and techniques of fractional calculus. For example, wide and fruitful applications can be found in rheology, viscoelasticity, acoustics, optics, chemical and statistical physics, robotics, control theory, electrical and mechanical engineering, bioengineering, etc..In fact, one could argue that real world processes are fractional order systems in general. The main reason for the success of (**FC**) applications is that these new fractional-order models are often more accurate than integer-order ones, i.e. there are more degrees of freedom in the fractional order model than in the corresponding classical one. One of the intriguing beauties of the subject is that fractional derivatives (and integrals) are not a local (or point) quantities. All fractional operators consider the entire history of the process being considered, thus being able to model the non-local and distributed effects often encountered in natural and technical phenomena. Fractional calculus is therefore an excellent set of tools for describing the memory and hereditary properties of various materials and processes.

1.1.2 Brief History and Background about PDEs stabilization

The Wave equation describes many physical phenomena (propagation of light waves, sound) and can also be considered as a simplified version of the elastodynamic system. In this more general context, a common problem (one can think for example of a beam embedded in a wall) consists in knowing how the solutions of the elastodynamic system behave if a part of the material has a part of its boundary put at rest and another on which an external force can act. From a mathematical point of view, the "simplified" case of waves is written in the form of a problem for the displacement u of the confined material

$$\begin{cases} u_{tt} - \Delta u = 0 \quad \text{in } \Omega \times \mathbb{R}^+, \\ u = 0 \quad \text{in } \partial\Omega_D \times \mathbb{R}^+, \\ \partial_\nu u = F \quad \text{in } \partial\Omega_N \times \mathbb{R}^+, \\ u(.,0) = u_0,\ u_t(.,0) = u_1 \quad \text{in } \Omega. \end{cases},$$

with F the exterior force exercising on the system, Ω_D the Dirichlet part of the bound stakes at rest, Ω_N the Neumann part of the bound on which the force exercises and (u_0, u_1) the Initial conditions. Let's go back to our example of the previous situation and consider the problem of whether the vibrations of a recessed beam can be attenuated in order to bring it back into a state of rest. To know if one can manage to bring the state back to equilibrium in a finite time from a state given initial is a control problem while know if we can mitigate the vibrations in controlling the speed at which one does it is a stabilization problem. A little observation shows us that these issues of stabilization and control are not limited to this framework. One can indeed think of the temperature of a room that can be modeled by the equation of heat and the problems of thermostats relating to the management of this temperature. These questions stimulated the study of the associated mathematical problems. The study of control and stabilization of partial differential equations has undergone a very important development since the work of Jacques-Louis Lions and David Russell in the late 1970s. A method for gain control and stabilization of the waves was the multiplier method developed in [35].

There exists several degrees of stability that one can study. The first degree consists at analyze merely the decreasing of the energy of the solutions towards zero, *i.e.* :

$$E(t) \to 0 \quad \text{when} \quad t \to +\infty.$$

For the second, one Study intermediate situations in which the solutions decreases of the polynomial type for example:

$$E(t) \leq \frac{C}{t^{\alpha}}, \quad \text{for } t > 0,$$

Where C And α Are positive constants with C depends on the initial data. In this case, one must take initial data more regular in the operator's domain.
As for the third, one is been interested in the decreasing of the fastest energy, namely when this one tends to 0 in an exponential manner i.e. :

$$E(t) \leq Ce^{-\delta t} \quad \text{for } t > 0,$$

Where C and δ are positive constants with C depends on the initial data.

1.2 Work plan

This thesis is composed of five principle chapters and an Annex. The first one is an Introduction, it contains in particular a brief history on the fractional calculus and its applications in life science and engineering, then a background about PDEs stabilization with Citing a simple example, this first chapter is ended by a third section devoted to the plane work of this thesis.
The second chapter is devoted to some notations and preliminaries needed, especially the first section treats the fractional integration and derivation in Caputo sense, proof of some principal inequalities used in proof of lemmas and theorem in this thesis, also we recall the definition of the fractional Laplacian, his chapter is finished by the notion of blow-up where we have introduce in particular what do peoples mean by blow-up.
The third chapter is divided in two section, the first one is devoted for the nonexistence of global solution to the problem (3.3), the the second section is consecrated for the nonexistence of global solution to the problem (3.14), in the two sections we have used the test function method by using a suitable test function, to prove the nonexistence of global solution, under some conditions. The next chapter, number four , is devoted to study the nonexistence of global solution to the problem (4.1) witch concerns system of semi-linear wave models with fractional damping, we have generalized some results obtained by Hakem [28] and F. Sun and M. Wang [53]. The last chapter contains two sections, the first one is devoted to study the exponential

decay of solutions to the problem (5.1)-(5.5), using the perturbed energy method, and finished it by obtaining the maximum rate of energy, in the second section we also study the exponential stabilization to the problem (5.13)-(5.17), using the multipliers technique, and finished it by obtaining the maximum rate of energy too.

Chapter 2

Notations and Preliminaries

In this chapter, we will introduce and state without proofs some important materials needed in the proof of our results,

2.1 Banach Spaces-Definition and properties

We first review some basic facts from calculus in the most important class of linear spaces " Banach spaces".

Definition 2.1.1. . *A Banach space is a complete normed linear space X. Its dual space X' is the linear space of all continuous linear functional $f : X \to \mathbb{R}$.*

Proposition 2.1.2. *X' equipped with the norm $\|.\|_{X'}$ defined by*

$$\|f\|_{X'} = \sup\{|f(u)| : \|u\| \leq 1\}, \tag{2.1}$$

is also a Banach space. We shall denote the value of $f \in X'$ at $u \in X$ by either $f(u)$ or $\langle f, u\rangle_{X',X}$.

2.1.3 Hilbert spaces

Now, we give some important results on these spaces here.

Definition 2.1.4. *A Hilbert space H is a vectorial space supplied with inner product $\langle u, v\rangle$ such that $\|u\| = \sqrt{\langle u, u\rangle}$ is the norm which let H complete.*

Theorem 2.1.5. *(Riesz). If $(H; \langle ., .\rangle)$ is a Hilbert space, $\langle ., .\rangle$ being a scalar product on H, then $H' = H$ in the following sense: to each $f \in H'$ there corresponds a unique $x \in H$ such that $f = \langle x, .\rangle$ and $\|f\|'_H = \|x\|_H$.*

Remark 2.1.6. *From this theorem we deduce that $H'' = H$. This means that a Hilbert space is reflexive.*

Theorem 2.1.7. *([50]) Let $(u_n)_{n\in N}$ is a bounded sequence in the Hilbert space H, it posses a sub-sequence which converges in the weak topology of H.*

Theorem 2.1.8. *([50]) In the Hilbert space, all sequence which converges in the weak topology is bounded.*

Theorem 2.1.9. *([50]) Let $(u_n)_{n\in N}$ be a sequence which converges to u, in the weak topology and $(\upsilon_n)_{n\in N}$ is an other sequence which converge weakly to υ, then*

$$\lim_{n\to\infty}\langle \upsilon_n, u_n\rangle = \langle \upsilon, u\rangle \tag{2.2}$$

Theorem 2.1.10. *([50]) Let X be a normed space, then the unit ball*

$$B' \equiv \{x \in X : \|x\| \leq 1\}, \tag{2.3}$$

of X' is compact in $\sigma(X', X)$.

2.2 Functional Spaces

2.2.1 The $L^p(\Omega)$ spaces

Definition 2.2.2. *Let $1 \leq p \leq \infty$ and let Ω be an open domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Define the standard Lebesgue space $L^p(\Omega)$ by*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |f(x)|^p dx < \infty \right\}. \tag{2.4}$$

Notation 2.2.3. *If $p = \infty$, we have*

$$L^\infty(\Omega) = \left\{ f : \Omega \to \mathbb{R} \text{ is measurable and there exists a constant } C \text{ such that } |f(x)| \leq C \text{ a.e in } \Omega \right\}.$$

Also, we denote by

$$\|f\|_\infty = \left\{ C,\ |f(x)| \leq C \text{ a.e in } \Omega \right\}. \tag{2.5}$$

Notation 2.2.4. *For $p \in \mathbb{R}$ and $1 \leq p \leq \infty$, we denote by q the conjugate of p i.e. $\frac{1}{p} + \frac{1}{q} = 1$.*

Theorem 2.2.5. *([50]) $L^p(\Omega)$ is a Banach space for all $1 \leq p \leq \infty$.*

Remark 2.2.6. *In particularly, when $p = 2$, $L^2(\Omega)$ equipped with the inner product*

$$\langle f, g\rangle_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx, \tag{2.6}$$

is a Hilbert space.

Theorem 2.2.7. *([50]) For $1 < p < \infty$, $L^p(\Omega)$ is a reflexive space.*

2.2.8 Some integral inequalities

We will give here some important integral inequalities. These inequalities play an important role in applied mathematics and also, it is very useful in our next chapters.

Theorem 2.2.9. *([50] Hölder's inequality). Let $1 \leq p \leq \infty$. Assume that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then $fg \in L^p(\Omega)$ and*

$$\int_\Omega |fg|dx \leq \|f\|_p \|g\|_q.$$

Lemma 2.2.10. *([50] Young's inequality). Let $f \in L^p(\mathbb{R})$ and $g \in L^g(\mathbb{R})$ with $1 < p < \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \geq 0$. Then $f * g \in L^r(\mathbb{R})$ and*

$$\|f * g\|_{L^r(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})} \|g\|_{L^q(\mathbb{R})}.$$

Lemma 2.2.11. *([50]) Let $1 \leq p \leq r \leq q$, $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$, and $1 \leq \alpha \leq 1$. Then*

$$\|u\|_{L^r} \leq \|u\|_{L^p}^{\alpha} \|u\|_{L^q}^{1-\alpha}.$$

Lemma 2.2.12. *([50]) If $\mu(\Omega) < \infty$, $1 \leq p \leq q \leq \infty$, then $L^q \hookrightarrow L^p$ and*

$$\|u\|_{L^p} \leq \mu(\Omega)^{\frac{1}{p} - \frac{1}{q}} \|u\|_{L^q}.$$

2.2.13 The $W^{m,p}(\Omega)$ spaces

Proposition 2.2.14. *Let Ω be an open domain in $\mathbb{R}^N$. Then the distribution $T \in D'(\Omega)$ is in $L^p(\Omega)$ if there exists a function $f \in L^p(\Omega)$ such that*

$$\langle T, \varphi\rangle = \int_\Omega f(x)\varphi(x)dx, \text{ for all } \varphi \in D(\Omega),$$

where $1 \leq p \leq \infty$ and it's well-known that f is unique.

Now, we will introduce the Sobolev spaces: The Sobolev space $W^{k,p}(\Omega)$ is defined to be the subset of L^p such that function f and its weak derivatives up to some order k have a finite L^p norm, for given $p \geq 1$.

$$W^{k,p}(\Omega) = \Big\{ f \in L^p(\Omega); D^\alpha f \in L^p(\Omega). \ \forall \alpha; \quad |\alpha| \leq k \Big\}.$$

With this definition, the Sobolev spaces admit a natural norm:

$$f \longrightarrow \|f\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq m} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{1/p}, \text{ for } p < +\infty$$

and

$$f \longrightarrow \|f\|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha| \leq m} \|D^\alpha f\|_{L^\infty(\Omega)}, \text{ for } p = +\infty$$

Space $W^{k,p}(\Omega)$ equipped with the norm $\| \ . \ \|_{W^{k,p}}$ is a Banach space. Moreover is a reflexive space for $1 < p < \infty$ and a separable space for $1 \leq p < \infty$. Sobolev spaces with $p = 2$ are especially important because of their connection with Fourier series and because they form a Hilbert space. A special notation has arisen to cover this case:

$$W^{k,2}(\Omega) = H^k(\Omega)$$

the H^k inner product is defined in terms of the L^2 inner product:

$$(f, g)_{H^k(\Omega)} = \sum_{|\alpha| \leq k} (D^\alpha f, D^\alpha g)_{L^2(\Omega)}.$$

The space $H^m(\Omega)$ and $W^{k,p}(\Omega)$ contain $\mathcal{C}^\infty(\overline{\Omega})$ and $\mathcal{C}^m(\overline{\Omega})$. The closure of $\mathcal{D}(\Omega)$ for the $H^m(\Omega)$ norm (respectively $W^{m,p}(\Omega)$ norm) is denoted by $H^m_0(\Omega)$ (respectively $W^{k,p}_0(\Omega)$).
Now, we introduce a space of functions with values in a space X (a separable Hilbert space). The space $L^2(a, b; X)$ is a Hilbert space for the inner product

$$(f, g)_{L^2(a,b;X)} = \int_a^b (f(t), g(t))_X \, dt$$

We note that $L^\infty(a, b; X) = (L^1(a, b; X))'$. Now, we define the Sobolev spaces with values in a Hilbert space X. For $k \in N$, $p \in [1, \infty]$, we set:

$$W^{k,p}(a, b; X) = \left\{ v \in L^p(a, b; X); \frac{\partial v}{\partial x_i} \in L^p(a, b; X). \ \forall i \leq k \right\},$$

The Sobolev space $W^{k,p}(a,b;X)$ is a Banach space with the norm

$$\begin{aligned} \|f\|_{W^{k,p}(a,b;X)} &= \left(\sum_{i=0}^{k} \left\|\frac{\partial f}{\partial x_i}\right\|^p_{L^p(a,b;X)}\right)^{1/p}, \text{ for } p < +\infty \\ \|f\|_{W^{k,\infty}(a,b;X)} &= \sum_{i=0}^{k} \left\|\frac{\partial v}{\partial x_i}\right\|_{L^\infty(a,b;X)}, \quad \text{for } p = +\infty \end{aligned}$$

The spaces $W^{k,2}(a,b;X)$ form a Hilbert space and it is noted $H^k(0,T;X)$. The $H^k(0,T;X)$ inner product is defined by:

$$(u,v)_{H^k(a,b;X)} = \sum_{i=0}^{k} \int_a^b \left(\frac{\partial u}{\partial x^i}, \frac{\partial v}{\partial x^i}\right)_X dt .$$

Lemma 2.2.15. *(Sobolev-Poincarés inequality)*

$$\text{If} \quad 2 \leq q \leq \frac{2n}{n-2}, n \geq 3 \quad \text{and} \quad q \geq 2, n = 1,2,$$

then

$$\|u\|_q \leq C(q,\Omega)\|\nabla u\|_2, \quad \forall u \in H_0^1(\Omega).$$

Remark 2.2.16. *For all $\varphi \in H^2(\Omega)$, $\Delta\varphi \in L^2(\Omega)$ and for Γ sufficiently smooth, we have*

$$\|\varphi(t)\|_{H^2(\Omega)} \leq C\|\Delta\varphi(t)\|_{L^2(\Omega)}.$$

Proposition 2.2.17. *([50] Green's formula) For all $u \in H^2(\Omega)$, $\upsilon \in H^1(\Omega)$ we have*

$$-\int_\Omega \Delta u \upsilon dx = \int_\Omega \nabla u \nabla \upsilon dx - \int_{\partial\Omega} \frac{\partial u}{\partial \eta} \upsilon d\sigma,$$

where $\frac{\partial u}{\partial \eta}$ is a normal derivation of u at Γ.

2.2.18 The $L^p(0,T,X)$ spaces

Let X be a Banach space, denote by $L^p(0,T,X)$ the space of measurable functions

Definition 2.2.19.

$$\begin{aligned} f: \ &]0,T[\to X \\ & t \to f(t). \end{aligned} \tag{2.7}$$

such that

$$\left(\int_0^T \|f(t)\|_X^p dt\right)^{\frac{1}{p}} = \|f\|_{L^p(0,T,X)} < \infty, for\ 1 \leq p < \infty. \tag{2.8}$$

If $p = \infty$,

$$\|f\|_{L^p(0,T,X)} = \sup_{t\in]0,T[} ess\|f(t)\|_X. \tag{2.9}$$

Theorem 2.2.20. *([50]) The space $L^p(0,T,X)$ is complete.*

We denote by $D'(0,T,X)$ the space of distributions in $]0,T[$ which take its values in X and let us define

$$D'(0,T,X) = £(D]0,T[,X),$$

where $£(\phi,\varphi)$ is the space of the linear continuous applications of ϕ *to* φ. Since $u \in D'(0,T,X)$, we define the distribution derivation as

$$\frac{\partial u}{\partial t}(\varphi) = -u\left(\frac{d\varphi}{dt}\right), \quad \forall \varphi \in D(]0,T[),$$

and since $u \in L^p(0,T,X)$, we have

$$u(\varphi) = \int_0^T u(t)\varphi(t)dt, \quad \forall \varphi \in D(]0,T[),$$

2.2.21 Some Algebraic inequalities

Since our study based on some known algebraic inequalities, we want to recall few of them here.

Lemma 2.2.22. *([50] The Cauchy-Schwartz's inequality) Every inner product satisfies the Cauchy-Schwartz's inequality*

$$\langle x_1, x_2\rangle \leq \|x_1\|\|x_2\|. \tag{2.10}$$

The equality sign holds if and only if x_1 and x_1 are dependent.

Lemma 2.2.23. *([50] Young's inequalities). For all $a, b \in \mathbb{R}^+$, we have*

$$ab \leq \alpha a^2 + \frac{1}{4\alpha}b^2 \tag{2.11}$$

where α is any positive constant.

Lemma 2.2.24. *([50]) For $a, b \geq 0$, the following inequality holds*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{2.12}$$

where, $\frac{1}{p} + \frac{1}{q} = 1$.

2.3 A result of exponential decay

The estimation of the energy decay for some dissipative problems is based on the following lemma:

Lemma 2.3.1. *([26]) Let $E : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a non-increasing function and assume that there is a constant $A > 0$ such that*

$$\forall t \geq 0, \qquad \int_t^{+\infty} E(\tau)\, d\tau \leq \frac{1}{A} E(t). \tag{2.13}$$

Then we have

$$\forall t \geq 0, \qquad E(t) \leq E(0)\, e^{1-At}. \tag{2.14}$$

Proof of Lemma 2.3.1.
The inequality (2.14) is verified for $t \leq \frac{1}{A}$, this follows from the fact that E is a decreasing function. We prove that (2.14) is verified for $t \geq \frac{1}{A}$. Introduce the function

$$h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \qquad h(t) = \int_t^{+\infty} E(\tau)\, d\tau.$$

It is non-increasing and locally absolutely continuous. Differentiating and using (2.13) we find that

$$\forall t \geq 0, \quad h'(t) + Ah(t) \leq 0.$$

Let

$$T_0 = \sup\{t,\ h(t) > 0\}. \tag{2.15}$$

For every $t < T_0$, we have

$$\frac{h'(t)}{h(t)} \leq -A,$$

thus

$$h(0) \leq e^{-At} \leq \frac{1}{A} E(0)\, e^{-At}, \qquad \text{for} \quad 0 \leq t < T_0. \tag{2.16}$$

Since $h(t) = 0$ if $t \geq T_0$, this inequality holds in fact for every $t \in \mathbb{R}_+$. Let $\varepsilon > 0$. As E is positive and decreasing, we deduce that

$$\forall t \geq \varepsilon, \quad E(t) \leq \frac{1}{\varepsilon} \int_{t-\varepsilon}^{t} E(\tau)\, d\tau \leq \frac{1}{\varepsilon} h(t-\varepsilon) \leq \frac{1}{A\varepsilon} E(0)\, e^{\varepsilon t}\, e^{-At}.$$

Choosing $\varepsilon = \frac{1}{A}$, we obtain

$$\forall t \geq 0, \qquad E(t) \leq E(0)\, e^{1-At}.$$

The proof of Lemma 2.3.1 is now completed.

2.4 Proof of some inequalities

2.4.1 Fatou's Lemma

The following lemma will enable us to establish several criteria to justify passage of the limit under the integral sign.

