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Intitulée

Etude De La Stabilité De Certains Problèmes D'Evolution Linéaire Par Analyse Spectrale.

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Abstract

Title : Stabilization of some linear evolution problems with spectral analysis.

This thesis is devoted to the study of the stabilisation of some linear evolution problems, In particular, the the Lamé system under some dissipations of fractional derivative type. First, we consider a Lamé system damped by a fractional boundary feedback of Neumann type, we prove stabilization by using multipliers method and Rellich type relation combined with the frequency domain method. Next, we consider a Lamé system with internal fractional delay and a dissipative damping, we prove that the frictional damping is strong enough to uniformly stabilize the system even in the presence of time delay. Lastly, our interest is to analyse the asymptotic behaviour of a Lamé system with an internal fractional delay and a boundary damping of Neumann type, we introduce a Lyapunov functional that gives the exponential decay.

Keywords:

Lamé system, Fractional feedback, Polynomial stability, Semigroup theory, Fractional delay term, Uniform stability, Optimal decay rate, Bessel functions.

Résumé

Titre : Etude de Stabilité de certains problèmes d'évolution linéaires par analyse spectrale.

Cette thèse est consacrée à l'étude de la stabilisation de certains problèmes d'évolution linéaires, en particulier le système Lamé en présence de termes dissipatifs de type fractionnaires. Nous considérons, d'abord, un système de Lamé avec un contrôle au frontière de type fractionnaire au sens de Caputo, nous établissons un taux de décroissance polynomiale de l'énergie du système. Ensuite, nous nous intéressons à l'étude de la stabilisation du système de Lamé avec un retard interne de type fractionnaire et un terme dissipatif, nous concentrons notre étude sur le comportement asymptotique des solutions où le terme dissipatif est suffisamment fort pour stabiliser le système, même en présence d'un retard. Enfin, notre intérêt est d'analyser le comportement asymptotique du système de Lamé avec un retard fractionnaire interne et un terme dissipatif au frontière, nous introduisons une fonction de Lyapunov qui donne la décroissance exponentielle.

Mots Clés:

Système de Lamé, Contrôle aux limites du type fractionnaire, Stabilité forte, Stabilité polynomiale, théorie des semi-groupes, Taux de décroissance optimale, Fonctions de Bessel.

Contents

Introduction

Many physical phenomena in nature can be described by partial differential equations and the control of such equations is a quite recent and very active field of investigation. The aim of this dissertation is to survey several issues related to the study of the Lamé system under fractional controls.

The problem of well-posedness and stability for elasticity systems in general, and the Lamé system in particular, has attracted considerable attention in recent years, where diverse types of dissipative mechanisms have been introduced and several stability and boundedness results have been obtained. The main problem concerning the stability of solutions is to determine and estimate the best decay rate for solutions.

Real progress has been realized during the last three decades, Let us recall here some known results addressing problems of existence, uniqueness and asymptotic behavior of solutions.

In particular, in the works of Guesmia **[23]**, **[24]**, considering the problem of observability, exact controllability and stability of general elasticity systems with variable coeffcients depending on both time and space variables in bounded domains, the results hold under linear or nonlinear, global or local feedbacks, and they generalize and improve, in some cases, the decay rate obtained by Alabau and Komornik **[4]**. In **[28]**, Lagnese proved some uniform stability results of elasticity systems with linear feedback under some technical assumptions on the elasticity tensor. In particular, these results do not hold in the linear homogeneous isotropic case for which the elasticity tensor depends on two parameters called Lamé constants. In **[30]**, Lagnese obtained uniform stability estimates for linear homogeneous isotropic and bidimensional elasticity systems under a linear boundary feedback. Martinez **[36]** generalized the results of Komornik **[30]** to the case of elasticity systems of cubic crystals under a nonlinear boundary feedback. For these systems, the elasticity tensor depends on three parameters.

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ of class C^2. We assume that $\Gamma = \Gamma_0 \cup \Gamma_1$, where Γ_0 and Γ_1 are closed subsets of Γ with $\Gamma = \Gamma_0 \cap \Gamma_1 = \emptyset$. ν is the unit outward normal to Γ and $u = (u_1, u_2, ..., u_n)^T$. μ, λ are Lamé constants which satisfy the conditions: $\mu > 0$, and $\mu + \lambda \geq 0$.. $u = (u_1, u_2, ..., u_n)^T$.

This thesis focuses on fractional calculus which has been applied successfully in various areas to modify many existing models of physical processes such as heat conduction, diffusion, viscoelasticity, wave propagation, electronics etc. Caputo and Mainardi [10] have established the relation between fractional derivative and theory of viscoelasticity. The feedback under consideration here is of fractional type and is described by the following fractional derivative:

$$\partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s)\, ds, \qquad \eta \geq 0. \tag{1}$$

The order of the derivative is between 0 and 1. In addition to being nonlocal, fractional derivative involves singular and non-integrable kernels ($t^\alpha, 0 < \alpha < 1$). It has been shown (see **[12]**) that, as ∂_t, the fractional derivative ∂_t^α forces the system to become dissipative and the solution to approach the equilibrium state.

Furthermore, This thesis intended also to state the well-posedness result for the Lamé system using the theory of semigroups. Linear semigroup theory received considerable attention in the 1930s as a new approach in the study of linear partial differential equations. Note that the linear semigroup theory has been later developed as an independent theory, with applications in some other fields, such as ergodic theory, the theory of Markov processes, etc.

Outline: This dissertation is split into four chapters.

