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Intitulée

A study of some fractional order differential equations and inclusions with impulses and delay

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1. R. P. Agarwal, B. Hedia and M. Beddani, Structure of solution sets for impulsive fractional differential equations, *J. Fract. Calc. Appli.* **9** (1) (2018), 15-34.

2. M. Beddani and B. Hedia, Solution set for fractional differential inclusions, *J. Fract. Calc. Appli.* **9** (2) (2019), 273-289.

3. M. Beddani and B. Hedia, Boundary value problems for fractional differential equations in special Banach Space, *Int. J. Anal. Appl.* **19** (2) (2021), 239-251.

4. M. Beddani, Solution set for impulsive fractional differential inclusions, *Kragujevav J. Math.* **46** (1) (2022), 49-64.

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Abstract

The content of this thesis is the study of a class of impulsive fractional differential equations and inclusions with Riemman-Liouville fractional derivative, Our starting point will be the property:

$$[(I^{\alpha} \circ^{RL} D^{\alpha})f](t) = f(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)} \lim_{t\to 0}(I^{1-\alpha}f)(t).$$

The value of $\lim_{t\to 0}(I^{1-\alpha}f)(t)$ stands to be value initial and this value is zero if f is continuous on the interval $[0, b]$. Hence the space of functions in which we consider the differential equations will be a space where this limit exists and finite without necessarily being zero. This idea of this thesis was recalled in the area of the impulsive differential equations on a bounded interval and in the case of differential inclusions with or without pulses. In another hand we have been led to study these equations where the interval is unbounded, we had a lack of compactness which we remedied by calling for a suitable choice of a measure of non-compactness.

Key words and phrases: Fractional differential equations, fractional differential inclusions, fixed points theorems, Riemann-Liouville fractional derivative, measure of non-compactness, unbounded domain, solution sets, topological structure.

Résumé

Le contenu de cette thèse est l'étude d'une classe d'equations et d'inclusions differentielles fractionnaires au sens de Riemman-Liouville. Notre point de départ sera la propriété suivante

$$[(I^{\alpha} \circ^{RL} D^{\alpha})f](t) = f(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)} \lim_{t\to 0}(I^{1-\alpha}f)(t).$$

La valeur de $\lim_{t\to 0}(I^{1-\alpha}f)(t)$ tient lieu de valeur initiale et cette valeur est nulle si f est continue sur un intervalle $[0, b]$. De là l'espace des fonctions dans lequel nous allons considérer ces équations différentielles sera un espace ou cette limite existe et finie sans etre forcément nulle. Cette idée de travail nous l'avons évoqué dans le cadre des equations différentielle impulsives sur un intervalle borné ensuite dans des inclusions différentielles avec ou sans impulsions. d'autre part nous avons été amené a étudié ces équations ou l'intevalle est non borné, nous avons eu un défaut de compacité auquel on a remedié en faisant appel à un choix propice d'une mesure de non compacité.

Mots clés: Equations différentielles fractionnaires, inclusions différentielles fractionnaires, théorèmes de points fixes, dérivée de Riemann-Liouville, mesure de non-compacité, intervalle non borné, ensemble de solutions, structure topologique.

Introduction

Fractional differential equations and inclusions has recently received much attention due to its important applications in modeling phenomena of science and engineering. The employment of differential equations and inclusions with fractional order allows to deal with many problems in numerous areas including fluid flow, rheology, electrical networks, viscoelasticity, electrochemistry, etc. For complete references, we refer to some significant works, e.g., see the famous works [34], Agur [4] Wagner [60] and E. Krug research's in [46, 47], Metzler [49], Klein [33], Delbosco [24] see also. In the past few years, there has been a great contribution in fractional differential inclusions, let us refer to some relevant works in [63, 64].

There are two measures which are the most important ones. the Kuratowski measure of noncompactness $\alpha(B)$ of a bounded set B in a metric space is defined as infimum of numbers $r > 0$ such that B can be covered with a finite number of sets of diamiter smaller than r, the Hausdorf measure of noncompactness $\chi(B)$ defined as infimum of numbers r suth that B can be covered with a finite number of balls of radius smaller than r. Several authors have studied the measures of noncompactness in Banach spaces. See, for example, the books such as [5, 10] and the articles [1, 14, 16, 17], and references therein.

Impulsive ordinary differential inclusions and functional differential inclusions with different conditions have been intensely studied by many mathematicians. At present the foundations of the general theory are already laid, and many of them are investigated in detail see[28, 22, 38].

Topological structure of the solution set for ordinary differential equations and inclusions is developed recently, see for example [6, 7, 18, 20, 32, 30, 58], and the monograph [28]. In 1923, Kneser's was the first to extend Peano's result concerning the existence of solutions to study the topological properties. He prove that the solution sets is compact and connected. After, in 1942, Aronszajn extend Kneser's theorem by showing that the set of all solutions is even an R_δ-set, the monograph [29], Chapter 4 is an excellent reference to study Aronszajn-type results.

We have organized this thesis as follows:

Chapter 1, groups together the definitions of the concepts used throughout this manuscript, we introduce some important notions for fractional calculus, set-valued maps, noncompactness measure and fixed point theory.

In Chapter 2, we discuss and establish the existence of solutions and the structure of the solution set for initial value problems for impulsive differential equation with fractional order of the form

$$\begin{cases} {}^{RL}D^{\alpha}y(t) = f(t, y(t)), \quad t \in J = (0, T], \ t \neq t_k, \\ \Delta^* y|_{t_k} = I_k(y(t_k^-)), \\ \lim\limits_{t \longrightarrow 0^+} t^{1-\alpha} y(t) = c_0, \end{cases}$$

where $k = 1, \ldots, m$, $0 < \alpha \leq 1$, ${}^{RL}D^{\alpha}$ is the standard Riemann-Liouville fractional derivative, $f : J \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given function , $c_0 \in \mathbb{R}$, $I_k : \mathbb{R} \to \mathbb{R}$ are continuous functions , $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T$ and $\Delta^* y|_{t_k} = y^*(t_k^+) - y(t_k^-)$, where $y^*(t_k^+) = \lim_{t \longrightarrow t_k^+} (t - t_k)^{1-\alpha} y(t)$ and $y(t_k^-) = \lim_{t \longrightarrow t_k^-} y(t)$. We first start by showing the existence of the solutions using the Leray-Schauder alternative and then we show that the set of solutions is a R_δ-set by the theorem of Lasota York [28, 35].

In Chapter 3, we are concerned by the study of initial value problem for a impulsive fractional differential inclusions defined by

$$\begin{cases} {}^{RL}D^{\alpha}y(t) \in F(t, y(t)), \quad \text{a.e. } t \in J = (0, T], \ t \neq t_k, \\ \lim\limits_{t \longrightarrow 0^+} t^{1-\alpha} y(t) = c, \\ \Delta^* y|_{t_k} = I_k(y(t_k^-)), \end{cases}$$

where $k = 1, \ldots, m$, $0 < \alpha \leq 1$, ${}^{RL}D^{\alpha}$ is the standard Riemann-Liouville fractional derivative, $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a given multivalued function ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$) and $c \in \mathbb{R}$. $I_k : \mathbb{R} \to \mathbb{R}$ are continuous functions, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T$ and $\Delta^* y|_{t_k} = y^*(t_k^+) - y(t_k^-)$, where $y^*(t_k^+) = \lim\limits_{t \longrightarrow t_k^+} (t - t_k)^{1-\alpha} y(t)$ and $y(t_k^-) = \lim\limits_{t \longrightarrow t_k^-} y(t)$.
We prove the existence of solutions, by using Covid Nadler Theorem fixed point for contraction multivalued maps [22]. We finish by showing that the solution set is compact and contractible.

In Chapter 4, we discuss and establish the existence of solutions and the structure of the solution set for fractional order differential inclusions of the form

$$\begin{cases} {}^{RL}D^{\alpha}y(t) \in F(t, y(t)), \ \text{a.e. } t \in J = (0, T], \quad 0 < \alpha \leq 1, \\ \lim\limits_{t \longrightarrow 0^+} t^{1-\alpha} y(t) = c, \end{cases}$$

where ${}^{RL}D^{\alpha}$ is the standard Riemann-Liouville fractional derivative, $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a given multivalued function ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$) and $c \in \mathbb{R}$. In section 4.3, we prove the existence and compactness of the solution sets, in the case where the set-valued maps is upper semi-continuous, by using a nonlinear alternative for multivalued maps combined with a compactness argument. In section 4.4, we prove the existence of the solution sets, in the case where the set-valued maps is Lipschitz. Our consideration is based on Covid Nadler Theorem with tools of fractional analysis.

In Chapter 5, we discuss and establish the existence of solutions for fractional differential equations on the half-line in a Banach space of the form

$$\begin{cases} {}^{RL}D^{\alpha}_{0^+}y(t) = f(t, y(t)), \quad t \in J = (0, +\infty), \\ I^{2-\alpha}_{0^+}y(0^+) = y_0, \\ {}^{RL}D^{\alpha-1}_{0^+}y(\infty) = y_\infty, \end{cases}$$

where ${}^{RL}D^{\alpha}$ denote the Riemann-Liouville fractional derivative, $1 < \alpha \leq 2$. The state $y(\cdot)$ takes value in a Banach space E, $f : (0, \infty) \times E \to E$ will be specified in later sections and $(y_0, y_\infty) \in E^2$. We prove the existence of solutions, by using Mönch's fixed point theorem combined with the technique of measure of noncompactness.

In Chapter 6, we are concerned by the study of initial value problem for a fractional differential equation with a special derivative

$$\begin{cases} {}^{RL}D^{\alpha}_{0^+}y'(t) = f(t, y(t), y'(t)), \quad t \in J = (0, T], \\ I^{1-\alpha}_{0^+}y'(0) = a, \\ y(0) = b, \end{cases}$$

where $0 < \alpha < 1$ and ${}^{RL}D^{\alpha}_{0^+}$ denote the Riemann-Liouville fractional derivative. The operator $I^{1-\alpha}_{0^+}$ denotes the Riemann-Liouville fractional integral, E is a Banach space with the norme $\|.\|$, and $f : (0, T] \times E \times E \to E$ a function satisfying some specified conditions (see, section 6.3).
We prove the existence of solutions, by using a combination of Mönch's fixed point theorem and a suitable Measure of non-compactness.

Chapter 1

Preliminaries

We introduce in this chapter notations, definitions, fixed point theorems and preliminary facts from multivalued analysis which are used throughout this thesis.

1.1 Notations and definitions

Let $J = [a, b]$, $a, b > 0$ and $(E, \|.\|)$ be a real Banach space. $C(J, E)$ the space of E-valued continuous functions on J with the norm

$$\|y\|_{\infty} = \sup_{t \in J} \|y(t)\|.$$

$L^1(J, \mathbb{R})$ the space of Lebesgue integrable functions on J with the norm

$$\|f\|_{L^1} = \int_a^b |f(t|dt.$$

$L^1(J, E)$ the space of E-valued Bochner integrable functions on J with the norm

$$\|f\|_{L^1} = \int_a^b \|f(t)\|dt.$$

We denote by $\mathcal{AC}(J, E)$ the space of E-valued absolutely continuous functions in J. For n $\in \mathbb{N}^*$, we denote by $\mathcal{AC}^n(J, E)$ the Banach space of functions from the interval J into E which is defined as:

$$\mathcal{AC}^n(J, E) = \{y : J \to E : y \in C^{n-1}(J, E) \text{ with } D^{n-1}y \in \mathcal{AC}(J, E)\}.$$

Consider the following spaces

$$\begin{aligned}\mathcal{PC}(J, E) = \quad & \{y : J \to E, y_k \in C(J_k, E), \text{ there exist } y(t_k^-), y(t_k^+) \\ & \text{with } y(t_k) = y(t_k^-)\},\end{aligned}$$

where y_k is the restriction of y to $J_k = (t_k, t_{k+1}]$, $k = 0, \dots, m$.
$\mathcal{PC}(J, E)$ is a Banach space with the norm

$$\|y\|_{\mathcal{PC}} = \max_{k=1,\dots m.} \|y_k\|_{\infty}.$$

Definition 1.1 *A function $f : J \times E \to E$ is said to be an L^1-Carathéodory function if it satisfies:*

(i) for each $t \in J$ the function $f(t, .) : E \to E$ is continuous,

(ii) for each $y \in E$ the function $f(., y) : J \to E$ is measurable,

(iii) for every positive integer k there exists $h_k \in L^1(J; \mathbb{R}^+)$ such that

$$\|f(t, y)\| \leq h_k(t), \text{ for all } \|y\| \leq k, \text{ and almost each } t \in J.$$

Definition 1.2 *[43] Let E be a Banach space. A sequence $(v_n)_{n\in\mathbb{N}} \subset L^1(J, E)$ is said to be semi-compact if*

(a) it is integrably bounded, i.e. there exists $q \in L^1(J, \mathbb{R}^+)$ such that

$$\|v_n(t)\|_E \leq q(t), \text{ for a.e. } t \in J \text{ and every } n \in \mathbb{N},$$

(b) the image sequence $(v_n(t))_{n\in\mathbb{N}}$ is relatively compact in E, for a.e. $t \in J$.

The following important result follows from the Dunford-Pettis theorem (see [43, Proposition 4.2.1]).

Lemma 1.3 *Let E be a Banach space. Then, every semi-compact sequence is weakly compact in $L^1(J, E)$.*

Lemma 1.4 *[?] Let $v : [0, b] \longrightarrow [0, +\infty)$ be a real function and $w(.)$ is a nonnegative, locally integrable function on $[0, b]$. Assume that there are constants $a > 0$ and $0 < \beta < 1$ such that*

$$v(t) \leq w(t) + a \int_0^t \frac{v(s)}{(t - s)^\beta} ds,$$

then there exists a constant $K = K(\beta)$ such that

$$v(t) \leq w(t) + Ka \int_0^t \frac{w(s)}{(t - s)^\beta} ds, \text{ for every } t \in [0, b].$$

1.2 Fractional integrals and derivatives

We begin with some definitions and lemmas of the theory of fractional calculus. Let $J = [a, b]$, $a, b > 0$ and $(E, \|.\|)$ be a real Banach space.

Definition 1.5 *[44] Let $h \in L^1(J, E)$. The Riemann-Liouville fractional integral of order $\alpha > 0$ of the function h is defined almost everywhere in $[a, b]$ by*

$$I^\alpha_{0^+} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} h(s) ds,$$

where Γ is the gamma function. When $a = 0$, we write $I^\alpha h(t) = [h * \varphi_\alpha](t)$, where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$ and $\varphi_\alpha(t) = 0$ for $t \leq 0$. The equality holds everywhere if $h \in C([a, b], E)$.

Definition 1.6 *[44] Let $h \in L^1(J, E)$, $\alpha > 0$ and n be the smallest integer greater than or equal to α and $h : [a, b] \to E$ be a function such that $I^{n-\alpha}h \in \mathcal{AC}^n([a, b], E)$. Then, the Riemann-Liouville fractional derivative of order α of the function h is defined almost every where in $[a, b]$ by*

$$^{RL}D_{a^+}^{\alpha}h(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt}\left(\int_a^t (t-s)^{n-\alpha-1}h(s)ds\right).$$

Lemma 1.7 *[62] Let $\alpha > 0$, then the differential equation*

$$^{RL}D_{a^+}^{\alpha}h(t) = 0,$$

has solutions $h(t) = c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + \ldots + c_n(t-a)^{\alpha-n}$, for some $c_i \in \mathbb{R}$, $i = 1 \ldots n$, where $n = [\alpha] + 1$.

Lemma 1.8 *[62] Let $\alpha > 0$, then*

$$I^{\alpha RL}D_{a^+}^{\alpha}h(t) = h(t) + c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + \ldots + c_n(t-a)^{\alpha-n},$$

for some $c_i \in \mathbb{R}$, $i = 0, \ldots, n$, where $n = [\alpha] + 1$.

Remark 1.9 *[44] For $\alpha > 0$, $k > -1$, we have*

$$I_{0+}^{\alpha}t^k = \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)}t^{\alpha+k} \text{ and } {}^{RL}D_{0+}^{\alpha}t^k = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}t^{k-\alpha}, \; t > 0,$$

giving in particular $^{RL}D_{0+}^{\alpha}t^{\alpha-m} = 0$, $m = 1, \ldots, n$, where n is the smallest integer greater than or equal to α.

Remark 1.10 *If h is suitabe function (see for instance [44, 55]), we have*

$$^{RL}D_{0+}^{\alpha}I_{0+}^{\alpha}h(t) = h(t), \; \alpha > 0 \text{ and } {}^{RL}D_{0+}^{\alpha}I_{0+}^{k}h(t) = I_{0+}^{k-\alpha}h(t), \; k > \alpha > 0, \; t > 0.$$

1.3 Some properties of set-valued maps

Let X be a metric space. Define $\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\}$, $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$ and $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact and convex}\}$. Consider the Hausdorff pseudo-metric distance

$$H_d : \mathcal{P}(X) \to \mathbb{R}^+ \cup \{\infty\},$$

defined by

$$H_d(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\right\},$$

where $d(a, B) = \inf_{b\in B} d(a, b)$ and $d(A, b) = \inf_{a\in A} d(a, b)$. From this definitions, it's clear that $(\mathcal{P}_{cl,b}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space see [45].

Let X and Y be metric spaces. A multivalued map F of X into Y is a correspondence which associates to every $x \in X$ a nonempty subset $F(x) \subset Y$, called the value of x. We write this correspondence as $F : X \to \mathcal{P}(Y)$. If $D \subset X$, then the set $F(D) = \cup_{x \in D} F(x)$ is called the image of D under F. The set $\mathfrak{gr}(F) \subset X \times Y$, defined by $\mathfrak{gr}(F) = \{(x, y) \in X \times Y,\ x \in F(y)\}$, is the graph of F.

Definition 1.11 *A multivalued map $F : X \to \mathcal{P}(X)$ is called*
(a) γ-Lipschitz if there exists $\gamma > 0$ such that

$$H_d(F(x), F(y)) \le \gamma d(x, y),\ \forall x,\ y \in X,$$

(b) a contraction if it is γ-Lipschitz with $\gamma < 1$.

Notice that if N is γ-Lipschitz and X is a Banach space, then for every $\gamma' > \gamma$:

$$F(x) \subset F(y) + \gamma' d(x, y) B(0, 1),\ \forall x,\ y \in X,$$

where $B(0, 1)$ refers to the unit ball in X.

Definition 1.12 *Let $F : X \to \mathcal{P}(Y)$ be a multivalued map and D be a subset of Y. The complete preimage $F^{-1}(D)$ of a set D is the set*

$$F^{-1}(D) = \{x \in X : F(x) \bigcap D \ne \emptyset\}.$$

Definition 1.13 *A multivalued map $F : X \to \mathcal{P}(Y)$ is said to be*

(i) closed if its graph $\mathfrak{gr}(F)$ is closed subset of the space $X \times Y$,

(ii) upper semicontinuous (shortly, u.s.c.) if the set $F^{-1}(D)$ is closed for every closed set $D \subset Y$.

Definition 1.14 *A multivalued map $F : X \to \mathcal{P}(Y)$ is*

(i) compact if its range $F(X)$ is relatively compact in Y, i.e., $\overline{F(X)}$ is compact in Y,

(ii) locally compact if every point $x \in X$ has a neighborhood $V(x)$ such that the restriction of F to $V(x)$ is compact,

(iii) quasicompact if $F(B)$ is relatively compact for each compact set B of X.

It is clear that $(i) \Longrightarrow (ii) \Longrightarrow (iii)$. The following facts will be used.

Lemma 1.15 *[43] Let X and Y be metric spaces and $F : X \to \mathcal{P}_{cl}(Y)$ an u.s.c. multivalued map. Then F is closed.*

The inverse relation between u.s.c. maps and closed ones is expressed in the following lemma.

Lemma 1.16 *[43] Let X and Y be metric spaces and $F : X \to \mathcal{P}_{cp}(Y)$ a closed quasicompact multivalued map. Then F is u.s.c.*

Let us consider some properties of closed and u.s.c. multivalued map.

Lemma 1.17 *[43] Let X and Y be metric spaces and $F : X \to \mathcal{P}_{cl}(Y)$ be a closed multivalued map. If $B \subseteq X$ is a compact set then its image $F(B)$ is a closed subset of Y.*

Lemma 1.18 *[43] Let X and Y be metric spaces and $F : X \to \mathcal{P}_{cl}(Y)$ be an u.s.c. multivalued map. If $B \subseteq X$ is a compact set then its image $F(B)$ is a compact subset of Y.*

When the nonlinearity takes convex values, Mazur's Lemma may be useful:

Lemma 1.19 *[52] Let X be a normed space and $(x_k)_{k\in\mathbb{N}} \subset X$ a sequence weakly converging to a limit $x \in X$. Then there exists a sequence of convex combinations $y_m = \sum_{k=m}^{k=n} \alpha_{mk} x_k$ with $\alpha_{mk} \geq 0$ for $k = m, \ldots, n$ and $\sum_{k=m}^{k=n} \alpha_{mk} = 1$ which converges strongly to x.*

Finally, the following results are easily deduced from the theoretical limit set properties.

Lemma 1.20 *[9] Let $(K_n)_{n\in\mathbb{N}} \subset K \subset X$ be a sequence of subsets where K is a compact subset of a separable Banach space X. Then*

$$\overline{co}(\lim_{n\longrightarrow\infty} \sup K_n) = \bigcap_{N>0} \overline{co}(\bigcup_{n\geq N} K_n),$$

where $\overline{co}A$ refers to the closure of the convex hull of A.