Lemma 2.4.2. *Let $\{f_n\}$ be a sequence of non-negative measurable functions on E.*
If $\{f_n\} \to f$ point-wise a.e. on E, then

$$\int_E f \leq \liminf \int_E f_n.$$

2.4.3 The Lebesgue Dominated Convergence Theorem

Theorem 2.4.4. *Let $\{f_n\}$ be a sequence of measurable functions on E. Suppose there is a function g that is integrable over E and dominates $\{f_n\}$ on E in the sense that $|f_n| \leq g$ on E for all n.*
If $f_n \to f$ point-wise a.e. on E, then f is integrable over E and

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Proof. Since $|f_n| \leq g$ on E and $|f| \leq g$ a.e. on E and g is integrable over E, by the integral comparison test, f and each f_n also are integrable over E. The function $g - f$ and for each n, the function $g - f_n$, are properly defined, non-negative and measurable. Moreover, the sequence $\{g - f_n\}$ converges point-wise a.e. on E to $g - f$. Fatou's Lemma tells us that

$$\int_E (g - f) = \liminf \int_E (g - f_n).$$

, Thus, by the linearity of integration for integrable functions,

$$\int_E g - \int_E f = \int_E (g - f) \leq \liminf \int_E (g - f_n) = \int_E g - \limsup \int_E f_n,$$

that is,

$$\limsup \int_E f_n \leq \limsup \int_E f.$$

Similarly, considering the sequence $\{g + f_n\}$, we obtain

$$\int_E f \leq \liminf \int_E f_n.$$

The proof is complete. □

2.4.5 Green's formulas

We remember at last some formulas of Green who generalize to the case multi-dimensional the formula of integration by parts of one dimension. These write themselves in the following way:

Proposition 2.4.6. *1. If $f \in C^1(\overline{\Omega})$ and $g \in C^1(\overline{\Omega})$, we have, for all $i \in \mathbb{N}^\star$:*

$$\int_\Omega \frac{\partial f}{\partial x_i} g(x)dx = -\int_\Omega \frac{\partial g}{\partial x_i} f(x)dx + \int_\Gamma f(x)g(x)\nu_i d\Gamma(x), \tag{2.17}$$

2. If $f \in C^2(\overline{\Omega})$ and $g \in C^1(\overline{\Omega})$, we have,

$$\int_\Omega \Delta f(x)g(x)dx = -\int_\Omega \nabla f(x).\nabla g(x)dx + \int_\Gamma \frac{\partial f}{\partial \nu}(x)g(x)d\Gamma(x), \tag{2.18}$$

3. If $f \in C^2(\overline{\Omega})$ and $g \in C^2(\overline{\Omega})$, we have,

$$\int_\Omega \Delta f(x)g(x)dx = -\int_\Omega f(x).\nabla g(x)dx + \int_\Gamma \Big[\frac{\partial f}{\partial \nu}(x)g(x) - \frac{\partial g}{\partial \nu}(x)f(x)\Big]d\Gamma(x), \tag{2.19}$$

Proof. All the expressions above deduce from the following Stokes's formula:

$$\int_\Omega DivU(x)dx = \int_\Gamma (U.V)(x)d\Gamma(x), \tag{2.20}$$

where U is a function with vectorial values, *i.e.* $U : x \in \mathbb{R}^n \longrightarrow U(x) \in \mathbb{R}^n$, and Div is the *divergence* operator defined by:

$$DivU = \sum_{i=1}^{n} \frac{\partial U_i}{\partial x_i} = \nabla.U,$$

we obtain (2.17) by applying (2.20) for U such that $U_i = fg$, the others components of U are all vanish, while (2.18) deduces from (2.20) by taking $U = (\nabla f)g$ (we have then $DivU = \Delta fg + \nabla f.\nabla g$). Exchanging afterward the role of f and g in (2.19), then docking the thus gotten relation off with (2.18), we deduce then (2.19). □

2.4.7 Poincaré's inequality

Let $\Omega \in (0, L) \times \mathbb{R}^{N-1}$, for $u \in C^\infty \subset (\Omega)$ we have the estimate

$$\int_\Omega |u|^2 \, dx \leq L^2 \int_\Omega |\nabla u|^2 \, dx.$$

Proof. Extend u with $u(x) = 0 \notin \Omega$. For $x = (x^1, x') \in \Omega$ estimate

$$\begin{aligned}\left|u(x^1, x')\right|^2 = \left|\int_0^{x^1} \frac{\partial u}{\partial x^1}(s, x')\, ds\right|^2 &\leq \left(\int_0^L \left|\frac{\partial u}{\partial x^1}(s, x')\, ds\right|\right)^2 \\ &\leq L \int_0^L |\nabla u(s, x')|^2 \, ds.\end{aligned}$$

Then it follows

$$\begin{aligned}\int_\Omega |u|^2 \, dx &\leq \int_{\mathbb{R}^{n-1}} \int_0^L |u(x_1, x')|^2 \, dx_1 \, dx' \\ &\leq \int_{\mathbb{R}^{n-1}} L^2 \int_0^L |\nabla u(s, x')|^2 \, ds \, dx' \\ &\leq L^2 \int_\Omega |\nabla u|^2 \, dx.\end{aligned}$$

□

2.4.8 Gronwall's inequality

1. **The differential shape of the Gronwall's inequality :**

   - Let $\eta(.)$ An absolutely continuing, positive function on $[0; T]$, satisfies a.e. The differential inequality

     $$\eta'(t) \leq \phi(t)\eta(t) + \psi(t), \tag{2.21}$$

     where $\phi(t)$ et $\psi(t)$ Are summable, positive functions on $[0; T]$. Then

     $$\eta(t) \leq e^{\int_0^t \phi(s) ds} \left[\eta(0) + \int_0^t \psi(s) ds\right] \tag{2.22}$$

     for all $0 \leq t \leq T$.

   - In particular, if

     $$\eta' \leq \phi\eta \ \text{ sur } \ [0; T] \ \text{ et } \ \eta(0) = 0,$$

     then

     $$\eta \equiv 0 \ \text{ sur } \ [0; T].$$

Proof: De (B.0.1) We see

$$\frac{d}{ds}\left(\eta(s)e^{-\int_0^s \phi(r)dr}\right) = e^{-\int_0^s \phi(r)dr}\left(\eta'(s) - \phi(s)\eta(s)\right) \leq e^{-\int_0^s \phi(r)dr}\psi(s)$$

for a.e. $0 \leq s \leq T$. Therefore for each $0 \leq t \leq T$, we have

$$\eta(t)e^{-\int_0^s \phi(r)dr} \leq \eta(0) + \int_0^t e^{-\int_0^s \phi(r)dr}\psi(s)ds \leq \eta(0) + \int_0^t \psi(s)ds.$$

This involves the inequality (B.0.2)

2. **The integral shape of the Gronwall's inequality :**

- Soit $\zeta(t)$ is a summable, positive function on $[0;T]$ satisfies a.e. the integral inequality

$$\zeta(t) \leq C_1 \int_0^t \zeta(s)ds + C_2 \tag{2.23}$$

  for the constants

$$\zeta(t) \leq C_2\left(1 + C_1 t e^{-C_1 t}\right) \tag{2.24}$$

  for a.e. $0 \leq t \leq T$.

- In particular, if

$$\zeta(t) \leq C_1 \int_0^t \zeta(s)ds$$

  pour p.p. $0 \leq t \leq T$,then

$$\zeta(t) = 0 \quad \text{a.e.}$$

Proof: Let $\eta(t) := \int_0^t \zeta(s)ds$; then $\eta \leq C_1\eta + C_2$ $[0;T]$. According to the differential shape of the Gronwall's inequality above :

$$\eta(t) \leq e^{C_1 t}\left(\eta(0) + C_2 t\right) = C_2 t e^{C_1 t}.$$

then (B.0.3) involves

$$\zeta(t) \leq C_1\eta(t) + C_2 \leq C_2\left(1 + C_1 t e^{C_1 t}\right).$$

2.5 The Schwartz Space and the Fourier Transform

2.5.1 The Schwartz Space:

Definition 2.5.2. *The* Schwartz *class of rapidly decreasing functions is defined as*

$$\mathcal{S}(\mathbb{R}^N) = \left\{ \phi \in C^\infty(\mathbb{R}^N) : \sup_{x\in\mathbb{R}^N} \left| x^\beta \partial^\alpha \phi(x) \right| < \infty, \forall \alpha, \beta \in \mathbb{N}_0^n \right\}.$$

We shall simply say that ϕ is a *Schwartz function* if $\phi \in \mathcal{S}(\mathbb{R}^N)$.

2.5.3 The Space of Tempered Distributions:

The algebraic dual of $\mathcal{S}(\mathbb{R}^N)$ is the vector space

$$\mathcal{S}'(\mathbb{R}^N) := \left\{ T : \mathcal{S}(\mathbb{R}^N) \to \mathbb{C} : T \text{is linear and continuous} \right\}.$$

Recall that a linear functional $T : \mathcal{S}(\mathbb{R}^N) \to \mathbb{C}$ is continuous if only if there exist $m, k \in \mathbb{N}_0^n$, and a finite constant $C > 0$, such that

$$|T(\phi) \leq| C \sup_{\alpha,\ \beta\in\mathbb{N}_0^n,\ |\alpha|\leq m,\ |\beta|\leq k} \ \sup_{x\in\mathbb{R}^N} \left| x^\beta \partial^\alpha \phi(x) \right|, \quad \forall \phi \in \mathcal{S}(\mathbb{R}^N).$$

2.5.4 The Fourier Transform:

Recall that if $u \in L^1(\mathbb{R}^N)$ then the *Fourier transform* of u is the mapping $\mathcal{F}u : \mathbb{R}^N \to \mathbb{C}$ defined by

$$\mathcal{F}u(\zeta) := \int_{\mathbb{R}^N} e^{-ix.\zeta} u(x)\, dx \quad \text{for each} \quad \zeta \in \mathbb{R}^N.$$

Proposition 2.5.5. *For all $u \in \mathcal{S}(\mathbb{R}^N)$:*

$$\mathcal{F}(D_x^\alpha u) = \zeta^\alpha \mathcal{F}u(\zeta).$$

Theorem 2.5.6. ***(inverse Fourier transform formula)***
Let $u \in \mathcal{S}(\mathbb{R}^N)$, for all $x \in \mathbb{R}^N$, we have the identity:

$$u(x) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{ix.\zeta} \mathcal{F}u(\zeta)\, d\zeta.$$

2.6 The differential and the pseudo-differential operators

2.6.1 The differential operator

Definition 2.6.2. *A differential linear operator of degree m is given by the shape:*

$$P(x, D) = P = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha,$$

where $a_\alpha(x)$ called the coefficients of the operator.

If we replace the D^α by the monomial ζ^α in $\mathbb{R}^N$, then we obtain the so-called symbol

$$P(x, \zeta) = \sum_{|\alpha| \le m} a_\alpha(x) \zeta^\alpha$$

of the operator.

Proposition 2.6.3. *Using the inverse* Foureir *transform, we have*

$$(Pf)(x) = \sum_{|\alpha| \le m} a_\alpha(x) (2\pi)^{-N} \int_{\mathbb{R}^N} e^{ix.\zeta} \zeta^\alpha \mathcal{F} f(\zeta)\, d\zeta.$$

2.6.4 The pseudo-differential operator of $\mathcal{S}$ and $\mathcal{S}'$

Most pseudo-differential operators write under the shape:

$$Au(x) = (2\pi)^{-N} \int \int_{\mathbb{R}^{2N}} e^{i\langle x-y, \zeta \rangle} a(x, y, \zeta) u(y)\, dy\, d\zeta,$$

where $a \in S^m$(the set of the symbols for order m) and $u \in \mathcal{D}(\mathbb{R}^N)$.

Remark 2.6.5. *The differential operator P is a pseudo-differential operator.*

Example 2.6.6. *Th differential operator Δ (Laplacien) is a pseudo-differential operator.*

2.7 Some notations

Most notations used throughout this thesis are standard. So, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of natural, real and complex numbers, respectively, and $\mathbb{R}_+ := [0, \infty)$.
A multi-index is an element $\alpha := (\alpha_1, \alpha_2, \cdots, \alpha_n)$ of $\mathbb{N}^n$.

The module of a multi-index α is denoted $|\alpha|$ and defined as

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$$

The factorial of α which is denoted $\alpha!$ is defined by

$$\alpha! = \alpha_1!\alpha_2!\cdots\alpha_n!$$

For a multi-index $\alpha := (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{N}^n$, ∂_x^α denotes the totality of all the α-*th* order derivatives with respect to $x \in \mathbb{R}^N$

$$\partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}}$$

$L^p(\mathbb{R}^N)$ denotes the space of all (equivalent class) of measurable functions $f : \mathbb{R}^N \to \mathbb{R}$, such that $|f|^p$ is integrable. The norm of a such f which is denoted $\|f\|_p$ instead of $\|f\|_{L^p}$ is defined as

$$\|f\|_p = (\int_{\mathbb{R}^N} |f(x)|^p dx)^{\frac{1}{p}}.$$

By $C_0^\infty(\mathbb{R}^N)$ consists of all infinitely differentiable functions with compact support. Let s be a non-negative integer. Then $W^{s,p}(\mathbb{R}^N)$ denotes the Sobolev space of L^p functions, equipped with the norm

$$\|f\|_{W^{s,p}} := \Big(\sum_{k=0}^{s} \|\partial_x^k f\|_{L_p}^p\Big)^{\frac{1}{p}}.$$

$H^s(\mathbb{R}^N)$ is the homogeneous Sobolev space of order s defined by

$$H^s(\mathbb{R}^N) = \left\{u \in \mathcal{S}'(\mathbb{R}^N); (-\Delta)^{s/2}u \in L^2(\mathbb{R}^N)\right\},$$

if $s \notin \mathbb{N}$, and by

$$H^s(\mathbb{R}^N) = \left\{u \in L^2(\mathbb{R}^N); (-\Delta)^{s/2}u \in L^2(\mathbb{R}^N)\right\},$$

if $s \in \mathbb{N}$, where $\mathcal{S}'(\mathbb{R}^N)$ is the space of Schwartz's distributions and $(-\Delta)^{s/2}$ is the fractional Laplacian operator defined by

$$(-\Delta)^{s/2}u(x) := \mathcal{F}^{-1}(|\xi|^s\mathcal{F}(u)(\xi))(x)$$

for every $u \in D((-\Delta)^{s/2}) = H^s(\mathbb{R}^N)$. Where $\mathcal{F}$ stands for the Fourier transform defined by

$$\mathcal{F}(f)(\xi) := \int_{\mathbb{R}^N} e^{-i\xi.x} f(x)dx, \xi \in \mathbb{R}^N.$$

and $\mathcal{F}^{-1}$ its inverse.

2.8 Fractional integration and differentiation

Let $\alpha > 0$, $m = [\alpha]$ and $I = (0, T)$ for some $T > 0$. For the sake of brevity we introduce, for $\beta > 0$, the following function:

$$g_\beta(t) := \begin{cases} \frac{1}{\Gamma(\beta)} t^{\beta-1} \text{ if } t > 0, \\ 0 \text{ if } t \leq 0. \end{cases} \tag{2.25}$$

where $\Gamma(\beta)$ is the Gamma function. Note that $g_0(t) = 0$, since $\Gamma(0)^{-1} = 0$ verify this!. These functions satisfy the semi group property

$$g_\alpha * g_\beta = g_{\alpha+\beta}. \tag{2.26}$$

The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined as follows:

$$J_t^\alpha f(t) := (g_\alpha * f)(t), f \in L^1(I), t > 0. \tag{2.27}$$

Set $J_t^0 f(t) := f(t)$. Thanks to (2.26) and the associativity of the convolution we obtain that the operators of fractional integration obey the semi group property

$$J_t^\alpha J_t^\beta = J_t^{\alpha+\beta}, \alpha, \beta \geq 0. \tag{2.28}$$

The Riemann-Liouville fractional derivative of order α is defined for all f satisfying

$$f \in L^1(I) \text{ and } g_{m-\alpha} * f \in W^{1,m}(I) \tag{2.29}$$

by

$$D_t^\alpha f(t) := D_t^m (g_{m-\alpha} * f)(t) = D_t^m J_t^{m-\alpha} f(t), \tag{2.30}$$

where $D_t^m := \frac{d^m}{dt^m}, m \in \mathbb{N}$. As in the case of differentiation and integration of integer order, D_t^α is the left inverse of J_t^α, but in general it is not right inverse. More precisely, we have the following theorem

Theorem 2.8.1. *Let $\alpha > 0$, $m = [\alpha]$. Then for any $f \in L^1(I)$*

$$D_t^\alpha J_t^\alpha f = f.$$

If moreover (2.27) holds then

$$J_t^\alpha D_t^\alpha f = f(t) - \sum_{k=0}^{m-1} (g_{m-\alpha} * f)^{(k)}(0) g_{\alpha+k+1-m}(t). \tag{2.31}$$

In particular case $g_{m-\alpha} * f \in W_0^{1,m}(I)$, we have $J_t^\alpha D_t^\alpha f = f$.

In particular, if $\alpha \in (0, 1)$, and if $g_{1-\alpha} * f \in W^{1,1}(I)$ then (2.31) reads

$$J_t^\alpha D_t^\alpha f = f(t) - (g_{1-\alpha} * f)(0) g_\alpha(t).$$

If $f \in W^{m,1}(I)$ (which implies (2.27)), then $D_t^\alpha f$ may be represented in the form

$$D_t^\alpha f = \sum_{k=0}^{m-1} f^{(k)}(0) g_{k-\alpha+1}(t) + J_t^{m-\alpha} D_t^\alpha f(t). \tag{2.32}$$

It follows from the representation (2.32) of the elements of $W^{m,1}(I)$ and the definition of D_t^α. In many cases it is more convenient to use the second term in the right-hand side of (2.32) as a definition of fractional derivative of order α. The usefulness of such a definition in the mathematical analysis is demonstrated in [18]. Later, this alternative definition of fractional derivative was introduced by Caputo [4], and adopted by Caputo and Mainardi [39] in the framework of the theory of linear viscoelasticity. So the Caputo fractional derivative of order $\alpha > 0$ is defined by

$$\mathbf{D}_t^\alpha f(t) := J_t^{m-\alpha} D_t^m f(t). \tag{2.33}$$

Some simple but relevant results valid for $\alpha, \beta, t > 0$ are:

$$J_t^\alpha g_\beta = g_{\alpha+\beta}, \quad D_t^\alpha g_\beta = g_{\beta-\alpha}, \quad \beta \geq \alpha. \tag{2.34}$$

In particular, $D_t^\alpha g_\alpha = 0$. We also note $D_t^\alpha 1 = g_{1-\alpha}$, $\alpha \leq 1$, while $\mathbf{D}_t^\alpha 1 = 0$ for all $\alpha > 0$. If instead $f \in W^{m,1}(I)$ we have only (2.29) and $f \in C^{m-1}(I)$, then we can use the following equivalent representation, which follows from (2.32), (2.33) and (2.34):

$$\mathbf{D}_t^\alpha f(t) = D_t^\alpha \Big(f(t) - \sum_{k=0}^{m-1} f^{(k)}(0) g_{k+1}(t) \Big). \tag{2.35}$$

The Caputo derivative $\mathbf{D}_t^\alpha$ is again a left inverse of J_t^α but in general not right inverse, that is

$$\mathbf{D}_t^\alpha J_t^\alpha f = f,\ J_t^\alpha \mathbf{D}_t^\alpha f = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0) g_{k+1}(t). \tag{2.36}$$

The first identity is valid for all $f \in L^1(I)$, the second for $f \in C^{m-1}(I)$, such that (2.29) is satisfied. In particular, if $\alpha \in (0,1)$, $g_{1-\alpha} * f \in W^{1,1}(I)$ and $f \in C^{m-1}(I)$, then $J_t^\alpha \mathbf{D}_t^\alpha f = f(t) - f(0)$. Applying the properties of the Laplace transform and since $\tilde{g}_\alpha(\lambda) = \lambda^{-\alpha}$, we obtain

$$\widetilde{D_t^\alpha f}(\lambda) = \lambda^\alpha \tilde{f}(\lambda) - \sum_{k=0}^{m-1} (g_{m-\alpha} * f)^{(k)}(0) \lambda^{m-1-k}, \tag{2.37}$$

and

$$\widetilde{\mathbf{D}_t^\alpha f}(\lambda) = \lambda^\alpha \tilde{f}(\lambda) - \sum_{k=0}^{m-1} f^{(k)}(0) \lambda^{\alpha-1-k}. \tag{2.38}$$

The left-handed derivative and the right-handed derivative in the Riemann-Liouville sense are defined ,for $\Psi \in L^1(0,T)$ and $0 < \alpha < 1$ as follows:

$$D_{0|t}^\alpha \Psi t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{\Psi(\sigma)}{(t-\sigma)^\alpha} d\sigma,$$

where the symbol Γ stands for the usual Euler's gamma function, and

$$D_{t|T}^\alpha \Psi(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T \frac{\Psi(\sigma)}{(\sigma-t)^\alpha} d\sigma,$$

respectively. If one try to compare the Caputo fractional derivative with the Riemann-Liouville one, one can find that the Caputo fractional derivative

$$\mathbf{D}_{0|t}^\alpha \Psi(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\Psi'(\sigma)}{(t-\sigma)^\alpha} d\sigma,$$

requires $\Psi' \in L^1(0,T)$. Clearly we have

$$D_{0|t}^\alpha g(t) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{g(0)}{t^\alpha} + \int_0^t \frac{g'(\sigma)}{(t-\sigma)^\alpha} d\sigma \right]$$

and

$$D_{t|T}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{f(T)}{(T-t)^\alpha} - \int_t^T \frac{f'(\sigma)}{(t-\sigma)^\alpha} d\sigma \right].$$

Therefore the Caputo derivative is related to the Riemann-Liouville derivative by

$$\mathbf{D}^{\alpha}_{0|t}\Psi(t) = D^{\alpha}_{0|t}\left[\Psi(t) - \Psi(0)\right]$$

we have the formula of integration by parts see ([13] p.46.):

$$\int_0^T f(t)\left(D^{\alpha}_{0|t}g\right)(t)dt = \int_0^T g(t)\left(D^{\alpha}_{t|T}f)(t\right)dt.$$

2.9 The fractional Laplacian

The integral representation of the fractional Laplacian in the N-dimensional space is

$$(-\Delta)^{\beta/2}\psi(x) = -c_N(\beta)\int_{\mathbb{R}^N}\frac{\psi(x+z)-\psi(x)}{|z|^{N+\beta}}dz, \quad \forall x \in \mathbb{R}^N, \tag{2.39}$$

where $c_N(\beta) = \Gamma((N+\beta)/2)/(2\pi^{N/2+\beta}\Gamma(1-\beta/2))$, and Γ denotes the gamma function.
Note that The fractional Laplacian $\left((-\Delta)^{\beta/2}\right)$ [48] with $\alpha \in (0;2]$ is a pseudo-differential operator defined by:

$$(-\Delta)^{\beta/2}u(x) = \mathcal{F}^{-1}\{|\zeta|^{\beta}\,\mathcal{F}(u)(\zeta)\}(x) \ \text{ for all } \quad x \in \mathbb{R}^N,$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are Fourier transform and its inverse, respectively.

2.10 Notion of Well Posedness

Definition 2.10.1. *We say that a problem of PDE is well posed in the sense of Hadamard if the solution depends continuously on the initial data, that is for a problem well posed, we have*

1. *Existence of solutions,*
2. *uniquness of solution,*
3. *Stabilty of solution.*

2.11 Notion of Blow-up

Sometimes, we are interesting by the behaviour of solution of a given PDE evolution problem, especially, if this PDE describe a concrete phenomena e.g. propagation of pollutant in the air, if we denote by $u(t,x)$ for the concentration of this pollutant in the point x at the time t then it is reasonable that one has $\lim_{t\to\infty} u(t,x) = 0$. From this point we start, and we have the following definition

Definition 2.11.1. *Let $\Omega \subset \mathbb{R}^N$ and $u = u(t,x)$ be a solution of a given evolution PDE on the set $\Omega := [0,T] \times A$. We say that u blows up in finite time T if such that*

$$\lim_{t\to T^-} |u(t,x)| = +\infty$$

In this case one has

$$\sup_{x\in\Omega} |u(t,x)| = +\infty$$

and T is called the time of Blow-up.

Case of ODE

The simplest example to show the blow-up phenomena in the case of ordinary differential equations is the following (non-linear) Cauchy problem

$$x'(t) = x^2(t), \quad t > 0, \quad x(0) = x_0.$$

One can show immediately that if $x_0 > 0$ then the above Cauchy problem admits in the interval $]0,T[$ the unique solution $x(t) = \frac{1}{T-t}$. This solution is a smooth function on $]0,T[$ and satisfies in particular at $\lim_{t\to T^-} x(t) = +\infty$.
This means that the solution blows up in finite time.
One can thought to generalize this remark as a main phenomena of ODEs and PDEs.

Case of PDE

The Blow-up's phenomena appears in particular when the unknown function in the posed problem depends not only on time, but also on the spacial variable, especially in the reaction-diffusion problems, propagation evolution problems, the famous example is the Fujita's equation.