CHAPTER 1: PRELIMINARIES

In this chapter, we present some well known results, definitions, properties and theorems that are used throughout the dissertation. Firstly, we recall some basic knowledge on linear operators and semigroups without proofs, including some theorems on strong, exponential and polynomial stability of C_0-semigroups. Next, we display a brief historical introduction to fractional derivatives and we define the fractional derivative operator in the sense of Caputo. After that, we introduce some preliminary facts on the Bessel functions and lastly, we define two different types of geometric conditions.

CHAPTER 2: ASYMPTOTIC STABILITY FOR THE LAMÉ SYSTEM WITH FRACTIONAL BOUNDARY DAMPING

This Chapter is devoted to the study of following the Lamé system damped by a fractional boundary feedback of Neumann type:

$$(P1) \qquad \begin{cases} u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla(\mathrm{div}\ \ u) = 0 & \text{in } \Omega \times (0,+\infty) \\ u = 0 & \text{in } \Gamma_0 \times (0,+\infty) \\ \mu\dfrac{\partial u}{\partial \nu} + (\mu+\lambda)(\mathrm{div}\ \ u)\nu = -\gamma \partial_t^{\alpha,\eta} u & \text{in } \Gamma_1 \times (0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & \text{in } \Omega \end{cases}$$

where γ is a positive constant.

We start by stating the well-posedness result for problem $(P1)$ using the theory of semigroups. Next, we show the lack of exponential stability by spectral analysis. Moreover, we prove that the stability of our system holds with fractional damping, indeed, regarding the strong asymptotic stability of solutions, we use a recent result of Borichev and Tomilov which relate resolvent bounds and decay rates. Lastly, we obtain an almost optimal polynomial energy decay rate depending on parameter α using multipliers method and Rellich type relation combined with the frequency domain method.

CHAPTER 3: STABILITY RESULT OF THE LAMÉ SYSTEM WITH A DELAY TERM IN THE INTERNAL FRACTIONAL FEEDBACK:

In this Chapter, we consider the initial boundary value problem for the Lamé system with a delay term in the internal fractional feedback. The system is given by:

$$(P2) \qquad \begin{cases} u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla(\text{div } u) + a_1\partial_t^{\alpha,\eta}u(x,t-\tau) + a_2 u_t(x,t) = 0 & \text{in } \Omega\times(0,+\infty), \\ u = 0 & \text{in } \Gamma\times(0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau) & \text{in } \Omega\times(0,\tau), \end{cases}$$

where $a_1 > 0$, $a_2 > 0$ and the constant $\tau > 0$ is the time delay.

We state a well-posedness result for problem $(P2)$ by means of the semigroup theory under a certain condition between the weight of the delay term in the fractional feedback and the weight of the term without delay, then, we prove the strong asymptotic stability of solutions. Furthermore, we obtain the exponential stability using the classical theorem of Gearhart, Huang and Pruss.

CHAPTER 4: EXPONENTIAL DECAY FOR THE LAMÉ SYSTEM WITH FRACTIONAL TIME DELAY AND BOUNDARY FEEDBACK:

This chapter focuses on the study of well-posedness and boundary stabilization of the following Lamé system :

$$(P3)\qquad \begin{cases} u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla(\text{div } u) + a_1\partial_t^{\alpha,\eta}u(x,t-\tau) = 0 & \text{in } \Omega\times(0,+\infty),\\ u = 0 & \text{in } \Gamma_0\times(0,+\infty),\\ \mu\dfrac{\partial u}{\partial\nu} + (\mu+\lambda)(\text{div } u)\nu = -a_2u_t(x,t) & \text{in } \Gamma_1\times(0,+\infty),\\ u(x,0) = u_0(x),\quad u_t(x,0) = u_1(x) & \text{in } \Omega,\\ u_t(x,t-\tau) = f_0(x,t-\tau) & \text{in } \Omega\times(0,\tau). \end{cases}$$

where $a_1 > 0$, $a_2 > 0$ and the constant $\tau > 0$ is the time delay.

The idea here is to prove that a damping with time delay does not destroy the stability if there is another boundary dissipative damping in which contrasts appropriately with the previous one. (i.e., by giving the control in the feedback form $-a_2u_t(x,t,)\ \ x\in\Gamma_1,\ \ t>0$). We will show that system $(P3)$ is exponentially stable for a_1 sufficiently small.

First, we deal with the well-posedness result of the problem $(P3)$ using the semigroup theory. Moreover, we obtain exponential stability results by constructing an appropriate Lyapunov functional as in [**5**].

Chapter 1

PRELIMINARIES

1.1 Linear operators

Definition 1.1.1 *Let X and Y be two Banach spaces. A linear mapping: $\mathcal{A} : D(\mathcal{A})(\subset X) \to Y$ is called a linear operator. The $D(\mathcal{A}) \subset X$ is called the domain of $\mathcal{A}$ and $\mathcal{R}(\mathcal{A}) \subset Y$ is called the range of $\mathcal{A}$:*

$$\mathcal{R}(\mathcal{A}) = \{\mathcal{A}x \mid x \in D(\mathcal{A})\}.$$

$\mathcal{A}$ is said to be invertible (or injective) if $\mathcal{A}x = 0$ if and only if $x = 0$; $\mathcal{A}$ is said to be onto (or surjective) if $\mathcal{R}(\mathcal{A}) = Y$; $\mathcal{A}$ is said to be densely defined if $\overline{D(\mathcal{A})} = X$.