Lemma 1.21 *[9] Let X, Y be two metric spaces. If $F : X \to \mathcal{P}_{cp}(Y)$ is u.s.c., then for each $x_0 \in X$,*

$$\lim_{x\longrightarrow x_0} \sup F(x) = F(x_0).$$

We end these ingredients of multivalued analysis with some definitions and a result regarding the measurability of multivalued maps. Let $J = [0, b]$, $b > 0$ and $F : [0, b]\times X \to \mathcal{P}_{cl}(Y)$ be a multivalued map. For each $x \in X$, define the set of selections of F by

$$S_{F,x} = \{v \in L^1(J, Y) : v(t) \in F(t, x(t)) \text{ a.e. } t \in J\}.$$

Lemma 1.22 *[36] Let $(X, \mathcal{A})$ be a measurable space, (Y, d) a separable, complete metric space (Polish space) and $F : X \to \mathcal{P}(Y)$ a multivalued map with nonempty closed values. If F is measurable, then it has a measurable selection.*

Definition 1.23 *A multivalued map $G : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be L^1-Carathéodory if*

(a) $t \to G(t, u)$ is measurable for each $u \in \mathbb{R}$,

(b) $t \to G(t, u)$ is upper semi-continuous for almost all $t \in J$,

(c) for each $q > 0$, there exists $\varphi_q \in L^1(J, \mathbb{R}^+)$ such that

$$\|G(t, u)\| = \sup\{|v| : v \in G(t, u)\} \leq \varphi_q(t), \text{ for all } |u| < q, \text{ a.e. } t \in J.$$

The following definitions can be found in [8, 36]

Definition 1.24 *A single-valued map $f : [0, b] \times X \to Y$ is said to be measurable locally Lipschitz (mLL) if $f(., x)$ is measurable for every $x \in X$ and for every $x \in X$, there exists a neighborhood V_x of $x \in X$ and an integrable function*

$$L_x : [0, b] \to [0, \infty)$$

such that

$$d'(f(t, x_1), f(t, x_2)) \leq L_x(t)d(x_1, x_2) \text{ for a.e. } t \in [0, b] \text{ and } x_1,\ x_2 \in V_x.$$

Definition 1.25 *A mapping $F : [0, b] \times X \to \mathcal{P}(Y)$ is mLL-selectionable provided there exists a measurable, locally-Lipchitzian map*

$$f : [0, b] \times X \to Y \text{ and } f(t, y(t))) \in F(t, y(t)) \text{ for a.e. } t \in [0, b].$$

For further reading and details on multivalued analysis, we refer the reader to the books of Andres and Górniewicz [6], Aubin and Cellina [8], Aubin and Frankowska [9], Deimling [23], Górniewicz [36], Kamenskii *et al* [43], Hu and Papageorgiou [41, 42].

1.4 Measure of noncompactness

We recall here some definitions and properties of measure of noncompactness. For more details, we refer the reader to Deimling [23] and Kamenskii [43].

Definition 1.26 *Let Y^+ be the positive cone of an ordered Banach space $(Y, \leq)$. A function m defined on the set of all bounded subsets of the Banach space X with values in Y^+ is called a measure of noncompactness (MNC) on X, if $m(\overline{co}\Omega) = m(\Omega)$, for all bounded subsets $\Omega \subset X$.*

Definition 1.27 *A measure of noncompactness m is called:*

(i) monotone if $A, B \in \mathcal{P}_b(X)$, $A \subset B$ implies $m(A) \leq m(B)$,

(ii) nonsingular if $m(\{a\} \cup A) = m(A)$ for every $a \in X$, $A \in \mathcal{P}_b(X)$,

(iii) regular if $m(A) = 0$ is equivalent to the relative compactness of Ω.

One of the most important examples of MNC is Hausdorff MNC χ defined on each bounded subset Ω of X by

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon-\text{net}\}.$$

Without confusion, Kuratowski MNC α is defined on each bounded subset Ω of X by

$$\alpha(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ admits a finite cover by sets of diameter } \leq \varepsilon\}.$$

It is well known that Hausdorff MNC χ and Kuratowski MNC α enjoy the above properties $(i) - (iii)$ and other properties.

(iv) $m(A + B) \leq m(A) + m(B)$,

(v) $m(c.B) \leq |c|m(B),\ c \in \mathbb{R}$,

(vi) $m(coB) = m(B)$,

(vii) the function $m : \mathcal{P}_b(X) \to \mathbb{R}^+$ is continuous with respect to the metric H_d on $\mathcal{P}_b(E)$.

where $m : \mathcal{P}_b(X) \to \mathbb{R}^+$ be either α or χ.

Remark 1.28 *For every $A \in \mathcal{P}_b(X)$, we have $\chi(A) \leq \alpha(A) \leq 2\chi(A)$.*

In the following, several examples of useful measures of noncompactness in spaces of continuous functions are presented.

Example 1.29 *We consider the general example of MNC in the space of continuous functions $C([a,b],X)$ defined for all $\Omega \subset C([a,b],X)$ by*

$$m(\Omega) = \sup_{t\in[a,b]} \chi(\Omega(t)),$$

where χ is Hausdorff MNC in X and $\Omega(t) = \{y(t) : y \in \Omega\}$.

Example 1.30 *Consider another useful MNC in the space $C([a,b],X)$ defined for all bounded subset $\Omega \subset C([a,b],X)$ by*

$$\nu(\Omega) = \max(\sup \alpha(\Omega(t)), mod_C(\Omega)),$$

here, the modulus of equicontinuity of the set of functions $\Omega \subset C([a,b],X)$ has the following form:

$$mod_C(\Omega) = \lim_{\delta\to 0} \sup_{x\in\Omega} \max_{|t_2-t_1|\leq\delta} |x(t_2) - x(t_1)|. \tag{1.1}$$

Example 1.31 *We consider one more MNC in the space $C([a,b],X)$ defined for all bounded subset $\Omega \subset C([a,b],X)$ by*

$$\nu(\Omega) = \max_{\Omega\in\Delta(\Omega)} (\sup e^{-Lt}\alpha(\Omega(t)), mod_C(D)),$$

where $\Delta(\Omega)$ is the collection of all denumerable subsets of Ω, L is a constant, and $mod_C(\Omega)$ is given in formula (1.1).

Lemma 1.32 *[59] Let $J = [0,\infty)$. If $H \subset C(I,X)$ is bounded and equicontinuous, then $\alpha(H(.))$ is continuous on I, and*

$$\alpha_C(H) = \max_{t\in I} \alpha(H(t))\ ,\ \alpha\left(\int_I x(t)dt, x \in H\right) \leq \int_I \alpha(H(t))dt,$$

where $H(t) = \{x(t), x \in H\}$, $t \in I$, I is a compact interval of J and α is the Kuratowski non-compactness measure on the space X.

1.5 Some fixed point theorems

The following theorem is due to Mönch's.

Theorem 1.33 *[3] Let D be a bounded, closed and convex subset of a Banach space E such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv} N(V) \ \text{or} \ V = N(V) \cup \{0\} \Rightarrow \gamma(V) = 0,$$

holds for every subset V of D, then N has a fixed point, where γ is the Kuratowski non-compactness measure on the space E.

The following fixed point theorem for multivalued maps is due to Covitz and Nadler.

Lemma 1.34 *[22] Let (X, d) be a complete metric space. If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

The following fixed point theorem it is nonlinear alternative of Leray-Schauder for multi-valued maps.

Lemma 1.35 *[37] Let X be a normed space and $N : X \to \mathcal{P}_{cl,c}(X)$ be a completely continuous, u.s.c. multivalued map. Then one of the following conditions holds:*

1. *N has at least one fixed point in X,*
2. *the set $M = \{x \in X, x \in \lambda N(x), \lambda \in (0, 1)\}$ is unbounded.*

1.6 Topological structure

In the study of the topological structure of the solution sets for differential equations and inclusions, an important aspect is the R_δ-property. Recall that a subset D of a metric space is an R_δ-set if there exists a decreasing sequence $\{D_n\}_{n=1}^{\infty}$ of compact and contractible sets such that $D = \bigcap_{n=1}^{\infty} D_n$ (see Definition 1.37 below). This means that an R_δ-set is acyclic (in particular, nonempty, compact, and connected) and may not be a singleton but, from the point of view of algebraic topology, it is equivalent to a point, in the sense that it has the same homology groups as one point space.

Definition 1.36 *Let X be a metric space and $A \in \mathcal{P}(X)$. The set A is called a contractible space provided there exists a continuous homotopy $H : A \times [0, 1] \longrightarrow A$ and $x_0 \in A$ such that*

(a) $H(x, 0) = x$, for every $x \in A$,

(b) $H(x, 1) = x_0$, for every $x \in A$,

namely if the identity map is homotopic to a constant map, A is homotopically equivalent to a point.

Note that if $A \in \mathcal{P}_{cp,c}(X)$, then A is contractible, but the class of contractible sets is much larger than the class of compact, convex sets.

Definition 1.37 *A compact nonempty space D of a metric space is called an $R_\delta-$set provided there exists a decreasing sequence of compact nonempty contractible spaces $\{D_n\}_{n\in N}$ such that $D = \bigcap_{n=1}^{\infty} D_n$.*

Let us recall the well-known Lasota-Yorke approximation lemma, [28, 35].

Lemma 1.38 *Let E be a normed space, X a metric space and $F : X \longrightarrow E$ be a continuous map. Then for each $\varepsilon > 0$ there is a locally Lipschitz map $F_\varepsilon : X \longrightarrow E$ such that*

$$\|F(x) - F_\varepsilon(x)\| < \varepsilon, \text{ for every } x \in X.$$

Next, we present a result about the topological structure of the solution set of some nonlinear functional equations due to N. Aronszajn and developed by Browder and Gupta [6, 18].

Theorem 1.39 *Let (X, d) be a metric space, $(E, \|.\|)$ a Banach space and $F : X \longrightarrow E$ a proper map, i.e., F is continuous and for every compact $K \subset E$, the set $F^{-1}(K)$ is compact. Assume further that for each $\varepsilon > 0$, a proper map $F_\varepsilon : X \longrightarrow E$ is given, and the following two conditions are satisfied*

(a) $\|F_\varepsilon(x) - F(x)\| < \varepsilon$, *for every* $x \in X$,

(b) for every $\varepsilon > 0$ and $u \in E$ in a neighborhood of the origin such that $\|u\| \leq \varepsilon$, the equation $F_\varepsilon(x) = u$ has exactly one solution x_ε,

then the set $S = F^{-1}(0)$ is an $R_\delta - set$.

Lemma 1.40 *Let E be a Banach space, $C \subset E$ be a nonempty closed bounded subset of E and $F : C \longrightarrow E$ is an completely continuous map, then $G = Id - F$ is a proper.*

Chapter 2

Structure of solution sets for impulsive fractional differential equations

2.1 Introduction

In this chapter, we are concerned by the structure of solutions set for impulsive fractional differential equations, we consider the following initial value problems for impulsive differential equation with fractional order

$$^{RL}D^{\alpha}y(t) = f(t, y(t)) \quad \text{a.e} \quad t \in J = (0, T], \; t \neq t_k, \tag{2.1}$$

$$\Delta^* y|_{t_k} = I_k(y(t_k^-)), \tag{2.2}$$

$$\lim_{t \longrightarrow 0^+} t^{1-\alpha} y(t) = c_0, \tag{2.3}$$

where $k = 1, \ldots, m$, $0 < \alpha \leq 1$, $^{RL}D^{\alpha}$ is the standard Riemann-Liouville fractional derivative, $f : J \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given function, $c_0 \in \mathbb{R}$, $I_k : \mathbb{R} \to \mathbb{R}$ are continuous functions, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T$ and $\Delta^* y|_{t_k} = y^*(t_k^+) - y(t_k^-)$, where $y^*(t_k^+) = \lim_{t \longrightarrow t_k^+} (t - t_k)^{1-\alpha} y(t)$ and $y(t_k^-) = \lim_{t \longrightarrow t_k^-} y(t)$.

2.2 Main results

In this section, we give our main result for problem (2.1)-(2.3), before starting and proving this result, we give following notations. We consider the following space

$$\mathcal{PC}_*([0,T], \mathbb{R}) = \Big\{ y : [0,T] \to \mathbb{R} : y_k \in C(t_k, t_{k+1}], \; k = 0, \ldots, m \quad \text{and} \\ \text{there exist } y^*(t_0^+), y(t_k^-), y^*(t_k^+), k = 1, \ldots, m, \text{ with } y(t_k) = y(t_k^-) \Big\},$$

$\mathcal{PC}_*([0,T], \mathbb{R})$ is a Banach space with the norm

$$\|y\|_{\mathcal{PC}_*} = \max_{k=1,\ldots m.} \|y_k\|_*,$$

where y_k is the restriction of y to $J_k = (t_k, t_{k+1}],\ k = 0, \dots, m$ and

$$\|y_k\|_* = \sup_{t \in [t_k, t_{k+1}]} |(t - t_k)^{1-\alpha} y_k(t)|, \quad \text{for every } k = 1 \dots m.$$

For A a subset of the space $\mathcal{PC}_*([0,T],\mathbb{R})$, define $\mathcal{A}_\alpha$ by

$$\mathcal{A}_\alpha = \{y_\alpha,\ y \in A\},$$

where

$$y_\alpha/_{[t_k,t_{k+1}]}(t) = \begin{cases} (t - t_k)^{1-\alpha} y(t), & \text{if } t \in (t_k, t_{k+1}] \\ \lim\limits_{t \to t_k} (t - t_k)^{1-\alpha} y(t), & \text{if } t = t_k. \end{cases}$$

Theorem 2.1 *Let A be a bounded set in $\mathcal{PC}_*([0,T],\mathbb{R})$. Assume that $\mathcal{A}_\alpha$ is equicontinuous on $\mathcal{PC}([0,T],\mathbb{R})$, then A is relatively compact in $\mathcal{PC}_*([0,T],\mathbb{R})$.*

Proof. Let $\{y_n\}_{n=1}^{\infty} \subset A$, then $\{(y_\alpha)_n\}_{n=1}^{\infty} \subset \mathcal{PC}([0,T],\mathbb{R})$, from a version of Arzela-Ascoli theorem, the set

$$K_0 = \{(y_\alpha)_n : n \in \mathbb{N}^*\},$$

is relatively compact in $\mathcal{PC}([0,T],\mathbb{R})$, thus there exists a subsequence of $(y_\alpha)_{n\in\mathbb{N}}$, still denoted by $(y_\alpha)_{n\in\mathbb{N}}$, which converges to $y \in (\mathcal{PC}([0,T],\mathbb{R}), \|.\|_{\mathcal{PC}})$.
Hence

$$\|y_n - z\|_* = \sup_{t \in [t_k, t_{k+1}]} (t - t_k)^{1-\alpha} |y_n(t) - (t - t_k)^{\alpha-1} y(t)| \to 0 \text{ as } n \to +\infty,$$

where $z(t) = (t - t_k)^{\alpha-1} y(t)$ on $(t_k, t_{k+1}]$. Therefore

$$\{y_n\}_{n=1}^{\infty} \longrightarrow z \text{ on } \mathcal{PC}_*([0,T],\mathbb{R}).$$

□

Let us define what we mean by a solution of problem (2.1)-(2.3).
Let

$$J' := J \backslash \{t_1, \dots, t_m\}.$$

Definition 2.2 *A function $y \in \mathcal{PC}_*([0,T],\mathbb{R})$ is said to be a solution of problem* (2.1) − (2.3) *if y satisfies the equation ${}^{RL}D^\alpha y(t) = f(t, y(t))$ on J' and conditions* (2.2)-(2.3) *hold.*

Lemma 2.3 *Let $0 < \alpha < 1$ and let h be a continuous function. Then, y satisfies the following equation*

$$
y(t)=\begin{cases}
t^{\alpha-1}c_0+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s)ds & \textit{if } t\in(0,t_1],\\
(t-t_1)^{\alpha-1}t_1^{\alpha-1}c_0+\frac{(t-t_1)^{\alpha-1}}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}h(s)ds & \\
+\frac{(t-t_1)^{\alpha-1}}{\Gamma(\alpha)}I_1(y(t_1^-))+\frac{1}{\Gamma(\alpha)}\int_{t_1}^t(t-s)^{\alpha-1}h(s)ds & \textit{if } t\in(t_1,t_2],\\
(t-t_k)^{\alpha-1}\prod_{i=1}^{i=k}(t_i-t_{i-1})^{\alpha-1}c_0 & \\
+\frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}h(s)ds\right. & \\
\left.+\displaystyle\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}h(s)ds\right] & \\
+\frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[I_k(y(t_k^-))\right. & \\
\left.+\displaystyle\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}I_i(y(t_i^-))\right] & \\
+\frac{1}{\Gamma(\alpha)}\int_{t_k}^t(t-s)^{\alpha-1}h(s)ds, & \textit{if } t\in(t_k,t_{k+1}]\\
 & k=2,\dots,m,
\end{cases} \tag{2.4}
$$

if and only if y satisfies the following problem

$$
{}^{RL}D^{\alpha}y(t)=h(t),\quad t\in J', \tag{2.5}
$$

$$
\Delta^{*}y|_{t_k}=I_k(y(t_k^-)),\ k=1\dots m, \tag{2.6}
$$

$$
\lim_{t\to0}t^{1-\alpha}y(t)=c_0. \tag{2.7}
$$

Proof. Assume y satisfies (2.5)-(2.7). If $t\in(0,t_1]$, then ${}^{RL}D^{\alpha}y(t)=h(t)$. Lemma 1.8 implies

$$
y(t)=t^{\alpha-1}c_1+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s)ds.
$$

Hence $c_1=c_0$. Thus

$$
y(t)=t^{\alpha-1}c_0+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s)ds.
$$

If $t\in(t_1,t_2]$, then Lemma 1.8 implies

$$
\begin{aligned}
y(t)&=(t-t_1)^{\alpha-1}y^{*}(t_1^+)+\frac{1}{\Gamma(\alpha)}\int_{t_1}^t(t-s)^{\alpha-1}h(s)ds\\
&=(t-t_1)^{\alpha-1}\left(I_1(y(t_1^-)+y(t_1^-)\right)+\frac{1}{\Gamma(\alpha)}\int_{t_1}^t(t-s)^{\alpha-1}h(s)ds\\
&=(t-t_1)^{\alpha-1}t_1^{\alpha-1}c_0+\frac{(t-t_1)^{\alpha-1}}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}h(s)ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^t(t-s)^{\alpha-1}h(s)ds+(t-t_1)^{\alpha-1}I_1(y(t_1^-)).
\end{aligned}
$$

If $t \in (t_2, t_3]$, then Lemma 1.8 implies

$$y(t) = (t - t_2)^{\alpha-1} y^*(t_2^+) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1} h(s) ds$$

$$\begin{aligned}
y(t) &= (t - t_2)^{\alpha-1} [y(t_2^-) + I_2(y(t_2^-))] + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1} h(s) ds \\
&= (t - t_2)^{\alpha-1} \left((t_2 - t_1)^{\alpha-1} t_1^{\alpha-1} c_0 + \frac{(t_2 - t_1)^{\alpha-1}}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} h(s) ds \right) \\
&+ \frac{(t - t_2)^{\alpha-1}}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} h(t) ds + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1} h(s) ds \\
&+ (t - t_2)^{\alpha-1} \left[(t_2 - t_1)^{\alpha-1} I_1(y(t_1^-)) + I_2(y_2(t_2^-)) \right].
\end{aligned}$$

If $t \in (t_k, t_{k+1}]$, then again from Lemma 1.8, we get (2.4).
Conversely, assume that y satisfies the impulsive equation (2.4). If $t \in (0, t_1]$ then $\lim_{t \to 0} t^{1-\alpha} y(t) = c_0$ and Applying ${}^{RL}D^{\alpha}$ to both side of (2.4), we have

$${}^{RL}D^{\alpha} y(t) = h(t), \quad \text{for each } t \in (0, t_1].$$

If $t \in (t_k, t_{k+1}]$, $k = 1, \dots, m$ and applying ${}^{RL}D^{\alpha}$ to both side of (2.4), we get

$${}^{RL}D^{\alpha} y(t) = h(t), \text{for each } t \in (t_k, t_{k+1}].$$

Also, we can easily show that

$$\Delta^* y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m.$$

□

We assume the following hypotheses:

(H_1) $F : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

(H_2) There exist $p, q \in C([0,T], \mathbb{R}_+)$ such that

$$|f(t,u)| \le p(t)|u| + q(t), \quad \text{for all } t \in J \quad \text{and } u \in \mathbb{R}.$$

(H_3) There exist constants $a_k, b_k \in \mathbb{R}_+$ such that

$$|I_k(u)| \le a_k |u| + b_k, \quad \text{for all } u \in \mathbb{R}.$$

2.2.1 Existence of solutions

Our result is based on the nonlinear alternative of Leray-Schauder type.

Theorem 2.4 *Under assumptions $(H_1) - (H_3)$, then, the problem (2.1)-(2.3) has at least one solution.*

Proof. Transform the problem (2.1)-(2.3) into a fixed point problem. Consider the operator $N : \mathcal{PC}_*([0,T],\mathbb{R}) \longrightarrow \mathcal{PC}_*([0,T],\mathbb{R})$ defined by

$$\begin{aligned}
N(y)(t) &= (t-t_k)^{\alpha-1} \prod_{t_0<t_{i+1}<t} (t_i-t_{i-1})^{\alpha-1} c_0 \\
&+ \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)} \int_{t_{k-1}}^{t_k} (t_k-s)^{\alpha-1} f(s,y(s))ds \\
&+ \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)} \sum_{0<t_{i+1}<t} \Big(\prod_{t_i<t_j<t} (t_j-t_{j-1})^{\alpha-1} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1} f(s,y(s))ds \Big) \\
&+ \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)} \left(I_k(y(t_k^-)) + \sum_{0<t_{i+1}<t} \Big(\prod_{t_i<t_j<t} (t_j-t_{j-1})^{\alpha-1} I_i(y(t_i^-)) \Big) \right) \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f(s,y(s))ds.
\end{aligned}$$

Clearly, from Lemma 2.3, the fixed points of N are solutions to (2.1)-(2.3). We shall show that N satisfies the assumptions of the nonlinear alternative of Leray-Schauder type [37]. The proof will be given in several steps.