Critical exponent of Fujita

Consider the following Cauchy problem of Fujita's equation

$$\begin{cases} u_t = \Delta u + u^p \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^N \end{cases} \tag{2.40}$$

Where the unknown function $u = u(t, x)$ is real valued, $t > 0$, $p > 1$ and Δ is the usual Laplace operator.
This equation is studied by Fujita in 1966, in particular, he showed that if $1 < p < 1 + \frac{2}{N}$ then all solutions in a given class blow-up in finite time.

Definition 2.11.2. *The upper bound $1 + \frac{2}{N}$, of the parameter p, is called the exponent of Fujita, it is denoted p^* or sometimes p_{Fuj}.That is $p^* = 1 + \frac{2}{N}$.*
It is characterized by the following:

1. *if $1 < p < p^*$ then all solution blows up in finite time.*
2. *If $p > p^*$ all solution is global in time, that is defined on $(0, \infty) \times \Omega$.*
3. *if $p = p^*$ this is a critical case!*

Remark 2.11.3. *Throughout this thesis, the constants will be denoted C and are, in general, different from line to line and even in the same line from one place to another one.*

Chapter 3

Nonexistence of Global Solutions to Some Evolution Problems

3.1 Introduction

The problem of global existence of solutions for nonlinear hyperbolic equations with a damping term have been studied by many researchers in several contexts (see [22], [28], [29] , [43], [54], [59]), for example, the following Cauchy problem:

$$\begin{cases} u_{tt} - \Delta u + u_t = |u|^p, & (t,x) \in (0,\infty) \times \mathbb{R}^N \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^N, \end{cases} \tag{3.1}$$

Todorova-Yordanov [54] showed that, if $p_c < p \leq \frac{N}{N-1}$, then (3.1) admits a unique global solution, and they proved that if $1 < p < 1 + \frac{2}{N}$, then the solution u blows up in a finite time.
Fino-Ibrahim and Wehbe [22] generalized the results of Ogawa-Takeda [43] by proving the blow-up of solutions of (3.1) under weaker assumptions on the initial data and they extended this results to the critical case $p_c = 1 + \frac{2}{N}$.
Qi. Zhang [59] studied the case $1 < p < 1 + \frac{2}{N}$, when $\int u_i(x)dx > 0, i = 0, 1$, he proved that global solution of (3.1) does not exist. Therefore, he showed that $p = 1 + \frac{2}{N}$ belongs to the blow-up case.
A. Hakem [28] treated the same type of (3.1), then he extended this result to the case of a system

:

$$\begin{cases} u_{tt} - \Delta u + g(t)u_t = |v|^p, & (t,x) \in (0,+\infty) \times \mathbb{R}^N \\ v_{tt} - \Delta v + f(t)v_t = |u|^q, & (t,x) \in (0,+\infty) \times \mathbb{R}^N \\ u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x) \\ v(0,x) = v_0(x), \;\; v_t(0,x) = v_1(x), \end{cases} \tag{3.2}$$

$g(t)$ and $f(t)$ are functions behaving like t^β and t^α, respectively, where $0 \le \beta, \alpha < 1$.
Hakem [28] showed that, if

$$\frac{N}{2} \le \frac{1}{pq-1} \max\left[1-\beta+p(1-\alpha), 1-\alpha+q(1-\beta)\right] - \max(\alpha,\beta),$$

then the problem (3.2) has only the trivial solution.
Our purpose of the first section is to generalize some of the above results, so with the suitable choice of the test function, we prove the non-existence of nontrivial non-negative global weak solution of (3.3). And in the second section, by combining the works of the above authors with those of Kirane *et al.* [31] and Escobido *et al.* [20], we were able to prove a nonexistence result to (3.14) in the weak formulation.

3.2 Nonexistence of Global Solutions to Semi-Linear Fractional Evolution Equation (Accepted)

In this section, we are concerned with the following problem:

$$\begin{cases} u_{tt} + (-\Delta)^{\frac{\beta}{2}} u + D^{\alpha}_{0|t} u = h(t,x)\,|u|^p, & (t,x) \in (0,+\infty) \times \mathbb{R}^N \\ \\ u(0,x) = u_0(x) \ge 0,, \;\; u_t(0,x) = u_1(x) \ge 0,, & x \in \mathbb{R}^N, \end{cases} \tag{3.3}$$

where $p > 1, 0 < \alpha < 1, 0 < \beta \le 2$ are constants.
The function h is a non-negative and assumed to satisfy the condition

$$h(t,x) \ge Ct^\nu\,|x|^\mu, \text{where } \; C > 0, \; \nu \ge 0, \; \mu \ge 0. \tag{3.4}$$

$D^{\alpha}_{0/t}$ denotes the derivatives of order α in the sense of Caputo and $(-\Delta)^{\frac{\beta}{2}}$ is the fractional power of the $(-\Delta)$.

3.2.1 Preliminaries

Set $\Sigma_T = (0,T)\times(\mathbb{R}^N)$. The results of our research are based on the following definitions:

Definition 3.2.2. *Let $0<\alpha<1$ and $\zeta' \in L^1(0,T)$. The left-sided and respectively right-sided Caputo derivatives of order α for ζ are defined as:*

$$D^{\alpha}_{0|t}\zeta(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\zeta'(s)}{(t-s)^{\alpha}}ds \quad and \quad D^{\alpha}_{t|T}\zeta(t) = -\frac{1}{\Gamma(1-\alpha)}\int_t^T \frac{\zeta'(s)}{(s-t)^{\alpha}}ds,$$

where Γ denotes the gamma function (see [44] p 79).

Definition 3.2.3. *We say that $u\geq 0$ is a local weak solution to (3.3), defined in Σ_T, $0<T<+\infty$, if u is a locally integrable function such that $u^p h \in L^1_{loc}(\Sigma_T)$ and*

$$\int_{\Sigma_T} h\,|u|^p\,\Psi\,dx\,dt + \int_{\mathbb{R}^N} u_0(x)\Psi(0,x)\,dx + \int_{\mathbb{R}^N} u_1(x)\Psi(0,x)\,dx - \int_{\mathbb{R}^N} u_0(x)\Psi_t(0,x)\,dx$$
$$= \int_{\Sigma_T} u\Psi_{tt}\,dx\,dt + \int_{\Sigma_T} uD^{\alpha}_{t|T}\Psi\,dx\,dt + \int_{\Sigma_T} u(-\Delta)^{\frac{\beta}{2}}\Psi\,dx\,dt,$$

is satisfied for any $\Psi \in C^{2,2}_{t,x}\left(\Sigma_T\right)$ such that $\Psi(T,.) = \Psi_t(T,.) = 0$

Definition 3.2.4. *We say that $u\geq 0$ is global weak solution to (3.3) if it is a local solution to (3.3) defined in Σ_T for any $T>0$.*

Now, we recall the following integration by parts formula (see [51] p 46):

$$\int_0^T \phi(t)(D^{\alpha}_{0|t}\psi)(t)dt = \int_0^T (D^{\alpha}_{t|T}\phi)(t)\psi(t)dt. \tag{3.5}$$

We notice that, in all steps of proof , $C>0$ is a real positive number which may change from line to line.

3.3 Main results

Our main result reads as follows:

Theorem 3.3.1. *Assume that $p > 1, 0 < \alpha < 1, 0 < \beta \leq 2$ and the conditions (3.4) are satisfied, if*

$$p \leq \frac{\alpha(N+\mu)+\beta(1+\nu)}{\alpha N+\beta(1-\alpha)}, \tag{3.6}$$

then the problem (3.3) has no nontrivial global weak solutions.

Proof. Since the principle of the method is the right choice of the test function, we choose it as follows:

$$\Psi(t,x) = \Phi\Big(\frac{t^2+|x|^{\frac{2\beta}{\alpha}}}{R^2}\Big), \quad R > 0,$$

where Φ is a cut-off no increasing function satisfying

$$\Phi(r) = \begin{cases} 0, & \text{if } r \geq 2, \\ 1, & \text{if } r \leq 1, \end{cases}$$

and

$$0 \leq \Phi \leq 1, \quad \text{for all } r > 0.$$

Now multiplying the equation (3.3) by Ψ and integrating by parts on $\Sigma_T = (0,T)\times(\mathbb{R}^N)$, we get

$$\begin{aligned} \int_{\Sigma_T} h\,|u|^p\,\Psi\,dx\,dt &+ \int_{\mathbb{R}^N} u_0(x)\Psi(0,x)\,dx + \int_{\mathbb{R}^N} u_1(x)\Psi(0,x)\,dx - \int_{\mathbb{R}^N} u_0(x)\Psi_t(0,x)\,dx \\ &= \int_{\Sigma_T} u\Psi_{tt}\,dx\,dt - \int_{\Sigma_T} uD^{\alpha}_{0|t}\Psi\,dx\,dt + \int_{\Sigma_T} u(-\Delta)^{\frac{\beta}{2}}\Psi\,dx\,dt. \end{aligned} \tag{3.7}$$

The fact that

$$\Psi_t(t,x) = 2tR^{-2}\Phi'\Big(\frac{t^2+|x|^{\frac{2\beta}{\alpha}}}{R^2}\Big),$$

we see easily that $\Psi_t(0,x) = 0$. By using (3.5), the formula (3.7) will be on the shape

$$\begin{aligned} \int_{\Sigma_T} h\,|u|^p\,\Psi\,dx\,dt &+ \int_{\mathbb{R}^N} u_0(x)\Psi(0,x)\,dx + \int_{\mathbb{R}^N} u_1(x)\Psi(0,x)\,dx \\ &= \int_{\Sigma_T} u\Psi_{tt}\,dx\,dt + \int_{\Sigma_T} uD^{\alpha}_{t|T}\Psi\,dx\,dt + \int_{\Sigma_T} u((-\Delta)^{\frac{\beta}{2}}\Psi)\,dx\,dt. \end{aligned} \tag{3.8}$$

To estimate

$$\int_{\Sigma_T} u\Psi_{tt}\,dx\,dt,$$

we observe that

$$\int_{\Sigma_T} u\Psi_{tt}\,dx\,dt = \int_{\Sigma_T} u(h\Psi)^{\frac{1}{p}}\Psi_{tt}(h\Psi)^{\frac{-1}{p}}\,dx\,dt,$$

we have also

$$\int_{\Sigma_T} uD^{\alpha}_{t|T}\Psi\,dx\,dt = \int_{\Sigma_T} u(h\Psi)^{\frac{1}{p}}D^{\alpha}_{t|T}\Psi(h\Psi)^{\frac{-1}{p}}\,dx\,dt,$$

and

$$\int_{\Sigma_T} u((-\Delta)^{\frac{\beta}{2}}\Psi)\,dx\,dt = \int_{\Sigma_T} u(h\Psi)^{\frac{1}{p}}((-\Delta)^{\frac{\beta}{2}}\Psi)(h\Psi)^{\frac{-1}{p}}\,dx\,dt.$$

An application of the following ϵ-Young's inequality

$$ab \le \epsilon a^p + C(\epsilon)b^q,\ a>0,\ b>0, \epsilon>0,\ pq = p+q \text{ and } C(\epsilon) = (\epsilon p)^{\frac{-q}{p}}q^{-1},$$

to the first integral of the right hand side of (3.8), we obtain

$$\int_{\Sigma_T} u\Psi_{tt}\,dx\,dt \le \epsilon\int_{\Sigma_T}|u|^p\,h\Psi\,dx\,dt + C(\epsilon)\int_{\Sigma_T}|\Psi_{tt}|^{\frac{p}{p-1}}\,(h\Psi)^{\frac{-1}{p-1}}\,dx\,dt,$$

and for the second integral of the right hand side of (3.8), we get

$$\int_{\Sigma_T} uD^{\alpha}_{t|T}\Psi\,dx\,dt \le \epsilon\int_{\Sigma_T}|u|^p\,h\Psi\,dx\,dt + C(\epsilon)\int_{\Sigma_T}\left(\left|D^{\alpha}_{t|T}\Psi\right|\right)^{\frac{p}{p-1}}(h\Psi)^{\frac{-1}{p-1}}\,dx\,dt.$$

Similarly for the third integral of the right hand side of (3.8), we have

$$\left|\int_{\Sigma_T} u(-\Delta)^{\frac{\beta}{2}}\Psi\,dx\,dt\right| \le \epsilon\int_{\Sigma_T}|u|^p\,h\Psi\,dx\,dt + C(\epsilon)\int_{\Sigma_T}\left|(-\Delta)^{\frac{\beta}{2}}(\Psi)\right|^{\frac{p}{p-1}}\,(h\Psi)^{\frac{-1}{p-1}}\,dx\,dt.$$

Finally, we get

$$\begin{aligned}\int_{\Sigma_T}|u|^p\,h\Psi\,dx\,dt \le C\int_{\Sigma_T}|\Psi_{tt}|^{\frac{p}{p-1}}\,(h\Psi)^{\frac{-1}{p-1}}\,dx\,dt + C\int_{\Sigma_T}\left|D^{\alpha}_{t|T}\Psi\right|^{\frac{p}{p-1}}\,(h\Psi)^{\frac{-1}{p-1}}\,dx\,dt\\ + C\int_{\Sigma_T}\left|(-\Delta)^{\frac{\beta}{2}}(\Psi)\right|^{\frac{p}{p-1}}\,(h\Psi)^{\frac{-1}{p-1}}\,dx\,dt.\end{aligned} \tag{3.9}$$

By the choice of Ψ, it is easy to show that

$$\begin{cases}\displaystyle\int_{\Sigma_T}|\Psi_{tt}|^{\frac{p}{p-1}}\,(h\Psi)^{\frac{-1}{p-1}}\,dx\,dt < \infty, \quad \int_{\Sigma_T}\left|D^{\alpha}_{t|T}\Psi\right|^{\frac{p}{p-1}}\,(h\Psi)^{\frac{-1}{p-1}}\,dx\,dt < \infty\\ \displaystyle\int_{\Sigma_T}\left|(-\Delta)^{\frac{\beta}{2}}(\Psi)\right|^{\frac{p}{p-1}}\,(h\Psi)^{\frac{-1}{p-1}}\,dx\,dt < \infty\end{cases}$$

At this stage, we introduce the scaled variables:

$$\tau = tR^{-1},\ \zeta = xR^{\frac{-\alpha}{\beta}}.$$

Using the fact that

$$dxdt = R^{\frac{N\alpha}{\beta}+1}d\zeta d\tau,\ \Psi_t = R^{-1}\Psi_\tau,\ \Psi_{tt} = R^{-2}\Psi_{\tau\tau}, (-\Delta)_x^{\frac{\beta}{2}}\Psi = R^{-\alpha}(-\Delta)_\zeta^{\frac{\beta}{2}}\Psi,\ D_{t|T}^\alpha\Psi = R^{-\alpha}D_{\tau|RT}^\alpha\Psi,$$

and setting

$$\Omega = \Big\{(\tau,\zeta) \in \mathbb{R}^+ \times \mathbb{R}^N; 1 \le \tau^2 + |\zeta|^{\frac{2\beta}{\alpha}} \le 2\Big\}, \quad \varphi(\tau,\zeta) = \tau^2 + |\zeta|^{\frac{2\beta}{\alpha}},$$

we arrive at

$$\begin{aligned}\int_{\Sigma_T} |u|^p\, h\Psi\, dx\, dt &\le CR^{\theta_1}\int_\Omega |(\Psi_{\tau\tau})(\varphi)|^{\frac{p}{p-1}}\,(h\Psi)^{\frac{-1}{p-1}}\, d\zeta\, d\tau \\ &+ CR^{\theta_2}\int_\Omega \Big|(D^\alpha_{\tau|RT})(\varphi)\Big|^{\frac{p}{p-1}}\,(h\Psi)^{\frac{-1}{p-1}}\, d\zeta\, d\tau + CR^{\theta_3}\int_\Omega \Big|(-\Delta)^{\frac{\beta}{2}}\Psi(\varphi)\Big|^{\frac{p}{p-1}}\,(h\Psi)^{\frac{-1}{p-1}}\, d\zeta\, d\tau.\end{aligned} \tag{3.10}$$

Where

$$\begin{cases} \theta_1 = \dfrac{N\alpha}{\beta} + 1 - \dfrac{2p}{p-1} - \dfrac{1}{p-1}\Big(\dfrac{\alpha}{\beta}\mu + \nu\Big), \\[2ex] \theta_2 = \dfrac{N\alpha}{\beta} + 1 - \dfrac{\alpha p}{p-1} - \dfrac{1}{p-1}\Big(\dfrac{\alpha}{\beta}\mu + \nu\Big), \\[2ex] \theta_3 = \dfrac{N\alpha}{\beta} + 1 - \dfrac{\alpha p}{p-1} - \dfrac{1}{p-1}\Big(\dfrac{\alpha}{\beta}\mu + \nu\Big). \end{cases}$$

One can easily observe that: $\theta_1 < \theta_2 = \theta_3$, we infer that

$$\begin{aligned}\int_{\Sigma_T} |u|^p\, h\Psi\, dx\, dt \le CR^\theta\Big[&\int_\Omega |(\Psi_{\tau\tau})(\varphi)|^{\frac{p}{p-1}}\,(h\Psi)^{\frac{-1}{p-1}}\, d\zeta\, d\tau + \int_\Omega \Big|(D^\alpha_{\tau|RT})(\varphi)\Big|^{\frac{p}{p-1}}\,(h\Psi)^{\frac{-1}{p-1}}\, d\zeta\, d\tau \\ &+ \int_\Omega \Big|(-\Delta)^{\frac{\alpha}{2}}\Psi(\varphi)\Big|^{\frac{p}{p-1}}\,(h\Psi)^{\frac{-1}{p-1}}\, d\zeta\, d\tau\Big],\end{aligned} \tag{3.11}$$

where $R > 0$, large and

$$\theta := \theta_2 = \frac{1}{\beta(p-1)}\Big\{[\alpha N + \beta(1-\alpha)]p - \alpha(N+\mu) - \beta(1+\nu)\Big\}.$$

It is clear that $\alpha N + \beta(1-\alpha) > 0$, thus, we distinguish two cases:

- If
$$\theta < 0 \Leftrightarrow p < \frac{\alpha(N+\mu)+\beta(1+\nu)}{\alpha N+\beta(1-\alpha)},$$
then the right-hand side of (3.11) goes to 0 when R tends to infinity, we pass to the limit in the left hand side, as R goes to $+\infty$; we get
$$\lim_{R\to+\infty}\int_{\Sigma_T} h\,|u|^p\,\Psi\,dx\,dt = 0.$$
Using the Lebesgue dominated convergence theorem, the continuity in time and space of u and the fact that $\Psi(t,x)\to 1$ as $R\to+\infty$, we infer that
$$\int_{\mathbb{R}^+\times\mathbb{R}^N} h\,|u|^p\;dx\,dt = 0.$$
Therefore, if u exists then necessarily $u\equiv 0$ *a. e.* on $\mathbb{R}^+\times\mathbb{R}^N$.

- If
$$\theta = 0 \Leftrightarrow p = \frac{\alpha(N+\mu)+\beta(1+\nu)}{\alpha N+\beta(1-\alpha)},$$
then we have
$$\int_{\mathbb{R}^+\times\mathbb{R}^N} |u|^p\,h\,dx\,dt < +\infty. \tag{3.12}$$
By using (3.8) we obtain
$$\begin{aligned}\int_{\Sigma_T} h\,|u|^p\,\Psi\,dx\,dt + \int_{\mathbb{R}^N} u_0(x)\Psi(0,x)\,dx + \int_{\mathbb{R}^N} u_1(x)\Psi(0,x)\,dx \\ \le \int_{\Sigma_T} u(h\Psi)^{\frac{1}{p}}\,|\Psi_{tt}|\,(h\Psi)^{\frac{-1}{p}}\,dx\,dt + \int_{\Sigma_T} u(h\Psi)^{\frac{1}{p}}\left|D^{\alpha}_{t|T}\right|(h\Psi)^{\frac{-1}{p}}\,dx\,dt \\ + \int_{\Sigma_T} u(h\Psi)^{\frac{1}{p}}\left|(-\Delta)^{\frac{\beta}{2}}\Psi\right|(h\Psi)^{\frac{-1}{p}}\,dx\,dt.\end{aligned} \tag{3.13}$$

Accordingly, using Hölder's inequality in the right hand side of (3.13), yields
$$\begin{aligned}\int_{\Sigma_T} h\,|u|^p\,\Psi\,dx\,dt \le \left(\int_{\Sigma_T} u^p h\Psi\,dx\,dt\right)^{\frac{1}{p}}\left(\int_{\Sigma_T} |\Psi_{tt}|^{\frac{p}{p-1}}\,(h\Psi)^{\frac{-1}{p-1}}\,dx\,dt\right)^{\frac{p-1}{p}} \\ + \left(\int_{\Sigma_T} u^p h\Psi\,dx\,dt\right)^{\frac{1}{p}}\left(\int_{\Sigma_T} (\left|D^{\alpha}_{t|T}\Psi\right|)^{\frac{p}{p-1}}(h\Psi)^{\frac{-1}{p-1}}\,dx\,dt\right)^{\frac{p-1}{p}} \\ + \left(\int_{\Sigma_T} u^p h\Psi\,dx\,dt\right)^{\frac{1}{p}}\left(\int_{\Sigma_T} (\left|(-\Delta)^{\frac{\beta}{2}}\Psi\right|)^{\frac{p}{p-1}}(h\Psi)^{\frac{-1}{p-1}}\,dx\,dt\right)^{\frac{p-1}{p}}.\end{aligned}$$

We easily see that

$$\int_{\Sigma_T} h\,|u|^p\,\Psi\,dx\,dt \leq \left(\int_{\Sigma_T} u^p h\Psi\,dx\,dt\right)^{\frac{1}{p}}\Bigg[\left(\int_{\Sigma_T} |\Psi_{tt}|^{\frac{p}{p-1}}\,(h\Psi)^{\frac{-1}{p-1}}\,dx\,dt\right)^{\frac{p-1}{p}}$$
$$+\left(\int_{\Sigma_T} (\left|D^{\alpha}_{t|T}\right|)^{\frac{p}{p-1}}(h\Psi)^{\frac{-1}{p-1}}\,dx\,dt\right)^{\frac{p-1}{p}}+\left(\int_{\Sigma_T} (\left|(-\Delta)^{\frac{\beta}{2}}\Psi\right|)^{\frac{p}{p-1}}(h\Psi)^{\frac{-1}{p-1}}\,dx\,dt\right)^{\frac{p-1}{p}}\Bigg].$$

Because $\theta = 0$, we get from (3.12) that

$$\int_{\Sigma_T} h\,|u|^p\,\Psi\,dx\,dt \leq \left(\int_{\Omega_2} u^p h\Psi\,dx\,dt\right)^{\frac{1}{p}}\times\Bigg[\left(\int_{\Omega_1} |\Psi_{\tau\tau}(\varphi)|^{\frac{p}{p-1}}\,(h\Psi(\varphi))^{\frac{-1}{p-1}}\,d\zeta\,d\tau\right)^{\frac{p-1}{p}}$$
$$+\left(\int_{\Omega_1} (\left|D^{\alpha}_{\tau|RT}(\varphi)\right|)^{\frac{p}{p-1}}(h\Psi(\varphi))^{\frac{-1}{p-1}}\,d\zeta\,d\tau\right)^{\frac{p-1}{p}}+\left(\int_{\Omega_1} (\left|(-\Delta)^{\frac{\beta}{2}}\Psi(\varphi)\right|)^{\frac{p}{p-1}}(h\Psi(\varphi))^{\frac{-1}{p-1}}\,d\zeta\,d\tau\right)^{\frac{p-1}{p}}\Bigg],$$

where

$$\Omega_1 = \left\{(\tau,\zeta) \in \mathbb{R}^+ \times \mathbb{R}^N;\ 1 \leq \tau^2 + |\zeta|^{\frac{2\beta}{\alpha}} \leq 2\right\},$$

and

$$\Omega_2 = \left\{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^N;\ R^2 \leq t^2 + |x|^{\frac{2\beta}{\alpha}} \leq 2R^2\right\}.$$

Taking into account the fact that $\int_{\mathbb{R}^+\times\mathbb{R}^N} |u|^p\,h\,dx\,dt < +\infty$, we obtain

$$\lim_{R\to+\infty}\int_{\Omega_2} |u|^p\,h\Psi\,dx\,dt = 0,$$

hence, we conclude that

$$\int_{\mathbb{R}^+\times\mathbb{R}^N} |u|^p\,h\,dx\,dt = 0.$$

Whereupon $u \equiv 0$. We deduce that no nontrivial global solution is possible. This finishes the proof. □

Remark 3.3.2. *We observe that in the case* $\alpha = 1, \beta = 2,\ \mu = \nu = 0$, *we retrieve the Fujita's critical exponent* $p_c = 1 + \frac{2}{N}$.