Definition 1.1.2 *A linear operator $\mathcal{A}$ is said to be closed if for any $x_n \in D(\mathcal{A}),\ n \geq 1$,*

$$x_n \to x,\ \ \mathcal{A}x_n \to y,\ \ \text{as } n \to \infty,$$

it must have $x \in D(\mathcal{A})$ and $\mathcal{A}x = y$. $\mathcal{A}$ is said to be bounded if $D(\mathcal{A}) = X$ and $\mathcal{A}$ maps a bounded set of X into a bounded set of Y. A linear operator is bounded if and only if it is continuous, that is,

$$x_n \to x_0 \in X \Longrightarrow \mathcal{A}x_n \to \mathcal{A}x_0 \in Y$$

for any $x_n \subset X$.

Obviously, any operator which has bounded inverse must be closed. All the bounded operators from X to Y are denoted by $\mathcal{L}(X, Y)$. In particular, when $X = Y,\ \mathcal{L}(X, Y)$ is abbreviated as $\mathcal{L}(X)$.

Theorem 1.1.1 *Let X and Y be Banach spaces. Then $\mathcal{L}(X, Y)$ is a Banach space with the norm*

$$\|\mathcal{A}\| = \sup\{\|\mathcal{A}x\| \mid x \in X,\ \|x\| = 1\}.$$

Definition 1.1.3 *Let X be a Banach space. If $Y = \mathbb{R}$ or $Y = \mathbb{C}$, then the operator in $\mathcal{L}(X,Y)$ is called a linear functional on X. A bounded functional is also denoted by f.*

By Theorem 1.1.1, all linear bounded functionals on X consist of a Banach space which is called the dual of the space X, denoted by X^*.

A bounded operator is called compact operator if $\mathcal{A}$ maps any bounded set into a relatively compact set which is a compact set but not necessarily closed. For a closed operator $\mathcal{A}$, we can define the graph space $[D(\mathcal{A})]$ where the norm is defined by

$$\|x\|_{[D(\mathcal{A})]} = \|x\| + \|\mathcal{A}x\|, \quad \forall x \in D(\mathcal{A}). \tag{1.1}$$

Theorem 1.1.2 [Open mapping theorem]
Let X and Y be Banach spaces and let $\mathcal{A}$ be a bounded operator from X to Y. If $\mathcal{R}(\mathcal{A}) = Y$, then $\mathcal{A}$ maps an open set of X into an open set of Y.

Theorem 1.1.3 [Closed graph theorem]
Suppose that $\mathcal{A}$ is a closed operator in a Banach space X. Then $\mathcal{A}$ must be bounded provided $D(\mathcal{A}) = X$.

Theorem 1.1.4 [Lax Milgram theorem]
Let $a(x,y)$ be a bilinear form, that is, it is linear in x and conjugate linear in y, and satisfies

- *there is an $M > 0$ such that $|a(x,y)| \le M\|x\|\|y\|$ for all $x, y \in H$;*
- *there is a $\delta > 0$ such that for any $x \in H$, $|a(x,x)| \ge \delta\|x\|^2$.*

Then there exists a unique $\mathcal{A} \in \mathcal{L}(H)$ which is bounded invertible and satisfies

$$a(x,y) = \langle x, Ay\rangle, \ \forall x, y \in H.$$

Definition 1.1.4 *A linear operator in a Hilbert space is said to be symmetric if*

$$\mathcal{A}^* = \mathcal{A} \text{ on } D(\mathcal{A}) \text{ and } D(\mathcal{A}^*) \supseteq D(\mathcal{A})$$

A symmetric operator is said to be self-adjoint, if $\mathcal{A}^* = \mathcal{A}$.

For bounded operators, the symmetric and self-adjoint are the same. But for unbounded operators, they are different.

Definition 1.1.5 *A linear operator $\mathcal{B}$ in a Hilbert space H is said to be A-bounded if*

- *$D(\mathcal{B}) \supset D(\mathcal{A})$, and*
- *there are $a, b > 0$ such that*

$$\|\mathcal{B}x\| \le a\|\mathcal{A}x\| + b\|x\|, \ \forall x \in D(\mathcal{A}).$$

Definition 1.1.6 *Let $\mathcal{A} \in \mathcal{L}(H)$ be a self-adjoint operator in a Hilbert space H. $\mathcal{A}$ is said to be positive if*

$$\langle \mathcal{A}x, x\rangle \geq 0, \ \forall x \in H. \tag{1.2}$$

A positive operator is denoted by $\mathcal{A} \geq 0$; $\mathcal{A}$ is said to be positive definite if the equality in 1.2 holds true only if $x = 0$, which is denoted by $\mathcal{A} > 0$; A positive operator $\mathcal{A}$ is said to be strictly positive if there exists an $m > 0$ such that

$$\langle \mathcal{A}x, x\rangle \geq m\|x\|^2, \ \forall x \in D(\mathcal{A}). \tag{1.3}$$

1.2 The spectrum of linear operators

Definition 1.2.1 *Suppose that X is a Banach space and $\mathcal{A} : D(\mathcal{A})(\subset X) \to X$ is a linear operator. The resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$ is an open set in the complex plane, which is defined by*

$$\rho(\mathcal{A}) = \left\{\lambda \in \mathbb{C} \mid (\lambda - \mathcal{A})^{-1} \in \mathcal{L}(X)\right\}.$$

when $\lambda \in \rho(\mathcal{A})$, the operator $R(\lambda, \mathcal{A}) = (\lambda - \mathcal{A})^{-1}$ is called the resolvent of $\mathcal{A}$. If one of resolvents is compact, then any of the resolvents must be compact. This comes from the following resolvent formula:

$$(\lambda - \mathcal{A})^{-1} - (\mu - \mathcal{A})^{-1} = (\mu - \lambda)(\lambda - \mathcal{A})^{-1}(\mu - \mathcal{A})^{-1}, \ \forall \lambda, \mu \in \rho(\mathcal{A}).$$

The spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ is the supplement set of the resolvent set in the complex plane, that is,

$$\sigma(\mathcal{A}) = \mathbb{C} \setminus \rho(\mathcal{A}).$$

Generally, the spectrum $\sigma(\mathcal{A})$ is decomposed into three parts:

$$\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) \cup \sigma_c(\mathcal{A}) \cup \sigma_r(\mathcal{A})$$

where

- *the point spectrum*

$$\sigma_p(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid \text{there exists a } 0 \neq x \in X \text{ so that } \mathcal{A}x = \lambda x\};$$

- *the continuous spectrum*

$$\sigma_c(\mathcal{A}) = \left\{\lambda \in \mathbb{C} \mid (\lambda - \mathcal{A}) \text{ is invertible and } \overline{\mathcal{R}(\lambda - \mathcal{A})} = X\right\};$$

- *the residual spectrum*

$$\sigma_r(\mathcal{A}) = \left\{\lambda \in \mathbb{C} \mid (\lambda - \mathcal{A}) \text{ is invertible and } \overline{\mathcal{R}(\lambda - \mathcal{A})} \neq X\right\};$$

When $\lambda \in \sigma_p(\mathcal{A})$, any nonzero vector x satisfying $\mathcal{A}x = \lambda x$ is said to be an eigenvector (it is also called eigenfunction if the space is the function space) of $\mathcal{A}$. For a matrix in $\mathbb{C}^n$, the spectrum is just the set of eigenvalues.

1.3 Semigroups of linear operators

Definition 1.3.1 *Semigroup theory is aiming to solve the following linear differential equation in Banach space X:*

$$\begin{cases} \dot{u}(t) = \mathcal{A}u(t), \ t > 0, \\ u(0) = x \in X, \end{cases} \tag{1.4}$$

where $\mathcal{A} : D(\mathcal{A})(\subset X) \to X$ is a linear operator.

Eq (1.4) is said to be well-posed (for bounded A) If:

- *for any initial value $x \in D(\mathcal{A}) = X$, there exists a solution $u(x,t)$ to (1.4) which is differentiable for $t > 0$, continuous at $t = 0$ and $u(x,t)$ satisfies (1.4) for $t > 0$,*
- *$u(x,t)$ depends continuously on the initial condition x, that is:*

$$x \to 0 \ \textit{implies} \ u(x,t) \to 0 \ \textit{for each} \ t > 0;$$

- *$u(x,t)$ is unique for each $x \in D(\mathcal{A}) = X$.*

We can then define an operator $T(t)$ by $T(t)x = u(x,t)$ for each $t \geq 0$. From the existence and uniqueness of the solution $u(x,t)$, we know that $T(t), t \geq 0$ is well defined on X.

Definition 1.3.2 *Let X be a Banach space and $T(t) : X \to X$ be a family of linear bounded operators, for $t \geq 0$, $T(t)$ is called a semigroup of linear bounded operators, or simply a semigroup, on X if*

- *$T(0) = I$;*
- *$T(t+s) = T(t)T(s), \ \forall t \geq 0, \ s \geq 0$*

A semigroup $T(t)$ is called uniformly continuous if

$$\lim_{t\to 0} \|T(t) - I\| = 0,$$

and is called strongly continuous, (or C_0-semigroup for short), if

$$\lim_{t\to 0} T(t)x - x = 0, \ \forall x \in X$$

Definition 1.3.3 *Let $T(t)$ be a C_0-semigroup on a Banach space X. The operator $\mathcal{A}$ is defined as*

$$\begin{cases} \mathcal{A}x = \lim\limits_{t\longrightarrow 0} \dfrac{T(t)x - x}{t}, \qquad \forall x \in D(\mathcal{A}), \\ D(\mathcal{A}) = \left\{ x \in X \mid \lim\limits_{t\longrightarrow 0} \dfrac{T(t)x - x}{t} \qquad \textit{exists} \right\} \end{cases}$$

is called the infinitesimal generator of the C_0-semigroup $T(t)$.

Theorem 1.3.1 *Let X be a Banach space. For any bounded linear operator $\mathcal{A}$ on X, $T(t) = e^{\mathcal{A}t}$ is a uniformly continuous semigroup and $\mathcal{A}$ is the infinitesimal generator of $T(t)$ with $D(\mathcal{A}) = X$.*

Theorem 1.3.2 *Let $T(t)$ be a C_0-semigroup on a Banach space X, then the following holds*

- *There exists constants $M > 1$ and $\omega \geq 0$ such that*
$$\|T(t)\| \leq Me^{\omega t}, \ \forall t \geq 0$$
- *Suppose that $\mathcal{A}$ is the generator of $T(t)$. Then*
$$\{\lambda \in \mathbb{C} \mid Re(\lambda) > \omega\} \subset \rho(\mathcal{A}).$$
- *In addition, if $Re(\lambda) > \omega$, then*
$$\mathcal{R}(\lambda, \mathcal{A})x = (\lambda - \mathcal{A})^{-1}x = \int_0^\infty e^{-\lambda t}T(t)x dt, \ \forall x \in X.$$
- *$T(t)$ is strongly continuous on X. i.e. for any $x \in X$, the map $t \to T(t)x$ is continuous.*