Step 1. N is continuous.
Let $\{y_n\}$ be a sequence such that, $y_n \longrightarrow y$ in $\mathcal{PC}_*([0,T],\mathbb{R})$, then

$$\begin{aligned}
&|(t-t_k)^{1-\alpha} N(y_n)(t) - (t-t_k)^{1-\alpha} N(y)(t)| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_{t_{k-1}}^{t_k} (t_k-s)^{\alpha-1} |f(s,y_n(s)) - f(s,y(s))| ds \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_{i+1}<t} \Big(\prod_{t_i<t_j<t} (t_j-t_{j-1})^{\alpha-1} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1} |f(s,y_n(s)) - f(s,y(s))| ds \\
&+ \frac{1}{\Gamma(\alpha)} |I_i(y_n(t_i^-)) - I_i(y(t_i^-))| \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_{i+1}<t} \Big(\prod_{t_i<t_j<t} (t_j-t_{j-1})^{\alpha-1} |I_i(y_n(t_i^-)) - I_i(y(t_i^-))| \\
&+ \frac{(t-t_k)^{1-\alpha}}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |f(s,y_n(s)) - f(s,y(s))| ds, \\
&\leq \frac{1}{\Gamma(\alpha)} \int_{t_{k-1}}^{t_k} (t_k-s)^{\alpha-1} |f(s,y_n(s)) - f(s,y(s))| ds \\
&+ \frac{d^{m(\alpha-1)}}{\Gamma(\alpha)} \sum_{0<t_{i+1}<t} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1} |f(s,y_n(s)) - f(s,y(s))| ds \\
&+ \frac{1}{\Gamma(\alpha)} |I_i(y_n(t_i^-)) - I_i(y(t_i^-))| + \frac{d^{m(\alpha-1)}}{\Gamma(\alpha)} \sum_{0<t_{i+1}<t} |I_i(y_n(t_i^-)) - I_i(y(t_i^-))|
\end{aligned}$$

$$+\frac{T^{1-\alpha}}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|f(s,y_n(s))-f(s,y(s))|ds.$$

Thus, from $(H_1),(H_2)$ and the continuity of I_k, we get

$$\|N(y_n)-N(y)\|_{\mathcal{PC}_*}\longrightarrow 0 \text{ as } n\longrightarrow +\infty.$$

Step 2. N maps bounded sets into bounded sets in $\mathcal{PC}_*([0,T],\mathbb{R})$.
Indeed, it is enough to show that there exists a positive constant l such that, for each $y\in B_\eta=\{y\in\mathcal{PC}_*([0,T],\mathbb{R}):\|y\|_{\mathcal{PC}_*}\le\eta\}$, we have $\|N(y)\|_{\mathcal{PC}_*}\le l$. Let $y\in B_\eta$, then from (H_2) and (H_3), we have

$$\begin{aligned}
&|(t-t_k)^{1-\alpha}N(y)(t)|\le\prod_{t_0<t_i<t}(t_i-t_{i-1})^{\alpha-1}c_0\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}f(s,y(s))ds\\
&+\frac{1}{\Gamma(\alpha)}\sum_{0<t_i<t}(\prod_{t_i<t_j<t}(t_j-t_{j-1})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}|f(s,y(s))|ds\\
&+\frac{1}{\Gamma(\alpha)}\left(|I_k(y(t_k^-)|+\sum_{0<t_i<t}(\prod_{t_i<t_j<t}(t_j-t_{j-1})^{\alpha-1}|I_i(y(t_i^-))|)\right)\\
&+\frac{(t-t_k)^{1-\alpha}}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|f(s,y(s))|ds\\
&\le d^{m(\alpha-1)}c_0+\frac{1}{\Gamma(\alpha)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|f(s,y(s))|ds\\
&+\frac{d^{m(\alpha-1)}}{\Gamma(\alpha)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}|f(s,y(s))|ds+\frac{d^{m(\alpha-1)}}{\Gamma(\alpha)}\\
&\times\sum_{0<t_i<t}|I_i(y(t_i^-))|+\frac{1}{\Gamma(\alpha)}|I_k(y(t_k^-)|+\frac{T^{1-\alpha}}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|f(s,y(s))|ds\\
&\le d^{m(\alpha-1)}c_0+\frac{\|q\|_\infty T^\alpha}{\Gamma(\alpha+1)}+\frac{\eta\|p\|_\infty}{\Gamma(\alpha)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}(s-t_{k-1})^{\alpha-1}ds\\
&+\frac{d^{m(\alpha-1)}}{\Gamma(\alpha)}\left[\frac{mT^\alpha\|q\|_\infty}{\alpha}+\eta\|p\|_\infty\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}(s-t_{i-1})^{\alpha-1}ds\right]\\
&+\frac{d^{m(\alpha-1)}}{\Gamma(\alpha)}\sum_{0<t_i<t}\left(a_i\eta d^{\alpha-1}+b_i\right)+\frac{1}{\Gamma(\alpha)}\left(a_i\eta d^{\alpha-1}+b_i\right)\\
&+\frac{T\|q\|_\infty}{\Gamma(\alpha+1)}+\frac{\eta\|p\|_\infty T^{1-\alpha}}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}(s-t_k)^{\alpha-1}ds.
\end{aligned}$$

Thus,

$$\|N(y)\|_{\mathcal{PC}_*} \leq d^{m(\alpha-1)}c_0 + \frac{\left(md^{m(\alpha-1)}T^{2\alpha-1} + T^\alpha + T^{2\alpha-1}\right)\eta\|q\|_\infty}{\Gamma(\alpha)}B(\alpha,\alpha)$$
$$+ \frac{T^\alpha(md^{m(\alpha-1)}+1)+T}{\Gamma(\alpha+1)}\|q\|_\infty + \frac{(md^{m(\alpha-1)}+1)(d^{\alpha-1}a^*\eta+b^*)}{\Gamma(\alpha)} := l,$$

where $a^* = \max_{i=\overline{1,m}} a_i$, $b^* = \max_{i=\overline{1,m}} b_i$ and $d = \min\limits_{i=1\ldots m}(t_i - t_{i-1})$ who should check $d \leq 1$.

Step 3. N maps bounded set into equicontinuous sets of $\mathcal{PC}_*([0,T],\mathbb{R})$.

Let $\tau_1, \tau_2 \in (0,T]$, $\tau_1 < \tau_2$ and B_η be a bounded set of $\mathcal{PC}_*([0,T],\mathbb{R})$ as in step 2. Let $y \in B_\eta$, we have

$$\begin{aligned}
&|(\tau_2 - t_k)^{1-\alpha}N(y)(\tau_2) - (\tau_1 - t_k)^{1-\alpha}N(y)(\tau_1)| \\
&\leq \prod_{t_0<t_i<\tau_2-\tau_1}(t_i - t_{i-1})^{\alpha-1}c \\
&+ \frac{1}{\Gamma(\alpha)}\sum_{0<t_i<\tau_2-\tau_1}\Big(\prod_{t_i<t_j<\tau_2-\tau_1}(t_j - t_{j-1})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i - s)^{\alpha-1}|f(s,y(s))|ds\Big) \\
&+ \frac{|(\tau_2 - t_k)^{1-\alpha} - (\tau_1 - t_k)^{1-\alpha}|}{\Gamma(\alpha)}\int_{t_k}^{\tau_1}(\tau_2 - s)^{\alpha-1}|f(s,y(s))|ds \\
&+ \frac{(\tau_1 - t_k)^{1-\alpha}}{\Gamma(\alpha)}\int_{t_k}^{\tau_1}[(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}]|f(s,y(s))|ds \\
&+ \frac{(\tau_2 - t_k)^{1-\alpha}}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}(\tau_2 - s)^{\alpha-1}|f(s,y(s))|ds \\
&+ \frac{1}{\Gamma(\alpha)}\sum_{0<t_i<\tau_2-\tau_1}\Big(\prod_{t_i<t_j<\tau_2-\tau_1}(t_j - t_{j-1})^{\alpha-1}|I_i(y(t_i^-))|\Big) \\
&\leq \prod_{t_0<t_i<\tau_2-\tau_1}(t_i - t_{i-1})^{\alpha-1}c_0 \\
&+ \frac{\|q\|_\infty}{\Gamma(\alpha)}\sum_{0<t_i<\tau_2-\tau_1}\Big(\prod_{t_i<t_j<\tau_2-\tau_1}(t_j - t_{j-1})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i - s)^{\alpha-1}ds\Big) + \frac{\eta\|p\|_\infty}{\Gamma(\alpha)} \\
&\times \sum_{0<t_i<\tau_2-\tau_1}\Big(\prod_{t_i<t_j<\tau_2-\tau_1}(t_j - t_{j-1})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i - s)^{\alpha-1}(s - t_i)^{\alpha-1}ds\Big) \\
&+ \frac{[(\tau_2 - t_k)^{1-\alpha} - (\tau_1 - t_k)^{1-\alpha}]\|q\|_\infty}{\Gamma(\alpha)}\int_{t_k}^{\tau_1}(\tau_2 - s)^{\alpha-1}ds \\
&+ \frac{[(\tau_2 - t_k)^{1-\alpha} - (\tau_1 - t_k)^{1-\alpha}]\eta\|p\|_\infty}{\Gamma(\alpha)}\int_{t_k}^{\tau_1}(\tau_2 - s)^{\alpha-1}(s - t_k)^{\alpha-1}ds \\
&+ \frac{(\tau_1 - t_k)^{1-\alpha}\|q\|_\infty}{\Gamma(\alpha)}\int_{t_k}^{\tau_1}[(\tau_1 - s)^{\alpha-1} - (\tau_2 - s)^{\alpha-1}]ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(\tau_1-t_k)^{1-\alpha}\eta\|p\|_\infty}{\Gamma(\alpha)}\int_{t_k}^{\tau_1}[(\tau_1-s)^{\alpha-1}-(\tau_2-s)^{\alpha-1}](s-t_k)^{\alpha-1}ds\\
&+\frac{(\tau_2-t_k)^{1-\alpha}\|q\|_\infty}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}ds\\
&+\frac{(\tau_2-t_k)^{1-\alpha}\eta\|p\|_\infty}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}(s-t_k)^{\alpha-1}ds\\
&+\frac{d^{\alpha-1}}{\Gamma(\alpha)}(a^*\eta+b^*)\sum_{0<t_i<\tau_2-\tau_1}\Big(\prod_{t_i<t_j<\tau_2-\tau_1}(t_j-t_{j-1})^{\alpha-1}\Big)\\
&\le\prod_{t_0<t_i<\tau_2-\tau_1}(t_i-t_{i-1})^{\alpha-1}c_0+\frac{B(\alpha,\alpha)(\|p\|_\infty\eta+T^{1-\alpha}\|q\|_\infty)}{\Gamma(\alpha)}\\
&\times\sum_{0<t_i<\tau_2-\tau_1}\Big((t_i-t_{i-1})^{2\alpha-1}\prod_{t_i<t_j<\tau_2-\tau_1}(t_j-t_{j-1})^{\alpha-1}\Big)\\
&+\frac{(\tau_2-t_k)^{2\alpha-1}B(\alpha,\alpha)(\|p\|_\infty\eta+T^{1-\alpha}\|q\|_\infty)}{\Gamma(\alpha)}[(\tau_2-t_k)^{1-\alpha}-(\tau_1-t_k)^{1-\alpha}]\\
&+\frac{T^{1-\alpha}B(\alpha,\alpha)(\|p\|_\infty\eta+T^{1-\alpha}\|q\|_\infty)}{\Gamma(\alpha)}[(\tau_1-t_k)^{2\alpha-1}-(\tau_2-t_k)^{2\alpha-1}]\\
&+\frac{T^{1-\alpha}(\tau_1-t_k)^{\alpha-1}(\|p\|_\infty\eta+T^{1-\alpha}\|q\|_\infty)}{\Gamma(\alpha+1)}[2(\tau_2-\tau_1)^\alpha]\\
&+\frac{a^*d^{\alpha-1}\eta+b^*}{\Gamma(\alpha)}\sum_{0<t_i<\tau_2-\tau_1}\Big(\prod_{t_i<t_j<\tau_2-\tau_1}(t_j-t_{j-1})^{\alpha-1}\Big).
\end{aligned}
$$

As $\tau_2 \longrightarrow \tau_1$ the right-hand side of the above inequality tends to zero, then $N(B_\eta)$ is equicontinuous. As a consequence of steps 1 to 3 together wit Theorem 2.1, we can conclude that $N : \mathcal{PC}_*([0,T],\mathbb{R}) \longrightarrow \mathcal{PC}_*([0,T],\mathbb{R})$ is completely continuous.

Step 4. A priori bounds on solutions.
Let $y = \lambda N(y)$ for some $0 < \lambda < 1$. This implies by (H_2) and (H_3), for $t \in (0,t_1]$, we have

$$
|y(t)| \le |c_0|t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(\|p\||_\infty y(s)| + \|q\|_\infty\right)ds,
$$

and then

$$
\begin{aligned}
|y(t)| &\le |c_0|t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(\|p\|_\infty|y(s)| + \|q\|_\infty\right)ds\\
&\le \left(|c_0| + \frac{T\|q\|_\infty}{\Gamma(\alpha+1)}\right)t^{\alpha-1} + \frac{\|p\|_\infty}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(|y(s)|\right)ds.
\end{aligned}
$$

From Lemma 1.4, there exists $K_0(\alpha)$ such that

$$
|y(t)| \le L_0 t^{\alpha-1} + \frac{\|p\|K_0(\alpha)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}L_0 ds,
$$

where

$$L_0 = |c_0| + \frac{T\|q\|_\infty}{\Gamma(\alpha+1)},$$

then

$$\sup_{t\in[0,t_1]} t^{1-\alpha}|y(t)| \leq L_0 + \frac{L_0 T^\alpha \|p\|_\infty K(\alpha)\Gamma(\alpha)}{\Gamma(2\alpha)} =: M_0.$$

We continue this process taking into account that $t \in (t_m, T]$, then

$$\begin{aligned}
|y(t)| &\leq |y^*(t_m^+)|(t-t_m)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_{t_m}^t (t-s)^{\alpha-1}|f(s,y(s))|ds \\
&\leq \left[|y(t_m^-)| + |I_m(y(t_m^-))|\right](t-t_m)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_{t_m}^t (t-s)^{\alpha-1}|f(s,y(s))|ds \\
&\leq \left[d^{\alpha-1}(a_m+1)M_{m-1} + b_m + \frac{T\|q\|_\infty}{\Gamma(\alpha+1)}\right](t-t_m)^{\alpha-1} \\
&+ \frac{\|p\|_\infty}{\Gamma(\alpha)}\int_{t_m}^t (t-s)^{\alpha-1}|y(s)|ds.
\end{aligned}$$

From Lemma 1.4, there exists $K_m(\alpha)$ such that

$$|y(t)| \leq L_m(t-t_m)^{\alpha-1} + \frac{\|p\|_\infty K_m(\alpha)}{\Gamma(\alpha)}\int_{t_m}^t (t-s)^{\alpha-1}(t-t_m)^{\alpha-1}L_m ds,$$

where

$$L_m = d^{\alpha-1}(a_m+1)M_{m-1} + b_m + \frac{T\|q\|_\infty}{\Gamma(\alpha+1)},$$

then

$$\sup_{t\in(t_m,T]} (t-t_m)^{1-\alpha}|y(t)| \leq L_m + \frac{L_m T^\alpha \|p\|_\infty K_m(\alpha)\Gamma(\alpha)}{\Gamma(2\alpha)} =: M_m.$$

Define M by

$$M = \max_{k=0,\ldots,m.} M_k,$$

let

$$U = \{y \in \mathcal{PC}_*([0,T],\mathbb{R}) : \|y\|_{\mathcal{PC}_*} < M+1\},$$

and consider the operator $N : \overline{U} \longrightarrow \mathcal{PC}_*([0,T],\mathbb{R})$. From the choice of U, there is no $y \in \partial U$ such that $y = \lambda N(y)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type, we deduce that N has a fixed point y in U which is a solution of the problem $(2.1)-(2.3)$. □

2.2.2 Structure of the solutions set

Theorem 2.5 *Under assumptions $(H_1)-(H_3)$, then, the solutions set of Problem $(2.1)-(2.3)$ is an R_δ-set.*

Proof. First of all, we show that the set

$$S(f,c_0)=\{y\in\mathcal{PC}_*([0,T],\mathbb{R}):y \text{ is a solution of } (2.1)-(2.3)\} \text{ is compact.}$$

Let $(y_n)_{n\in N}$ be a sequence in $S(f,c_0)$. We put $B=\{y_n : n\in\mathbb{N}\}\subseteq\mathcal{PC}_*([0,T],\mathbb{R})$. Then from earlier parts of the proof of this theorem, we conclude that B is bounded and equicontinuous. Then, from Theorem 2.1, we can conclude that B is relatively compact. Consider the equation

$$^{RL}D^{\alpha}y(t)=f(t,y(t)),\quad \text{a.e }\ t\in J=(0,t_1], \tag{2.8}$$

$$\lim_{t\longrightarrow 0^+}t^{1-\alpha}y(t)=c_0. \tag{2.9}$$

Recall that $J_0=(0,t_1]$ and $J_k=(t_k,t_{k+1}]$, $k=1,\dots,m$,

$$C_{k,*}([t_k,t_{k+1}],\mathbb{R})=\{y\in C(J_k,\mathbb{R}),\quad \text{with } y^*(t_k^+) \text{ exists}\}.$$

Hence,

$y_n/_{J_0}$ has a subsequence $(y_{n_m})_{n_m\in N}$ converges to y with

$(y_{n_m})_{n_m\in N}\subset S_1=\{y\in C_{0,*}([0,t_1],\mathbb{R}):y \text{ is a solution of } (2.8)-(2.9)\}$.

Let

$$z_0(t)=t^{\alpha-1}c_0+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,y(s))ds,$$

and

$$|y_{n_m}(t)-z_0(t)|\le\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|f(s,y_{n_m}(s))-f(s,y(s))|ds.$$

As $n_m\to+\infty$, $y_{n_m}(t)\to z_0(t)$, and then

$$y(t)=t^{\alpha-1}c_0+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,y(s))ds.$$

Consider now

$$^{RL}D^{\alpha}y(t)=f(t,y(t)),\quad \text{a.e }\ t\in J_1=(t_1,t_2], \tag{2.10}$$

$$y^*(t_k^+)=y(t_k^-)+I_k(t_k^-), \tag{2.11}$$

$y_n/_{J_1}$ has a subsequence relabeled as $(y_{n_m})\subset S_2$ converging to y in $C_{1,*}([t_1,t_2],\mathbb{R})$ where

$$S_2=\{y\in C_{1,*}([t_1,t_2],\mathbb{R}):y \text{ is a solution of } (2.8)-(2.9)\}.$$

Let

$$z_1(t)=(t-t_1)^{\alpha-1}t_1^{\alpha-1}c_0+\frac{(t-t_1)^{\alpha-1}}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}h(s)ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_1}^t(t-s)^{\alpha-1}f(s,y(s))ds+(t-t_1)^{\alpha-1}I_1(y(t_1^-)),$$

and

$$|y_{n_m}(t) - z_1(t)| \le \frac{(t-t_1)^{\alpha-1}}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} |f(s, y_{n_m}(s)) - f(s, y(s))| ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} |f(s, y_{n_m}(s)) - f(s, y(s))| ds$$
$$+ (t-t_1)^{\alpha-1} |I_1(y_{n_m}(t_1^-) - y(t_1^-))|.$$

As $n_m \to +\infty$, $y_{n_m}(t) \to z_1(t)$, and then

$$y(t) = (t-t_1)^{\alpha-1} t_1^{\alpha-1} c_0 + \frac{(t-t_1)^{\alpha-1}}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} h(s) ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} h(s) ds + (t-t_1)^{\alpha-1} I_1(y(t_1^-)).$$

We continue this process, we conclude that $\{y_n \mid n \in \mathbb{N}\}$ has subsequence converging to

$$y(t) = (t-t_m)^{\alpha-1} \prod_{t_0 < t_i < t} (t_i - t_{i-1})^{\alpha-1} c_0$$
$$+ \frac{(t-t_m)^{\alpha-1}}{\Gamma(\alpha)} \int_{t_m}^{T} (T-s)^{\alpha-1} f(s, y(s)) ds$$
$$+ \frac{(t-t_m)^{\alpha-1}}{\Gamma(\alpha)} \sum_{t_1 < t_{i+1} < t} \Big(\prod_{t_i < t_j < t} (t_j - t_{j-1})^{\alpha-1} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} f(s, y(s)) ds \Big)$$
$$+ \frac{(t-t_m)^{\alpha-1}}{\Gamma(\alpha)} \left(I_m(y(t_m^-)) + \sum_{t_1 < t_{i+1} < t} \Big(\prod_{t_i < t_j < t} (t_j - t_{j-1})^{\alpha-1} I_i(y(t_i^-)) \Big) \right)$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_m}^{t} (t-s)^{\alpha-1} f(s, y(s)) ds.$$