3.4 Nonexistence of global solutions to system of semi-linear fractional evolution equations (Published)

in this section we are concerned with the following Cauchy problem:

$$\begin{cases} u_{tt} + (-\Delta)^{\frac{\beta_1}{2}} u + D^{\alpha_1}_{0|t} u = f(t,x)\,|u|^{p_1}\,|v|^{q_1}, \qquad (t,x) \in (0,+\infty) \times \mathbb{R}^N \\ \\ v_{tt} + (-\Delta)^{\frac{\beta_2}{2}} v + D^{\alpha_2}_{0|t} v = g(t,x)\,|u|^{p_2}\,|v|^{q_2}, \qquad (t,x) \in (0,+\infty) \times \mathbb{R}^N \end{cases} \tag{3.14}$$

subjected to the conditions

$$u(0,x) = u_0(x) \geq 0,, \quad u_t(0,x) = u_1(x) \geq 0,$$

$$v(0,x) = v_0(x) \geq 0,, \quad v_t(0,x) = v_1(x) \geq 0,.$$

where $p_1 \geq 0, q_2 \geq 0, p_2 > 1, q_1 > 1, 0 < \alpha_i < 1 \leq \beta_i \leq 2, i = 1,2$ are constants. $D^{\alpha_i}_{0/t}$ denotes the derivatives of order α_i in the sense of Caputo and $(-\Delta)^{\frac{\beta_i}{2}}$ is the fractional power of the $(-\Delta)$. The functions f and g are non-negatives and assumed to satisfy the conditions

$$f(t,x) \geq C_1 t^{\nu_1}\,|x|^{\mu_1}\,, g(t,x) \geq C_2 t^{\nu_2}\,|x|^{\mu_2}\,, \text{ where } \nu_i \geq 0,\ \mu_i \geq 0,\ i = 1,2. \tag{3.15}$$

3.4.1 Preliminaries

Let us start by introducing the definitions concerning fractional derivatives in the sense of Caputo and the weak local solution to problem (3.14).

Definition 3.4.2. *Let $0 < \alpha < 1$ and $\zeta' \in L^1(0,T)$. The left-sided and respectively right-sided Caputo derivatives of order α for ζ are defined as:*

$$D^{\alpha}_{0|t}\zeta(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\zeta'(s)}{(t-s)^{\alpha}} ds,$$

and

$$D^{\alpha}_{t|T}\zeta(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^T \frac{\zeta'(s)}{(s-t)^{\alpha}} ds,$$

where Γ denotes the gamma function (see [44] p 79).

Definition 3.4.3. *Let $Q_T = (0,T) \times \mathbb{R}^N$, $0 < T < +\infty$.*
We say that $(u,v) \in (L^1_{loc}(Q_T))^2$ is a local weak solution to problem (3.14) on Q_T,

if $(fu^{p_1}v^{q_1}, gu^{p_2}v^{q_2}) \in (L^1_{loc}(Q_T))^2$, *and it satisfies*

$$\int_{Q_T} f\,|u|^{p_1}\,|v|^{q_1}\,\zeta_1\,dx\,dt + \int_{\mathbb{R}^N} u_0(x)\zeta_1(0,x)\,dx + \int_{\mathbb{R}^N} u_1(x)\zeta_1(0,x)\,dx - \int_{\mathbb{R}^N} u_0(x)\zeta_{1_t}(0,x)\,dx = \int_{Q_T} u\zeta_{1_{tt}}\,dx\,dt + \int_{Q_T} uD^{\alpha_1}_{t|T}\zeta_1\,dx\,dt + \int_{Q_T} u(-\Delta)^{\frac{\beta_1}{2}}\zeta_1\,dx\,dt. \tag{3.16}$$

and

$$\int_{Q_T} g\,|u|^{p_2}\,|v|^{q_2}\,\zeta_2\,dx\,dt + \int_{\mathbb{R}^N} v_0(x)\zeta_2(0,x)\,dx + \int_{\mathbb{R}^N} v_1(x)\zeta_2(0,x)\,dx - \int_{\mathbb{R}^N} v_0(x)\zeta_{2_t}(0,x)\,dx = \int_{Q_T} v\zeta_{2_{tt}}\,dx\,dt + \int_{Q_T} vD^{\alpha_2}_{t|T}\zeta_2\,dx\,dt + \int_{Q_T} v(-\Delta)^{\frac{\beta_2}{2}}\zeta_2\,dx\,dt. \tag{3.17}$$

for all test function $\zeta_j \in C^{2,2}_{t,x}(Q_T)$ *such as* $\zeta_j \geq 0$ *and* $\zeta_j(T,x) = \zeta_{j_t}(T,x) = \zeta_{j_t}(0,x) = 0,\ \ j = 1,2$

(see [21] p 5501).

Remark 3.4.4. *To get the definition 3.4.3, we multiplying the first equation in (3.14) by* ζ_1 *and the second equation by* ζ_2*, integrating by parts on* $Q_T = (0,T)\times\mathbb{R}^N$ *and using the definition 3.4.2*

The integrals in the above definition are supposed to be convergent.
If in the definition $T = +\infty$, the solution (u,v) is called global.
Now, we recall the following integration by parts formula:

$$\int_0^T \phi(t)(D^{\alpha}_{0|t}\psi)(t)dt = \int_0^T (D^{\alpha}_{t|T}\phi)(t)\psi(t)dt,$$

(see [51], p 46).

3.4.5 Main results

We now in position to announce our result.

Theorem 3.4.6. *Let* $p_2 > 1, q_1 > 1, 0 < \alpha_i < 1 \leq \beta_i \leq 2, i = 1,2,$ *and*

$$\mathcal{A} := \frac{\alpha_1 + \dfrac{\alpha_2}{p_2} - \Big(1 - \dfrac{1}{p_2q_1}\Big) - \dfrac{1}{p_2}\Big(\mu_2\dfrac{\alpha_1}{\beta_1} + \nu_2\Big) - \dfrac{1}{p_2q_1}\Big(\mu_1\dfrac{\alpha_2}{\beta_2} + \nu_1\Big)}{\dfrac{\alpha_1}{\beta_1\tilde{p}_2} + \dfrac{\alpha_2}{\beta_2p_2\tilde{q}_1}}$$

and

$$\mathcal{B} := \frac{\alpha_2 + \dfrac{\alpha_1}{q_1} - \Big(1 - \dfrac{1}{p_2 q_1}\Big) - \dfrac{1}{q_1}\Big(\mu_1 \dfrac{\alpha_2}{\beta_2} + \nu_1\Big) - \dfrac{1}{p_2 q_1}\Big(\mu_2 \dfrac{\alpha_1}{\beta_1} + \nu_2\Big)}{\dfrac{\alpha_2}{\beta_2 \tilde{q}_1} + \dfrac{\alpha_1}{\beta_1 q_1 \tilde{p}_2}}$$

where $p_2\tilde{p}_2 = p_2 + \tilde{p}_2$, $q_1\tilde{q}_1 = q_1 + \tilde{q}_1$,

and the conditions (3.15) are fulfilled.

If

$$N \le \max\{\mathcal{A}; \mathcal{B}\},$$

then the problem (3.14) admits no nontrivial global weak solutions.

Proof. We notice that, in all steps of proof , $C > 0$ is a real positive number which may change from line to line.

Set $\zeta_j(t,x) = \Phi\Big(\dfrac{t^2 + |x|^{2\theta_j}}{R^2}\Big)$, $j = 1,2$ such as Φ is a decreasing function $C_0^2(\mathbb{R}^+)$, satisfies

$$0 \le \Phi \le 1 \text{ and } \Phi(r) = \begin{cases} 1 & \text{if } 0 \le r \le 1, \\ 0 & \text{if } r \ge 2. \end{cases}$$

Where $R > 0$, $\theta_1 = \beta_1/\alpha_1$ and $\theta_2 = \beta_2/\alpha_2$ (see [31]).

Multiplying the first equation of (3.14) by ζ_1 and integrating by parts on $Q_T = (0,T) \times \mathbb{R}^N$, we get

$$\begin{aligned} \int_{Q_T} f\,|u|^{p_1}\,|v|^{q_1}\,\zeta_1\,dx\,dt + \int_{\mathbb{R}^N} u_0(x)\zeta_1(0,x)\,dx + \int_{\mathbb{R}^N} u_1(x)\zeta_1(0,x)\,dx - \int_{\mathbb{R}^N} u_0(x)\zeta_{1_t}(0,x)\,dx \\ = \int_{Q_T} u\zeta_{1_{tt}}\,dx\,dt - \int_{Q_T} uD^{\alpha_1}_{0|t}\zeta_1\,dx\,dt + \int_{Q_T} u(-\Delta)^{\frac{\beta_1}{2}}\zeta_1\,dx\,dt. \end{aligned} \tag{3.18}$$

It is clear that $\zeta_{j_t}(t,x) = 2R^{-2}t\Phi'\Big(\dfrac{t^2 + |x|^{2\theta_j}}{R^2}\Big)$, consequently $\zeta_{j_t}(0,x) = 0$, thus

$$\begin{aligned} \int_{Q_T} f\,|u|^{p_1}\,|v|^{q_1}\,\zeta_1\,dx\,dt + \int_{\mathbb{R}^N} u_0(x)\zeta_1(0,x)\,dx + \int_{\mathbb{R}^N} u_1(x)\zeta_1(0,x)\,dx \\ = \int_{Q_T} u\zeta_{1_{tt}}\,dx\,dt + \int_{Q_T} uD^{\alpha_1}_{t|T}\zeta_1\,dx\,dt + \int_{Q_T} u(-\Delta)^{\frac{\beta_1}{2}}\zeta_1\,dx\,dt. \end{aligned} \tag{3.19}$$

Hence,

$$\int_{Q_T} f\,|u|^{p_1}\,|v|^{q_1}\,\zeta_1\,dx\,dt \le \int_{Q_T} |u|\,\left|\zeta_{1_{tt}}\right|\,dx\,dt + \int_{Q_T} |u|\,\left|D^{\alpha_1}_{t|T}\zeta_1\right|\,dx\,dt + \int_{Q_T} |u|\,\left|(-\Delta)^{\frac{\beta_1}{2}}\zeta_1\right|\,dx\,dt. \tag{3.20}$$

We have also

$$\int_{Q_T} g\,|u|^{p_2}\,|v|^{q_2}\,\zeta_2\,dx\,dt \le \int_{Q_T} |v|\left|\zeta_{2tt}\right|\,dx\,dt + \int_{Q_T} |v|\left|D^{\alpha_2}_{t|T}\zeta_2\right|\,dx\,dt + \int_{Q_T} |v|\left|(-\Delta)^{\frac{\beta_2}{2}}\zeta_2\right|\,dx\,dt. \tag{3.21}$$

To estimate $\int_{Q_T} |u|\left|\zeta_{1tt}\right|\,dx\,dt$, we observe that it can be rewritten as

$$\int_{Q_T} |u|\left|\zeta_{1tt}\right|\,dx\,dt = \int_{Q_T} |u|\,(g\,|v|^{q_2}\,\zeta_2)^{\frac{1}{p_2}}\left|\zeta_{1tt}\right|(g\,|v|^{q_2}\,\zeta_2)^{\frac{-1}{p_2}}\,dx\,dt.$$

Using Hölder's inequality, we obtain

$$\int_{Q_T} |u|\left|\zeta_{1tt}\right|\,dx\,dt \le \left(\int_{Q_T} |u|^{p_2}\,(g\,|v|^{q_2}\,\zeta_2)\,dx\,dt\right)^{\frac{1}{p_2}}\left(\int_{Q_T}\left|\zeta_{1tt}\right|^{\frac{p_2}{p_2-1}}(g\,|v|^{q_2}\,\zeta_2)^{\frac{-1}{p_2-1}}\,dx\,dt\right)^{\frac{p_2-1}{p_2}}.$$

Proceeding as above, we have

$$\begin{aligned}\int_{Q_T} |u|\left|D^{\alpha_1}_{t|T}\zeta_1\right|\,dx\,dt \le &\left(\int_{Q_T} |u|^{p2}\,(g\,|v|^{q_2}\,\zeta_2)\,dx\,dt\right)^{\frac{1}{p_2}}\\ &\times\left(\int_{Q_T}\left|D^{\alpha_1}_{t|T}\zeta_1\right|^{\frac{p_2}{p_2-1}}(g\,|v|^{q_2}\,\zeta_2)^{\frac{-1}{p_2-1}}\,dx\,dt\right)^{\frac{p_2-1}{p_2}},\end{aligned}$$

and

$$\begin{aligned}\int_{Q_T} |u|\left|(-\Delta)^{\frac{\beta_1}{2}}\zeta_1\right|\,dx\,dt \le &\left(\int_{Q_T} |u|^{p_2}\,(g\,|v|^{q_2}\,\zeta_2)\,dx\,dt\right)^{\frac{1}{p_2}}\\ &\times\left(\int_{Q_T}\left|(-\Delta)^{\frac{\beta_1}{2}}\zeta_1\right|^{\frac{p_2}{p_2-1}}(g\,|v|^{q_2}\,\zeta_2)^{\frac{-1}{p_2-1}}\,dx\,dt\right)^{\frac{p_2-1}{p_2}}.\end{aligned}$$

Finally, we infer

$$\int_{Q_T} f\,|u|^{p_1}\,|v|^{q_1}\,\zeta_1\,dx\,dt \le \left(\int_{Q_T} |u|^{p_2}\,(g\,|v|^{q_2}\,\zeta_2)\,dx\,dt\right)^{\frac{1}{p_2}}\mathcal{K}_1, \tag{3.22}$$

where

$$\begin{aligned}\mathcal{K}_1 = &\left(\int_{Q_T}\left|\zeta_{1tt}\right|^{\frac{p_2}{p_2-1}}(g\,|v|^{q_2}\,\zeta_2)^{\frac{-1}{p_2-1}}\,dx\,dt\right)^{\frac{p_2-1}{p_2}} + \left(\int_{Q_T}\left|D^{\alpha_1}_{t|T}\zeta_1\right|^{\frac{p_2}{p_2-1}}(g\,|v|^{q_2}\,\zeta_2)^{\frac{-1}{p_2-1}}\,dx\,dt\right)^{\frac{p_2-1}{p_2}}\\ &+\left(\int_{Q_T}\left|(-\Delta)^{\frac{\beta_1}{2}}\zeta_1\right|^{\frac{p_2}{p_2-1}}(g\,|v|^{q_2}\,\zeta_2)^{\frac{-1}{p_2-1}}\,dx\,dt\right)^{\frac{p_2-1}{p_2}}.\end{aligned}$$

Arguing as above we have likewise

$$\int_{Q_T} g\,|u|^{p_2}\,|v|^{q_2}\,\zeta_2\,dx\,dt \le \left(\int_{Q_T} |v|^{q_1}\,(f\,|u|^{p_1}\,\zeta_1)\,dx\,dt\right)^{\frac{1}{q_1}} \mathcal{K}_2, \tag{3.23}$$

where

$$\mathcal{K}_2 = \left(\int_{Q_T} \left|\zeta_{2tt}\right|^{\frac{q_1}{q_1-1}}\,(f\,|u|^{p_1}\,\zeta_1)^{\frac{-1}{q_1-1}}\,dx\,dt\right)^{\frac{q_1-1}{q_1}} + \left(\int_{Q_T} \left|D^{\alpha_2}_{t|T}\zeta_2\right|^{\frac{q_1}{q_1-1}}\,(f\,|u|^{p_1}\,\zeta_1)^{\frac{-1}{q_1-1}}\,dx\,dt\right)^{\frac{q_1-1}{q_1}}$$
$$+ \left(\int_{Q_T} \left|(-\Delta)^{\frac{\beta_2}{2}}\zeta_2\right|^{\frac{q_1}{q_1-1}}\,(f\,|u|^{p_1}\,\zeta_1)^{\frac{-1}{q_1-1}}\,dx\,dt\right)^{\frac{q_1-1}{q_1}}.$$

Using inequalities (3.22) and (3.23), it yield

$$\left(\int_{Q_T} f\,|u|^{p_1}\,|v|^{q_1}\,\zeta_1\,dx\,dt\right)^{\frac{q_1p_2-1}{q_1p_2}} \le \mathcal{K}_1\mathcal{K}_2^{\frac{1}{p_2}}. \tag{3.24}$$

similarly, we get

$$\left(\int_{Q_T} g\,|u|^{p_2}\,|v|^{q_2}\,\zeta_2\,dx\,dt\right)^{\frac{q_1p_2-1}{q_1p_2}} \le \mathcal{K}_2\mathcal{K}_1^{\frac{1}{q_1}}. \tag{3.25}$$

Now, in $\mathcal{K}_1$ we consider the scale of variables:

$$t = \tau R, \quad x = yR^{\frac{\alpha_1}{\beta_1}},$$

while in $\mathcal{K}_2$ we use:

$$t = \tau R, \quad x = yR^{\frac{\alpha_2}{\beta_2}},$$

and use the fact that

$$dxdt = R^{\left(\frac{N\alpha_1}{\beta_1}+1\right)}dyd\tau,\ \zeta_{i_{tt}} = R^{-2}\zeta_{i_{\tau\tau}},\ D^{\alpha_i}_{0|t}\zeta_{i_t} = R^{-\alpha_i}D^{\alpha_i}_{0|\tau R}\zeta_{i_\tau},$$