Theorem 1.3.3 *Let $\mathcal{A}$ be the generator of a C_0-semigroup $T(t)$ on a Banach space X. we have the following*

- *$D(\mathcal{A})$ is dense in X*
- *$\mathcal{A}$ is a closed operator.*
- *For any $n \geq 1$, $D(\mathcal{A}^n)$ is dense in X. The set $D = \bigcap\limits_{n=1}^{\infty} D(\mathcal{A}^n)$ is also dense in X and is invariant under $T(t)$. i.e. for $x \in D$, $T(t)x \in D$ for $t \geq 0$. Moreover, if we define $D^\infty = \{x \in X | t \to T(t)x \in C^\infty\}$. then we have $D = D^\infty$*

Theorem 1.3.4 [Hille-Yosida]
Let X be a Banach space and let $\mathcal{A}$ be a linear (not necessirely bounded) operator in X. Then, $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions $T(t)$ on X, if and only if

- *$\mathcal{A}$ is closed and $D(\mathcal{A})$ is dense in X*
- *There exist positive constants M and ω verifying the property: for all $\lambda > \omega$, $\lambda \in \rho(\mathcal{A})$, the following holds*
$$\|\mathcal{R}(\lambda, \mathcal{A})^n\| \leq \frac{M}{(\lambda - \omega)^n}, \ \ n = 1, 2, ...$$

Definition 1.3.4 *Let $T(t)$ be a C_0-semigroup on a Banach space X and let $M \geq 1$ and $\omega \geq 0$.*

If $\omega = 0$, then we have $\|T(t)\| \leq M$ for $t \geq 0$ and $T(t)$ is called uniformly bounded.

Moreover, if we have $M = 1$, then $T(t)$ is called a contraction.

Corollary 1.3.1 *Let X be a Banach space and let $\mathcal{A}$ be a linear (not necessirely bounded) operator in X. Then, $\mathcal{A}$ is the infinitesimal generator of the C_0-semigroup of contractions $T(t)$ on X, if and only if the following holds.*

- *$\mathcal{A}$ is closed and $D(\mathcal{A})$ is dense in X*
- *For any $\lambda > 0$, $\lambda \in \rho(\mathcal{A})$ and*
$$\|\mathcal{R}(\lambda, \mathcal{A})\| \leq \frac{1}{\lambda}$$
.

Definition 1.3.5 *Let X be a Banach space and let $F(x)$ be the duality set. A linear operator $\mathcal{A}$ in X is said to be dissipative if for every $x \in D(\mathcal{A})$ there is an $x^* \in F(x)$ such that*

$$Re\langle \mathcal{A}x, x^*\rangle \leq 0$$

Definition 1.3.6 *A linear operator $\mathcal{A}$ in a Banach space X is called m-dissipative if $\mathcal{A}$ is dissipative and $\mathcal{R}(\lambda - \mathcal{A}) = X$, for some $\lambda > 0$.*

Remark 1.3.1 *In a Hilbert space H, the dissipativity of $\mathcal{A}$ simply means that*

$$Re\langle \mathcal{A}x, x\rangle \leq 0, \ \forall x \in D(\mathcal{A}).$$

Theorem 1.3.5 [Lümer-Phillips]
Let $\mathcal{A}$ be a linear operator in a Banach space X. Then $\mathcal{A}$ generates a C_0-semigroup of contractions on X if and only if

- *$\overline{D(\mathcal{A})} = X$.*
- *$\mathcal{A}$ is dissipative.*

Remark 1.3.2 *When X is reflexive, the condition $\overline{D(\mathcal{A})} = X$ can be removed in the Lümer-Phillips theorem.*

1.4 Stability of C_0-semigroups.

Definition 1.4.1 *Let $T(t)$ be a C_0-semigroup on a Banach space X.*

- *$T(t)$ is said to be exponentially stable, if there exist two positive constants $M, \omega > 0$ such that*
$$\|T(t)\| \leq Me^{-\omega t}, \ \forall t \geq 0.$$
- *$T(t)$ is said to be strongly or asymptotically stable, if*
$$\lim_{t\to+\infty} \|T(t)x\| = 0 \ \forall x \in X.$$

- *$T(t)$ is said to be weakly stable, if*

$$\langle T(t)x, y\rangle \to 0 \ \text{as} \ t \to \infty, \ \forall x \in X, \ y \in X^*.$$

- *$T(t)$ is said to be Polynomially stable if there exist two positive constants C and α such that*

$$\|T(t)\| \leq Ct^{-\alpha} \ \ \forall t > 0, \ \forall x \in X.$$

Theorem 1.4.1 [Spectral mapping theorem]
Let $T(t)$ be a C_0-semigroup on a Banach space X and $\mathcal{A}$ be its infinitesimal generator. Then

$$e^{t\sigma_p(\mathcal{A})} \subset \sigma_p(T(t)) \subset e^{t\sigma(\mathcal{A})} \cup \{0\}.$$

More precisely, if $\lambda \in \sigma_p(\mathcal{A})$. then $e^{\lambda t} \in \sigma_p(T(t))$. and if $e^{\lambda t} \in \sigma_p(T(t))$ then there exists an integer k such that $\lambda_k = \lambda + 2\pi ik/t \in \sigma_p(\mathcal{A})$.