Hence $S(f, c_0)$ is compact.
Next, define

$$\tilde{f}(t, y(t)) = \begin{cases} f(t, y(t)), & |y(t)| \le \tilde{M}, \\ f(t, \frac{My(t)}{|y(t)|}), & |y(t)| \ge \tilde{M}, \end{cases}$$

where $\tilde{M} > M$. Since f is continuous, the function $\widetilde{f}$ is continuous and it is bounded by (H_2), there exists $M_* > 0$ such that

$$|\widetilde{f}(t, y)| \le M_*, \; for \; a.e. \; t \text{ and all } \; y \in \mathbb{R}. \tag{2.12}$$

We consider the following modified problem,

$$\begin{cases} D^{\alpha}y(t)=\tilde{f}(t,y(t)), & t\in J=(0,T],\ t\neq t_k, \quad 0<\alpha\leq 1,\\ \Delta^{*}y|_{t_k}=I_k(y(t_k^-)), & k=1,\dots,m,\\ \lim\limits_{t\longrightarrow 0^+} t^{1-\alpha}y(t)=c_0. \end{cases}$$

We can easily prove that $S(f,c_0)=S(\tilde{f},c_0)=Fix\tilde{N}$, where

$$\tilde{N}:\mathcal{PC}_{*}([0,T],\mathbb{R})\longrightarrow \mathcal{PC}_{*}([0,T],\mathbb{R})$$

is defined by

$$\begin{aligned} \tilde{N}(y)(t) &= (t-t_k)^{\alpha-1}\prod_{t_0<t_i<t}(t_i-t_{i-1})^{\alpha-1}c_0\\ &+\frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\sum_{t_1<t_{i+1}<t}\left(\prod_{t_i<t_j<t}(t_j-t_{j-1})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\tilde{f}(s,y(s))ds\right)\\ &+\frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left(I_k(y(t_k^-)+\sum_{t_1<t_{i+1}<t}\prod_{t_i<t_j<t}(t_j-t_{j-1})^{\alpha-1}\tilde{I}_i(y(t_i^-))\right)\\ &+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\tilde{f}(s,y(s))ds. \end{aligned}$$

We have

$$\begin{aligned} (t-t_k)^{1-\alpha}|\tilde{N}(y)(t)| \leq d^{m(\alpha-1)}|c_0| &+ \frac{M_* d^{m(\alpha-1)}}{\Gamma(\alpha)}\\ &+\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}ds+\frac{M_*(d^{m(\alpha-1)}+1)}{\Gamma(\alpha)}\\ &+\frac{M_*T^{\alpha}}{\Gamma(\alpha+1)}, \end{aligned}$$

$$\|\tilde{N}y\|_{\mathcal{PC}_*}\leq d^{m(\alpha-1)}|c_0|+\frac{M_*\ T^{\alpha}\ (d^{m(\alpha-1)}+1)}{\Gamma(\alpha+1)}+\frac{M_*(d^{m(\alpha-1)}+1)}{\Gamma(\alpha)}=: M_{**}.$$

Finally, we have

$$\|\tilde{N}(y)\|_{\mathcal{PC}_*}\leq M_{**},$$

then $\tilde{N}$ is uniformly bounded, as in steps 1 to 2, we can prove that

$$\tilde{N}:\mathcal{PC}_{*}([0,T],\mathbb{R})\longrightarrow \mathcal{PC}_{*}([0,T],\mathbb{R}),$$

is compact which allows us to define the compact perturbation of the identity $\widetilde{G}(y)=y-\tilde{N}(y)$ which is a proper map. From the Lasota-Yorke approximation theorem, we can easily prove that all conditions of Theorem 1.39 are satisfied. Therefore the solution set $S(\tilde{f},c_0)=\tilde{G}^{-1}(0)$ is an R_δ-set so $S(f,c_0)$ is an R_δ-set.

□

2.3 Example

In this section, we consider the following fractional impulsive differential equation

$$^{RL}Dy(t) = f(t, y(t)),\ t \in J = (0, 1],\ t \neq \frac{1}{2},\ 0 < \alpha, \beta \leq 1, \tag{2.13}$$

$$\lim_{t \longrightarrow 0} t^{1-\alpha} y(t) = 0, \tag{2.14}$$

$$\Delta^* y|_{t=\frac{1}{2}} = \frac{1}{3} |y(\frac{1}{2}^-)| + 1, \tag{2.15}$$

where

$$k_{\alpha,\beta} = \frac{1}{\Gamma(1-\alpha)} - \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)},\ a > 0,$$

$$f(t, u) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)(t+a)^\alpha} u + \frac{k_{\alpha,\beta}}{(t+a)^\alpha},\ \text{for } (t, u) \in J \times \mathbb{R}$$

and

$$I_1(u) = \frac{1}{3}|u| + 1, \text{for all } u \in \mathbb{R}.$$

We have $t_1 = \frac{1}{2}$ and $c_0 = 0$, it clear that f is a continuous function and so condition (H_3) is satisfied.

Let

$$P(t) = \frac{\Gamma(\alpha)}{\Gamma(\beta-\alpha)(t+a)^\alpha},\quad \|P\|_\infty = \frac{\Gamma(\alpha)}{\Gamma(\beta-\alpha)a^\alpha},$$

$$q(t) = k_{\alpha,\beta},\quad \|q\|_\infty = \frac{k_{\alpha,\beta}}{a^\alpha},$$

we can easily prove that condition (H_2) yields.
Let also $a_1 = \frac{1}{3}$, $b_1 = 1$ and so condition (H_3) is satisfied. Therefore, the solutions set of the problem $(2.13) - (2.15)$ is not empty and it is interesting to study the topological properties of the solutions set in this case. Therefore, the solution set of $(2.13) - (2.15)$ is an R_δ-set.

Chapter 3

Solution set for impulsive fractional differential inclusions

3.1 Introduction

We consider the following initial value problem

$$\begin{cases} {}^{RL}D^{\alpha}y(t) \in F(t, y(t)), \quad \text{a.e. } t \in J = (0, T], \ t \neq t_k, \\ \lim\limits_{t \longrightarrow 0^+} t^{1-\alpha}y(t) = c, \\ \Delta^* y|_{t_k} = I_k(y(t_k^-)), \end{cases} \tag{3.1}$$

where $k = 1, \ldots, m$, $0 < \alpha \leq 1$, ${}^{RL}D^{\alpha}$ is the standard Riemann-Liouville fractional derivative, $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a given multivalued function ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$) and $c \in \mathbb{R}$. $I_k : \mathbb{R} \to \mathbb{R}$ are continuous functions, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T$ and $\Delta^* y|_{t_k} = y^*(t_k^+) - y(t_k^-)$, where $y^*(t_k^+) = \lim\limits_{t \longrightarrow t_k^+} (t - t_k)^{1-\alpha}y(t)$ and $y(t_k^-) = \lim\limits_{t \longrightarrow t_k^-} y(t)$.

3.2 Main results

Let $\mathcal{PC}_*([0, T], \mathbb{R})$ the Banach space defined as the second chapter. Consider the Banach space

$$\Omega = \mathcal{PC}_*([0, T], \mathbb{R}) \cap (\cup_{k=1}^{m} \mathcal{AC}([t_k, t_{k+1}], \mathbb{R})),$$

with the norm

$$\|y\|_{\Omega} = \max_{k=1,\ldots m.} \|y_k\|_*,$$

where y_k is the restriction of y to $J_k = (t_k, t_{k+1}]$, $k = 0, \ldots, m$, and

$$\|y_k\|_* = \sup_{t \in [t_k, t_{k+1}]} |(t - t_k)^{1-\alpha}y_k(t)|, \quad \text{for every } k = 1 \ldots m.$$

Let $J' = J \backslash \{t_1, \ldots, t_m\}$.

Definition 3.1 *A function $y \in \Omega$ is said to be a solution of problem (3.1) if there exists $v \in L^1(J,\mathbb{R})$ such that $v(t) \in F(t, y(t))$, a.e. on J', $\lim\limits_{t\longrightarrow 0} t^{1-\alpha}y(t) = c$, $\Delta^* y|_{t_k} = I_k(t_k^-)$ and ${}^{RL}D^\alpha y(t) = v(t)$, a.e. $t \in J'$.*

The aim of our work is to present an overall existence result for the problem (3.1) by using a fixed point theorem for multivalued maps due to Covitz and Nadler and prove the compactness and contractibly of the solution set for the problem (3.1).

Theorem 3.2 *Let $F : J \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ be a multivalued map that is mLL-selectionable. Assume that:*

(H_1) there exist constants $\overline{a}$ and $\overline{b} \in \mathbb{R}^+$ such that

$$\|F(t,x)\|_{\mathcal{P}} \leq \overline{a}|x| + \overline{b}, \text{ for a.e. } t \in J \text{ and each } x \in \mathbb{R},$$

(H_2) there exist a function $p \in C([0,T],\mathbb{R}^+)$ such that

$$H_d(F(t,z_1), F(t,z_2)) \leq p(t)|z_1 - z_2|, \text{ for all } z_1,\ z_2 \in \mathbb{R}, \text{ and}$$
$$d(0, F(t,0)) \leq p(t),\ t \in J,$$

(H_3) there exist constant $L > 0$ such that

$$|I_k(z_1) - I_k(z_2)| \leq L|z_1 - z_2|, \text{ for all } z_1,\ z_2 \in \mathbb{R}.$$

If

$$\frac{T^\alpha \|p\|_\infty \Gamma(\alpha)(1 + mT_0^{\alpha-1})}{\Gamma(2\alpha)} + \frac{mT_0^{\alpha-1}L}{\Gamma(\alpha)} < 1,$$

where $T_0 = \min_{i=\overline{0,m}}(t_{i+1} - t_i) \leq 1$, then the problem (3.1) has at least one solution. If further $F : J \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is a Carathéodory multivalued map, then the solution set is a compact and contractible.

Proof.
Existence of solutions: Consider the operator $P : \Omega \to \mathcal{P}(\Omega)$ defined for $y \in \Omega$ by

$$\begin{aligned}
P(y) = \quad & \{h \in \Omega : h(t) = (t - t_k)^{\alpha-1} \prod_{i=1}^{i=k} (t_i - t_{i-1})^{\alpha-1} c \\
& + \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)} \left[\int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} h(s)ds \right. \\
& \left. + \sum_{i=1}^{k-1} \prod_{j=1}^{k-i} (t_{k-j+1} - t_{k-j})^{\alpha-1} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} v(s)ds \right] \\
& + \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)} \left[I_k(y(t_k^-)) + \sum_{i=1}^{k-1} \prod_{j=1}^{k-i} (t_{k-j+1} - t_{k-j})^{\alpha-1} I_i(y(t_i^-)) \right] \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} v(s)ds\},
\end{aligned}$$

where $v \in S_{F,y} = \{v \in L^1(J, \mathbb{R}) : v(t) \in F(t, y(t)),\ \text{a.e. } t \in J\}$.
Clearly, from Lemma 2.3 the fixed points of P are solutions to (3.1). Now we show that the operator P satisfies the assumptions of Lemma 1.34. To show that $P(y) \in \mathcal{P}_{cl}(\Omega)$, for each $y \in \Omega$, let $\{u_n\}_{n \geq 0} \in P(y)$ be such that $u_n \longrightarrow u$ as $n \longrightarrow \infty$ in Ω. Then $u \in \Omega$ and there exists $v_n \in S_{F,y}$ such that, for each $t \in (0, T]$.

$$\begin{aligned} u_n(t) &= (t-t_k)^{\alpha-1}\prod_{i=1}^{i=k}(t_i-t_{i-1})^{\alpha-1}c \\ &+ \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}v_n(s)ds\right. \\ &+ \left.\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}v_n(s)ds\right] \\ &+ \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[I_k(y_n(t_k^-)) + \sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}I_i(y_n(t_i^-))\right] \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}v_n(s)ds. \end{aligned}$$

As F has compact values and from (H_1), we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1([0,T], \mathbb{R})$. Thus $v \in S_{F,y}$ and for each $t \in (0,T]$, we have

$$\begin{aligned} u_n(t) \longrightarrow u(t) &= (t-t_k)^{\alpha-1}\prod_{i=1}^{i=k}(t_i-t_{i-1})^{\alpha-1}c \\ &+ \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}v(s)ds\right. \\ &+ \left.\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}v(s)ds\right] \\ &+ \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[I_k(y(t_k^-)) + \sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}I_i(y(t_i^-))\right] \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}v(s)ds. \end{aligned}$$

Hence, $u \in P(y)$.
Next, we show that there exists $\delta < 1$ such that

$$H_d(P(x), P(y)) \leq \delta \|x-y\|_\Omega, \text{ for each } x,\ y \in \Omega.$$

Let $x, y \in \Omega$ and $h_1 \in P(x)$. Then, there exists $v_1(t) \in F(t, x(t))$ such that, for each

$t \in (0,T]$.

$$\begin{aligned}
h_1(t) &= (t-t_k)^{\alpha-1}\prod_{i=1}^{i=k}(t_i-t_{i-1})^{\alpha-1}c \\
&+ \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}v_1(s)ds\right. \\
&+ \left.\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}v_1(s)ds\right] \\
&+ \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[I_k(x(t_k^-))+\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}I_i(x(t_i^-))\right] \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}v_1(s)ds.
\end{aligned}$$

By (H_2), we have

$$H_d(F(t,x(t)),F(t,y(t))) \le p(t)|x(t)-y(t)|.$$

So, there exists $w \in F(t,y(t))$ such that

$$|v_1(t)-w| \le p(t)|x(t)-y(t)|,\ t \in (0,T].$$

Define $U:(0,T] \to \mathcal{P}(\mathbb{R})$ by

$$U(t) = \{w(t) \in \mathbb{R} : |v_1(t)-w| \le p(t)|x(t)-y(t)|.$$

Since the multivalued operator $U(t) \cap F(t,y(t))$ is measurable [[21], Proposition III.4], there exists a function $v_2(.)$ which is a measurable selection for $U(.) \cap F(.,y(.))$. So, $v_2(t) \in F(t,y(t))$ and for each $t \in (0,T]$, we have

$$|v_1(t)-v_2(t)| \le p(t)|x(t)-y(t)|.$$

For each $t \in (0,T]$, let us define

$$\begin{aligned}
h_1(t) &= (t-t_k)^{\alpha-1}\prod_{i=1}^{i=k}(t_i-t_{i-1})^{\alpha-1}c \\
&+ \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}v_2(s)ds\right. \\
&+ \left.\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}v_2(s)ds\right] \\
&+ \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[I_k(y(t_k^-))+\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}I_i(y(t_i^-))\right] \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}v_2(s)ds.
\end{aligned}$$

Thus,

$$
\begin{aligned}
|(t-t_k)^{1-\alpha}h_1(t) &- (t-t_k)^{1-\alpha}h_2(t)| \\
&\leq \frac{1}{\Gamma(\alpha)}\left[\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|v_2(s)-v_1(s)|ds\right. \\
&+ \left.\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}|v_2(s)-v_1(s)|ds\right] \\
&+ \frac{1}{\Gamma(\alpha)}\left[|I_k(y(t_k^-))-I_k(x(t_k^-))|\right. \\
&+ \left.\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}|I_i(y(t_i^-))-I_i(x(t_i^-))|\right] \\
&+ \frac{(t-t_k)^{1-\alpha}}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|v_2(s)-v_1(s)|ds.
\end{aligned}
$$

Hence,

$$\|h_1-h_2\|_\Omega \leq \left[\frac{T^\alpha\|p\|_\infty\Gamma(\alpha)(1+mT_0^{\alpha-1})}{\Gamma(2\alpha)}+\frac{mT_0^{\alpha-1}L}{\Gamma(\alpha)}\right]\|x-y\|_\Omega.$$

Analogously, interchanging the roles of x and y, we obtain

$$H_d(P(x),P(y)) \leq \delta\|x-y\|_\Omega,$$

where $\delta = \frac{T^\alpha\|p\|_\infty\Gamma(\alpha)(1+mT_0^{\alpha-1})}{\Gamma(2\alpha)}+\frac{mT_0^{\alpha-1}L}{\Gamma(\alpha)} < 1$. So, P is a contraction. Hence it follows by Lemma 1.34, P has a fixed point y which is a solution of (3.1).

Structure of the solutions set: Let

$$S_F(c) = \{y \in \Omega : \ y \text{ is solution of (3.1) }\}.$$

First of all, we prove that $S_F(c)$ is compact in Ω. Let $\{y_n\}_{n\in\mathbb{N}} \in S_F(c)$, then, there exists

$v_n \in S_{F,y_n}$ such that, for a.e. $t \in J$, we have

$$
\begin{aligned}
y_n(t) &= (t-t_k)^{\alpha-1}\prod_{i=1}^{i=k}(t_i-t_{i-1})^{\alpha-1}c \\
&+ \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\Bigg[\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}v_n(s)ds \\
&+ \sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}v_n(s)ds\Bigg] \\
&+ \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[I_k(y_n(t_k^-)) + \sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}I_i(y_n(t_i^-))\right] \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}v_n(s)ds.
\end{aligned}
$$

From (H_1) and (H_3), we can easily prove that there exists an $M_1 > 0$ such that $\|y_n\||_\Omega \leq M_1$, for every $n \geq 1$ and the set $\{y_n : n \geq 1\}$ is equicontinuous in Ω. Thus, by using Theorem 2.1, we can conclude that, there exists a subsequence (denoted again by $\{y_n\}$ of $\{y_n\}$) such that y_n converges to y in Ω. We shall show that there exist $v(t) \in F(t, y(t))$, a.e. $t \in J$ such that

$$
\begin{aligned}
y(t) &= (t-t_k)^{\alpha-1}\prod_{i=1}^{i=k}(t_i-t_{i-1})^{\alpha-1}c \\
&+ \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\Bigg[\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}v(s)ds \\
&+ \sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}v(s)ds\Bigg] \\
&+ \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[I_k(y(t_k^-)) + \sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}I_i(y(t_i^-))\right] \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}v(s)ds.
\end{aligned}
$$

Since $F(.,.)$ is upper semi-continuous, then for every $\varepsilon > 0$, there exist $n_0 \geq 0$ such that, for every $n \geq n_0$, we have

$$v_n(t) \in F(t, y_n(t)) \subset F(t, y(t)) + \varepsilon B(0,1), \text{ a.e.} t \in J.$$

Using the compactness of $F(.,.)$, we get the subsequence $v_{nm}(.)$ such that

$$v_{nm}(.) \longrightarrow v(.) \text{ and } v(t) \in F(t, y(t)), \text{ a.e. } t \in J.$$

From (H_1), we have

$$v_{nm}(t) \leq \overline{a}(t-t_k)^{\alpha-1}M_1 + \overline{b}, \text{ a.e. } t \in (t_k, t_{k+1}].$$

By Lebesgue's dominated convergence theorem, we conclude that $v \in L^1(J, \mathbb{R})$, which implies that $v \in S_{F,y}$. Thus, since I_k are continuous functions, for a.e. $t \in J$, we have

$$\begin{aligned}
y(t) &= (t-t_k)^{\alpha-1}\prod_{i=1}^{i=k}(t_i-t_{i-1})^{\alpha-1}c \\
&+ \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}v(s)ds\right. \\
&+ \left.\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}v(s)ds\right] \\
&+ \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[I_k(y(t_k^-)) + \sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}I_i(y(t_i^-))\right] \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}v(s)ds.
\end{aligned}$$

Then, $S_F(c)$ is compact.
Next, let $f \in F$ be measurable and locally Lipschitz. Consider the single-valued problem

$$\begin{cases} {}^{RL}D^\alpha y(t) = f(t, y(t)), \quad \text{a.e. } t \in J = (0, T],\ t \neq t_k, \\ \lim\limits_{t\longrightarrow 0^+} t^{1-\alpha}y(t) = c, \\ \Delta^* y|_{t_k} = I_k(y(t_k^-)). \end{cases} \tag{3.2}$$

By the Banach fixed point theorem, we can prove that the problem (3.2) has exactly one solution $\overline{x}$. We define the homotopy $h : S_F(c) \times [0, 1] \to S_F(c)$ by

$$h(y, \lambda)(t) = \begin{cases} y(t), & \text{for } 0 < t \leq \lambda T, \\ \overline{x}(t), & \text{for } \lambda T < t \leq T. \end{cases}$$

Where $\overline{x}$ is exactly one solution of the problem (3.2). Note that

$$h(y, \lambda) = \begin{cases} y, & \text{for } \lambda = 1 \text{ and } y \in S_F(c), \\ \overline{x}, & \text{for } \lambda = 0. \end{cases}$$

Now, we prove that h is a continuous homotopy. Let $(y_n, \lambda_n) \in S_F(c) \times [0, 1]$ such that $(y_n, \lambda_n) \longrightarrow (y, \lambda)$. We shall show that $h(y_n, \lambda_n) \longrightarrow h(y, \lambda)$. We have

$$h(y_n, \lambda_n)(t) = \begin{cases} y_n(t), & \text{for } t \in (0, \lambda_n T], \\ \overline{x}(t), & \text{for } (\lambda_n T, T]. \end{cases}$$

(a) If $\lim\limits_{n\longrightarrow\infty} \lambda_n = 0$, then $h(y, 0)(t) = \overline{x}(t)$, for $t \in (0, T]$, hence

$$\|h(y_n, \lambda_n) - h(y, \lambda)\|_\Omega \to 0 \text{ as } n \longrightarrow +\infty,$$

the case when $\lim\limits_{n\longrightarrow\infty} \lambda_n = 1$ is treated similarly.