$(-\Delta)_x^{\frac{\beta_i}{2}}\zeta_i = R^{-\alpha_i}(-\Delta)_y^{\frac{\beta_i}{2}}\zeta_i,\ i = 1, 2,$
we arrive at

$$\left(\int_{Q_T} f\,|u|^{p_1}\,|v|^{q_1}\,\zeta_1\,dx\,dt\right)^{\frac{q_1p_2-1}{q_1p_2}} \le C\Big[R^{\gamma_1} + R^{\gamma_2} + R^{\gamma_3}\Big] \times \Big[R^{\lambda_1} + R^{\lambda_2} + R^{\lambda_3}\Big]^{\frac{1}{p_2}}, \tag{3.26}$$

similarly, we have

$$\left(\int_{Q_T} g\,|u|^{p_2}\,|v|^{q_2}\,\zeta_2\,dx\,dt\right)^{\frac{q_1p_2-1}{q_1p_2}} \leq C\Big[R^{\lambda_1}+R^{\lambda_2}+R^{\lambda_3}\Big]\times\Big[R^{\gamma_1}+R^{\gamma_2}+R^{\gamma_3}\Big]^{\frac{1}{q_1}}, \tag{3.27}$$

where
$$\begin{cases} \gamma_1 = \Big(\frac{N\alpha_1}{\beta_1}+1\Big)\Big(\frac{p_2-1}{p_2}\Big) - 2 - \Big(\frac{\mu_2\alpha_1}{\beta_1}+\nu_2\Big)\frac{1}{p_2} \\ \gamma_2 = \Big(\frac{N\alpha_1}{\beta_1}+1\Big)\Big(\frac{p_2-1}{p_2}\Big) - \alpha_1 - \Big(\frac{\mu_2\alpha_1}{\beta_1}+\nu_2\Big)\frac{1}{p_2} \\ \gamma_3 = \Big(\frac{N\alpha_1}{\beta_1}+1\Big)\Big(\frac{p_2-1}{p_2}\Big) - \alpha_1 - \Big(\frac{\mu_2\alpha_1}{\beta_1}+\nu_2\Big)\frac{1}{p_2} \end{cases}$$

and
$$\begin{cases} \lambda_1 = \Big(\frac{N\alpha_2}{\beta_2}+1\Big)\Big(\frac{q_1-1}{q_1}\Big) - 2 - \Big(\frac{\mu_1\alpha_2}{\beta_2}+\nu_1\Big)\frac{1}{q_1} \\ \lambda_2 = \Big(\frac{N\alpha_2}{\beta_2}+1\Big)\Big(\frac{q_1-1}{q_1}\Big) - \alpha_2 - \Big(\frac{\mu_1\alpha_2}{\beta_2}+\nu_1\Big)\frac{1}{q_1} \\ \lambda_3 = \Big(\frac{N\alpha_2}{\beta_2}+1\Big)\Big(\frac{q_1-1}{q_1}\Big) - \alpha_2 - \Big(\frac{\mu_1\alpha_2}{\beta_2}+\nu_1\Big)\frac{1}{q_1} \end{cases}$$

we observe that $\gamma_1 < \gamma_2 = \gamma_3$ and $\lambda_1 < \lambda_2 = \lambda_3$, hence

$$\left(\int_{Q_T} f\,|u|^{p_1}\,|v|^{q_1}\,\zeta_1\,dx\,dt\right)^{\frac{q_1p_2-1}{q_1p_2}} \leq CR^{\gamma_2+\frac{\lambda_2}{p_2}} \tag{3.28}$$

and

$$\left(\int_{Q_T} g\,|u|^{p_2}\,|v|^{q_2}\,\zeta_2\,dx\,dt\right)^{\frac{q_1p_2-1}{q_1p_2}} \leq CR^{\lambda_2+\frac{\gamma_2}{q_1}}. \tag{3.29}$$

with the fact that

$$\frac{1}{p_2}+\frac{1}{\tilde{p}_2} = 1 \text{ and } \frac{1}{q_1}+\frac{1}{\tilde{q}_1} = 1 \tag{3.30}$$

by a simple computation,

$$\gamma_2+\frac{\lambda_2}{p_2} = N\Big(\frac{\alpha_1}{\beta_1\tilde{p}_2}+\frac{\alpha_2}{\beta_2p_2\tilde{q}_1}\Big) - \Big(\alpha_1+\frac{\alpha_2}{p_2}\Big) + \frac{1}{\tilde{p}_2} + \frac{1}{p_2\tilde{q}_1} + \frac{1}{p_2}\Big(\mu_2\frac{\alpha_1}{\beta_1}+\nu_2\Big) + \frac{1}{p_2q_1}\Big(\mu_1\frac{\alpha_2}{\beta_2}+\nu_1\Big)$$

and

$$\lambda_2+\frac{\gamma_2}{q_1} = N\Big(\frac{\alpha_2}{\beta_2\tilde{q}_1}+\frac{\alpha_1}{\beta_1q_1\tilde{p}_2}\Big) - \Big(\alpha_2+\frac{\alpha_1}{q_1}\Big) + \frac{1}{\tilde{q}_1} + \frac{1}{q_1\tilde{p}_2} + \frac{1}{q_1}\Big(\mu_1\frac{\alpha_2}{\beta_2}+\nu_1\Big) + \frac{1}{p_2q_1}\Big(\mu_2\frac{\alpha_1}{\beta_1}+\nu_2\Big)$$

also, using (3.30) we have

$$\frac{1}{\tilde{p}_2}+\frac{1}{p_2\tilde{q}_1}=1-\frac{1}{p_2}+\frac{1}{p_2\tilde{q}_1}=1-\frac{1}{p_2}\Big(1-\frac{1}{\tilde{q}_1}\Big)=1-\frac{1}{p_2q_1}$$

and

$$\frac{1}{\tilde{q}_1}+\frac{1}{q_1\tilde{p}_2}=1-\frac{1}{q_1}+\frac{1}{q_1\tilde{p}_2}=1-\frac{1}{q_1}\Big(1-\frac{1}{\tilde{p}_2}\Big)=1-\frac{1}{p_2q_1}$$

we obtain

$$\gamma_2+\frac{\lambda_2}{p_2}=N\Big(\frac{\alpha_1}{\beta_1\tilde{p}_2}+\frac{\alpha_2}{\beta_2p_2\tilde{q}_1}\Big)-\Big(\alpha_1+\frac{\alpha_2}{p_2}\Big)+1-\frac{1}{p_2q_1}+\frac{1}{p_2}\Big(\mu_2\frac{\alpha_1}{\beta_1}+\nu_2\Big)+\frac{1}{p_2q_1}\Big(\mu_1\frac{\alpha_2}{\beta_2}+\nu_1\Big)$$

and

$$\lambda_2+\frac{\gamma_2}{q_1}=N\Big(\frac{\alpha_2}{\beta_2\tilde{q}_1}+\frac{\alpha_1}{\beta_1q_1\tilde{p}_2}\Big)-\Big(\alpha_2+\frac{\alpha_1}{q_1}\Big)+1-\frac{1}{p_2q_1}+\frac{1}{q_1}\Big(\mu_1\frac{\alpha_2}{\beta_2}+\nu_1\Big)+\frac{1}{p_2q_1}\Big(\mu_2\frac{\alpha_1}{\beta_1}+\nu_2\Big)$$

We conclude that

- If $\gamma_2+\dfrac{\lambda_2}{p_2}<0$, it yield

$$N<\frac{\alpha_1+\frac{\alpha_2}{p_2}-\Big(1-\frac{1}{p_2q_1}\Big)-\frac{1}{p_2}\Big(\mu_2\frac{\alpha_1}{\beta_1}+\nu_2\Big)-\frac{1}{p_2q_1}\Big(\mu_1\frac{\alpha_2}{\beta_2}+\nu_1\Big)}{\frac{\alpha_1}{\beta_1\tilde{p}_2}+\frac{\alpha_2}{\beta_2p_2\tilde{q}_1}}.$$

  Then the right hand side of (3.28) goes to 0, when R tends to infinity, we pass to the limit in the left hand side, as R goes to $+\infty$; we get

$$\lim_{R\to+\infty}\left(\int_{Q_T}f\,|u|^{p_1}\,|v|^{q_1}\,\zeta_1\,dx\,dt\right)^{\frac{q_1p_2-1}{q_1p_2}}=0.$$

  Using the Lebesgue dominated convergence theorem, the continuity in time and space of u, v and the fact that $\zeta_1(t,x)\to 1$ as $R\to+\infty$, we infer that

$$\int_{\mathbb{R}^+\times\mathbb{R}^N}f\,|u|^{p_1}\,|v|^{q_1}\;dx\,dt=0$$

  This implies that $v\equiv 0$ or $u\equiv 0$.

Similarly, if $\lambda_2 + \dfrac{\gamma_2}{q_1} < 0$, it yield

$$N < \frac{\alpha_2 + \dfrac{\alpha_1}{q_1} - \Big(1 - \dfrac{1}{p_2 q_1}\Big) - \dfrac{1}{q_1}\Big(\mu_1 \dfrac{\alpha_2}{\beta_2} + \nu_1\Big) - \dfrac{1}{p_2 q_1}\Big(\mu_2 \dfrac{\alpha_1}{\beta_1} + \nu_2\Big)}{\dfrac{\alpha_2}{\beta_2 \tilde{q}_1} + \dfrac{\alpha_1}{\beta_1 q_1 \tilde{p}_2}},$$

by using also (3.29) to proceeding as above, we obtain $u \equiv 0$ or $v \equiv 0$.

- If $\gamma_2 + \frac{\lambda_2}{p_2} = 0$, we get

$$\int_{\mathbb{R}^+ \times \mathbb{R}^N} f\,|u|^{p_1}\,|v|^{q_1}\;dx\,dt < +\infty.$$

Using again Hölder's inequality, we obtain

$$\int_{Q_T} g\,|u|^{p_2}\,|v|^{q_2}\,\zeta_2\,dx\,dt \leq \bigg(\int_{B_R} |v|^{q_1}\,(f\,|u|^{p_1}\,\zeta_1)\,dx\,dt\bigg)^{\frac{1}{q_1}} \mathcal{K}_2,$$

where

$$B_R = \Big\{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^N;\; R^2 \leq t^2 + |x|^{2\theta_1} \leq 2R^2\Big\}.$$

Since,

$$\int_{\mathbb{R}^+ \times \mathbb{R}^N} f\,|u|^{p_1}\,|v|^{q_1}\;dx\,dt < +\infty,$$

we get

$$\lim_{R \to +\infty} \int_{B_R} f\,|u|^{p_1}\,|v|^{q_1}\;dx\,dt = 0,$$

hence, we infer that

$$\int_{\mathbb{R}^+ \times \mathbb{R}^N} g\,|u|^{p_2}\,|v|^{q_2}\;dx\,dt = 0,$$

this implies that $v \equiv 0$ or $u \equiv 0$.
Similarly, if $\lambda_2 + \frac{\gamma_2}{q_1} = 0$, proceeding as above, we infer that $u \equiv 0$ or $v \equiv 0$.
We deduce that no global weak solution is possible other than the trivial one, which ends the proof.

□

Remark 3.4.7. *In the case $\alpha_i = 1$, $\beta_i = 2, \nu_i = \mu_i = 0, p_1 = q_2 = 0$, $i = 1, 2$, we recover the case who studied by A. Hakem (see [28]), when $\alpha = \beta = 0$.*

Chapter 4

Nonexistence of global solution to system of semi-linear wave models with fractional damping (Submitted)

4.1 Introduction

in this chapter we are concerned with the following Cauchy problem:

$$\begin{cases} u_{tt} - \Delta u + D_{0|t}^{\alpha_1+1} u + (-\Delta)^{\frac{\beta_1}{2}} u_t = h(t,x)\,|v|^q\,, & (t,x) \in (0,+\infty) \times \mathbb{R}^N \\ \\ v_{tt} - \Delta v + D_{0|t}^{\alpha_2+1} v + (-\Delta)^{\frac{\beta_2}{2}} v_t = k(t,x)\,|u|^p\,, & (t,x) \in (0,+\infty) \times \mathbb{R}^N \end{cases} \tag{4.1}$$

subjected to the conditions

$$u(0,x) = u_0(x) > 0,, \quad u_t(0,x) = u_1(x) > 0,$$

$$v(0,x) = v_0(x) > 0,, \quad v_t(0,x) = v_1(x) > 0,.$$

where $p > 1, q > 1, 0 < \alpha_i < 1, 0 < \beta_i \leq 2, i = 1,2$ are constants.
$D_{0/t}^{\alpha_i}$ denotes the derivatives of order α_i in the sense of Caputo and $(-\Delta)^{\frac{\beta_i}{2}}$ is $\frac{\beta_i}{2}-$ fractional power of the $(-\Delta)$.
The functions h and k are non-negatives and assumed to satisfy the conditions

$$h(tR^{\frac{2}{\alpha_1+1}}, xR) = R^{\mu} h(t,x), \quad k(tR^{\frac{2}{\alpha_2+1}}, xR) = R^{\nu} k(t,x), \text{ where } \nu \geq 0,\ \mu \geq 0. \tag{4.2}$$

In the beginning of this work we note that Chen and Holm [10] studied the equation

$$\nabla^2 p = \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} + \frac{2\alpha_0}{c_0^{1-y}}\frac{\partial}{\partial t}(-\nabla^2)^{\frac{y}{2}} p,$$

where $0 \le y \le 2$ and $(-\nabla^2)^{\frac{y}{2}}$ is $\frac{y}{2}-$fractional Laplacian which generalize the two cases:

- when $y = 2$ the equation

$$\nabla^2 p = \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} + \frac{\mu}{c_0^2}\frac{\partial}{\partial t}(-\nabla^2) p,$$

  which governs the propagation of sound through a viscous fluid, where c_0 is the small signal sound speed, and $\mu = 2\alpha_0 c_0^3$ the collective thermoviscous coefficient.

- when $y = 0$ the standard damped wave equation

$$\nabla^2 p = \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} + \frac{2\alpha_0}{c_0}\frac{\partial p}{\partial t},$$

  which describes the frequency-independent attenuation.

The problem of global existence of solutions for nonlinear hyperbolic equations with a damping term have been studied by many researchers in several contexts [22, 28, 29, 43, 54, 59] , for example, the following Cauchy problem:

$$\begin{cases} u_{tt} - \Delta u + u_t = |u|^p, & (t,x) \in (0,\infty) \times \mathbb{R}^N \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^N, \end{cases} \tag{4.3}$$

Todorova-Yordanov [54] showed that, if $p_c < p \le \frac{N}{N-1}$, then (4.3) admits a unique global solution, and they proved that if $1 < p < 1 + \frac{2}{N}$, then the solution u blows up in a finite time.
Fino-Ibrahim and Wehbe [22] generalized the results of Ogawa-Takeda [43] by proving the blow-up of solutions of (4.3) under weaker assumptions on the initial data and they extended this results to the critical case $p_c = 1 + \frac{2}{N}$.
Qi. Zhang [59] studied the case $1 < p < 1 + \frac{2}{N}$, when $\int u_i(x)dx > 0, i = 0, 1$, he proved that global solution of (4.3) does not exist. Therefore, he showed that $p = 1 + \frac{2}{N}$ belongs to the blow-up case.
A. Hakem [28] treated the same type of (4.3), then he extended this result to the case of a system

:

$$\begin{cases} u_{tt} - \Delta u + g(t)u_t = |v|^p, & (t,x) \in (0,+\infty) \times \mathbb{R}^N \\ v_{tt} - \Delta v + f(t)v_t = |u|^q, & (t,x) \in (0,+\infty) \times \mathbb{R}^N \\ u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x) \\ v(0,x) = v_0(x), \;\; v_t(0,x) = v_1(x), \end{cases} \tag{4.4}$$

$g(t)$ and $f(t)$ are functions behaving like t^β and t^α, respectively, where $0 \le \beta, \alpha < 1$.

Hakem [28] showed that, if

$$\frac{N}{2} \le \frac{1}{pq-1} \max\left[1-\beta+p(1-\alpha), 1-\alpha+q(1-\beta)\right] - \max(\alpha,\beta),$$

then the problem (4.4) has only the trivial solution.

F. Sun and M. Wang [53] studied the same type of (4.4) in the case $g(t) = f(t) = 1$, they showed that if $\max\left\{\frac{1+p}{pq-1}, \frac{1+q}{pq-1}\right\} \ge \frac{N}{2}$ for $N \ge 1$ where $p, q \ge 1$ and satisfy $pq > 1$, then every solution with initial data having positive average value does not exist globally.

Our purpose of this work is to generalize some of the above results, so with the suitable choice of the test function, we were able to prove a nonexistence result to (4.1) in the weak formulation.

4.2 Preliminaries

Let us start by introducing the definitions concerning fractional derivatives in the sense of Caputo and the weak local solution to problem (4.1).

Definition 4.2.1. *Let $0 < \alpha < 1$ and $\zeta' \in L^1(0,T)$. The left-sided and respectively right-sided Caputo derivatives of order α for ζ are defined as:*

$$D^\alpha_{0|t}\zeta(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\zeta'(s)}{(t-s)^\alpha} ds,$$

and

$$D^\alpha_{t|T}\zeta(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^T \frac{\zeta'(s)}{(s-t)^\alpha} ds,$$

where Γ denotes the gamma function (see [44] p 79).

In general

$$D^\alpha_{0|t}\zeta(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\zeta^n(s)}{(t-s)^{-n+\alpha+1}} ds,$$

where $n = [\alpha] + 1, \alpha > 0$.
By using the property

$$D^{\alpha}_{0|t}(D^m \zeta(t)) = D^{\alpha+m}_{0|t}\zeta(t)$$

where $m \in \mathbb{N}$ and $n - 1 < \alpha < n$.
We have, in particular

$$D^{\alpha+1}_{0|t}\zeta(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\zeta_{tt}(s)}{(t-s)^{\alpha}}ds, \quad (0 < \alpha < 1).$$

Definition 4.2.2. *Let $Q_T = (0, T) \times \mathbb{R}^N$, $0 < T < +\infty$.*
We say that $(u, v) \in (L^1_{loc}(Q_T))^2$ is a local weak solution to problem (4.1) on Q_T, if $(hv^q, ku^p) \in (L^1_{loc}(Q_T))^2$, and it satisfies

$$\begin{aligned}
\int_{Q_T} h\,|v|^q\,\zeta_1\,dx\,dt &+ \int_{\mathbb{R}^N} u_1(x)\zeta_1(0,x)\,dx + \int_{\mathbb{R}^N} u_0(x) D^{\alpha_1}_{t|T}\zeta_1(0,x)\,dx \\
&+ \int_{\mathbb{R}^N} u_1(x) D^{\alpha_1}_{t|T}\zeta_1(0,x)\,dx + \int_{\mathbb{R}^N} \zeta_1(0,x)(-\Delta)^{\frac{\beta_1}{2}} u_0(x)\,dx \\
&= \int_{Q_T} u\zeta_{1_{tt}}\,dx\,dt - \int_{Q_T} u\Delta\zeta_1\,dx\,dt \\
&- \int_{Q_T} u D^{\alpha_1+1}_{t|T}\zeta_1\,dx\,dt - \int_{Q_T} u(-\Delta)^{\frac{\beta_1}{2}}\zeta_{1_t}\,dx\,dt.
\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
\int_{Q_T} k\,|u|^p\,\zeta_1\,dx\,dt &+ \int_{\mathbb{R}^N} v_1(x)\zeta_2(0,x)\,dx + \int_{\mathbb{R}^N} v_0(x) D^{\alpha_2}_{t|T}\zeta_2(0,x)\,dx \\
&+ \int_{\mathbb{R}^N} v_1(x) D^{\alpha_2}_{t|T}\zeta_2(0,x)\,dx + \int_{\mathbb{R}^N} \zeta_2(0,x)(-\Delta)^{\frac{\beta_2}{2}} v_0(x)\,dx \\
&= \int_{Q_T} u\zeta_{2_{tt}}\,dx\,dt - \int_{Q_T} u\Delta\zeta_2\,dx\,dt \\
&- \int_{Q_T} v D^{\alpha_2+1}_{t|T}\zeta_2\,dx\,dt - \int_{Q_T} v(-\Delta)^{\frac{\beta_2}{2}}\zeta_{2_t}\,dx\,dt.
\end{aligned} \tag{4.6}$$

for all test function $\zeta_i \in C^{2,2}_{t,x}(Q_T)$ such as $\zeta_i \geq 0$ and $\zeta_i(T,x) = \zeta_{i_t}(T,x) = 0,\ i = 1, 2$

(see [29]).

Remark 4.2.3. *To get the definition 4.2.2, we multiplying the first equation in (4.1) by ζ_1 and the second equation by ζ_2, integrating by parts on $Q_T = (0, T) \times \mathbb{R}^N$ and using the definition 4.2.1*

The integrals in the above definition are supposed to be convergent.
If in the definition $T = +\infty$, the solution (u, v) is called global.

Now, we recall the following integration by parts formula:

$$\int_0^T \phi(t)(D^{\alpha}_{0|t}\psi)(t)dt = \int_0^T (D^{\alpha}_{t|T}\phi)(t)\psi(t)dt,$$

(see [51], p 46).

4.3 Main results

We now in position to announce our result.

Theorem 4.3.1. *Let* $p > 1, q > 1, 0 < \alpha_i < 1, 0 \leq \beta_i \leq 2, i = 1, 2,$ *and*

$$N_1 := \frac{-\frac{2}{\alpha_1+1}(pq-q) - \frac{2}{\alpha_2+1}(q-1) + pq\rho + q\sigma + q\nu + \mu}{pq-1}$$

and

$$N_2 := \frac{-\frac{2}{\alpha_2+1}(pq-p) - \frac{2}{\alpha_1+1}(p-1) + pq\sigma + p\rho + p\mu + \nu}{pq-1}$$

and the conditions (4.2) are fulfilled.
If the initial data satisfies

$$\begin{aligned} &\int_{\mathbb{R}^N} u_i(x)\,dx > 0, \int_{\mathbb{R}^N} v_i(x)\,dx > 0, \int_{\mathbb{R}^N} (-\Delta)^{\frac{\beta_1}{2}} u_0(x)\,dx > 0, \\ &\int_{\mathbb{R}^N} (-\Delta)^{\frac{\beta_2}{2}} v_0(x)\,dx > 0, \quad i = 0, 1 \end{aligned} \tag{4.7}$$

$$N \leq \max\{N_1; N_2\},$$

then, every weak solution of the problem (4.1) does not exist globally in time .

Proof. We notice that, in all steps of proof , $C > 0$ is a real positive number which may change from line to line.
Set $\zeta_i(t,x) = \Phi^{\ell}\left(\frac{t^{2(\alpha_i+1)}}{R^4}\right)\Phi^{\ell}\left(\frac{|x|^2}{R^2}\right)$, $i = 1, 2$ such as Φ is a decreasing function $C^2_0(\mathbb{R}^+)$, satisfies

$$0 \leq \Phi \leq 1 \text{ and } \Phi(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq 1, \\ 0 & \text{if } r \geq 2. \end{cases}$$

$R > 0$, and $\ell \geq 2\max\{\tilde{p}, \tilde{q}\}$, where $p\tilde{p} = p + \tilde{p}$ and $q\tilde{q} = q + \tilde{q}$.