Theorem 1.4.2 *Let $T(t)$ be a C_0-semigroup on a Banach space with generator $\mathcal{A}$. Then*

$$e^{t\sigma(\mathcal{A})} \subset \sigma(T(t)).$$

Proposition 1.4.1 *Let $X = H$ be a Hilbert space. Suppose that $T(t)$ is a weakly stable C_0-semigroup on H. i.e. $\langle T(t)x, y\rangle \to 0$ as $t \to \infty$ for all $x, y \in H$. If its infinitesimal generator $\mathcal{A}$ has compact resolvent, then $T(t)$ is asymptotically stable. i.e. $\|T(t)z\| \to 0$ as $t \to \infty$ for all $z \in H$.*

Theorem 1.4.3 *Let $T(t)$ be a uniformly bounded C_0-semigroup on a Banach space X and let $\mathcal{A}$ be its generator. Then*

- *If $T(t)$ is asymptotically stable then $\sigma(\mathcal{A}) \cap i\mathbb{R} \subset \sigma_c(\mathcal{A})$.*
- *If $\sigma(\mathcal{A}) \cap i\mathbb{R} \subset \sigma_c(\mathcal{A})$. and $\sigma_c(\mathcal{A})$ is countable. then $T(t)$ is asymptotically stable.*
- *If $\mathcal{R}(\lambda, \mathcal{A})$ is compact, then $T(t)$ is asymptotically stable if and only if $Re\lambda < 0$ for all $\lambda \in \sigma(\mathcal{A})$.*

Corollary 1.4.1 *Let $T(t)$ be a C_0-semigroup on a Banach space X and $\mathcal{A}$ be its generator. Suppose that $\sigma(\mathcal{A}) \cap i\mathbb{R} \subset \sigma_c(\mathcal{A})$ and $\sigma_c(\mathcal{A})$ is countable. then $T(t)$ is weakly stable if and only if $T(t)$ is asymptotically stable.*

Theorem 1.4.4 *Let $\mathcal{A}$ be the infinitesimal generator of a C_0-semigroup $T(t)$ on a Banach space X. If for some $p \geq 1$*

$$\int_0^\infty \|T(t)x\|^p dt < \infty, \ \ \text{for every} \ x \in X,$$

then $T(t)$ is exponentially stable.

Remark 1.4.1 *We say that $T(t)$ is exponentially asymptotically stable if for every $x \in X$, there exist $M_x, \omega_x > 0$ depending on x such that*

$$\|T(t)x\| \leq M_x e^{-\omega_x t}$$

Theorem 1.4.4 shows that a linear C_0-semigroup is exponentially asymptotically stable if and only if it is exponentially stable.

Theorem 1.4.5 *Let $T(t)$ be a C_0-semigroup with infinitesimal generator $\mathcal{A}$. The following statements are equivalent.*

- *$T(t)$ is exponentially stable, i.e. $\|T(t)x\| \leq Me^{-\omega t}$. for $M \geq 1$. $\omega > 0$;*
- $\lim\limits_{t\to\infty} \|T(t)\| = 0$
- *there exists a $t_0 > 0$ such that*
$$\|T(t_0)\| < 1.$$

We assume, that $X = H$ is a Hilbert space with the inner product $\langle ., . \rangle$ and the induced norm $\|.\|$. Recall that if $\mathcal{A}$ generates a C_0-semigroup $T(t)$ on H with $\|T(t)\| \leq Me^{wt}$, then for all λ with $Re\lambda > \omega$,

$$\mathcal{R}(\lambda, \mathcal{A})x = \int_0^\infty e^{-\lambda t} T(t)x \, dt$$

Theorem 1.4.6 *Let $T(t)$ be a C_0-semigroup on a Hilbert space H with generator $\mathcal{A}$. Then $T(t)$ is exponentially stable if and only if $\{\lambda, \ |, \ Re\lambda \geq 0\}, \ \subset \sigma(\mathcal{A})$ and*

$$\|\mathcal{R}(\lambda, \mathcal{A})\| \leq M$$

for all λ with $Re\lambda \geq 0$ and some constant $M > 0$.

1.5 Bessel functions

The second order differential equation given as

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y = 0$$

is known as Bessel′s differential equation which is often encountered when solving boundary value problems, especially when working in cylindrical or spherical coordinates. The constant ν, determines the order of the Bessel functions found in the solution to Bessel′s differential equation and can take on any real numbered value. For cylindrical problems the order of the Bessel function is an integer value ($\nu = n$) while for spherical problems the order is of half integer value ($\nu = n + 1/2$).

Since Bessel′s differential equation is a second-order equation, there must be two linearly independent solutions. Typically the general solution is given as:

$$y = AJ_\nu(x) + BY_\nu(x)$$

where A and B are arbitrary constants and the special functions $J_\nu(x)$ and $Y_\nu(x)$ are:

- Bessel functions of the first kind, $J_\nu(x)$, which are finite at $x = 0$ for all real values of ν
- Bessel functions of the second kind, $Y_\nu(x)$, (also known as Weber or Neumann functions) which are singular at $x = 0$.

The Bessel function of the first kind of order ν can be determined using an infinite power series expansion as follows:

$$\begin{aligned} J_\nu(x) &= \sum_{\kappa=0}^{\infty} \frac{(-1)^\kappa (x/2)^{\nu+2\kappa}}{\kappa!\Gamma(\nu+\kappa+1)} \\ &= \frac{1}{\Gamma(1+\nu)}\left(\frac{x}{2}\right)^\nu \left\{1 - \frac{(x/2)^2}{1(1+\nu)}\left(1 - \frac{(x/2)^2}{2(2+\nu)}\left(1 - \frac{(x/2)^2}{3(3+\nu)}(1 - ...)\right)\right)\right\} \end{aligned}$$

or by noting that $\Gamma(\nu + \kappa + 1) = (\nu + \kappa)!$, we can write

$$J_\nu(x) = \sum_{\kappa=0}^{\infty} \frac{(-1)^\kappa (x/2)^{\nu+2\kappa}}{\kappa!(\nu+\kappa)!}$$

Bessel Functions of the first kind of order $0, 1, 2$ are shown in Fig. 1.1.