(b) If $\lambda_n \neq 0$ and $0 < \lim_{n \longrightarrow \infty} \lambda_n = \lambda < 1$,

then, we may distinguish between two sub-cases:

(i) Since $y_n \in S_F(c)$, there exist $v_n \in S_{F,y_n}$ such that, for $t \in (0, \lambda_n T]$

$$\begin{aligned}
y_n(t) &= (t-t_k)^{\alpha-1}\prod_{i=1}^{i=k}(t_i-t_{i-1})^{\alpha-1}c \\
&+ \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}v_n(s)ds\right. \\
&+ \left.\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}v_n(s)ds\right] \\
&+ \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[I_k(y_n(t_k^-)) + \sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}I_i(y_n(t_i^-))\right] \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}v_n(s)ds.
\end{aligned}$$

Since y_n converges to y, there exists $M_1 > 0$ such that $\|y_n\|_\Omega \leq M_1$.

Since $F(.,.)$ is upper semi-continuous, then for every $\varepsilon > 0$, there exist $n_0 \geq 0$ such that for every $n \geq n_0$, we have

$$v_n(t) \in F(t, y_n(t)) \subset F(t, y(t)) + \varepsilon B(0,1), \text{ a.e.} t \in J.$$

Using the compactness of $F(.,.)$, we get the subsequence $v_{nm}(.)$ such that

$$v_{nm}(.) \longrightarrow v(.), \text{ and } v(t) \in F(t, y(t)), \text{ a.e. } t \in J.$$

From (H_1), we have

$$v_{nm}(t) \leq \overline{a}(t-t_k)^{\alpha-1}M_1 + \overline{b}, \text{ a.e. } t \in (t_k, t_{k+1}].$$

From the Lebesgue ś dominated convergence theorem, we have $v \in L^1(J, \mathbb{R})$, so

$v \in S_{F,y}$ and since I_k are continuous functions, for $t \in J$, we have

$$\begin{aligned}
y(t) &= (t-t_k)^{\alpha-1}\prod_{i=1}^{i=k}(t_i-t_{i-1})^{\alpha-1}c \\
&+ \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}v(s)ds\right. \\
&+ \left.\sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}v(s)ds\right] \\
&+ \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left[I_k(y(t_k^-)) + \sum_{i=1}^{k-1}\prod_{j=1}^{k-i}(t_{k-j+1}-t_{k-j})^{\alpha-1}I_i(y(t_i^-))\right] \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}v(s)ds.
\end{aligned}$$

(ii) $t \in (\lambda_n T, T]$, then

$$h(y_n,\lambda_n)(t) = h(y,\lambda)(t) = \overline{x}.$$

Thus,

$$\|h(y_n,\lambda_n) - h(y,\lambda)\|_\Omega \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Therefore h is a continuous function, proving that $S_F(c)$ is contractible to the point $\overline{x}$.

□

3.3 An Example

As an application of the main results, we consider the impulsive fractional differential inclusion

$$\begin{cases} {}^{RL}D^{\frac{1}{2}}y(t) \in F(t,y(t)), & \text{a.e. } t \in J=(0,1],\ t \neq \frac{1}{2}, \\ \lim\limits_{t\longrightarrow 0^+} t^{1-\alpha}y(t) = \dfrac{1}{4}, \\ \Delta^* y|_{t=\frac{1}{2}} = \frac{1}{9}|y(\frac{1}{2}^-)| + 1, \end{cases} \tag{3.3}$$

where $T=1, m=1, t_1=\frac{1}{2}$,

$$F(t,x) = [0,\ \frac{1}{9}\sin x + \frac{|x|}{t+9} + \frac{1}{9}].$$

And

$$I_1(u) = \frac{1}{9}|\sin(u)| + 1,\ \text{for } u \in \mathbb{R}.$$

Clearly

$$\sup\{|v| : v \in F(t,x)\} \leq \frac{1}{9}|\sin x| + \frac{|x|}{t+9} + \frac{1}{9},$$

$$H_d(F(t,x),F(t,y)) \leq [\frac{1}{9}+\frac{1}{t+9}]|x-y|,$$

and

$$|I_1(u)| \leq \frac{1}{9}|\sin(u)|+1,\ \ |I_1(u)-I_1(v)| \leq \frac{1}{9}|u-v|.$$

Let $p(t)=\frac{1}{9}+\frac{1}{t+9}$. Then $\|p\||_\infty=\frac{2}{9}$ and $\frac{T^\alpha\|p\|_\infty\Gamma(\alpha)(1+mT_0^{\alpha-1})}{\Gamma(2\alpha)}+\frac{mT_0^{\alpha-1}L}{\Gamma(\alpha)} \simeq 0,73965<1$. Hence from Theorem 3.2, the problem (3.3) has at least one solution. It is clear that F is a mLL-selectionable multivalued map (the fonction $f(t,u)=\frac{1}{9}\sin u+\frac{|u|}{t+9}+\frac{1}{9}$ is measurable, locally-Lipchitzian) with compact convex values. Then the solution set is a compact and contractible.

Chapter 4

Solutions set for fractional differential inclusions

4.1 Introduction

We consider the following initial value problem for fractional order differential inclusion

$$^{RL}D^{\alpha}y(t) \in F(t, y(t)), \text{ a.e. } t \in J = (0, T], \quad 0 < \alpha \leq 1, \tag{4.1}$$

$$\lim_{t \longrightarrow 0^+} t^{1-\alpha}y(t) = c, \tag{4.2}$$

where $^{RL}D^{\alpha}$ is the standard Riemann-Liouville fractional derivative, $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a Carathéodory multivalued function ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$) and $c \in \mathbb{R}$.

4.2 The upper semi-continuous case

In this section, we present a global existence result and prove the compactness of the solution set for problem (4.1) − (4.2) by using a nonlinear alternative for multivalued maps combined with a compactness argument. Consider the Banach space

$$\mathcal{C}_{\alpha}([0, T], \mathbb{R}) = \{y \in C((0, T], \mathbb{R}) : \lim_{t \longrightarrow 0} t^{1-\alpha}y(t) \text{ exists }\}.$$

Endowed with the norm

$$\|y\|_{\alpha} = \sup\{t^{1-\alpha}|y(t)| : t \in (0, T]\},$$

$\mathcal{C}_{\alpha}$ is a Banach space. For A a subset of the space $\mathcal{C}_{\alpha}([0, T], \mathbb{R})$, define A_{α} by $A_{\alpha} = \{y_{\alpha} : y \in A\}$, where

$$y_{\alpha}(t) = \begin{cases} t^{1-\alpha}y(t), & t \in (0, T], \\ \lim\limits_{t \longrightarrow 0} t^{1-\alpha}y(t), & t = 0. \end{cases}$$

Lemma 4.1 *Let A be a bounded set in $\mathcal{C}_{\alpha}([0, T], \mathbb{R})$. Assume that A_{α} is equicontinuous on $C([0, T], \mathbb{R})$. Then A is relatively compact in $\mathcal{C}_{\alpha}([0, T], \mathbb{R})$.*

Proof. Let $\{y_n\}_{n=1}^{\infty} \subset A$, then $\{(y_\alpha)_n\}_{n=1}^{\infty} \subset C([0,T],\mathbb{R})$. From Arzelá-Ascoli theorem, the set $K_0 = \{(y_\alpha)_n : n \in \mathbb{N}^*\}$ is relatively compact in $C([0,T],\mathbb{R})$, thus there exists a subsequence of $\{(y_\alpha)_n\}_{n\in\mathbb{N}}$, still denoted by $\{(y_\alpha)_n\}_{n=1}^{\infty}$, which converges to y, where $y \in (C([0,T],\mathbb{R}), \|.\|_\infty)$.
Hence

$$\|(y_\alpha)_n - z\| = \sup\{|t^{1-\alpha}y_n(t) - t^{1-\alpha}t^{\alpha-1}y(t)|, t \in (0,T]\} \longrightarrow 0.$$

Therefore

$$\{y_n\}_{n=1}^{\infty} \longrightarrow z \text{ on } \mathcal{C}_\alpha([0,T],\mathbb{R}).$$

□

Let us define what we mean by a solution of problem (4.1) − (4.2).

Definition 4.2 *A function $y \in \mathcal{C}_\alpha$ is said to be a solution of problem* (4.1) − (4.2) *if there exists $v \in L^1(J,\mathbb{R})$ such that $v(t) \in F(t,y(t))$ a.e. $t \in J$ satisfies ${}^{RL}D^\alpha y(t) = v(t)$ on J and condition $\lim_{t\longrightarrow 0} t^{1-\alpha}y(t) = c$, is satisfied.*

As a consequence of Lemma 1.7 and Lemma 1.8, we have the following result which is useful in what follows.

Lemma 4.3 *Let $0 < \alpha \leq 1$ and let $\rho \in L^1(J,\mathbb{R})$. Then, y satisfies the following equation*

$$y(t) = t^{\alpha-1}c + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\rho(s)ds, \ \text{a.e. } t \in J,$$

if and only if y satisfies the following problem

$${}^{RL}D^\alpha y(t) = \rho(t), \ \text{for each } t \in J,$$

$$\lim_{t\longrightarrow 0} t^{1-\alpha}y(t) = c.$$

We assume that the multi-valued map F is a compact and convex valued which satisfies following hypotheses:

(H_1) $F : J \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is a Carathéodory multi-valued map,

(H_2) there exist nonnegative constants $a,\ b \in \mathbb{R}^+$ such that

$$\|F(t,x)\|_{\mathcal{P}} \leq at^{1-\alpha}|x| + b, \ \text{for a.e. } t \in J \text{ and each } x \in \mathbb{R}.$$

Theorem 4.4 *Under Assumptions $(H_1) - (H_2)$, the initial-value problem* (4.1) − (4.2) *has at least one solution. Moreover, the solution set $S_F(c)$ is compact.*

Proof.
Existence results: Consider the operator $N : \mathcal{C}_\alpha \to \mathcal{P}(\mathcal{C}_\alpha)$ defined for $y \in \mathcal{C}_\alpha$ by

$$N(y) = \{h \in \mathcal{C}_\alpha : h(t) = t^{\alpha-1}c + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds, \ \text{a.e. } t \in J\},$$

where $v \in S_{F,y} = \{v \in L^1(J,\mathbb{R}) : v(t) \in F(t,y(t)) \text{ a.e. } t \in J\}$. Note that from [60, Theorem 5.10], the set $S_{F,y}$ is nonempty if and only if the mapping $t \to \inf\{\|v(t)\| :$

$v(t) \in F(t, y(t))\}$ belongs to $L^1(J)$. It is further bounded if and only if the mapping $t \to \|F(t, y(t))\|_{\mathcal{P}} = \sup\{\|v(t)\| : v(t) \in F(t, y(t))\}$ belongs to $L^1(J)$, this particularly holds true when F satisfies (H_1).
Clearly, from Lemma 4.3, the fixed points of N are solutions to $(4.1) - (4.2)$. We shall prove that N satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof is given in several steps.

Step 1. *$N(y)$ is convex for each $y \in \mathcal{C}_\alpha$.*

Indeed, if $h_1,\ h_2$ belong to $N(y)$, then there exist $v_1, v_2 \in S_{F,y}$, such that for each $t \in J$, we have

$$h_i(t) \ = \ t^{\alpha-1}c + \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_i(s) ds, \quad i = 1, 2.$$

Let $0 \le d \le 1$. Then, for each $t \in J$, we have

$$\begin{array}{rcl} (dh_1 + (1-d)h_2)(t) & = & t^{\alpha-1}c \\ & + & \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[dv_1(s) + (1-d)v_2(s)]ds. \end{array}$$

Since $S_{F,y}$ is convex (because F has convex values), we have

$$dh_1 + (1-d)h_2 \in N(y).$$

Step 2. *N maps bounded sets into bounded sets in $\mathcal{C}_\alpha(J, \mathbb{R})$.*
Indeed, it is enough to show that there exists a positive constant l such that for each $y \in B_r = \{y \in \mathcal{C}_\alpha(J, \mathbb{R}) : \|y\|_\alpha \le r\}$, we have $\|N(y)\|_\alpha \le l$. Let $y \in B_r$. Then for each $h \in N(y)$, there exists $v \in S_{F,y}$ such that

$$h(t) = t^{\alpha-1}c + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) ds.$$

By (H_2), we have

$$\begin{array}{rcl} |t^{1-\alpha}h(t)| & \le & |c| + \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|v(s)|ds \\ & \le & |c| + \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(as^{1-\alpha}|y(s)| + b)ds \\ & \le & |c| + \frac{at^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|s^{1-\alpha}y(s)|ds + \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} b ds \\ & \le & |c| + \frac{T(ra+b)}{\Gamma(1+\alpha)} = l. \end{array}$$

Step 3. *N maps bounded sets into equicontinuous sets of $\mathcal{C}_\alpha([0, T], \mathbb{R})$.*
Let $\tau_1, \tau_2 \in (0, T],\ \tau_1 < \tau_2$ and B_r be a bounded set of $\mathcal{C}_\alpha([0, T], \mathbb{R})$ as Step 2, let $y \in B_r$ and $h \in N(y)$, then

$$\begin{array}{rcl} |\tau_2^{1-\alpha}h(\tau_2) & - & \tau_1^{1-\alpha}h(\tau_1)| \le \frac{\tau_2^{1-\alpha} - \tau_1^{1-\alpha}|}{\Gamma(\alpha)} \int_0^{\tau_1} (\tau_1 - s)^{\alpha-1}||v(s)|ds \\ & + & \frac{|(\tau_2^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\tau_1} |(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}||v(s)|ds \\ & + & \frac{\tau_2^{1-\alpha}}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1}|v(s)|ds \\ & \le & \frac{T^\alpha(ar+b)}{\Gamma(\alpha+1)}(\tau_2^{1-\alpha} - \tau_1^{1-\alpha}) + \frac{T^{1-\alpha}(ar+b)}{\Gamma(\alpha+1)}((\tau_2 - \tau_1)^\alpha - \tau_2^\alpha + \tau_1^\alpha) \\ & + & \frac{T^\alpha(ar+b)}{\Gamma(\alpha+1)}(\tau_2^\alpha - \tau_1^\alpha). \end{array}$$

As $\tau_2 \longrightarrow \tau_1$, the right-hand side of the above inequality tends to zero. Then $N(B_r)$ is equicontinuous.

As a consequence of Steps 1 to 3 together with Lemma 4.1, we can conclude that $N : \mathcal{C}_\alpha \to \mathcal{P}(\mathcal{C}_\alpha)$ is completely continuous.

Step 4. *N is u.s.c.*

To this end, it is sufficient to show that N has a closed graph. Let $h_n \in N(y_n)$ be such that $h_n \longrightarrow h$ and $y_n \longrightarrow y$ as $n \longrightarrow +\infty$.

Then there exists $M > 0$ such that $\|y_n\|_\alpha \leq M$. We shall prove that $h \in N(y)$. $h_n \in N(y_n)$ means that there exists $v_n \in S_{F,y_n}$ such that, for a.e. $t \in J$, we have

$$h_n(t) = t^{\alpha-1}c + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_n(s)ds,$$

(H_2) implies that $v_n(t) \in aM + bB(0,1)$. Then $(v_n)_{n\in\mathbb{N}}$ is integrably bounded in $L^1(J,\mathbb{R})$. It follows that $(v_n)_{n\in\mathbb{N}}$ is weakly compact . There exists a subsequence still denoted $(v_n)_{n\in\mathbb{N}}$, which converges weakly to some limit $v \in L^1(J,\mathbb{R})$. Furthermore, for a.e. $t \in J$, the mapping $\Gamma_t : L^1(J,\mathbb{R}) \to \mathbb{R}$ defined by

$$\Gamma_t(g) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds,$$

is a continuous linear operator. Then it remains continuous if these spaces are endowed with their weak topologies. Moreover, for a.e. $t \in J$, we have

$$h(t) = t^{\alpha-1}c + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)ds.$$

It remains to prove that $v(t) \in F(t, y(t))$, a.e. $t \in J$. Mazur's Lemma 1.19 yields the existence of $\alpha_i^n \geq 0$, $i = 1, \ldots, k(n)$ such that $\sum_{i=1}^{k(n)} \alpha_i^n = 1$ and the sequence of convex combinations $g_n(.) = \sum_{i=1}^{k(n)} \alpha_i^n v_i(.)$ converges strongly to v in L^1. Using Lemma 1.20, we obtain that

$$\begin{array}{rcl} v(t) & \in & \bigcap_{N\geq 1} \overline{\{g_N(t)\}} \text{ a.e. } t \in J \\ & \subset & \bigcap_{N\geq 1} \overline{co}\{v_n(t),\ n \geq N\} \\ & \subset & \bigcap_{N\geq 1} \overline{co}\left\{\bigcup_{n\geq N} F(t, y_N(t))\right\} \\ & = & \overline{co}\left\{\limsup F(t, y_n(t)))\right\}. \end{array} \tag{4.3}$$

However, the fact that the multivalued $x \to F(.,x)$ is u.s.c. and has compact values, together Lemma 1.21 implies that $\lim_{n\longrightarrow\infty} \sup F(t, y_n(t)) = F(t, y(t))$, a.e. $t \in J$, combining with (4.3) yields that $v(t) \in \overline{co}F(t, y(t))$, from the convexity and closedness of F it follows that $v(t) \in F(t, y(t))$, a.e. $t \in J$. Thus $h \in N(y)$, proving that N has a closed graph. Finally, with Lemma 1.16 and the compactness of N, we conclude that N is u.s.c.

Step 5 : *A priori bounds on solutions.*
Let $y \in \mathcal{C}_\alpha(J,\mathbb{R})$ be such that $y \in \lambda N(y)$ for some $\lambda \in (0,1)$. Then there exists

$v \in L^1(J,\mathbb{R})$ with $v \in S_{F,y}$ such that, for each $t \in J$.

$$y(t) = ct^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} v(s)ds.$$

From (H_2), we have

$$\begin{array}{rcl} |t^{1-\alpha}y(t)| & \leq & |c| + \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|v(s)|ds \\ & \leq & |c| + \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(as^{1-\alpha}|y(s)|+b)ds \\ & \leq & |c| + \frac{bT}{\Gamma(1+\alpha)} + \frac{aT^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|s^{1-\alpha}y(s)|ds. \end{array}$$

From Lemma 1.4, there exists $k(\alpha) > 0$ such that

$$|t^{1-\alpha}y(t)| \leq L + \frac{ak(\alpha)T^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} L ds,$$

where $L = |c| + \frac{bT}{\Gamma(1+\alpha)}$. Therefore

$$\|y\|_\alpha \leq L + \frac{ak(\alpha)T}{\Gamma(1+\alpha)} = \widetilde{M}.$$

Let

$$U := \{y \in \mathcal{C}_\alpha([0,T],\mathbb{R}) : \|y(t)\|_{\Omega_c} < \widetilde{M}+1\},$$

and consider the operator $N : U \longrightarrow \mathcal{P}_{cv,cp}(\mathcal{C}_\alpha)$. From the choice of U, there is no $y \in \partial U$ such that $y \in \lambda N(y)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [37], we deduce that N has a fixed point y in $\overline{U}$ which is a solution of the problem (4.1)-(4.2).

Compactness of the solutions set:

Let

$$S_F(c) = \{y \in \mathcal{C}_\alpha(J,\mathbb{R}) : y \text{ is a solution of problem (4.1)-(4.2) }\}.$$

From the previous consideration, there exists $\widetilde{M}$ such that for every $y \in S_F(c)$, $\|y\|_\alpha \leq \widetilde{M}$. Since N is completely continuous, $N(S_F(c))$ is relatively compact in $\mathcal{C}_\alpha$. Let $y \in S_F(c)$, then $y \in N(y)$ and hence $S_F(c) \subset N(S_F(c))$. It remains to prove that $S_F(c)$ is a closed subset in $\mathcal{C}_\alpha$. Let $\{y_n : n \in \mathbb{N}\} \subset S_F(c)$ be such that the sequence $(y_n)_{n\in\mathbb{N}}$ converges to y. For every $n \in \mathbb{N}$, there exists v_n such that $v_n(t) \in F(t, y_n(t))$, a.e. $t \in J$ and

$$y_n(t) = t^{\alpha-1}c + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} v_n(s)ds.$$

Arguing as in **Step 4**, we can prove that there exists v such that $v(t) \in F(t, y(t))$ and

$$y(t) = t^{\alpha-1}c + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} v(s)ds.$$

Therefore $y \in S_F(c)$ which yields that $S_F(c)$ is closed, and hence compact in $\mathcal{C}_\alpha$.

□

4.3 Covitz Nadler approach

4.3.1 Existence results

We present now a result for the problem (4.1)-(4.2) with a non-convex valued right hand side. Our considerations are based on the fixed point theorem for contraction multivalued maps given by Covitz and Nadler.

Theorem 4.5 *Assume that the following hypothesis holds:*

(H_3) $F : J \times \mathbb{R} \longrightarrow \mathcal{P}_{cp}(\mathbb{R})$ *has the property that* $F(\cdot, u) : J \to \mathcal{P}_{cp}(\mathbb{R})$ *is measurable for each* $u \in \mathbb{R}$,

(H_4) there exist a function $p \in C([0,T], \mathbb{R}^+)$ such that

$$H_d(F(t,z_1), F(t,z_2)) \leq p(t)\|z_1 - z_2\|, \text{ for all } z_1,\ z_2 \in \mathbb{R},$$

and

$$d(0, F(t,0)) \leq p(t),\ t \in J.$$

If

$$\frac{T^\alpha \|p\|_\infty \Gamma(\alpha)}{\Gamma(2\alpha)} < 1, \tag{4.4}$$

then the problem (4.1)-(4.2) has at least one solution.