Multiplying the first equation of (4.1) by ζ_1 and integrating by parts on $Q_T = (0,T)\times\mathbb{R}^N$, we get

$$\begin{aligned}\int_{Q_T} h\,|v|^q\,\zeta_1\,dx\,dt &+ \int_{\mathbb{R}^N} u_1(x)\zeta_1(0,x)\,dx + \int_{\mathbb{R}^N} u_0(x)D^{\alpha_1}_{t|T}\zeta_1(0,x)\,dx\\ &+ \int_{\mathbb{R}^N} u_1(x)D^{\alpha_1}_{t|T}\zeta_1(0,x)\,dx + \int_{\mathbb{R}^N}\zeta_1(0,x)(-\Delta)^{\frac{\beta_1}{2}}u_0(x)\,dx\\ &= \int_{Q_T} u\zeta_{1tt}\,dx\,dt - \int_{Q_T} u\Delta\zeta_1\,dx\,dt\\ &- \int_{Q_T} uD^{\alpha_1+1}_{t|T}\zeta_1\,dx\,dt - \int_{Q_T} u(-\Delta)^{\frac{\beta_1}{2}}\zeta_{1t}\,dx\,dt.\end{aligned} \tag{4.8}$$

Hence,

$$\begin{aligned}\int_{Q_T} h\,|v|^q\,\zeta_1\,dx\,dt \le& \int_{Q_T}|u|\left|\zeta_{1tt}\right|dx\,dt + \int_{Q_T}|u|\,|\Delta\zeta_1|\,dx\,dt\\ &+ \int_{Q_T}|u|\left|D^{\alpha_1+1}_{t|T}\zeta_1\right|dx\,dt + \int_{Q_T}|u|\left|(-\Delta)^{\frac{\beta_1}{2}}\zeta_{1t}\right|dx\,dt.\end{aligned} \tag{4.9}$$

We have also

$$\begin{aligned}\int_{Q_T} k\,|u|^p\,\zeta_2\,dx\,dt \le& \int_{Q_T}|v|\left|\zeta_{2tt}\right|dx\,dt + \int_{Q_T}|v|\,|\Delta\zeta_2|\,dx\,dt\\ &+ \int_{Q_T}|v|\left|D^{\alpha_2+1}_{t|T}\zeta_2\right|dx\,dt + \int_{Q_T}|v|\left|(-\Delta)^{\frac{\beta_2}{2}}\zeta_{2t}\right|dx\,dt.\end{aligned} \tag{4.10}$$

To estimate $\int_{Q_T}|u|\left|\zeta_{1tt}\right|\,dx\,dt$, we observe that it can be rewritten as

$$\int_{Q_T}|u|\left|\zeta_{1tt}\right|\,dx\,dt = \int_{Q_T}|u|\,(k\zeta_2)^{\frac{1}{p}}\left|\zeta_{1tt}\right|(k\zeta_2)^{\frac{-1}{p}}\,dx\,dt.$$

Using Hölder's inequality, we obtain

$$\int_{Q_T}|u|\left|\zeta_{1tt}\right|\,dx\,dt \le \left(\int_{Q_T}|u|^p\,(k\zeta_2)\,dx\,dt\right)^{\frac{1}{p}}\left(\int_{Q_T}\left|\zeta_{1tt}\right|^{\frac{p}{p-1}}(k\zeta_2)^{\frac{-1}{p-1}}\,dx\,dt\right)^{\frac{p-1}{p}}.$$

Proceeding as above, we have

$$\begin{aligned}\int_{Q_T}|u|\,|\Delta\zeta_1|\,dx\,dt \le& \left(\int_{Q_T}|u|^p\,(k\zeta_2)\,dx\,dt\right)^{\frac{1}{p}}\\ &\times\left(\int_{Q_T}|\Delta\zeta_1|^{\frac{p}{p-1}}\,(k\zeta_2)^{\frac{-1}{p-1}}\,dx\,dt\right)^{\frac{p-1}{p}}.\end{aligned}$$

and

$$\int_{Q_T} |u| \left|D^{\alpha_1+1}_{t|T}\zeta_1\right| \, dx\, dt \leq \left(\int_{Q_T} |u|^p \,(k\zeta_2)\, dx\, dt\right)^{\frac{1}{p}} \times \left(\int_{Q_T} \left|D^{\alpha_1+1}_{t|T}\zeta_1\right|^{\frac{p}{p-1}} (k\zeta_2)^{\frac{-1}{p-1}} \, dx\, dt\right)^{\frac{p-1}{p}},$$

and

$$\int_{Q_T} |u| \left|(-\Delta)^{\frac{\beta_1}{2}}\zeta_1\right| \, dx\, dt \leq \left(\int_{Q_T} |u|^p \,(k\zeta_2)\, dx\, dt\right)^{\frac{1}{p}} \times \left(\int_{Q_T} \left|(-\Delta)^{\frac{\beta_1}{2}}\zeta_1\right|^{\frac{p}{p-1}} (k\zeta_2)^{\frac{-1}{p-1}} \, dx\, dt\right)^{\frac{p-1}{p}}.$$

Finally, we infer

$$\int_{Q_T} h\,|v|^q\, \zeta_1\, dx\, dt \leq \left(\int_{Q_T} |u|^p \,(k\zeta_2)\, dx\, dt\right)^{\frac{1}{p}} \mathcal{A}, \tag{4.11}$$

where

$$\begin{aligned}\mathcal{A} = &\left(\int_{Q_T} \left|\zeta_{1tt}\right|^{\frac{p}{p-1}} (k\zeta_2)^{\frac{-1}{p-1}} \, dx\, dt\right)^{\frac{p-1}{p}} + \left(\int_{Q_T} \left|\Delta\zeta_1\right|^{\frac{p}{p-1}} (k\zeta_2)^{\frac{-1}{p-1}} \, dx\, dt\right)^{\frac{p-1}{p}} \\ &+ \left(\int_{Q_T} \left|D^{\alpha_1+1}_{t|T}\zeta_1\right|^{\frac{p}{p-1}} (k\zeta_2)^{\frac{-1}{p-1}} \, dx\, dt\right)^{\frac{p-1}{p}} + \left(\int_{Q_T} \left|(-\Delta)^{\frac{\beta_1}{2}}\zeta_1\right|^{\frac{p}{p-1}} (k\zeta_2)^{\frac{-1}{p-1}} \, dx\, dt\right)^{\frac{p-1}{p}}.\end{aligned}$$

Arguing as above we have likewise

$$\int_{Q_T} k\,|u|^p\, \zeta_2\, dx\, dt \leq \left(\int_{Q_T} |v|^q \,(h\zeta_1)\, dx\, dt\right)^{\frac{1}{q}} \mathcal{B}, \tag{4.12}$$

where

$$\begin{aligned}\mathcal{B} = &\left(\int_{Q_T} \left|\zeta_{2tt}\right|^{\frac{q}{q-1}} (h\zeta_1)^{\frac{-1}{q-1}} \, dx\, dt\right)^{\frac{q-1}{q}} + \left(\int_{Q_T} \left|\Delta\zeta_2\right|^{\frac{q}{q-1}} (h\zeta_1)^{\frac{-1}{q-1}} \, dx\, dt\right)^{\frac{q-1}{q}} \\ &+ \left(\int_{Q_T} \left|D^{\alpha_2+1}_{t|T}\zeta_2\right|^{\frac{q}{q-1}} (h\zeta_1)^{\frac{-1}{q-1}} \, dx\, dt\right)^{\frac{q-1}{q}} + \left(\int_{Q_T} \left|(-\Delta)^{\frac{\beta_2}{2}}\zeta_2\right|^{\frac{q}{q-1}} (h\zeta_1)^{\frac{-1}{q-1}} \, dx\, dt\right)^{\frac{q-1}{q}}.\end{aligned}$$

By the choice of ζ_i, it is easy to show that $\mathcal{A}$ and $\mathcal{B}$ are finite.
Using inequalities (4.11) and (4.12), it yield

$$\left(\int_{Q_T} h\,|v|^q\, \zeta_1\, dx\, dt\right)^{\frac{pq-1}{pq}} \leq \mathcal{A}\mathcal{B}^{\frac{1}{p}}. \tag{4.13}$$

similarly, we get

$$\left(\int_{Q_T} k\,|u|^p\,\zeta_2\,dx\,dt\right)^{\frac{pq-1}{pq}} \leq \mathcal{B}\mathcal{A}^{\frac{1}{q}}. \tag{4.14}$$

Now, in $\mathcal{A}$ we consider the scale of variables:

$$t = \tau R^{\frac{2}{\alpha_1+1}}, \quad x = yR,$$

while in $\mathcal{B}$ we use:

$$t = \tau R^{\frac{2}{\alpha_2+1}}, \quad x = yR.$$

We define the set Ω and the functions ϕ_i by

$$\Omega := \left\{(\tau, y) \in \mathbb{R}_+ \times \mathbb{R}^N; \tau^{2(\alpha_i+1)} \leq 2, |y|^2 \leq 2\right\}$$

and

$$\zeta_i(t,x) = \zeta_i(\tau R^{\frac{2}{\alpha_i+1}}, Ry) := \phi_i(\tau, y)$$

and use the fact that

$$dxdt = R^{\left(N+\frac{2}{\alpha_i+1}\right)} dyd\tau,\ \zeta_{i_{tt}} = R^{\frac{-4}{\alpha_i+1}}\phi_{i_{\tau\tau}},\ \Delta_x\zeta_i = R^{-2}\Delta_y\phi_i$$

$$D^{\alpha_i+1}_{t|TR^{\frac{2}{\alpha_i+1}}}\zeta_i = R^{-2}D^{\alpha_i+1}_{\tau|T}\phi_\tau,\ (-\Delta)_x^{\frac{\beta_i}{2}}\zeta_{i_t} = R^{-\left(\beta_i+\frac{2}{\alpha_i+1}\right)}(-\Delta)_y^{\frac{\beta_i}{2}}\phi_{i_\tau},\ i = 1, 2.$$

Thus,

$$\left(\int_{Q_T} \left|\zeta_{1_{tt}}\right|^{\frac{p}{p-1}} (k\zeta_2)^{\frac{-1}{p-1}}\,dx\,dt\right)^{\frac{p-1}{p}} = R^{\gamma_1}\left(\int_\Omega \left|\phi_{1_{tt}}\right|^{\frac{p}{p-1}} (k\phi_2)^{\frac{-1}{p-1}}\,dy\,d\tau\right)^{\frac{p-1}{p}}$$

$$\left(\int_{Q_T} |\Delta\zeta_1|^{\frac{p}{p-1}} (k\zeta_2)^{\frac{-1}{p-1}}\,dx\,dt\right)^{\frac{p-1}{p}} = R^{\gamma_2}\left(\int_\Omega |\Delta\phi_1|^{\frac{p}{p-1}} (k\phi_2)^{\frac{-1}{p-1}}\,dy\,d\tau\right)^{\frac{p-1}{p}}$$

$$\left(\int_{Q_T} \left|D^{\alpha_1+1}_{t|T}\zeta_1\right|^{\frac{p}{p-1}} (k\zeta_2)^{\frac{-1}{p-1}}\,dx\,dt\right)^{\frac{p-1}{p}} = R^{\gamma_3}\left(\int_\Omega \left|D^{\alpha_1+1}_{t|T}\phi_1\right|^{\frac{p}{p-1}} (k\phi_2)^{\frac{-1}{p-1}}\,dy\,d\tau\right)^{\frac{p-1}{p}}$$

$$\left(\int_{Q_T} \left|(-\Delta)^{\frac{\beta_1}{2}}\zeta_1\right|^{\frac{p}{p-1}} (k\zeta_2)^{\frac{-1}{p-1}}\,dx\,dt\right)^{\frac{p-1}{p}} = R^{\gamma_4}\left(\int_\Omega \left|(-\Delta)^{\frac{\beta_1}{2}}\phi_1\right|^{\frac{p}{p-1}} (k\phi_2)^{\frac{-1}{p-1}}\,dy\,d\tau\right)^{\frac{p-1}{p}},$$

and

$$\left(\int_{Q_T} \left|\zeta_{2_{tt}}\right|^{\frac{q}{q-1}} (h\zeta_1)^{\frac{-1}{q-1}}\,dx\,dt\right)^{\frac{q-1}{q}} = R^{\lambda_1}\left(\int_\Omega \left|\phi_{2_{tt}}\right|^{\frac{q}{q-1}} (h\phi_1)^{\frac{-1}{q-1}}\,dy\,d\tau\right)^{\frac{q-1}{q}}$$

$$\Big(\int_{Q_T}|\Delta\zeta_2|^{\frac{q}{q-1}}(h\zeta_1)^{\frac{-1}{q-1}}\,dx\,dt\Big)^{\frac{q-1}{q}}=R^{\lambda_2}\Big(\int_{\Omega}|\Delta\phi_2|^{\frac{q}{q-1}}(h\phi_1)^{\frac{-1}{q-1}}\,dy\,d\tau\Big)^{\frac{q-1}{q}}$$

$$\Big(\int_{Q_T}\Big|D_{t|T}^{\alpha_2+1}\zeta_2\Big|^{\frac{q}{q-1}}(h\zeta_1)^{\frac{-1}{q-1}}\,dx\,dt\Big)^{\frac{q-1}{q}}=R^{\lambda_3}\Big(\int_{\Omega}\Big|D_{t|T}^{\alpha_2+1}\phi_2\Big|^{\frac{q}{q-1}}(h\phi_1)^{\frac{-1}{q-1}}\,dy\,d\tau\Big)^{\frac{q-1}{q}}$$

$$\Big(\int_{Q_T}\Big|(-\Delta)^{\frac{\beta_2}{2}}\zeta_2\Big|^{\frac{q}{q-1}}(h\zeta_1)^{\frac{-1}{q-1}}\,dx\,dt\Big)^{\frac{q-1}{q}}=R^{\lambda_4}\Big(\int_{\Omega}\Big|(-\Delta)^{\frac{\beta_2}{2}}\phi_2\Big|^{\frac{q}{q-1}}(h\phi_1)^{\frac{-1}{q-1}}\,dy\,d\tau\Big)^{\frac{q-1}{q}},$$

where
$$\begin{cases}\gamma_1=\Big(N+\dfrac{2}{\alpha_1+1}\Big)\dfrac{1}{\tilde p}-\dfrac{4}{\alpha_1+1}-\dfrac{\nu}{p}\\[2ex]\gamma_2=\Big(N+\dfrac{2}{\alpha_1+1}\Big)\dfrac{1}{\tilde p}-2-\dfrac{\nu}{p}\\[2ex]\gamma_3=\Big(N+\dfrac{2}{\alpha_1+1}\Big)\dfrac{1}{\tilde p}-2-\dfrac{\nu}{p}\\[2ex]\gamma_4=\Big(N+\dfrac{2}{\alpha_1+1}\Big)\dfrac{1}{\tilde p}-\Big(\beta_1+\dfrac{2}{\alpha_1+1}\Big)-\dfrac{\nu}{p}\end{cases}$$

and
$$\begin{cases}\lambda_1=\Big(N+\dfrac{2}{\alpha_2+1}\Big)\dfrac{1}{\tilde q}-\dfrac{4}{\alpha_2+1}-\dfrac{\mu}{q}\\[2ex]\lambda_2=\Big(N+\dfrac{2}{\alpha_2+1}\Big)\dfrac{1}{\tilde q}-2-\dfrac{\mu}{q}\\[2ex]\lambda_3=\Big(N+\dfrac{2}{\alpha_2+1}\Big)\dfrac{1}{\tilde q}-2-\dfrac{\mu}{q}\\[2ex]\lambda_4=\Big(N+\dfrac{2}{\alpha_2+1}\Big)\dfrac{1}{\tilde q}-\Big(\beta_2+\dfrac{2}{\alpha_2+1}\Big)-\dfrac{\mu}{q}\end{cases}$$

we arrive at

$$\Big(\int_{Q_T}h\,|v|^q\,\zeta_1\,dx\,dt\Big)^{\frac{pq-1}{pq}}\le C\Big[R^{\gamma_1}+R^{\gamma_2}+R^{\gamma_3}+R^{\gamma_4}\Big]\times\Big[R^{\lambda_1}+R^{\lambda_2}+R^{\lambda_3}+R^{\lambda_4}\Big]^{\frac{1}{p}},\qquad(4.15)$$

similarly, we have

$$\Big(\int_{Q_T}k\,|u|^p\,\zeta_2\,dx\,dt\Big)^{\frac{pq-1}{pq}}\le C\Big[R^{\lambda_1}+R^{\lambda_2}+R^{\lambda_3}+R^{\lambda_4}\Big]\times\Big[R^{\gamma_1}+R^{\gamma_2}+R^{\gamma_3}+R^{\gamma_4}\Big]^{\frac{1}{q}},\qquad(4.16)$$

we observe that $\gamma_1<\gamma_2=\gamma_3$ and $\lambda_1<\lambda_2=\lambda_3$.

Set $\gamma = \Big(N + \dfrac{2}{\alpha_1+1}\Big)\dfrac{1}{\tilde{p}} - \rho - \dfrac{\nu}{p}$ and $\lambda = \Big(N + \dfrac{2}{\alpha_2+1}\Big)\dfrac{1}{\tilde{q}} - \sigma - \dfrac{\mu}{q}$,
where $\rho = \min\Big\{2, \beta_1 + \dfrac{2}{\alpha_1+1}\Big\}$ and $\sigma = \min\Big\{2, \beta_2 + \dfrac{2}{\alpha_2+1}\Big\}$.
Hence

$$\Big(\int_{Q_T} h\,|v|^q\,\zeta_1\,dx\,dt\Big)^{\frac{pq-1}{pq}} \leq C R^{\gamma+\frac{\lambda}{p}} \tag{4.17}$$

and

$$\Big(\int_{Q_T} k\,|u|^p\,\zeta_2\,dx\,dt\Big)^{\frac{pq-1}{pq}} \leq C R^{\lambda+\frac{\gamma}{q}}. \tag{4.18}$$

with the fact that

$$\frac{1}{p} + \frac{1}{\tilde{p}} = 1 \text{ and } \frac{1}{q} + \frac{1}{\tilde{q}} = 1 \tag{4.19}$$

by a simple computation,

$$\gamma + \frac{\lambda}{p} = N\Big(\frac{pq-1}{pq}\Big) + \frac{2}{\alpha_1+1}\Big(1 - \frac{1}{p}\Big) + \frac{2}{\alpha_2+1}\Big(\frac{1}{p} - \frac{1}{pq}\Big) - \rho - \frac{\sigma}{p} - \frac{\nu}{p} - \frac{\mu}{pq}$$

and

$$\lambda + \frac{\gamma}{q} = N\Big(\frac{pq-1}{pq}\Big) + \frac{2}{\alpha_2+1}\Big(1 - \frac{1}{q}\Big) + \frac{2}{\alpha_1+1}\Big(\frac{1}{q} - \frac{1}{pq}\Big) - \sigma - \frac{\rho}{q} - \frac{\mu}{q} - \frac{\nu}{pq}$$

We conclude that

- If $\gamma + \dfrac{\lambda}{p} < 0$, it yield

$$N < \frac{-\frac{2}{\alpha_1+1}(pq-q) - \frac{2}{\alpha_2+1}(q-1) + pq\rho + q\sigma + q\nu + \mu}{pq-1}.$$

Then the right hand side of (4.17) goes to 0, when R tends to infinity, we pass to the limit in the left hand side, as R goes to $+\infty$; we get

$$\lim_{R\to+\infty}\Big(\int_{Q_T} h\,|v|^q\,\zeta_1\,dx\,dt\Big)^{\frac{pq-1}{pq}} = 0.$$

Using the Lebesgue dominated convergence theorem, the continuity in time and space of v and the fact that $\zeta_1(t,x) \to 1$ as $R \to +\infty$, we infer that

$$\Big(\int_{\mathbb{R}^+\times\mathbb{R}^N} h\,|v|^q\;dx\,dt\Big)^{\frac{pq-1}{pq}} = 0.$$

This implies that $v \equiv 0$ *a. e.* on $\mathbb{R}^+ \times \mathbb{R}^N$

Similarly, if $\lambda + \dfrac{\gamma}{q} < 0$, it yield

$$N < \frac{-\dfrac{2}{\alpha_2+1}(pq-p) - \dfrac{2}{\alpha_1+1}(p-1) + pq\sigma + p\rho + p\mu + \nu}{pq-1},$$

by using also (4.18) to proceeding as above, we obtain $u \equiv 0$ *a. e.* on $\mathbb{R}^+ \times \mathbb{R}^N$, which is a contradiction with (4.7).

- If $\gamma + \frac{\lambda}{p} = 0$, we have

$$\int_{\mathbb{R}^+\times\mathbb{R}^N} h\,|v|^q \; dx\,dt < \infty$$

Define

$$\Sigma = \left\{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^N;\, t^{2(\alpha_1+1)} \leq 2R^4, |x|^2 \leq 2R^2\right\}$$

From (4.17) we can get

$$\int_{\mathbb{R}^+\times\mathbb{R}^N} h\,|v|^q \; dx\,dt \leq C\left(\int_\Sigma h\,|v|^q\,\zeta_1\,dx\,dt\right)^{\frac{1}{pq}}$$

we have

$$\lim_{R\to+\infty}\int_\Sigma h\,|v|^q\,\zeta_1\,dx\,dt = 0,$$

hence, we infer that

$$\int_{\mathbb{R}^+\times\mathbb{R}^N} h\,|v|^q \; dx\,dt = 0,$$

this implies that $v \equiv 0$.
Similarly, if $\lambda + \frac{\gamma}{q} = 0$, proceeding as above, we infer that $u \equiv 0$, we arrive again at a contradiction with (4.7).
We deduce that no global weak solution is possible , which ends the proof.

□

Remark 4.3.2. *When* $\alpha_i \to 0$, $\beta_i \to 0$ *and* $\nu = \mu = 0$ (i.e. $h = k = 1$), *we recover the case who studied by A. Hakem (see [28]), when* $\alpha = \beta = 0$. *Also we retrieve the same result obtained by F. Sun & M. Wang (see [53]).*

Chapter 5

Exponential Stabilization of Some Evolution Problems

5.1 Exponential Stabilization of Solutions of the 1-D Transmission Wave Equation With Boundary Feedback (Published)

5.1.1 Introduction

in this section we are concerned with the following system:

$$u_{tt} = a^2 u_{xx} \quad \text{in} \quad (0, L/2) \times (0, \infty), \tag{5.1}$$

$$v_{tt} = b^2 v_{xx} \quad \text{in} \quad (L/2, L) \times (0, \infty), \tag{5.2}$$

$$u(0,t) = 0; \quad b^2 v_x(L,t) = -\lambda v_t(L,t); \quad t \geq 0, \tag{5.3}$$

$$u(L/2,t) = v(L/2,t); \quad a^2 u_x(L/2,t) = b^2 v_x(L/2,t); \quad t \geq 0, \tag{5.4}$$

$$u(x,0) = u_0(x); \;\; v(x,0) = v_0(x); \;\; u_t(x,0) = u_1(x); \;\; v_t(x,0) = v_1(x), \tag{5.5}$$

Where a, b, λ are positive constants.
The well-posed of problem ((5.1)- (5.5)) is by now well known in the case where $a = b$ (see [7], [34]), and can be similarly treated without any difficulty in the case where $a \neq b$. Define

the energy functional $E(t)$ (see [41])by

$$E(t) = \frac{1}{2}\int_0^{L/2}\Big(|u_t(x,t)|^2 + a^2|u_x(x,t)|^2\Big)dx + \frac{1}{2}\int_{L/2}^{L}\Big(|v_t(x,t)|^2 + b^2|v_x(x,t)|^2\Big)dx,$$

we construct the following perturbed energy functional E_ϵ

$$F(t) = 2\int_0^{L/2} xu_t(x,t)u_x(x,t)dx + 2\int_{L/2}^{L} xv_t(x,t)v_x(x,t)dx, \tag{5.6}$$

$$E_\epsilon(t) = E(t) + \epsilon F(t), \tag{5.7}$$

where ϵ is a positive constant, choosing sufficiently small.

5.1.2 Preliminaries

Before proving the below main result theorem, we first establish the following lemmas.