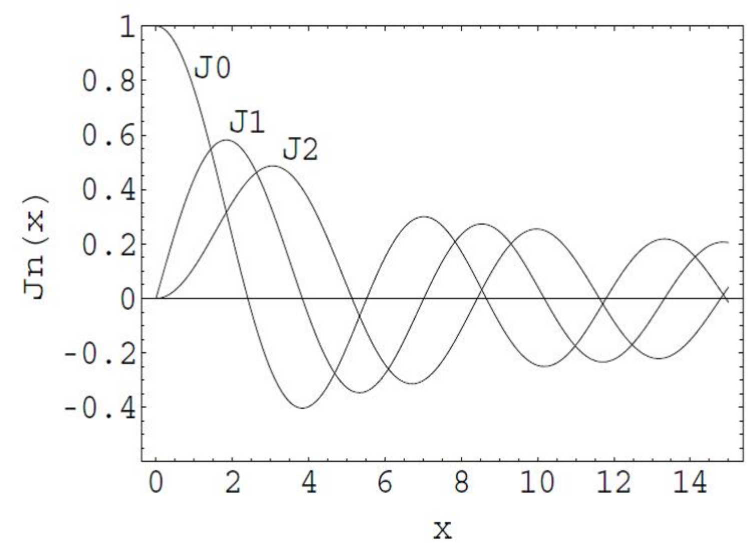

Figure 1.1: Plot of the Bessel Functions of the First Kind, Integer Order.

The Bessel function of the second kind, $Y_\nu(x)$ is sometimes referred to as a Weber function or a Neumann function (which can be denoted as $N_\nu(x)$). It is related to the Bessel function of the first kind as follows:

$$Y_\nu(x) = \frac{J_\nu(x)cos(\nu\pi) - J_{-\nu}(x)}{sin(\nu\pi)}$$

where we take the limit $\nu \rightarrow n$ for integer values of ν.

For integer order ν, $J_\nu, J_{-\nu}$ are not linearly independent:

$$J_{-\nu}(x) = (-1)^\nu J_\nu(x)$$

$$Y_\nu(x) = (-1)^\nu Y_\nu(x)$$

in which case Y_ν is needed to provide the second linearly independent solution of Bessel's equation. In contrast, for non-integer orders, J_ν and $J_{-\nu}$ are linearly independent and Y_ν is redundant.

The Bessel function of the second kind of order ν can be expressed in terms of the Bessel function of the first kind as follows:

$$\begin{aligned} Y_\nu(x) &= \frac{2}{\pi} J_\nu(x) \left(\ln\frac{x}{2} + \gamma \right) - \frac{1}{\pi} \sum_{\kappa=0}^{\nu-1} \frac{(\nu-\kappa-1)!}{\kappa!} \left(\frac{x}{2}\right)^{2\kappa-\nu} + \\ &+ \frac{1}{\pi} \sum_{\kappa=0}^{\infty} \frac{(-1)^{\kappa-1} \left[\left(1 + \frac{1}{2} + \ldots + \frac{1}{\kappa}\right) + \left(1 + \frac{1}{2} + \ldots + \frac{1}{\kappa+\nu}\right) \right]}{\kappa!(\kappa+\nu)!} \left(\frac{x}{2}\right)^{2\kappa+\nu} \end{aligned}$$

Bessel Functions of the second kind of order $0, 1, 2$ are shown in Fig. 1.2.

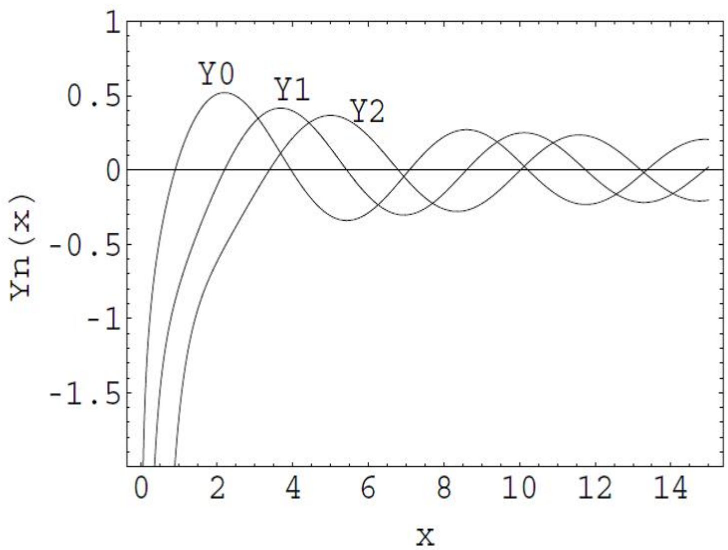

Figure 1.2: Plot of the Bessel Functions of the Second Kind, Integer Order.