Remark 4.6 *For each $y \in \mathcal{C}_\alpha$ the set $S_{F,y}$ is nonempty since by (H_3), F has a measurable selection (see [21], Theorem III.6).*

Proof. We shall show that N satisfies the assumptions of Lemma 1.34. The proof will be given in two steps.

Step 1: $N(y) \in \mathcal{P}_{cl}(\mathcal{C}_\alpha(J, \mathbb{R}))$ *for each* $y \in \mathcal{C}_\alpha(J, \mathbb{R})$.

Indeed, let $(y_n)_{n\geq 0} \in N(y)$ such that $y_n \longrightarrow \tilde{y}$ in $\mathcal{C}_\alpha(J, \mathbb{R})$. Then, $\tilde{y} \in \mathcal{C}_\alpha(J, \mathbb{R})$ and there exists $v_n \in S_{F,y}$ such that, for each $t \in J$,

$$y_n(t) = t^{\alpha-1}c + \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_n(s)ds.$$

Using the fact that F has compact values and from (H_4), we may pass to a subsequence if necessary to get that v_n converges to v in $L^1(J, \mathbb{R})$. Moreover, for a.e. $t \in J$, we have

$$y_n(t) \longrightarrow \tilde{y}(t) = t^{\alpha-1}c + \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)ds.$$

So, $\tilde{y} \in N(y)$.
Step 2: *There exists $\gamma < 1$ such that*

$$H_d(N(y), N(\overline{y})) \leq \gamma \|y - \overline{y}\|_\alpha \text{ for each, } y, \overline{y} \in \mathcal{C}_\alpha.$$

Let $y,\ \overline{y} \in \mathcal{C}_\alpha$ and $h_1 \in N(y)$. Then there exists $v_1(t) \in F(t, y(t))$ such that for each $t \in J$

$$h_1(t) = t^{\alpha-1}c + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v_1(s)ds.$$

From (H_4) it follows that

$$H_d(F(t,y(t)), F(t,\overline{y}(t))) \leq p(t)|y(t) - \overline{y}(t)|.$$

Hence, there exists $w \in F(t,\overline{y}(t))$ such that

$$|v_1(t) - w| \leq p(t)|y(t) - \overline{y}(t)|, \ t \in J.$$

Consider $U : J \to \mathcal{P}(\mathbb{R})$ given by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq p(t)|y(t) - \overline{y}(t)|\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t,\overline{y}(t))$ is measurable (see Proposition III.4 in [21]), there exists a function $v_2(t)$ which is a measurable selection for V. So, $v_2(t) \in F(t,\overline{y}(t))$, and for each $t \in J$,

$$|v_1(t) - v_2(t)| \leq p(t)|y(t) - \overline{y}(t)|.$$

Let us define for each $t \in J$

$$h_2(t) = t^{\alpha-1}c + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v_2(s)ds.$$

Then for $t \in J$

$$\begin{array}{rcl} |t^{1-\alpha}h_1(t) - t^{1-\alpha}h_2(t)| & \leq & \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|v_2(s) - v_1(s)|ds \\ & \leq & \frac{t^{1-\alpha}\|p\|_\infty}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|y(s) - \overline{y}(s)|ds \\ & \leq & \frac{t^{1-\alpha}\|p\|_\infty}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}\|y - \overline{y}\|_\alpha ds. \end{array}$$

Thus,

$$\|h_1 - h_2\|_\alpha \leq \frac{T^\alpha\|p\|_\infty\Gamma(\alpha)}{\Gamma(2\alpha)}\|y - \overline{y}\|_\alpha.$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$H_d(N(y), N(\overline{y})) \leq \frac{T^\alpha\|p\|_\infty\Gamma(\alpha)}{\Gamma(2\alpha)}\|y - \overline{y}\|_\alpha.$$

So, by (4.4), N is a contraction and thus, by Lemma 1.34, N has a fixed point y which is solution to (4.1)-(4.2).

□

4.3.2 Structure of the solutions set

Theorem 4.7 *Let $F : J \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R})$ be a Carathéodory and mLL-selectionable multivalued which satisfies conditions (H_2). Then for every $c \in \mathbb{R}$, the solution set $S_F(c)$ is compact and contractible.*

Proof. Let $f \in F$ be measurable and locally Lipschitz selection. Consider the single-valued problem

$$^{RL}D^{\alpha}y(t) = f(t, y(t)), \text{ a.e. } t \in J = (0, T], \ 0 < \alpha \leq 1, \tag{4.5}$$

$$\lim_{t \longrightarrow 0^+} t^{1-\alpha}y(t) = c. \tag{4.6}$$

Arguing as in Theorem 4.5, we can prove that the (4.5)-(4.6) has exactly one local solution $\overline{x}$ for every $c \in \mathbb{R}$. Theorem 4.4 implies that $S_F(c)$ is nonempty and compact. We define the homotopy $H : S_F(c) \times [0,1] \to S_F(c)$ by

$$H(y, \lambda)(t) = \begin{cases} y(t), & 0 < t \leq \lambda T, \\ \overline{x}(t) & \lambda T < t \leq T, \end{cases}$$

where $\overline{x}$ is the unique solution of problem $(4.5) - (4.6)$. In particular

$$H(y, \lambda) = \begin{cases} y, & \text{for } \lambda = 1, \\ \overline{x}, & \text{for } \lambda = 0. \end{cases}$$

We prove that H is a continuous homotopy. Let $(y_n, \lambda_n) \in S_F(c) \times [0,1]$ be such that $(y_n, \lambda_n) \longrightarrow (y, \lambda)$, as $n \longrightarrow +\infty$.
We shall prove that $H(y_n, \lambda_n) \longrightarrow H(y, \lambda)$, we have

$$H(y_n, \lambda_n)(t) = \begin{cases} y_n(t), & \text{for } t \in (0, \lambda_n T], \\ \overline{x}(t), & \text{for } t \in (\lambda_n T, T]. \end{cases}$$

We consider several cases,

(a) if $\lim_{n \longrightarrow +\infty} \lambda_n = 0$,

$$H(y, 0)(t) = \overline{x}(t), \ t \in (0, T].$$

Hence

$$\|H(y_n, \lambda_n) - H(y, \lambda)\|_{\alpha} \text{ tends to 0 as } n \longrightarrow +\infty.$$

(b) If $\lambda_n \neq 0$ and $0 < \lim_{n \longrightarrow \infty} \lambda_n < 1$, two cases must be treated,

If $t \in (0, \lambda_n T]$, then $H(y_n, \lambda_n)(t) - H(y, \lambda)(t) = y_n(t) - y(t)$,
since $y_n \in S_F(c)$, there exist $v_n \in S_{F, y_n}$ such that

$$y_n(t) = t^{\alpha-1}c + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_n(s) ds, \ t \in (0, \lambda_n T].$$

We must show that there exists $v \in S_{F,y}$ such that, for each $t \in J$,

$$y(t) = t^{\alpha-1}c + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) ds, \ t \in (0, \lambda T].$$

Since $F(t, \cdot)$ is upper semi-continuous, then for every $\varepsilon > 0$, there exist $n_0(\epsilon) \geq 0$ such that for every $n \geq n_0$, we have

$$v_n(t) \in F(t, y_n(t)) \subset F(t, y(t)) + \varepsilon B(0, 1), \text{ a.e. } t \in J.$$

Since $F(\cdot,\cdot)$ has compact values, then there exists a subsequence $v_{n_m}(\cdot)$ such that

$$v_{n_m}(\cdot) \to v(\cdot) \text{ as } m \to \infty$$

and

$$v(t) \in F(t, y(t)), \text{ a.e. } t \in J.$$

Since y_n converges to y, there exists $M > 0$ such that $\|y_n\|_\alpha \leq M$.
Hence, from (H_2), we have

$$|v_n(t)| \leq \overline{a}M + \overline{b}, \text{ a.e. } t \in J,$$

which implies

$$v_n(t) \in \overline{a}M + \overline{b}B(0,1).$$

From the Lebesgue dominated convergence theorem, yields

$$y(t) = t^{\alpha-1}c + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds, \ t \in (0, \lambda T].$$

If $t \in (\lambda_n T, T]$, then

$$H(y_n, \lambda_n)(t) = H(y, \lambda)(t).$$

Thus

$$\|H(y_n, \lambda_n) - H(y, \lambda)\|_\alpha \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

(c) If $\lambda = 1$, we have

$$H(y_n, \lambda_n) = H(y, \lambda) = y.$$

Therefore H is a continuous function, proving that $S_F(c)$ is contractible to the point $\overline{x}$.

□

4.4 Examples

Consider the problem

$$^{RL}D^{\frac{1}{2}}y(t) \in F_1(t, y(t)), \text{ a.e. } t \in J = (0,1], \ \alpha = \frac{1}{2}, \tag{4.7}$$

$$\lim_{t \longrightarrow 0^+} t^{\frac{1}{2}}y(t) = 4. \tag{4.8}$$

Let $F_1 : (0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map given by

$$F_1(t,x) = \left[\frac{3x^2}{3x^2+1} + t^3, \ t^{1-\alpha}|x| + \frac{|x|}{|x|+2} + t^2 + 2\right]. \tag{4.9}$$

It is clear that F_1 is a Carathéodory multivalued map with compact, convex values and for $f \in F_1$, we have

$$|f(t)| \leq t^{1-\alpha}|x| + \frac{|x|}{|x|+2} + t^2 + 2| \leq 4 + t^{1-\alpha}|x|, \ x \in \mathbb{R}.$$

Thus,

$$\|F_1(t,x)\|_{\mathcal{P}} = \sup\{|y| : \ y \in F(t,x)\} \leq at^{1-\alpha}|x| + b, \ x \in \mathbb{R},$$

with $a = 1$ and $b = 4$. Hence, by Theorem 4.4, the problem (4.7)-(4.8) with F given by (4.9) has at least one solution and the solution set $S_{F_1}(4)$ is compact.

Consider the multivalued map $F_2 : (0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ given by

$$F_2(t,x) = \left[0, \ \frac{t^{1-\alpha}|x|}{t+9} + \frac{1}{9}\right], \tag{4.10}$$

and the fractional differential inclusion defined by

$${}^{RL}D^{\frac{1}{2}}y(t) \in F_2(t,y(t)), \ \text{a.e. } t \in J = (0,1], \ \alpha = \frac{1}{2}, \tag{4.11}$$

$$\lim_{t \longrightarrow 0^+} t^{\frac{1}{2}}y(t) = 4. \tag{4.12}$$

Clearly

$$\begin{aligned} \|F_2(t,x)\| &= \sup\{|v| : v \in F_2(t,x)\} \\ &\leq \frac{t^{1-\alpha}|x|}{t+9} + \frac{1}{9} \\ &\leq \frac{t^{1-\alpha}}{9}|x| + \frac{1}{9}, \end{aligned}$$

and

$$H_d(F_2(t,x), F_2(t,y)) \leq \frac{1}{t+9}|x-y|,$$

where $p(t) = \frac{1}{t+9}$. Then $\|p\||_\infty = \frac{1}{9}$ and $\frac{T^\alpha \|p\|_\infty \Gamma(\alpha)}{\Gamma(2\alpha)} \approx 0.1969 < 1$. Hence by Theorem 4.7, the problem (4.11)-(4.12) with F_2 given by (4.10) has at least one solution. It is clear that F_2 is a Carathéodory multivalued and mLL-selectionable map with compact, convex values and satisfies the growth condition (H_2). Then the solution set $S_{F_2}(4)$ is a compact and contractible set.

Chapter 5

Boundary value problems for fractional differential equation in Banach space

5.1 Introduction

We consider the following boundary- value problem

$$ {}^{RL}D^{\alpha}_{0^+}y(t) = f(t, y(t)), \quad t \in J = (0, +\infty), \tag{5.1}$$

$$ I^{2-\alpha}_{0^+}y(0^+) = y_0, \tag{5.2}$$

$$ {}^{RL}D^{\alpha-1}_{0^+}y(\infty) = y_\infty. \tag{5.3}$$

${}^{RL}D^{\alpha}_{0^+}$ denote Riemann-Liouville fractional derivative, $1 < \alpha \leq 2$. The operator $I^{2-\alpha}_{0^+}$ denotes the left-sided Riemann-Liouville fractional integral, the state $y(\cdot)$ takes value in a Banach space E, $f : (0, \infty) \times E \to E$ will be specified in later sections and $(y_0, y_\infty) \in E \times E$.

5.2 Main result

We consider the space of functions

$$ C_\alpha([0, \infty), E) = \{y \in C(J, E) : \lim_{t \to 0^+} t^{2-\alpha}y(t) \text{ exists and finite}\}.$$

For $y \in C_\alpha((0, \infty), E)$, we define y_α by

$$ y_\alpha(t) = \begin{cases} t^{2-\alpha}y(t), & t \in (0, \infty), \\ \lim_{t \longrightarrow 0} t^{2-\alpha}y(t), & t = 0. \end{cases}$$

It is clear that $y_\alpha \in C([0, \infty), E)$.
We consider the following Banach space

$$ X_\alpha([0, \infty), E) = \{y \in C_\alpha([0, \infty), E) : \lim_{t \to \infty} \frac{t^{2-\alpha}y(t)}{1 + t^\alpha} \text{ exists and finite}\}.$$

A norm in this space is given by

$$\|y\|_\alpha = \sup_{t\in J} \frac{t^{2-\alpha}\|y(t)\|}{1+t^\alpha}.$$

Let

$$B = \{y \in X_\alpha([0,\infty), E) : \|y\|_\alpha \leq R\}.$$

We will need to introduce the following hypotheses which are assumed here after.

(H_1) There exists a nonnegative functions $a, b \in C(J, \mathbb{R}^+)$ such that

$$\|f(t,u)\| \leq a(t) + t^{2-\alpha} b(t)\|u\|, \text{ for all } t \in J \text{ and } u \in E,$$

where

$$\int_0^\infty (1+t^\alpha) b(t)dt < \Gamma(\alpha), \ \int_0^\infty a(t)dt < \infty.$$

(H_2) For all $(0, c] \subset J$, there exists a constant $A > 0$ such that

$$\forall t \in (0, c], \forall x, y \in E : \|f(t,x) - f(t,y)\| \leq \frac{t^{2-\alpha} A}{1+t^\alpha}\|x - y\|.$$

(H_3) There exists nonnegative function $\ell \in L^1(J, \mathbb{R}^+)$ such that, for each nonempty, bounded set $\Omega \subset X_\alpha([0,\infty), E)$

$$\gamma(f(t, \Omega(t))) \leq t^{2-\alpha}\ell(t)\gamma(\Omega(t)), \quad \text{for all } t \in J,$$

where

$$\int_0^\infty (1+t^\alpha)\ell(t)dt \leq \Gamma(\alpha).$$

(H_4)

$$R > \frac{\|y_\infty\| + (\alpha-1)\|y_0\| + \int_0^\infty a(t)dt}{\Gamma(\alpha) - \int_0^\infty (1+t^\alpha)b(t)dt}.$$

Definition 5.1 *A function $y \in X_\alpha([0,+\infty), E)$ is said to be solution of the problem (5.1)-(5.3) if y satisfies the equation ${}^{RL}D^\alpha_{0^+}y(t) = f(t, y(t))$ and the conditions* (5.2) – (5.3).

Lemma 5.2 *Let $1 < \alpha < 2$ and let $h : J \to E$ be continuous. Then, y satisfies the following equation*

$$y(t) = \frac{1}{\Gamma(\alpha)}[y_\infty - \int_0^\infty h(t)dt]t^{\alpha-1} + \frac{y_0}{\Gamma(\alpha-1)}t^{\alpha-2} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds \quad (5.4)$$

if and only if y satisfies the following problem

$$ {}^{RL}D^\alpha_{0^+}y(t) = h(t), \quad t \in J = (0,+\infty), \quad (5.5)$$

$$I^{2-\alpha}_{0^+}y(0^+) = y_0, \quad (5.6)$$

$$ {}^{RL}D^{\alpha-1}_{0^+}y(\infty) = y_\infty. \quad (5.7)$$

Proof. Assume that y satisfies the problem (5.5)-(5.7). We may apply Lemma 1.8 to reduce equation (5.5) to an equivalent integral equation

$$y(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + I_{0^+}^{\alpha} h(t), \tag{5.8}$$

for some $c_1,\ c_2 \in \mathbb{R}$. Applying $I_{0^+}^{2-\alpha}$ to both side of (5.8), we have

$$I_{0^+}^{2-\alpha} y(t) = c_1 I_{0^+}^{2-\alpha} t^{\alpha-1} + c_2 I_{0^+}^{2-\alpha} t^{\alpha-2} + I_{0^+}^{2-\alpha} I_{0^+}^{\alpha} h(t).$$

From Remark 1.9, we then get

$$I_{0^+}^{2-\alpha} y(t) = \frac{c_1\Gamma(\alpha)}{\Gamma(2)} t + c_2\Gamma(\alpha-1) + \frac{1}{\Gamma(2)} \int_0^t (t-s)h(s)ds.$$

As $t \longrightarrow 0$, we obtain

$$c_2 = \frac{y_0}{\Gamma(\alpha-1)}.$$

Applying ${}^{RL}D_{0^+}^{\alpha-1}$ to both side of (5.8), we have

$${}^{RL}D_{0^+}^{\alpha-1} y(t) = c_1^{RL} D_{0^+}^{\alpha-1} t^{\alpha-1} + c_2^{RL} D_{0^+}^{\alpha-1} t^{\alpha-2} +^{RL} D_{0^+}^{\alpha-1} I_{0^+}^{\alpha} h(t).$$

From Remark 1.9 and Remark 1.10, we then get

$${}^{RL}D_{0^+}^{\alpha-1} y(t) = c_1\Gamma(\alpha) + \frac{1}{\Gamma(1)} \int_0^t h(s)ds.$$

Hence

$$c_1 = \frac{1}{\Gamma(\alpha)} (y_\infty - \int_0^\infty h(t)dt).$$

Thus, we have

$$y(t) = \frac{1}{\Gamma(\alpha)} [y_\infty - \int_0^\infty h(t)dt] t^{\alpha-1} + \frac{y_0}{\Gamma(\alpha-1)} t^{\alpha-2} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds.$$

Conversely. The proof is simple. □

Consider the operator $N : X_\alpha([0,\infty), E) \to X_\alpha([0,\infty), E)$ defined by

$$\begin{aligned} N(y)(t) = {} & \frac{y_\infty}{\Gamma(\alpha)} t^{\alpha-1} + \frac{y_0}{\Gamma(\alpha-1)} t^{\alpha-2} - \frac{1}{\Gamma(\alpha)} \int_0^t [t^{\alpha-1} - (t-s)^{\alpha-1}] f(s, y(s))ds \\ & - \frac{1}{\Gamma(\alpha)} \int_t^\infty t^{\alpha-1} f(s, y(s))ds. \end{aligned}$$

The following several Lemmas present some properties of the operator N, which are necessary for the proof of our main result.