Lemma 5.1.3. *The total energy $E(t)$ of the system (5.1)-(5.5) is decreasing function for all $t \geq 0$.*

Proof. We examine the derivative of the energy

$$\frac{dE}{dt} = \frac{1}{2}\int_0^{L/2}\frac{\partial}{\partial t}\Big(u_t^2(x,t)\Big) + a^2\frac{\partial}{\partial t}\Big(u_x^2(x,t)\Big)dx + \frac{1}{2}\int_{L/2}^{L}\frac{\partial}{\partial t}\Big(v_t^2(x,t)\Big) + b^2\frac{\partial}{\partial t}\Big(v_x^2(x,t)\Big)dx,$$

using the identities

$$u_tu_{tt} = \frac{1}{2}\frac{\partial}{\partial t}(u_t^2), \quad \text{and} \quad u_tu_{xx} = \frac{\partial}{\partial x}(u_xu_t) - \frac{1}{2}\frac{\partial}{\partial t}(u_x^2),$$

we get

$$\begin{aligned}\frac{dE}{dt} &= \int_0^{L/2} u_tu_{tt} + a^2(\frac{\partial}{\partial x}(u_xu_t) - u_tu_{xx})dx + \int_{L/2}^{L} v_tv_{tt} + b^2(\frac{\partial}{\partial x}(v_xv_t) - v_tv_{xx})dx\\ &= \int_0^{L/2} u_t(u_{tt} - a^2u_{xx})dx + \int_{L/2}^{L} v_t(v_{tt} - b^2v_{xx})dx\\ &+ a^2\Big(u_x(L/2,t)u_t(L/2,t) - u_x(0,t)u_t(0,t)\Big) + b^2\Big(v_x(L,t)v_t(L,t) - v_x(L/2,t)v_t(L/2,t)\Big)\end{aligned}$$

using (5.1)-(5.2) we get

$$\frac{dE}{dt} = a^2\Big(u_x(L/2,t)u_t(L/2,t) - u_x(0,t)u_t(0,t)\Big) + b^2\Big(v_x(L,t)v_t(L,t) - v_x(L/2,t)v_t(L/2,t)\Big)$$

finally, using (5.3)-(5.4) it yield

$$\frac{dE}{dt} = -\lambda \left|v_t(L,t)\right|^2 \leq 0, \tag{5.8}$$

and then the energy is decreasing with time, *i.e.*,

$$E(t) \leq E(0) \quad \text{for all} \quad t \geq 0.$$

□

Lemma 5.1.4. *The perturbed energy satisfies*

$$\Big(1 - \frac{L\epsilon}{\min(a,b)}\Big)E(t) \leq E_\epsilon(t) \leq \Big(1 + \frac{L\epsilon}{\min(a,b)}\Big)E(t), \tag{5.9}$$

where ϵ is small enough, such that $0 < \epsilon < \dfrac{\min(a,b)}{L}$.

Proof. We have

$$\begin{aligned}
|F(t)| &= \left|2\int_0^{L/2} xu_t(x,t)u_x(x,t)dx + 2\int_{L/2}^{L} xv_t(x,t)v_x(x,t)dx\right| \\
&\leq \left|2\int_0^{L/2} xu_t(x,t)u_x(x,t)dx\right| + \left|2\int_{L/2}^{L} xv_t(x,t)v_x(x,t)dx\right| \\
&\leq \frac{1}{a}\int_0^{L/2} 2\,|x|\,|u_t(x,t)|\,|au_x(x,t)|\,dx + \frac{1}{b}\int_{L/2}^{L} 2\,|x|\,|v_t(x,t)|\,|bv_x(x,t)|\,dx \\
&\leq \frac{L}{2a}\int_0^{L/2} 2\,|u_t(x,t)|\,|au_x(x,t)|\,dx + \frac{L}{2b}\int_{L/2}^{L} 2\,|v_t(x,t)|\,|bv_x(x,t)|\,dx
\end{aligned}$$

by applying Young's inequality, we derive that

$$\begin{aligned}
|F(t)| &\leq \frac{L}{2a}\int_0^{L/2} |u_t(x,t)|^2 + a^2\,|u_x(x,t)|^2\,dx + \frac{L}{2b}\int_{L/2}^{L} |v_t(x,t)|^2 + b^2\,|v_x(x,t)|^2\,dx \\
&\leq \frac{L}{\min(a,b)}\Big(\frac{1}{2}\int_0^{L/2} |u_t(x,t)|^2 + a^2\,|u_x(x,t)|^2\,dx + \frac{1}{2}\int_{L/2}^{L} |v_t(x,t)|^2 + b^2\,|v_x(x,t)|^2\,dx\Big) \\
&= \frac{L}{\min(a,b)}E(t),
\end{aligned}$$

it therefore follows that

$$E_\epsilon(t) \leq E(t) + \epsilon\,|F(t)| \leq \Big(1 + \frac{L\epsilon}{\min(a,b)}\Big)E(t),$$

and

$$E_\epsilon(t) \geq E(t) - \epsilon\,|F(t)| \geq \Big(1 - \frac{L\epsilon}{\min(a,b)}\Big)E(t),$$

finally, we get

$$\Big(1 - \frac{L\epsilon}{\min(a,b)}\Big)E(t) \leq E_\epsilon(t) \leq \Big(1 + \frac{L\epsilon}{\min(a,b)}\Big)E(t).$$

□

5.1.5 Main results

We now in position to announce our result.

Theorem 5.1.6. *Assume that $b \leq a$, then there exist constants $M, \omega > 0$ such that the solution of (5.1)-(5.5)satisfies*

$$E(t) \leq ME(0)e^{-\omega t} \quad for \quad t \geq 0.$$

Proof. Differentiating (5.6) with respect to t, we obtain

$$\frac{dF}{dt} = \int_0^{L/2} 2xu_{tt}u_x dx + \int_0^{L/2} 2xu_t u_{xt} dx + \int_{L/2}^{L} 2xv_{tt}v_x dx + \int_{L/2}^{L} 2xv_t v_{xt} dx.$$

Moreover, by (5.1) it yield

$$\begin{aligned}\int_0^{L/2} 2xu_{tt}u_x dx &= \int_0^{L/2} 2xa^2 u_{xx}u_x dx\\ &= \int_0^{L/2} a^2 x\frac{\partial}{\partial x}(u_x^2)dx,\end{aligned}$$

by integrating by parts, we obtain

$$\int_0^{L/2} 2xu_{tt}u_x dx = a^2\frac{L}{2}u_x^2(L/2,t) - a^2\int_0^{L/2} u_x^2 dx,$$

and

$$\begin{aligned}\int_0^{L/2} 2xu_t u_{xt} dx &= \int_0^{L/2} x\frac{\partial}{\partial x}(u_t^2)dx\\ &= \frac{L}{2}u_t^2(L/2,t) - \int_0^{L/2} u_t^2 dx.\end{aligned}$$

Similarly, we have

$$\int_{L/2}^{L} 2xv_{tt}v_x dx = b^2 Lv_x^2(L,t) - b^2\frac{L}{2}v_x^2(L/2,t) - b^2\int_{L/2}^{L} v_x^2 dx,$$

and

$$\int_{L/2}^{L} 2xv_t v_{xt} = Lv_t^2(L,t) - \frac{L}{2}v_t^2(L/2,t) - \int_{L/2}^{L} v_t^2 dx,$$

then

$$\begin{aligned}\frac{dF}{dt} &= \frac{L}{2}\Big(a^2u_x^2(L/2,t) - b^2v_x^2(L/2,t)\Big) + \frac{L}{2}\Big(u_t^2(L/2,t) - v_t^2(L/2,t)\Big)\\ &\quad + L\Big(b^2v_x^2(L,t) + v_t^2(L,t)\Big) - 2\Big[\frac{1}{2}\int_0^{L/2}\Big(a^2u_x^2 + u_t^2\Big)dx + \frac{1}{2}\int_{L/2}^{L}\Big(b^2v_x^2 + v_t^2\Big)dx\Big].\end{aligned}$$

By (5.4)-(5.3), we infer

$$\frac{dF}{dt} = L\Big(1 + \frac{\lambda^2}{b^2}\Big)v_t^2(L,t) - 2E(t),$$

with the fact that

$$\frac{dE_\epsilon}{dt} = \frac{dE}{dt} + \epsilon\frac{dF}{dt}, \quad \text{and} \quad \frac{dE}{dt} = -\lambda\left|v_t(L,t)\right|^2,$$

we get

$$\begin{aligned}\frac{dE_\epsilon(t)}{dt} &= -\lambda v_t^2(L,t) + \epsilon L\Big(1 + \frac{\lambda^2}{b^2}\Big)v_t^2(L,t) - 2\epsilon E(t)\\ &= -2\epsilon E(t) - \lambda\Big[1 - \epsilon\frac{L(b^2+\lambda^2)}{\lambda b^2}\Big]v_t^2(L,t)\\ &\le -2\epsilon E(t),\end{aligned}$$

for all $0 < \epsilon < \min\Big(\frac{b}{L}, \frac{\lambda b^2}{L(b^2+\lambda^2)}\Big)$.
It then follows from (5.9) with $b \le a$ that

$$\begin{aligned}\frac{dE_\epsilon(t)}{dt} &\le -2\epsilon\Big(1 + \frac{L\epsilon}{b}\Big)\Big(1 - \frac{L\epsilon}{b}\Big)E(t)\\ &\le -2\epsilon\Big(1 - \frac{L\epsilon}{b}\Big)E_\epsilon(t)\end{aligned} \tag{5.10}$$

by using Gronwall's inequality or multiplying (5.10) by $e^{\omega t}$ and integrating from zero to t, we obtain

$$E_\epsilon(t) \le E_\epsilon(0)e^{-\omega t}, \quad \text{where } \omega = 2\epsilon\Big(1 - \frac{L\epsilon}{b}\Big),$$

from (5.9) we have

$$\begin{aligned}E(t) &\le \frac{1}{1 - \frac{L\epsilon}{b}}E_\epsilon(0)e^{-\omega t}\\ &\le \frac{1}{1 - \frac{L\epsilon}{b}}\Big(1 + \frac{L\epsilon}{b}\Big)E(0)e^{-\omega t}\\ &= \frac{b + L\epsilon}{b - L\epsilon}E(0)e^{-\omega t}.\end{aligned}$$

We deduce that

$$E(t) \leq ME(0)e^{-\omega t},$$

where

$$M = \frac{b + L\epsilon}{b - L\epsilon}, \quad \omega = 2\epsilon\Big(1 - \frac{L\epsilon}{b}\Big) \text{ such that, } \quad 0 < \epsilon < \min\Big(\frac{b}{L}, \frac{\lambda b^2}{L(b^2 + \lambda^2)}\Big).$$

□

Finally, we can say that the wave equation is *exponentially stabilizable* by boundary feedback.
The maximum decay rate: The constant λ called control gain and ω represents the decay rate of energy.
Let the functions $\Psi(\lambda) = \dfrac{\lambda b^2}{L(b^2 + \lambda^2)}$, and $\Phi(\epsilon) = 2\epsilon\big(1 - \frac{L\epsilon}{b}\big)$.
Because $\Psi'(\lambda) = \frac{b^2}{L}\big(\frac{b^2-\lambda^2}{(b^2+\lambda^2)^2}\big)$, and $\Phi'(\epsilon) = 2 - \frac{4L\epsilon}{b}$, then $\Psi(\lambda)$ attains the maximum $\frac{b}{2L}$, at $\lambda = b$, $\Phi(\epsilon)$ attains the maximum $\frac{b}{2L}$, at $\epsilon = \frac{b}{2L}$.
We infer that the decay rate ω achieve $\frac{b}{2L}$ when the control gain $\lambda = b$.

5.2 Exponential Stabilization of Solutions for Internally Damped Wave Equation Using Linear Boundary Feedback

5.2.1 Introduction and preliminaries

Let Ω a bounded open set in $\mathbb{R}^n$ having a boundary Γ of class C^2. We shall denote by ν the outward unit normal to Γ. Fix a point $x_0 \in \mathbb{R}^n$, set

$$\mathbf{m}(x) := x - x_0, \quad x, \in \mathbb{R}^n \tag{5.11}$$

and fix an open subset Γ^0 of Γ such that setting $\Gamma^1 = \Gamma \setminus \Gamma^0$ we have

$$\mathbf{m}(x).\nu > 0 \text{ on } \Gamma^1 \text{ and } \mathbf{m}(x).\nu \leq 0 \text{ on } \Gamma^0. \tag{5.12}$$

Consider the following feedback system:

$$u_{tt} = \Delta u + \alpha\Delta u_t \quad \text{in } \Omega \times \mathbb{R}_+, \tag{5.13}$$

$$u = 0 \quad \text{on } \Gamma^0 \times \mathbb{R}_+, \tag{5.14}$$

$$\frac{\partial u_t}{\partial \nu} = 0 \quad \text{on } \Gamma^1 \times \mathbb{R}_+, \tag{5.15}$$

$$\frac{\partial u}{\partial \nu} = -(\mathbf{m}.\nu)(\beta u + u_t) \quad \text{on } \Gamma^1 \times \mathbb{R}_+, \tag{5.16}$$

$$u(0) = u_0 \ \text{ and } \ u_t(0) = u_1 \ \text{ in } \ \Omega. \tag{5.17}$$

Where $\alpha > 0$ is the small internal damping constant and β be non-negative number. This system is well-posed in the following sense ([61]), introducing the Hilbert space V by

$$V = \left\{v \in H^1(\Omega) : v = 0 \ \text{ on } \ \Gamma^0\right\},$$

for every $(u_0, u_1) \in L^2(\Omega)$ the system (5.13)-(5.17) has a unique satisfying

$$C(\mathbb{R}_+; V) \cap (\mathbb{R}_+; L^2(\Omega));$$

furthermore its energy $E : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$E = \frac{1}{2}\int_\Omega |u_t|^2 + |\nabla u|^2 dx + \frac{\beta}{2}\int_{\Gamma^1} (\mathbf{m}\,.\nu)\, |u|^2 \, d\Gamma \tag{5.18}$$

The question of uniform exponential decay of energy defined by

$$E = \frac{1}{2}\int_\Omega |u_t|^2 + |\nabla u|^2 \, dx \tag{5.19}$$

of the solution of the undamped wave equation in Ω has been studied by a number of authors Chen[7], Lagnese [32], Lasiecka and Triggiani [33], Triggiani [55], and Lions [34]. They considered the system

$$u_{tt} = \Delta u \quad \text{in } \Omega \times \mathbb{R}_+, \tag{5.20}$$

$$u = 0 \quad \text{on } \Gamma^0 \times \mathbb{R}_+, \tag{5.21}$$

$$\frac{\partial u}{\partial \nu} = -b(x) u_t \quad \text{on } \Gamma^1 \times \mathbb{R}_+, \tag{5.22}$$

$$u(0) = u_0 \ \text{ and } \ u_t(0) = u_1 \ \text{ in } \ \Omega. \tag{5.23}$$

, where $b(x) \geq b_0 > 0$. They proved a result of the type

$$E(t) \leq M e^{-\beta t}, \quad t \geq 0, \tag{5.24}$$

$M \geq 1$ and $\beta > 0$ being some constants. Later, Lagnese [32] and Komornik [57] obtained somewhat faster energy decay rates for certain forms of $b(x)$. Also Chen [11] demonstrated , faster energy decay rate than (5.24) when external damping $2uu_t$ is present in the left hand side of (5.19). In the method of treatment [11, 56, 32] adopt a direct method by constructing suitable functionals related to $E(t)$, where as [7, 55] employ semi-group theory , in as much as the underlying operator of the system generates a strongly continuous contraction semi-group. Our goal in this section is to prove that the linear boundary feedback controller exponentially stabilizes the equilibrium to zero of the system (5.13)-(5.17), i.e. the feedback leads to faster energy decay.

The construction of the functionals is motivated by the multiplier technique.

Proposition 5.2.2. *The energy* $E : \mathbb{R}_+ \to \mathbb{R}_+$ *of the system* (5.13)-(5.17) *is non-increasing function and*

$$E'(t) = -\alpha \int_\Omega |\nabla u_t|^2 \, dx - \int_{\Gamma^1} \boldsymbol{m}.\nu \, |u_t|^2 \, d\Gamma \leq 0. \tag{5.25}$$

Proof. We proceed by differentiating Eq.(5.18) with respect t, we have

$$E'(t) = \frac{1}{2} \int_\Omega 2u_t u_{tt} + 2\nabla u.\nabla u_t dx + \beta \int_\Gamma \mathbf{m}.\nu u u_t d\Gamma,$$

replacing u_{tt} by $\Delta u + \alpha \Delta u_t$ to obtain

$$\begin{aligned} E'(t) &= \int_\Omega u_t(\Delta u + \alpha \Delta u_t) + \nabla u.\nabla u_t dx + \beta \int_\Gamma \mathbf{m}.\nu u u_t d\Gamma \\ &= \int_\Omega (u_t \Delta u + \alpha u_t \Delta u_t + \nabla u.\nabla u_t) dx + \beta \int_\Gamma \mathbf{m}.\nu u u_t d\Gamma. \end{aligned}$$

Applying Green's formula we have

$$\begin{aligned} E'(t) &= \int_\Gamma u_t \frac{\partial u}{\partial \nu} d\Gamma - \int_\Omega \nabla u.\nabla u_t dx + \alpha \int_\Gamma u_t \frac{\partial u_t}{\partial \nu} d\Gamma \\ &- \alpha \int_\Omega |\nabla u_t|^2 \, dx + \int_\Omega \nabla u.\nabla u_t dx + \beta \int_\Gamma \mathbf{m}.\nu u u_t d\Gamma, \end{aligned}$$

using (5.15) and replacing $\frac{\partial u}{\partial \nu}$ by $-(\mathbf{m}.\nu)(\beta u + u_t)$ to obtain

$$\begin{aligned} E'(t) &= \beta \int_{\Gamma^1} \mathbf{m}.\nu u u_t d\Gamma - \int_{\Gamma^1} \mathbf{m}.\nu \, |u_t|^2 \, d\Gamma \\ &- \alpha \int_\Omega |\nabla u_t|^2 \, dx + \beta \int_{\Gamma^1} \mathbf{m}.\nu u u_t d\Gamma \\ &= -\alpha \int_\Omega |\nabla u_t|^2 \, dx - \int_{\Gamma^1} \mathbf{m}.\nu \, |u_t|^2 \, d\Gamma. \end{aligned}$$

Use of (5.12) yields

$$E'(t) \leq 0 \quad \text{for} \quad t \geq 0. \tag{5.26}$$

Hence energy is non-increasing with time, *i.e.,*

$$E(t) \le E(0) \quad \text{for} \quad t > 0. \tag{5.27}$$

□

5.2.3 Main results

We now in position to announce our result. The uniform exponential decay of $E(t)$ for the problem (5.13)-(5.17) follows from the theorem:

Theorem 5.2.4. *Let u be a regular solution of* (5.13)-(5.17)*. Then the energy satisfy*

$$E(t) \le Me^{-\omega t}E(0), \quad t \ge 0,$$

for some reals $M \ge 1,\ \omega > 0$.

Before proving the above main theorem, we first establish the following lemmas.

Lemma 5.2.5. *If u is regular solution of problem* (5.13)-(5.17)*, then the function* $F(t)$ *defined by*

$$F(t) = \frac{1}{2}\int_\Omega (|\nabla u_t|^2 + |\Delta u|^2)dx \tag{5.28}$$

is non-increasing for $t \ge 0$.

Proof. Differentiating (5.28) with respect to t we obtain

$$F'(t) = \int_\Omega (\nabla u_t.\nabla u_{tt} + \Delta u \Delta u_t)dx \tag{5.29}$$

by using Green's formula, we have

$$\int_\Omega \nabla u_t.\nabla u_{tt}dx = \int_\Gamma u_{tt}\frac{\partial u_t}{\partial \nu}d\Gamma - \int_\Omega u_{tt}\Delta u_t dx \tag{5.30}$$

then, using (5.13) in second integral of the right hand side of (5.30), we have

$$\int_\Omega \nabla u_t.\nabla u_{tt}dx = \int_\Gamma u_{tt}\frac{\partial u_t}{\partial \nu}d\Gamma - \alpha\int_\Omega |\Delta u_t|^2\,dx - \int_\Omega \Delta u \Delta u_t dx \tag{5.31}$$

we obtain

$$F'(t) = \int_\Gamma u_{tt}\frac{\partial u_t}{\partial \nu}d\Gamma - \alpha\int_\Omega |\Delta u_t|^2\,dx, \tag{5.32}$$

using the boundary condition (5.14) and (5.15) we have

$$F'(t) = -\alpha \int_\Omega |\Delta u_t|^2 \, dx \le 0. \tag{5.33}$$

Hence $F(t)$ is non-increasing for $t \ge 0$. We conclude that

$$F(t) \le F(0) \quad \text{for } t \ge 0. \tag{5.34}$$

□

Lemma 5.2.6. *Let u is regular solution of problem (5.13)-(5.17). If we define the function $G(t)$ by*

$$G(t) = \int_\Omega \nabla u_t.(\boldsymbol{m}.\Delta)\nabla u \, dx \tag{5.35}$$

then $|G(t)| \le KF(t)$ for $t \ge 0$, where $K \ge 1$ is a constant, independent of t.