Lemma 5.3 *Suppose that conditions (H_1) and (H_2) are valid. Then the operator N is bounded and continuous.*

Proof. For $y \in X_\alpha([0,\infty),E)$, from (H_1), it is easy to deduce that $N(y) \in X_\alpha([0,\infty),E)$. Furthermore, (H_1) guarantees that

$$\begin{aligned}
\frac{t^{2-\alpha}\|N(y)(t)\|}{1+t^\alpha} &\leq \frac{\|y_\infty\|}{\Gamma(\alpha)} + \frac{\|y_0\|}{\Gamma(\alpha-1)} + \frac{1}{\Gamma(\alpha)}\int_0^\infty \|f(s,y(s))\|ds \\
&\leq \frac{\|y_\infty\|}{\Gamma(\alpha)} + \frac{\|y_0\|}{\Gamma(\alpha-1)} + \frac{\|y\|_\alpha}{\Gamma(\alpha)}\int_0^\infty (1+t^\alpha)b(t)dt + \frac{1}{\Gamma(\alpha)}\int_0^\infty a(t)dt.
\end{aligned}$$

Hence, $N : X_\alpha([0,\infty),E) \to X_\alpha([0,\infty),E)$ is bounded. Next, we prove that N is continuous. Let $\{y_n\}_{n=1}^\infty \subset X_\alpha([0,\infty),E)$ and $y \in X_\alpha([0,\infty),E)$ such that $y_n \to y$ as $n \to \infty$. Then, $\{y_n\}_{n=1}^\infty$ is a bounded set of $X_\alpha([0,\infty),E)$, i.e. there exists $M > 0$ such that $\|y_n\|_\alpha \leq M$, for $n > 1$. We also have by taking limit that $\|y\|_\alpha \leq M$. In view of condition (H_1), for any $\varepsilon > 0$, there exists $L > 0$ such that

$$\int_L^\infty a(t)dt < \frac{\Gamma(\alpha)}{6}\varepsilon, \ \int_L^\infty (1+t^\alpha)b(t)dt < \frac{\Gamma(\alpha)}{6M}\varepsilon,$$

and from (H_2), there exists $\widetilde{N} \in \mathbb{N}$ such that, for all $n \geq \widetilde{N}$ and $t \in (0,L]$, we have

$$\|f(s,y_n(s)) - f(s,y(s))\| < \frac{\Gamma(\alpha)}{3L}\varepsilon.$$

Therefore, for all $t \in J$ and $n > \widetilde{N}$, we obtain

$$\begin{aligned}
\frac{t^{2-\alpha}}{1+t^\alpha}\|N(y_n)(t) - N(y)(t)\| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t \|f(s,y_n(s)) - f(s,y(s))\|ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_t^\infty \|f(s,y_n(s)) - f(s,y(s))\|ds.
\end{aligned}$$

If $t \leq L$ and $n > \widetilde{N}$, we have

$$\begin{aligned}
&\frac{t^{2-\alpha}}{1+t^\alpha}\|N(y_n)(t) - N(y)(t)\| \leq \frac{1}{\Gamma(\alpha)}\int_0^t \|f(s,y_n(s)) - f(s,y(s))\|ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_t^\infty \|f(s,y_n(s)) - f(s,y(s))\|ds \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t \|f(s,y_n(s)) - f(s,y(s))\|ds \\
&+ \frac{1}{\Gamma(\alpha)}\left[\int_t^L \|f(s,y_n(s)) - f(s,y(s))\|ds + \int_L^\infty \|f(s,y_n(s)) - f(s,y(s))\|ds\right] \\
&\leq \frac{2}{\Gamma(\alpha)}\int_0^L \|f(s,y_n(s)) - f(s,y(s))\|ds \\
&+ \frac{2M}{\Gamma(\alpha)}\int_L^\infty (1+s^\alpha)b(s)ds + \frac{2}{\Gamma(\alpha)}\int_L^\infty a(s)ds \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

The case when $t > L$ and $n > \widetilde{N}$ is treated similarly. Thus, we conclude that

$$\|y_n - y\|_\alpha \to 0 \text{ as } n \to \infty,$$

namely, N is continuous. $\square$

Lemma 5.4 *Let condition (H_1) be satisfied and B be a bounded subset of $X_\alpha([0,\infty), E)$. Then*

(i) $\frac{t^{2-\alpha}N(B)(t)}{1+t^\alpha}$ *is equicontinuous on any compact interval of J.*

(ii) For given $\varepsilon > 0$, there exists a constant $N_1 > 0$ such that $\|\frac{t_1^{2-\alpha}N(y)(t_1)}{1+t_1^\alpha} - \frac{t_2^{2-\alpha}N(y)(t_2)}{1+t_2^\alpha}\| < \varepsilon$*, for any $t_1, t_2 \geq N_1$ and $y(.) \in B$.*

Proof. We have

$$\begin{aligned} N(y)(t) = & \frac{y_\infty - \int_0^\infty f(t, y(t))dt}{\Gamma(\alpha)} t^{\alpha-1} + \frac{y_0}{\Gamma(\alpha-1)} t^{\alpha-2} \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s))ds. \end{aligned}$$

In view of condition (H_1) and the boundedness of B, there exists $M > 0$ such that

$$\int_0^\infty \|f(t, y(t))\|dt \leq M \text{ for any } y \in B. \tag{5.9}$$

In order to prove (i), let the constant r be such that $\|y\|_\alpha \leq r$, for any $y \in B$, and without loss of generality, let $[a, b] \subset J$ be a compact interval and $t_1, t_2 \in [a, b]$ with $t_1 < t_2$. Then

$$\begin{aligned} & \|\frac{t_1^{2-\alpha}N(y)(t_1)}{1+t_1^\alpha} - \frac{t_2^{2-\alpha}N(y)(t_2)}{1+t_2^\alpha}\| \leq \frac{\|y_\infty\| + M}{\Gamma(\alpha)} |\frac{t_1}{1+t_1^\alpha} - \frac{t_2}{1+t_2^\alpha}| \\ & + \frac{\|y_0\|}{\Gamma(\alpha-1)} \left| \frac{1}{1+t_1^\alpha} - \frac{1}{1+t_2^\alpha} \right| \\ & + \frac{1}{\Gamma(\alpha)} \left\| \int_0^{t_1} (t_1 - s)^{\alpha-1} f(s, y(s))ds - \int_0^{t_2} (t_2 - s)^{\alpha-1} f(s, y(s))ds \right\| \\ & \leq \frac{\|y_\infty\| + M}{\Gamma(\alpha)} |\frac{t_1}{1+t_1^\alpha} - \frac{t_2}{1+t_2^\alpha}| + \frac{\|y_0\|}{\Gamma(\alpha-1)} \left| \frac{1}{1+t_1^\alpha} - \frac{1}{1+t_2^\alpha} \right| \\ & + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} |(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}| \|f(s, y(s))\|ds \\ & + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \|f(s, y(s))\|ds \\ & \leq \frac{\|y_\infty\| + M}{\Gamma(\alpha)} |\frac{t_1}{1+t_1^\alpha} - \frac{t_2}{1+t_2^\alpha}| + \frac{\|y_0\|}{\Gamma(\alpha-1)} \left| \frac{1}{1+t_1^\alpha} - \frac{1}{1+t_2^\alpha} \right| \\ & + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} |(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}| a(s)ds \end{aligned}$$

$$
\begin{aligned}
&+\frac{r}{\Gamma(\alpha)}\int_0^{t_1}|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}|(1+s^\alpha)b(s)ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}a(s)ds+\frac{r}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}(1+s^\alpha)b(s)ds\\
&\leq\frac{\|y_\infty\|+M}{\Gamma(\alpha)}|\frac{t_1}{1+t_1^\alpha}-\frac{t_2}{1+t_2^\alpha}|+\frac{\|y_0\|}{\Gamma(\alpha-1)}\left|\frac{1}{1+t_1^\alpha}-\frac{1}{1+t_2^\alpha}\right|\\
&+\frac{a^*+b^*r}{\Gamma(\alpha)}\left(\int_0^{t_1}(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}ds\right)+\frac{a^*+b^*r}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\\
&+\frac{2b^*r}{\Gamma(\alpha)}\left(\int_0^{t_2}(t_2-s)^{\alpha-1}s^\alpha ds-\int_0^{t_1}(t_1-s)^{\alpha-1}s^\alpha ds\right)\\
&\leq\frac{\|y_\infty\|+M}{\Gamma(\alpha)}|\frac{t_1}{1+t_1^\alpha}-\frac{t_2}{1+t_2^\alpha}|+\frac{\|y_0\|}{\Gamma(\alpha-1)}\left|\frac{1}{1+t_1^\alpha}-\frac{1}{1+t_2^\alpha}\right|\\
&+\frac{a^*+b^*r}{\Gamma(1+\alpha)}\left(t_2^\alpha-t_1^\alpha-(t_2-t_1)^\alpha\right)+\frac{a^*+b^*r}{\Gamma(1+\alpha)}(t_2-t_1)^\alpha\\
&+\frac{2b^*r\mathcal{B}(\alpha,\alpha+1)}{\Gamma(\alpha)}(t_2^{2\alpha}-t_1^{2\alpha}),
\end{aligned}
$$

where $a^*=\max_{t\in[a,b]}a(t)$ and $b^*=\max_{t\in[a,b]}b(t)$. As $t_2\to t_1$, the right-hand side of the above inequality tends to zero. Then $\frac{t^{2-\alpha}N(B)(t)}{1+t^\alpha}$ is equicontinuous on $[a,b]$.
Next, we verify assertion (ii). Let $\varepsilon>0$, we heve

$$
\begin{aligned}
&\|\frac{t_1^{2-\alpha}N(y)(t_1)}{1+t_1^\alpha}-\frac{t_2^{2-\alpha}N(y)(t_2)}{1+t_2^\alpha}\|\leq\frac{\|y_\infty\|+M}{\Gamma(\alpha)}\left|\frac{t_1}{1+t_1^\alpha}-\frac{t_2}{1+t_2^\alpha}\right|\\
&+\frac{\|y_0\|}{\Gamma(\alpha-1)}\left|\frac{1}{1+t_1^\alpha}-\frac{1}{1+t_2^\alpha}\right|\\
&+\frac{1}{\Gamma(\alpha)}\left\|\int_0^{t_1}\frac{t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{1+t_1^\alpha}f(s,y(s))ds-\int_0^{t_2}\frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{1+t_2^\alpha}f(s,y(s))ds\right\|.
\end{aligned}
$$

It is sufficient to prove that

$$
\left\|\int_0^{t_1}\frac{t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{1+t_1^\alpha}f(s,y(s))ds-\int_0^{t_2}\frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{1+t_2^\alpha}f(s,y(s))ds\right\|\leq\varepsilon.
$$

Relation (5.9) yields that there exits $N_0>0$ such that

$$
\int_{N_0}^\infty\|f(t,y(t))\|dt\leq\frac{\varepsilon}{3},\ \text{for any } y\in B. \tag{5.10}
$$

On the other hand, since $\lim_{t\to\infty}\frac{t^{2-\alpha}(t-N_0)^{\alpha-1}}{1+t^\alpha}=0$, there exists $N_1>N_0$ such that, for any $t_1,t_2\geq N_1$ and $s\in[0,N_0]$, we have

$$
\left|\frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{1+t_2^\alpha}-\frac{t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{1+t_1^\alpha}\right|<\frac{\varepsilon}{3M}. \tag{5.11}
$$

Now taking $t_1, t_2 \geq N_1$, from (5.10), (5.11), we can arrive at

$$\begin{aligned}
&\left\| \int_0^{t_1} \frac{t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{1+t_1^\alpha} f(s,y(s))ds - \int_0^{t_2} \frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{1+t_2^\alpha} f(s,y(s))ds \right\| \\
&\leq \int_0^{N_1} \left| \frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{1+t_2^\alpha} - \frac{t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{1+t_1^\alpha} \right| \|f(s,y(s))\|ds \\
&+ \int_{N_1}^{t_1} \frac{t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{1+t_1^\alpha} \|f(s,y(s))\|ds + \int_{N_1}^{t_2} \frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{1+t_2^\alpha} \|f(s,y(s))\|ds \\
&< \frac{\varepsilon}{3M} \int_0^\infty \|f(s,y(s))\|ds + 2\int_{N_1}^\infty \|f(s,y(s))\|ds < \varepsilon.
\end{aligned}$$

Therefore, the proof of lemma 5.4 is completed. □

We denote by γ_{X_α} the Kuratowski measure of non-compactness defined on any bounded subset of $X_\alpha([0,\infty),E)$

Lemma 5.5 *[57] Suppose that condition (H_1) holds and B is a bounded subset of $X_\alpha([0,\infty),E)$. Then $\gamma_{X_\alpha}(N(B)) = \sup_{t\in J} \gamma \left(\frac{t^{2-\alpha}N(B)(t)}{1+t^\alpha} \right)$.*

Theorem 5.6 *Suppose that conditions (H_1), (H_2) (H_3) and (H_4) are valid. Then the problem (5.1)-(5.3) has at least one solution.*

Proof.
First, we transform the problem (5.1)-(5.3) into a fixed point problem. Consider the operator $N : X_\alpha([0,\infty),E) \to X_\alpha([0,\infty),E)$ defined by

$$\begin{aligned}
N(y)(t) = &\frac{y_\infty}{\Gamma(\alpha)} t^{\alpha-1} + \frac{y_0}{\Gamma(\alpha-1)} t^{\alpha-2} - \frac{1}{\Gamma(\alpha)} \int_0^t [t^{\alpha-1} - (t-s)^{\alpha-1}] f(s,y(s))ds \\
&- \frac{1}{\Gamma(\alpha)} \int_t^\infty t^{\alpha-1} f(s,y(s))ds.
\end{aligned}$$

From Lemma 5.2, the fixed points of N are solutions to (5.1)-(5.3). We shall show that N satisfies the assumptions of Mönch fixed point theorem.
Then, we can derive that $N : B \to B$. Indeed, for any $y \in B$, by condition (H_1), we get

$$\begin{aligned}
&\| \frac{t^{2-\alpha}N(y)(t)}{1+t^\alpha} \| \leq \frac{\|y_\infty\|}{\Gamma(\alpha)} + \frac{\|y_0\|}{\Gamma(\alpha-1)} + \frac{1}{\Gamma(\alpha)} \int_0^\infty \|f(t,y(t))\|dt \\
&\leq \frac{(1}{\Gamma(\alpha)} \left(\|y_\infty\| + (\alpha-1)\|y_0\| + \int_0^\infty a(t)dt + R\int_0^\infty (1+t^\alpha))b(t)dt \right) < R.
\end{aligned}$$

Hence, from (H_4), we have $\|Ny\|_\alpha \leq R$, so, we conclude that $N : B \to B$. Clearly B is a bounded, convex and closed subset of $X_\alpha([0,\infty),E)$, together with Lemma 5.3, we know that $N : B \to B$ is continuous. Finally, we need to prove the following implication

$$V \subset \overline{conv}\{N(V) \cup \{0\}\} \Longrightarrow \gamma_{X_\alpha}(V) = 0, \text{ for any } V \subset B.$$

Let $V \subset B$ such that $V \subset \overline{conv}\{N(V) \cup \{0\}\}$ and $t \in J$, we choose $\xi > 0$ and $n > 0$ such that $\xi < t < n$. For each $y \in V$, we consider

$$N_{\xi,n}(y)(t) = \frac{y_\infty}{\Gamma(\alpha)}t^{\alpha-1} + \frac{y_0}{\Gamma(\alpha-1)}t^{\alpha-2} + \frac{1}{\Gamma(\alpha)}\int_\xi^t [t^{\alpha-1} - (t-s)^{\alpha-1}]f(s,y(s))ds$$
$$+ \frac{1}{\Gamma(\alpha)}\int_t^n (t-s)^{\alpha-1}f(s,y(s))ds.$$

Then from (H_1), we obtain that

$$\frac{t^{2-\alpha}}{1+t^\alpha}\|N_{\xi,n}(y)(t) - N(y)(t)\| \le \frac{1}{\Gamma(\alpha)}\int_0^\xi \|f(t,y(t))\|dt + \frac{1}{\Gamma(\alpha)}\int_n^\infty \|f(t,y(t))\|dt$$
$$\le \frac{1}{\Gamma(\alpha)}\left(\int_0^\xi a(t)dt + R\int_0^\xi (1+t^\alpha))b(t)dt + \int_n^\infty a(t)dt + R\int_n^\infty (1+t^\alpha))b(t)dt\right),$$

this shows that $H_d\left(\frac{t^{2-\alpha}N_{\xi,n}(V)(t)}{1+t^\alpha}, \frac{t^{2-\alpha}N(V)(t)}{1+t^\alpha}\right) \to 0$ as $\xi \to 0$ and $n \to \infty,\ t \in J$, where H_d denotes the Hausdorff metric in space E. By the prorerty of noncompactness mearure, we get

$$\lim_{\xi\to 0,\ n\to\infty} \gamma\left(\frac{t^{2-\alpha}N_{\xi,n}(V)(t)}{1+t^\alpha}\right) = \gamma\left(\frac{t^{2-\alpha}N(V)(t)}{1+t^\alpha}\right). \tag{5.12}$$

From lemma 1.16, the set $\{\frac{t^{2-\alpha}N_{\xi,n}(V)(t)}{1+t^\alpha}\} \subset X_\alpha([0,\infty), E)$ is equicontinuous on any compact of J. By (H_1). Using Lemma 1.32, Lemma 1.18 and (H_3), we arrive at

$$\gamma\left(\frac{t^{2-\alpha}N_{\xi,n}V(t)}{1+t^\alpha}\right) \le \frac{1}{\Gamma(\alpha)}\int_\xi^n (1+t^\alpha)\ell(t)\gamma\left(\frac{t^{2-\alpha}V(t)}{1+t^\alpha}\right)dt$$
$$\le \frac{1}{\Gamma(\alpha)}\int_\xi^n (1+t^\alpha)\ell(t)\gamma\left(\frac{t^{2-\alpha}N(V)(t)}{1+t^\alpha}\right)dt$$
$$\le \frac{1}{\Gamma(\alpha)}\int_\xi^n (1+t^\alpha)\ell(t)\gamma_{X_\alpha}(N(V))dt.$$

From (5.12), we know that

$$\gamma\left(\frac{t^{2-\alpha}N(V)(t)}{1+t^\alpha}\right) \le \frac{1}{\Gamma(\alpha)}\int_0^\infty (1+t^\alpha)\ell(t)\gamma_{X_\alpha}(N(V))dt.$$

Thus,

$$\gamma_{X_\alpha}(N(V)) \le \frac{1}{\Gamma(\alpha)}\int_0^\infty (1+t^\alpha)\ell(t)\gamma_{X_\alpha}(N(V))dt.$$

Consequently, by condition (H_3). We get $\gamma_{X_\alpha}(N(V)) = 0$; that is $\gamma_{X_\alpha}(V) = 0$. From the theorem 1.33, we conclude that N has a fixed point $y \in B$ which is a solution of problem (5.1)-(5.3). $\square$

5.3 Example

We consider the following problem.

$$
{}^{RL}D^{\frac{3}{2}}y(t)=\left(\frac{\sqrt{t}y_n(t)}{(1+t^{\frac{3}{2}})e^{10t}}+\frac{2t}{(1+t^2)^2}\right)_{n=1}^{\infty},\quad t\in J=(0,+\infty),\tag{5.13}
$$

$$
I_{0^+}^{\frac{1}{2}}y(t)=y_0,\tag{5.14}
$$

$$
{}^{RL}D_{0^+}^{\frac{1}{2}}y(\infty)=y_\infty.\tag{5.15}
$$

Let

$$
E=\{(y_1,\ldots,y_n,\ldots):\sup|y_n|<\infty\},
$$

with the norm $\|y\|=\sup_n|y_n|$, then E is a Banach space and Problem (5.13)-(5.15) can be regarded as a problem of the form (5.1)-(5.3), with

$$
\alpha=\frac{3}{2}\text{ and }f(t,y(t))=(f(t,y_1(t)),\ldots,f(t,y_n(t)),\ldots),
$$

where

$$
f(t,y_n(t))=\frac{\sqrt{t}y_n(t)}{(1+t^{\frac{3}{2}})e^{10t}}+\frac{2t}{(1+t^2)^2},n\in\mathbb{N}^*.
$$

We shall verify the conditions $(H_1)-(H_3)$. Evidently, f is continuous in $J\times E$ and

$$
\|f(t,y(t))\|\le\frac{\sqrt{t}}{(1+t^{\frac{3}{2}})e^{10t}}\|y(t)\|+\frac{2t}{(1+t^2)^2}.
$$

With the aid of simple computation, we find that

$$
\int_0^\infty e^{-5t}dt=\frac{1}{10}<\Gamma(\frac{3}{2})\text{ and }\int_0^\infty\frac{2t}{(1+t^2)^2}dt=1<\infty.
$$

Finally, we verify condition (H_3). For any bounded set $B\subset E$, we have

$$
f(t,B(t))=\frac{\sqrt{t}}{(1+t^{\frac{3}{2}})e^{5t}}B(t)+\{\frac{2t}{(1+t^2)^2}\}.
$$

Then

$$
\gamma(f(t,B(t))\le\frac{\sqrt{t}}{(1+t^{\frac{3}{2}})e^{5t}}\gamma(B(t)).
$$

Since $\int_0^\infty e^{-10t}dt=0.1<\Gamma(\frac{3}{2})$, we conclude that condition (H_3) is satisfied. Therefore, Theorem 5.6 ensures that problem (5.13)-(5.15) has a solution.

Chapter 6

Existence result for a fractional differential equation involving a sequential derivative

6.1 Introduction

We consider the following initial value problem of fractional-ordinary differential equation:

$$(\mathbf{P}) \quad \begin{cases} {}^{RL}D^{\alpha}_{0^+}y'(t) = f(t, y(t), y'(t)), \quad t \in (0, T], \\ I^{1-\alpha}_{0^+}y'(0) = a, \\ y(0) = b, \end{cases}$$

where $0 < \alpha \leq 1$ and ${}^{RL}D^{\alpha}_{0^+}$ denote the left-sided Riemann-Liouville fractional derivative. The operator $I^{1-\alpha}_{0^+}$ denotes the left-sided Riemann-Liouville fractional integral, E is a Banach space with the norm $\|.\|$, $a, b \in E$ and $f : (0, T] \times E \times E \to E$ a function satisfying some specified conditions (see, section 6.3).

6.2 Background and basic results

We introduce in this section some notation and technical results which are used throughout this chapter. For all $0 < \alpha < 1$, let $C_{1-\alpha}([0, T], E)$ be the Banach spaces of functions from the interval $[0, T]$ into E which is defined as:

$$C_{1-\alpha}([0, T], E) = \{y \in C((0, T], E) : \lim_{t \to 0^+} t^{1-\alpha}y(t) \text{ exists and finite}\}.$$

A norm in this space is given by

$$\|y\|_{\alpha} = \sup_{t \in [0,T]} t^{1-\alpha}\|y(t)\|,$$

and

$$C^1_{\alpha}([0, T], E) = \{y : [0, T] \to E : y \in C([0, T], E) \text{ and } y' \in C_{1-\alpha}([0, T], E)\}$$

with the norm

$$\|y\|_{C^1_\alpha} = \sup_{t\in[0,T]} \|y(t)\| + \sup_{t\in[0,T]} t^{1-\alpha}\|y'(t)\|$$

For any subset N of $C_{1-\alpha}([0,T],E))$, we put $N_\alpha =: \{y_\alpha,\ y \in N\}$, where

$$y_\alpha(t) = \begin{cases} t^{1-\alpha}y(t), & \text{if } t \in (0,T] \\ \lim\limits_{t\to 0} t^{1-\alpha}y(t), & \text{if } t = 0. \end{cases}$$

Clearly the function y_α belongs to $C([0,T],E)$, and hence $N_\alpha \subseteq C([0,T],E)$.