Proof. From (5.35) we can write

$$|G(t)| \le \int_\Omega |\nabla u_t| \, |(\mathbf{m}.\Delta)\nabla u| \, dx \le \frac{1}{2}\int_\Omega |\nabla u_t|^2 + |(\mathbf{m}.\Delta)\nabla u|^2 \, dx, \tag{5.36}$$

where we have used the Young's inequality. We now define a constant $K \ge 1$ so that we can write

$$\int_\Omega |(\mathbf{m}.\Delta)\nabla u|^2 \, dx \le K \int_\Omega |\Delta u|^2. \tag{5.37}$$

It follows from (5.36) then

$$G(t) \le KF(t) \quad \text{for } t > 0.$$

□

Lemma 5.2.7. *For every $u \in H^1(\Omega)$,*

$$\int_\Omega [2u.(\boldsymbol{m}.\Delta)u + n\,|u|^2]dx = \int_\Gamma \boldsymbol{m}.\nu\,|u|^2. \tag{5.38}$$

Proof. We have

$$\begin{aligned}\int_\Omega [2u.(\mathbf{m}.\Delta)u + n\,|u|^2]dx &= \int_\Omega [(\mathbf{m}.\Delta\,|u|^2) + n\,|u|^2]dx \\ &= \int_\Omega div(\mathbf{m}\,|u|^2)dx \\ &= \int_\Gamma \mathbf{m}.\nu\,|u|^2\,dx.\end{aligned}$$

Hence the lemma. □

Lemma 5.2.8. *if u is a regular solution of* (5.13)-(5.17)*, then*

$$\begin{aligned}&\rho'(t) + \alpha\rho_0'(t) + 2\alpha G(t) + 2E(t)\\ &\quad\leq \int_{\Gamma^0} |\boldsymbol{m}.\nu|\,|\nabla u_t|^2\,d\Gamma + \int_{\Gamma^1} \boldsymbol{m}.\nu u_t^2 d\Gamma,\end{aligned} \tag{5.39}$$

where

$$\rho(t) = \int_\Omega [2u_t(\boldsymbol{m}.\nabla u) + (n-1)uu_t]dx \tag{5.40}$$

and

$$\rho_0(t) = \frac{n+1}{2}\int_\Omega |\nabla u|^2\,dx \tag{5.41}$$

Proof. Differentiating (5.40) with respect to t and replacing u_{tt} by $\Delta u + \alpha\Delta u_t$ we have

$$\begin{aligned}\rho'(t) = \int_\Omega &[2(\Delta u + \alpha\Delta u_t)(\mathbf{m}.\nabla u) + 2u_t(\mathbf{m}.\nabla u_t)\\ &\qquad + (n-1)(\Delta u + \alpha\Delta u_t)u + (n-1)u_t^2\,dx.\end{aligned}$$

Applying Green's formula we obtain

$$\begin{aligned}\rho'(t) = &\int_\Gamma [2(\mathbf{m}.\nabla u) + (n-1)u]\Big[\frac{\partial u}{\partial \nu} + \alpha\frac{\partial u_t}{\partial \nu}\Big]d\Gamma\\ &- \int_\Omega [2\nabla(\mathbf{m}.\nabla u) + (n-1)\nabla u].\nabla(u + \alpha u_t)dx\\ &+ \int_\Omega [2u_t(\mathbf{m}.\nabla u_t) + (n-1)u_t^2]dx.\end{aligned}$$

Using the boundary conditions (5.14)-(5.16) we have

$$\begin{aligned}
\rho'(t) &= \int_{\Gamma^0} 2(\mathbf{m}.\nabla u)\Big[\frac{\partial u}{\partial \nu} + \alpha\frac{\partial u_t}{\partial \nu}\Big]d\Gamma - (n-1)\int_{\Gamma^1} u(\mathbf{m}.\nu)(\beta u + u_t)d\Gamma \\
&\quad - \int_\Omega [2(\nabla u + (\mathbf{m}.\nabla)\nabla u) + (n-1)\nabla u].\nabla(u + \alpha u_t)dx \\
&\quad + \int_\Omega [2u_t(\mathbf{m}.\nabla u_t) + (n-1)u_t^2]dx \\
&= \int_{\Gamma^0} 2(\mathbf{m}.\nabla u)\Big[\frac{\partial u}{\partial \nu} + \alpha\frac{\partial u_t}{\partial \nu}\Big]d\Gamma - (n-1)\int_{\Gamma^1} u(\mathbf{m}.\nu)(\beta u + u_t)d\Gamma \\
&\quad - \int_\Omega [2(\nabla u.(\mathbf{m}.\nabla)\nabla u) + n\,|\nabla u|^2]dx \\
&\quad + \int_\Omega [2u_t(\mathbf{m}.\nabla)u_t + nu_t^2]dx - 2\alpha\int_\Omega (\nabla u_t.(\mathbf{m}.\nabla)\nabla u)dx \\
&\quad - \alpha(n+1)\int_\Omega (\nabla u.\nabla u_t)dx - \int_\Omega (|\nabla u|^2 + u_t^2)dx.
\end{aligned}$$

Applying Lemma 5.2.7, we obtain

$$\begin{aligned}
\rho'(t) &= \int_{\Gamma^0} 2(\mathbf{m}.\nabla u)\Big[\frac{\partial u}{\partial \nu} + \alpha\frac{\partial u_t}{\partial \nu}\Big]d\Gamma - (n-1)\int_{\Gamma^1} u(\mathbf{m}.\nu)(\beta u + u_t)d\Gamma \\
&\quad + \int_\Gamma \mathbf{m}.\nu(u_t^2 - |\nabla u|^2)d\Gamma - 2\alpha G(t) - \alpha\rho_0(t) - 2E(t).
\end{aligned} \tag{5.42}$$

Since $u = 0$ on Γ^0, $\nabla u = \nu\big(\frac{\partial u}{\partial \nu}\big)$ and $|\nabla u|^2 = \left|\frac{\partial u}{\partial \nu}\right|^2$ on Γ^0. Also $\mathbf{m}.\nu > 0$ on Γ^1. Hence we have from (5.42)

$$\begin{aligned}
&\rho'(t) + 2\alpha G(t) + \alpha\rho_0(t) + 2E(t) \\
&\quad \le \int_{\Gamma^0} \mathbf{m}.\nu\,|\nabla u|^2\,d\Gamma + 2\alpha\int_{\Gamma^0} \mathbf{m}.\nu(\nabla u.\nabla u_t)d\Gamma + \int_{\Gamma^1} \mathbf{m}.\nu u_t^2 d\Gamma \\
&\quad \le \int_{\Gamma^0} \mathbf{m}.\nu\,|\nabla u|^2\,d\Gamma + \int_{\Gamma^0} \mathbf{m}.\nu(|\nabla u|^2 + \alpha^2\,|\nabla u_t|^2)d\Gamma + \int_{\Gamma^1} \mathbf{m}.\nu u_t^2 d\Gamma \\
&\quad \le \alpha^2\int_{\Gamma^0} |\mathbf{m}.\nu|\,|\nabla u|^2\,d\Gamma + \int_{\Gamma^1} \mathbf{m}.\nu u_t^2 d\Gamma
\end{aligned}$$

since $\mathbf{m}.\nu < 0$ on Γ^0. Hence the lemma. □

We are now ready to prove the main result.

Proof of the Theorem. We define a function $H(t)$ by

$$H(t) = \lambda E(t) + \alpha[\rho(t) + \alpha\rho_0(t) + F(t)], \tag{5.43}$$

where λ is a positive constant defined by

$$\int_{\Gamma^1} \mathbf{m}.\nu u_t^2 d\Gamma \leq \lambda \int_{\Omega} |\nabla u_t|^2 \, dx \tag{5.44}$$

for all $u \in H^2_{\Gamma^0}(\Omega)$, where (5.44) follow from combination of the Poincaré's inequality with the Trace inequality in $H^2(\Omega)$.
We also define the positive constants λ_0, λ_1, and λ_2 by

$$E(t) \leq \lambda_0 F(t) \tag{5.45}$$

$$\int_{\Gamma^1} u^2 dx \leq \lambda_1 \int_{\Omega} |\nabla u|^2 \, dx \;\; (\lambda_1 > 1) \quad \text{(Poincaré's inequality)} \tag{5.46}$$

and

$$\int_{\Gamma^0} |\mathbf{m}.\nu| \, |\nabla u_t|^2 \, d\Gamma \leq \lambda_2 \int_{\Omega} |\Delta u_t|^2 \, dx \qquad \text{(Poincaré- Trace).} \tag{5.47}$$

Now we have from (5.40)

$$\begin{aligned} |\rho(t)| &\leq R_0 \int_{\Omega} (u_t^2 + |\nabla u|^2) dx + \frac{n-1}{2} \int_{\Omega} (u^2 + u_t^2) dx \quad \text{(use Young's inequality)} \\ &\leq [2R_0 + (n-1)\lambda_1] E(t) = C_0 E(t), \end{aligned} \tag{5.48}$$

where $R_0 = \sup\{|\mathbf{m}(x)| : x \in \Omega\}$ and $C_0 = [2R_0 + (n-1)\lambda_1]$.
From (5.41) we also have

$$0 \leq \rho_0(t) \leq (n+1)E(t). \tag{5.49}$$

thanks to (5.45), (5.48) and (5.49), it follows from (5.43) that

$$\begin{aligned} (\lambda + \frac{\alpha}{\lambda_0} - \alpha C_0) E(t) &\leq H(t) \\ &\leq [\lambda + \alpha(C_0 + \alpha(n+1))] E(t) + \alpha F(t). \end{aligned} \tag{5.50}$$

Now differentiating (5.43) with respect to t and applying (5.25), (5.33) and Lemma 5.2.7, we have

$$\begin{aligned} H'(t) &= \lambda E'(t) + \alpha[\rho'(t) + \alpha \rho_0'(t) + F'(t)] \\ &\leq -\lambda \alpha \int_{\Omega} |\nabla u_t|^2 \, dx + \alpha \Big[\alpha^2 \int_{\Gamma^0} |\mathbf{m}.\nu| \, |\nabla u_t|^2 \, d\Gamma + \int_{\Gamma^1} \mathbf{m}.\nu \nabla u_t^2 d\Gamma \\ &- 2G(t) - 2E(t) - \alpha \int_{\Omega} |\Delta u_t|^2 \, dx. \Big] \end{aligned} \tag{5.51}$$

Applying the inequalities (5.44), (5.47) and Lemma 5.2.6, we obtain

$$H'(t) \le \alpha^2(\alpha\lambda_2 - 1)\int_\Omega |\Delta u_t|^2\,dx + 2\alpha[\alpha K F(t) - E(t)]. \tag{5.52}$$

Now since $(u_0, u_1) \in H^2_{\Gamma 0}(\Omega) \times H^1(\Omega)$, therefore, from the inequalities (5.34) and (5.27), we have $F(t) \le F(0) < \infty$ and $E(t) \le E(0) < \infty$. Hence there exists a positive constant K_0 such that $F(t) \le K_0 E(t)$. Inequality (5.52) can then be written as

$$H'(t) \le \alpha^2(\alpha\lambda_2 - 1)\int_\Omega |\Delta u_t|^2\,dx + \alpha(2\alpha K K_0 - 1)E(t) - \alpha E(t). \tag{5.53}$$

Let

$$\alpha \le \min\Big\{\frac{1}{\lambda_2}, \frac{1}{2KK_0}, \frac{\lambda}{C_0}\Big\}, \tag{5.54}$$

which determines here an upper bound of the value of α consistent with stability. We then have from (5.54)

$$H'(t) \le \alpha E(t) \tag{5.55}$$

and at the same time we have from (5.50)

$$\begin{aligned}\frac{\alpha}{\lambda_0}E(t) \le H(t) &\le [\lambda + \alpha(C_0 + \alpha(n+1) + K_0)]E(t)\\ &= \alpha_0 E(t),\end{aligned} \tag{5.56}$$

where the positive constant

$$\alpha_0 = a\alpha^2 + b\alpha + c \tag{5.57}$$

is a quadratic function of α, and $a = n + 1,\ b = C_0 + K_0,\ c = \lambda$ are independent of it. Use of (5.56) in (5.55) yields

$$H'(t) + \omega H(t) < 0, \tag{5.58}$$

where $\omega = \dfrac{\alpha}{\alpha_0}$. Multiplying (5.58) by $e^{\omega t}$ and integrating from zero to t, we get

$$H(t) \le e^{-\omega t} H(0).$$

Thus it follows from (5.56)

$$E(t) \le M e^{-\omega t} E(0) \quad \text{for} \quad t \ge 0, \tag{5.59}$$

where

$$M = \frac{\alpha_0\lambda_0}{\alpha_0} \ge 1 \tag{5.60}$$

by virtue of (5.56).

Finally, we can say that the internally damped wave equation is *exponentially stabilizable* by linear boundary feedback.

The maximum decay rate: The number ω represents the decay rate of energy. Let the function $\omega(\alpha) =: \dfrac{\alpha}{(n+1)\alpha^2 + (C_0 + K_0)\alpha + \lambda}$.

Because $\omega'(\alpha) = \dfrac{-(n+1)\alpha^2 + \lambda}{[(n+1)\alpha^2 + (C_0 + K_0)\alpha + \lambda]^2}$, then the decay rate is maximum for $\alpha = \sqrt{\frac{\lambda}{n+1}}$.

From (5.54), the maximum decay rate is attained for $\alpha \leq \min\left\{\frac{1}{\lambda_2}, \frac{1}{2KK_0}, \frac{\lambda}{C_0}, \sqrt{\frac{\lambda}{n+1}}\right\}$.

Bibliography

[1] K. Ammari, *Derichlet boundary stabilization of the wave equation* , Asymptot. Anal. **30** (2002) 117-130.

[2] A. Benaissa & S. Messoudi , *Exponential decay of solutions of a non-linearly damped wave equation.* NoDEA Nonlinear Differential Equations Appl, **12** (4): 391-399, 2005.

[3] M. Berbiche, A. Hakem, *Necessary conditions for the existence and sufficient conditions for the nonexistence of solutions to a certain fractional telegraph equation.* Memoirs on Differential Equations and Mathematical physics. vol 56, 2012, 37-55.

[4] M. Caputo, *Linear models of dissipation whose Q is almost frequency independent-II*, Geophysical Journal of the Royal Astronomical Society **13** (1967), 529-539.

[5] T. Cazenave, F. Dickstein and F. D. Weissler, *An equation whose Fujita critical exponent is not given by scaling*, Nonlinear anal. 68(2008), pp. 862–874.

[6] T. Cazenave & A. Haraux, *Introduction aux problèmes d'évolution semi-linéaires*, Ellipses, Paris, (1990).

[7] G. Chen, *Energy decay estimates and exact boundary value controllability for the wave equation in a bounded domain* . J. Math. Pures Appl. **58**, 249-273 (1979)

[8] G. Chen, *control and stabilization for the wave equation in a bounded domain* . SIAM J. Control Optim. **17**, 66-81 (1979).

[9] G. Chen, *A note on the boundary stabilization of wave equation* , SIAM J. Control Optim.**19**, 106-113 (1981).

[10] W. Chen & S. Holm, *Fractional Laplacian time-space models for linear and nonlinear lossy media exhibiting arbitrary frequency power-law dependency* J. Acoust. Soc. Am., Vol. 115, No. 4, April 2004.

[11] G. Chen, *A note on the boundary stabilization of wave equation* . SIAM J. Control Optim. **19**, 106-113 (1981).

[12] G. Chen& D. L. Russell, *A mathematical model for linear elastic system with structural damping* , Quart. Appl. Math. **39** (1982) 433-454.

[13] R. Christensen, *Theory of Viscoelasticity*, Academic Press, New York, 1971.

[14] C. Deng & Y. Liu & W. Jiang & F. Huang, *Exponential decay rate for a wave equation with Dirichlet boundary control*, Applied Mathematics letters, **20** (2007) 861-865.

[15] M. Djilali & A. Hakem, *Nonexistence of global solutions to system of semi-linear fractional evolution equations.* Universal Journal of Mathematics and Applications, **1** (3) (2018) 171-177.

[16] M. Djilali & A. Hakem, *Exponential stabilization of solutions for the 1-D transmission wave equation with boundary feedback.* Advances in the Theory of Nonlinear Analysis and its Applications **2** (2018) No. 4, 217-223.

[17] M. Djilali & A. Hakem, *Nonexistence of global solutions to semi-linear fractional evolution equation.* International Journal of Maps in Mathematics. (**Accepted**) (12, 1^{st} 2018).

[18] M.M. Dzrbashjan, *Integral transforms and representations of functions in the complex domain*, Nauka, Moscow, 1966.

[19] L.C. Evans, *Partial Differential Equations*, Vol. 19, American Mathematical Society, 1997.

[20] M. Escobedo & H. A. Levine, *Critical blow up and global existence numbers of a weakly coupled system of reaction-diffusion equation.* Arch. Rational. Mech. Anal. 129 (1995),47-100.

[21] A. Z. Fino, *Critical exponent for damped wave equations with nonlinear memory* , Nonlinear Analysis, **74** (2011) 5495-5505.

[22] A. Z. Fino, H. Ibrahim & A. Wehbe, *A blow-up result for a nonlinear damped wave equation in exterior domain: The critical case*, Computers & Mathematics with Applications, Volume 73, Issue 11, (2017), pp. 2415-2420.

[23] H. Fujita, *On the blowing up of solutions of the problem for* $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci.Univ. Tokyo **13** (1966), 109 - 124.

[24] M. GUEDDA & M. KIRANE, *A note on nonexistence of global solutions to a nonlinear integral equation*, Bull. Belg. Math. Soc. Simon Stevin **6** (1999), 491 - 497.

[25] M. GUEDDA & M. KIRANE, *Local and global nonexistence of solutions to semilinear evolution equations*. Electronic Journal of Differential Equations, Conference **09** (2002), pp. 149-160.

[26] A. GUESMIA, *Nouvelles inégalités intégrales et applications a la stabilisation des systèmes distribués non dissipatifs*, CR, Math. ACAD, Sci. Paris 336(2003), 801-804.

[27] B. GUO, X. PU & F. HUANG, *Fractional Partial Differential Equations and Their Numerical Solutions*. World Scientific Publishing Co. Pte. Ltd. Beijing, China (2011).

[28] A. HAKEM, *Nonexistence of weak solutions for evolution problems on* $\mathbb{R}^N$, Bull. Belg. Math. Soc. **12** (2005), 73-82.

[29] A. HAKEM & M. BERBICHE, *On the blow-up of solutions to semi-linear wave models with fractional damping* . IAENG International Journal of Applied Mathematics, (2011) 41:3, IJAM-41-3-05.

[30] R. IKEHATA, *Small data global existence of solutions for dissipative wave equations in an exterior domain*, Funkcial. Ekvac. **44** (2002) 259-269.

[31] M. KIRANE, Y. LASKRI & N.-E.TATAR, *Critical exponents of fujita type for certain evolution equations and systems with spation-temporal fractional derivatives*.J. Math. Anal. Appl. 312 (2005) 488-501.OL

[32] J. LAGNESE, *Note on boundary stabilization of wave equations*, SIAM J. Control Optim. **26**(1988). 1250-1256.

[33] I. LASIECKA & R. TRIGIANI, *Uniform exponential energy decay of the wave equation in a bounded region with feedback control in the Dirichlet boundary conditions*, J. Differential Equations. **66** (1987) 340-390.

[34] J. L. LIONS, *Contrôlabilité exacte perturbation et stabilisation de systèmes distribués, Tome 1, Contrôlabilité exacte*. Masson, Paris (1988).

[35] J. L. LIONS, *Contrôlabilité exacte perturbation et stabilisation de systèmes distribués, Tome 2, Perturbation*. Masson, Paris (1988).

[36] W. LIU, *Stabilization and controllability for the transmission wave equation*. IEEE Transcation on Automatic Control **46**, 1900-1907 (2001).

[37] W. LIU & G. H. WILLIAMS, *The exponential stability of the problem of the transmission of the wave equation*. Bull Australian Math. Soc. **57** (1998), 305-327.

[38] W. LIU & E. ZUAZUA, *Decay rates for dissipative wave equations*. Ricerche di Matimatica. **48**, 61-75 (1999).

[39] F. MAINARDI, *Fractional Calculs and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*, Imperial College Press (2010).

[40] W. MINGXIN, *Global existence and finite time blow up for a reaction-diffusion system*. Z. Angew. Math. Phys. 51 (2000) 160-167.

[41] J. E. MUÑOZ RIVERA & H. P. OQUENDO, *The Transmission Problem of Viscoelastic Waves*. Acta Applicandae Mathematicae. **62**: 1-21, 2000.

[42] M. NAKAO, *Energy decay for the wave equation with nonlinear weak dissipation*. Differential Integral Equation, **8**, 681-688 (1995).

[43] T. OGAWA & H. TAKIDA, *Non-existence of weak solutions to nonlinear damped wave equations in exterior domains*, J. Nonliniear analysis **70** (2009), 3696-3701.

[44] I. PODLUBNY, *Fractional differential equations*. Mathematics in Science and Engineering, vol 198, Academic Press, New York, 1999.

[45] S.I. POHOZAEV & A. TESEI, Nonexistence of Local Solutions to Semilinear Partial Differential Inequalities, Nota Scientifica 01/28, Dip. Mat. Universitá "La Sapienza", Roma (2001).

[46] S. POHOZAEV & L. VERON, *Blow up results for nonlinear hyperbolic enequalities*, Ann. Scuola Norm. Sup. Pisa CI. Sci. (4) Vol. **XXIX** (2000), pp. 393-420.

[47] Y. POVSTENKO, *Linear Fractional Diffusion-Wave Equation for Scientists and Engineers.* Springer International Publishing Switzerland 2015.

[48] C. POZRIKIDIS, *The fractional Laplacian*, Taylor & Francis Group, LLC /CRC Press, Boca Raton (USA), (2016).

[49] J. Rauch & M. Taylor, *Exponential decay of solutions to hyperbolic equations in bounded domains*. India J. Math. **24** 79-83 (1974)

[50] W. Rudin, *Real and complex analysis*, Second edition, Mc Graw-Hill, Inc, New York, 1974.

[51] S. G. Samko, A. A. Kilbas & O. I. Marichev, *Fractional integrals and derivatives: Theory and application*, Gordan and Breach Sci. Publishers, Yverdon, 1993.

[52] B. Straughan, *The Energy Method, Stability, and Nonlinear Convection*, Springer Science +Business Media, New York, 2004.

[53] F. Sun & M. Wang, *Existence and nonexistence of global solutions for a nonlinear hyperbolic system with damping*, Nonlinear analysis **66** (2007) 2889-2910.

[54] G.Todorova & B.Yordanov, *Critical Exponent for a Nonlinear Wave Equation with Damping*. Journal of Differential Equations **174**, 464-489 (2001).

[55] R. Triggiani, *Wave equation on abounded domain with boundary dissipation: An operator approch*, J. Math. Anal. Appl. **137** 438-461 (1989).

[56] V. Komornik, *Rapid Boundary Stabilization of the Wave Equation*, SIAM J. Control Optim. **29** (1991), 197-208.

[57] V. Komornik, *On the Nonlinear Boundary Stabilization of the Wave Equation*, Chin. Ann. of Mths. **14B**: 2(1993), 153-164.

[58] Y. Yamauchi, *Blow-up results for a reaction-deffusion system* , Methods Appl. Anal. **13** (2006), 337 - 350.

[59] Q. S. Zhang, *A blow up result for a nonlinear wave equation with damping: the critical case*, C. R. Acad.Sci. paris, Volume **333**, no.2, (2001), 109-114.

[60] S-Mu. Zheng, *Nonlinear evolution equations*,
Chapman & Hall/CRC Press, Florida (USA), (2004).

[61] E. Zuazua, *Uniform stabilization of the wave equation by nonliniear boundary feedback*. SIAM J. Control and optim. **28** (1990) 466-478.

[62] E. Zuazua, *Exponential decay for the semi-linear wave equation with locally distributed damping*. Commun. in Partial Differential Equations **15**, 205-235 (1990).