Lemma 6.1 ([31]) *Let $\lambda,\nu,\omega > 0$. Then*

$$\int_0^t (t-s)^{\nu-1}s^{\lambda-1}e^{-\omega s}ds \le Mt^{\nu-1}, \quad t > 0,$$

where

$$M = \left(\frac{\Gamma(\lambda)(1+\lambda(1+\lambda)/\nu)}{\omega^\lambda}\right)\max\{2^{1-\nu},1\}.$$

Lemma 6.2 *[31] Let $\alpha > 0$ and $0 \le \gamma < 1$. If $\gamma \le \alpha$, then $I^\alpha_{0^+}$ is bounded from $C_\gamma([0,T],E)$ into $C([0,T],E)$.*

Lemma 6.3 *[31] Let $0 < \alpha < 1$, $0 \le \gamma < 1$. If $y \in C_\gamma([0,T],E)$ and $I^{1-\alpha}_{0^+}y \in C^1_\gamma([0,T],E)$ then*

$$I^\alpha_{0^+}D^\alpha_{0^+}y(t) = y(t) - \frac{I^{1-\alpha}_{0^+}y(0)}{\Gamma(\alpha)}t^{\alpha-1}, \ \textit{for all}\ t \in (0,T].$$

Lemma 6.4 *Let h a function of $C([0,T],E)$ and y be a function in $C^1_\alpha([0,T],E)$. Then, y satisfies the following equation*

$$y(t) = b + \frac{at^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)}\int_0^t\int_0^s (s-\tau)^{\alpha-1}h(\tau)d\tau ds. \tag{6.1}$$

If and only if y satisfies the following problem

$$(\mathbf{P}^*) \quad \begin{cases} {}^{RL}D^\alpha_{0^+}y'(t) = h(t), \quad t \in (0,T], \\ I^{1-\alpha}_0 y'(0) = a, \\ y(0) = b, \end{cases}$$

Proof.
Let $y \in C^1_\alpha([0,T],E)$ be a solution of (6.1), we have $y(0) = b$. Next, by applying $\frac{d}{dt}$ to both sides of (6.1), we get

$$y'(t) = \frac{at^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds. \tag{6.2}$$

Applying $I_{0^+}^{1-\alpha}$ to both sides of the equation (6.2) and utilizing Remak 1.9, we get

$$I_{0^+}^{1-\alpha}y'(t) = a + I_{0^+}^{1}h(t)$$

Taking $t \longrightarrow 0$, we get

$$I_{0^+}^{1-\alpha}y'(0^+) = a.$$

Conversely, let $y \in C_\alpha^1([0,T],E)$ be a solution of problem $(\mathbf{P}^*)$. We want to prove that y is a solution of (6.1). By the definition of $C_\alpha^1([0,T],E)$, Lemma 6.2 and Definition of $I_{0^+}^{1-\alpha}$, we have $I_{0^+}^{1-\alpha}y' \in C([0,T],E)$ and $\frac{d}{dt}\left(I_{0^+}^{1-\alpha}y'(t)\right) = {}^{RL}D_{0^+}^{\alpha}y'(t) \in C_{1-\alpha}([0,T],E)$. Thus, we have

$$I_{0^+}^{1-\alpha}y' \in C_\alpha^1([0,T],E).$$

Now, applying Lemma 6.3 to obtain

$$I_{0^+}^{\alpha}D_{0^+}^{\alpha}y'(t) = y'(t) - \frac{I_{0^+}^{1-\alpha}y'(0)}{\Gamma(\alpha)}t^{\alpha-1} = I_{0^+}^{\alpha}h(t).$$

Which implies

$$y'(t) = \frac{at^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds,$$

then

$$\begin{aligned} y(t) &= y(0) + \frac{at^{\alpha}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_0^t\int_0^s (s-\tau)^{\alpha-1}h(\tau)d\tau ds \\ &= b + \frac{at^{\alpha}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_0^t\int_0^s (s-\tau)^{\alpha-1}h(\tau)d\tau ds. \end{aligned}$$

Thus, y is a solution of (6.1). □

6.3 Existence of the solutions

Let

$$B = \{y \in C_\alpha^1([0,T],E) : \|y\|_{C_\alpha^1} \leq R\}.$$

We assume the following hypotheses:

($\mathbf{H_1}$) The function $f : (0,T] \times E \times E \to E$ is continuous and for all $x, y, u, v \in E$ and $t \in (0,T]$:

$$\|f(t,x,y) - f(t,u,v)\| \leq A\|x-u\| + t^{1-\alpha}B\|y-v\|,$$

where $A,\ B \in \mathbb{R}^+$.

($\mathbf{H_2}$) For $t \in (0,T]$ and $u, v \in E$,

$$\|f(t,u,v)\| \leq a(t)\|u\| + t^{\lambda}e^{-\sigma t}b(t)\|v\|,$$

where $\sigma > 0, \lambda \geq 1-\alpha$ and $a(.),\ b(.) : [0,\infty) \to \mathbb{R}^+$ are continuous functions,

($\mathbf{H_3}$) There exist two functions $\ell, \jmath \in C([0,T],\mathbb{R}^+)$ such that for each bounded nonempty subset $\Omega \subset C^1_\alpha([0,T],E)$,

$$\gamma(f(t,\Omega(t),\Omega'(t))) \le \ell(t)\gamma(\Omega_(t)) + \jmath(t)\gamma(t^{1-\alpha}\Omega'(t)), \quad \text{for all } t \in (0,T],$$

and

$$TF'(T) + F(T) < 1,$$

where $\Omega(t) = \{y(t) : y \in \Omega\}$, $\Omega'(t) = \{y'(t) : y \in \Omega\}$ and

$$F(t) = \frac{2}{\Gamma(\alpha)}\int_0^t\int_0^s (s-\tau)^{\alpha-1}(\ell(\tau)+\jmath(\tau))d\tau ds, \quad \text{for all } t \in (0,T],$$

($\mathbf{H_4}$) $\alpha(1+\alpha)b^*C + (1+\alpha)[a^*T + b^*CT^\alpha] + a^*T^{1+\alpha} < \Gamma(2+\alpha)$,
where $a^* = \sup_{[0,T]} a(t)$ and $b^* = \sup_{[0,T]} b(t)$.

($\mathbf{H_5}$)

$$R > \frac{\|b\|\Gamma(2+\alpha) + (1+\alpha)\|a\|T^\alpha + \alpha(1+\alpha)\|a\|}{\Gamma(2+\alpha) - [\alpha(1+\alpha)b^*C + (1+\alpha)[a^*T + b^*CT^\alpha] + a^*T^{1+\alpha}]}.$$

Define the operator $N : C^1_\alpha([0,T],E) \to C^1_\alpha([0,T],E)$ by

$$Ny(t) = b + \frac{at^\alpha}{\Gamma(1+\alpha)} + \frac{1}{\Gamma(\alpha)}\int_0^t\int_0^s (s-\tau)^{\alpha-1} f(\tau,y(\tau),y'(\tau))d\tau ds,$$

and the operator $N' : C_{1-\alpha}([0,T],E) \to C_{1-\alpha}([0,T],E)$ by

$$N'y(t) = (Ny)'(t) = \frac{at^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,y(s),y'(s))ds.$$

We will give some useful lemmas to prove our main result.

Lemma 6.5 *Suppose that* ($\mathbf{H_1}$) *and* ($\mathbf{H_2}$) *hold. Then*

(1) N is continuous and bounded.

(2) NB is equicontinuous for all bounded subset B of $C^1_\alpha([0,T],E)$.

Proof. Let us prove (1), we start to prove that N is bounded. Let $y \in C^1_\alpha([0,T],E)$, from ($\mathbf{H_2}$) it is easy to deduce that $Ny \in C^1_\alpha([0,T],E)$. Using ($\mathbf{H_2}$) and Lemma 1.15, for all $y \in B_\kappa = \{y \in C^1_\alpha([0,T],E) : \|y\|_{C^1_\alpha} < \kappa\}$ and $t \in (0,T]$ we get

$$\begin{aligned}\|Ny(t)\| &\le \|b\| + \frac{\|a\|t^\alpha}{\Gamma(1+\alpha)} + \frac{1}{\Gamma(\alpha)}\int_0^t\int_0^s (s-\tau)^{\alpha-1}\|f(\tau,y(\tau),y'(\tau))\|d\tau ds \\ &\le \|b\| + \frac{\|a\|T^\alpha}{\Gamma(\alpha+1)} + \frac{a^*\|y\|_\infty}{\Gamma(\alpha)}\int_0^t\int_0^s (s-\tau)^{\alpha-1}d\tau ds \\ &+ \frac{b^*\|y'\|_\alpha}{\Gamma(\alpha)}\int_0^t\int_0^s (s-\tau)^{\alpha-1}\tau^{\lambda+\alpha-1}e^{-\sigma\tau}d\tau ds.\end{aligned}$$

So,

$$\|Ny\|_\infty \le \|b\| + \frac{\|a\|T^\alpha}{\Gamma(1+\alpha)} + \frac{a^*\kappa T^{1+\alpha}}{\Gamma(2+\alpha)} + \frac{b^*\kappa C T^\alpha}{\Gamma(1+\alpha)} = M_1. \tag{6.3}$$

And, we have alos

$$\begin{aligned}
\|t^{1-\alpha}(Ny)'(t)\| &\le \frac{\|a\|}{\Gamma(\alpha)} + \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,y(s),y'(s))\|ds \\
&\le \frac{\|a\|}{\Gamma(\alpha)} + \frac{a^*\|y\|_\infty t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\
&+ \frac{b^*\|y'\|_{C_\alpha} t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\lambda+\alpha-1}e^{-\sigma s}ds.
\end{aligned}$$

So,

$$\|(Ny)'\|_\alpha \le \frac{\|a\|}{\Gamma(\alpha)} + \frac{a^*\kappa T}{\Gamma(\alpha+1)} + \frac{b^*\kappa C}{\Gamma(\alpha)} = M_2, \tag{6.4}$$

where

$$C = \max\{1, 2^{1-\alpha}\}\Gamma(\lambda+\alpha)[1 + (\lambda+\alpha)(\lambda+\alpha+1)/\alpha]\sigma^{-(\lambda+\alpha)}.$$

From (6.3) and (6.4), we get

$$\|(Ny)\|_{C^1_\alpha} \le M = M_1 + M_2.$$

Now we will prove that N is continuous. Let $\{y_n\}_{n\in\mathbb{N}} \to y$ in $C^1_\alpha([0,T],E)$. The hypothesis $(\mathbf{H_1})$ confirm the existence of an integer $m \in \mathbb{N}$ such that, for all $n \ge m$ and $t \in (0,T]$,

$$\|f(t,y_n(t),y_n'(t)) - f(t,y(t),y'(t))\| < \frac{\Gamma(1+\alpha)}{T}\left[\frac{1+\alpha}{1+\alpha+T^\alpha}\right]\epsilon. \tag{6.5}$$

Thus, $\forall t \in (0,T]$,

$$\begin{aligned}
&\|Ny_n(t) - Ny(t)\| + t^{1-\alpha}\|(Ny_n)'(t) - (Ny)'(t)\| \\
&\quad\le \frac{1}{\Gamma(\alpha)}\int_0^t\int_0^s (s-\tau)^{\alpha-1}\|f(\tau,y_n(\tau),y_n'(\tau)) - f(\tau,y(\tau),y'(\tau))\|d\tau ds \\
&\quad+ \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,y_n(s),y_n'(s)) - f(s,y(s),y'(s))\|ds.
\end{aligned}$$

From (6.5), we conclude that for all $n \ge m$:

$$\|Ny_n - Ny\|_{C^1_\alpha} < \epsilon.$$

To prove (2), it suffices to show that NB_κ and $(NB_\kappa)'$ are equicontinuous respectively in $C([0,T],E)$ and in $C_{1-\alpha}([0,T],E)$. Let $y \in B_\kappa$ and $t_1, t_2 \in (0,T]$ with $t_1 < t_2$. First of all, we have

$$\begin{aligned}
\|N(y)(t_2) - N(y)(t_1)\| &\le \frac{a(t_2^\alpha - t_1^\alpha)}{\Gamma(\alpha+1)} \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \int_0^s (s-\tau)^{\alpha-1} \|f(\tau, y(\tau), y'(\tau))\| d\tau ds \\
&\le \frac{a(t_2^\alpha - t_1^\alpha)}{\Gamma(\alpha+1)} + \frac{a^*\kappa}{\Gamma(\alpha)} \int_{t_1}^{t_2} \int_0^s (s-\tau)^{\alpha-1} d\tau ds \\
&+ \frac{b^*\kappa}{\Gamma(\alpha)} \int_{t_1}^{t_2} \int_0^s (s-\tau)^{\alpha-1} \tau^{\lambda+\alpha-1} e^{-\sigma\tau} d\tau ds \\
&\le \frac{a + b^*\kappa C}{\Gamma(\alpha+1)} (t_2^\alpha - t_1^\alpha) + \frac{a^*\kappa}{\Gamma(\alpha+2)} (t_2^{\alpha+1} - t_1^{\alpha+1}).
\end{aligned}$$

By taking t_2 tends to t_1, the right-hand side of the last inequality approaches to 0. Therefore NB_κ is equicontinuous in $C([0,T], E)$.
And, we also have

$$\begin{aligned}
&\|t_2^{1-\alpha} N'(y)(t_2) - t_1^{1-\alpha} N'(y)(t_1)\| \le \frac{t_1^{1-\alpha}}{\Gamma(\alpha)} \\
&\times \left\| \int_0^{t_2} (t_2 - s)^{\alpha-1} f(s, y(s), y'(s)) ds - \int_0^{t_1} (t_1 - s)^{\alpha-1} f(s, y(s), y'(s)) ds \right\| \\
&\le \frac{t_1^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}] \|f(s, y(s), y'(s))\| ds \\
&+ \frac{t_2^{1-\alpha} - t_1^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_1} (t_2 - s)^{\alpha-1} \|f(s, y(s), y'(s))\| ds \\
&+ \frac{t_2^{1-\alpha}}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \|f(s, y(s), y'(s))\| ds \\
&\le \frac{a^*\kappa t_1^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}] ds \\
&+ \frac{b^*\kappa t_1^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}] s^{\lambda+\alpha-1} ds \\
&+ \frac{a^*\kappa (t_2^{1-\alpha} - t_2^{1-\alpha})}{\Gamma(\alpha)} \int_0^{t_1} (t_2 - s)^{\alpha-1} ds \\
&+ \frac{b^*\kappa (t_2^{1-\alpha} - t_2^{1-\alpha})}{\Gamma(\alpha)} \int_0^{t_1} (t_2 - s)^{\alpha-1} s^{\lambda+\alpha-1} ds \\
&+ \frac{a^*\kappa t_2^{1-\alpha}}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds + \frac{a^*\kappa t_2^{1-\alpha}}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} s^{\lambda+\alpha-1} ds \\
&\le \frac{a^*\kappa T^{1-\alpha} + b^*\kappa T^\lambda}{\Gamma(\alpha+1)} [t_1^\alpha - t_2^\alpha + (t_2 - t_1)^\alpha]
\end{aligned}$$

$$+\frac{a^*\kappa T^{\alpha}+b^*\kappa T^{\lambda+2\alpha-1}}{\Gamma(\alpha+1)}(t_2^{1-\alpha}-t_2^{1-\alpha})+\frac{a^*\kappa T^{1-\alpha}+b^*\kappa T^{\lambda}}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha}.$$

By taking t_2 tends to t_1, the right-hand side of the last inequality approaches to 0, and hence $(NB_\kappa)'$ is equicontinuous in $C_{1-\alpha}([0,T],E)$. □

To present our main result, we recall here some definitions and notations. We denote by γ_α the Kuratowski measure of non-compactness defined on any bounded subset of $C^1_\alpha([0,T],E)$. We can easily show the following inequality

$$\gamma_\alpha(D)\leq\sup_{t\in[0,T]}\gamma(D(t))+\sup_{t\in(0,T]}\gamma(D'_\alpha(t))\leq 2\gamma_\alpha(D), \tag{6.6}$$

where $D(t)=\{y(t):y\in D\}$, $D'_\alpha(t)=\{y'_\alpha(t):y\in D\}$ and D is a bounded, equicontinuous subset of $C^1_\alpha([0,T],E)$.

Theorem 6.6 *Suppose that conditions* $(\mathbf{H_1})-(\mathbf{H_5})$ *are valid. Then Problem* $(\mathbf{P})$ *has at least one solution.*

Proof. Using Lemma 6.4, it is clear that the fixed points of the operator N, defined previously, are solutions of Problem $(\mathbf{P})$. We will verify that N satisfies the assumptions of Mönch fixed point theorem (Theorem 1.33).

First, we show that N is well-defined from B to B, indeed: let $y\in B$. By using the condition $(\mathbf{H_2})$ and after some calculations, we get

$$\begin{aligned}\|Ny(t)\|+\|t^{1-\alpha}(Ny)'(t)\|&\leq\|b\|+\frac{\|a\|T^{\alpha}}{\Gamma(\alpha+1)}+\frac{\|a\|}{\Gamma(\alpha)}\\&+\left[\frac{\alpha(\alpha+1)b^*C+(\alpha+1)[a^*T+b^*CT^{\alpha}]+a^*T^{\alpha+1}}{\Gamma(\alpha+2)}\right]R.\end{aligned}$$

From $(\mathbf{H_4})$ and the inequality $(\mathbf{H_5})$, we obtain

$$\forall y\in B:\|Ny\|_{C^1_\alpha}<R.$$

Note that B is bounded, convex and closed subset of $C^1_\alpha([0,T],E)$ and N is continuous on B. Next we need to prove the following implication

$$V\subset\overline{conv}\{N(V)\cup\{0\}\}\Longrightarrow\gamma_\alpha(V)=0,\ \text{for any } V\subset B.$$

Let $V\subset B$ such that $V\subset\overline{conv}\{N(V)\cup\{0\}\}$. From Lemmas 1.32, 6.5, $(\mathbf{H_3})$, the inequality (6.6) and the previous steps, we arrive at

$$\begin{aligned}\gamma(NV(t))+\gamma(t^{1-\alpha}(NV)'(t))&\leq\frac{1}{\Gamma(\alpha)}\int_0^T\int_0^s(s-\tau)^{\alpha-1}\ell(\tau)\gamma(NV(\tau))d\tau ds\\&+\frac{1}{\Gamma(\alpha)}\int_0^T\int_0^s(s-\tau)^{\alpha-1}\jmath(\tau)\gamma((NV)'_\alpha(\tau))d\tau ds\\&+\frac{T}{\Gamma(\alpha)}\int_0^T(t-s)^{\alpha-1}\ell(s)\gamma(NV(s))ds+\frac{T}{\Gamma(\alpha)}\int_0^T t-s)^{\alpha-1}\jmath(s)\gamma((NV)'_\alpha(s))ds\end{aligned}$$

Thus,

$$\gamma_\alpha(NV) \leq [F(T) + TF'(T)]\,\gamma_\alpha(NV).$$

By condition $(\mathbf{H_3})$. We get $\gamma_\alpha(N(V)) = 0$, that is $\gamma_\alpha(V) = 0$. From the theorem 1.33, N has a fixed point $y \in B$ which is a solution of Problem $(\mathbf{P})$. □

6.4 Example

We shall now consider the following fractional-ordinary differential equation

$$^{RL}D^{\frac{1}{2}}y'(t) = \left(\frac{\sin(t)}{20(1+t^2)}y_n(t) + \frac{\sqrt{t}}{20(1+t^2)e^{5\sqrt{2}t}}y'_n(t)\right)_{n=1}^{\infty}, \quad t \in (0,1], \tag{6.7}$$

$$I_{0^+}^{\frac{1}{2}}y'(0^+) = (1,0,\ldots,0,\ldots), \tag{6.8}$$

$$y(0) = (1,0,\ldots,0,\ldots). \tag{6.9}$$

Let

$$E = \{(y_1, y_2, \ldots, y_n, \ldots) : \sup_n |y_n| < \infty\},$$

with the norm $\|y\| = \sup_n |y_n|$, then $(E, \|.\|)$ consists a Banach space, by comparing with the equations of Problem $(\mathbf{P})$, we notice that

$$\alpha = \lambda = \frac{1}{2} \text{ and } f(t, y(t), y'(t)) = (f(t, y_1(t), y'_1(t)), \ldots, f(t, y_n(t), y'_n(t)), \ldots),$$

where

$$f(t, y_n(t), y'_n(t)) = \frac{\sin(t)}{20(1+t^2)}y_n(t) + \frac{\sqrt{t}}{20(1+t^2)e^{5\sqrt{2}t}}y'_n(t), n \in \mathbb{N}^*.$$

Clear that $f : (0,1] \times E \times E \to E$ is continuous and

$$\|f(t, y(t), y'(t))\| \leq \frac{1}{20(1+t^2)}\|y(t)\| + \frac{\sqrt{t}}{20(1+t^2)e^{5\sqrt{2}t}}\|y'(t)\|.$$

Hence $(\mathbf{H_1})$ and $(\mathbf{H_2})$ are satisfied. Next, For any bounded set $B \subset C^1_\alpha([0,1], E)$, we have

$$f(t, B(t), B'(t)) = \frac{\sin(t)}{20(1+t^2)}B(t) + \frac{\sqrt{t}}{20(1+t^2)}B'(t).$$

Then

$$\gamma(f(t, B(t), B'(t)) \leq \frac{\sin(t)}{20(1+t^2)}\gamma(B(t)) + \frac{1}{20(1+t^2)}\gamma(\sqrt{t}B'(t)),$$

since $F(1) + F'(1) \leq \frac{1}{5\Gamma(2.5)} + \frac{1}{5\Gamma(1.5)} \simeq 0.3760 < 1$. So, $(\mathbf{H_3})$ holds. We have $C = 1$, thus

$$\alpha(1+\alpha)b^*C + (1+\alpha)[a^*T + b^*CT^\alpha] + a^*T^{\alpha+1} = 0.475 < \Gamma(2+\alpha) = 1.3293.$$

So, $(\mathbf{H_4})$ holds. Therefore, Theorem 6.6 ensures that problem (6.7)-(6.9) has a solution.

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