Theorem 1.5.1 ([33]) *Asymptotic representations for the Bessel Functions for large $|z|$ of the first and second kinds:*

$$J_\nu(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cos\left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)\left[\sum_{\kappa=0}^{n}(-1)^\kappa(\nu, 2\kappa)(2z)^{-2\kappa} + O(|z|^{-2n-2})\right]$$
$$- \left(\frac{2}{\pi z}\right)^{1/2} \sin\left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)\left[\sum_{\kappa=0}^{n}(-1)^\kappa(\nu, 2\kappa+1)(2z)^{-2\kappa-1} + O(|z|^{-2n-3})\right],$$
$$|argz| \le \pi - \delta$$

and

$$Y_\nu(x) = \left(\frac{2}{\pi z}\right)^{1/2} \cos\left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)\left[\sum_{\kappa=0}^{n}(-1)^\kappa(\nu, 2\kappa+1)(2z)^{-2\kappa-1} + O(|z|^{-2n-3})\right]$$
$$+ \left(\frac{2}{\pi z}\right)^{1/2} \sin\left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)\left[\sum_{\kappa=0}^{n}(-1)^\kappa(\nu, 2\kappa)(2z)^{-2\kappa} + O(|z|^{-2n-2})\right],$$
$$|argz| \le \pi - \delta$$

1.6 Caputo's fractional derivative

There are various ways of defining the fractional derivative, but we will focus primarily on the Caputo fractional derivative defined by Podlubny **[40]** (chapter 2.4) who gave few formal definitions and theorems.

The approach suggested by Caputo is very useful for the formulation and solution of applied problems and their transparency. It allows the formulation of initial conditions for initial-value problems for fractional-order differential equations in a form involving the limit values of integer-order derivatives at the lower terminal (initial time) $t = a$, such as $y^{'}(a), y^{''}(a)$ etc.

The definition of the fractional derivative of the Reimann-Liouville type played an important role in the development of the theory of fractional derivatives and integrals and for its applications in pure mathematics (solution of integer-order differential equations, definitions of new function classes, summation of series, etc.). We define it by

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f(s)\, ds, \quad (n-1 \leq \alpha < n) $$

or

$$ {}_aD_t^{\alpha} f(t) = \frac{d^n}{dt^n} \left({}_aD_t^{-(n-\alpha)} f(t) \right), \quad (n-1 \leq \alpha < n) $$

Moreover, we see that for $\alpha = n \geq 1$ and $t > a$

$$ {}_aD_t^{\alpha} f(t) = \frac{d^n}{dt^n} \left({}_aD_t^{0} f(t) \right) = \frac{d^n f(t)}{dt^n} = f^n(t) $$

which means that for $t > a$ the Riemann-Liouville fractional derivative of order $\alpha = n > 1$ coincides with the conventional derivative of order n.

However, there have appeared a number of works, especially in the theory of viscoelasticity and in solid mechanics, where fractional derivatives are used for a better description of material properties. Mathematical modeling naturally leads to differential equations of fractional order, and to the necessity of the formulation of initial conditions to such equations. This means that the Riemann-Liouville is not the best definition to take when solving some problems, their solutions are practically useless because there is no known physical interpretation for such types of initial conditions, it is better to use a different definition, such as the Caputo definition which makes initial conditions for differential equations nicer.

Caputo's definition can be written as

$$ {}_a^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-n)} \int_a^t \frac{f^{(n)}(s)\, ds}{(t-s)^{\alpha+1-n}}, \quad (n-1 < \alpha < n). $$

Under natural conditions on the function $f(t)$, for $\alpha \to n$ the Caputo derivative becomes a conventional n^{th} derivative of the function $f(t)$. Indeed, let us assume that $0 \leq n-1 < \alpha < n$ and that the function $f(t)$ has $n+1$ continuous bounded derivatives in $[a, t]$ for every $t > a$, then

$$
\begin{aligned}
\lim_{\alpha \to n} {}_a^C D_t^\alpha f(t) &= \lim_{\alpha \to n} \left(\frac{f^{(n)}(a)(t-a)^{n-\alpha}}{\Gamma(n-\alpha+1)} + \frac{1}{\Gamma(n-\alpha+1)} \int_a^t (t-s)^{n-\alpha} f^{(n+1)}(s)\, ds \right) \\
&= f^{(n)}(a) + \int_a^t f^{(n+1)}(s)\, ds \\
&= f^n(t) \quad n = 1, 2, \ldots
\end{aligned}
$$

The main advantage of Caputo's approach is that the initial conditions for fractional differential equation with Caputo derivatives take on the same form as for integer-order differential equations, i.e. contain the limit values of integer-order derivatives of unknown functions at the lower terminal $t = a$.

Definition 1.6.1 *The fractional derivative of order α, $0 < \alpha < 1$, in sense of Caputo, is defined by*

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{df}{ds}(s) ds.$$

Definition 1.6.2 *The fractional integral of order α, $0 < \alpha < 1$, in sense Riemann-Liouville, is defined by*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds.$$

Remark 1.6.1 *From the above definitions, clearly*

$$D^\alpha f = I^{\alpha-1} Df, \;\; 0 < \alpha < 1.$$

Now, we give the definitions of the generalized Caputo's fractional derivative and the generalized fractional integral.

Definition 1.6.3 *The generalized Caputo's fractional derivative is given by*

$$D^{\alpha,\eta} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{df}{ds}(s)\, ds, \;\; 0 < \alpha < 1, \;\; \eta \geq 0.$$

Definition 1.6.4 *The generalized fractional integral in sense Riemann-Liouville, is given by*

$$I^{\alpha,\eta} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\eta(t-s)} f(s)\, ds, \;\; 0 < \alpha < 1, \eta \geq 0.$$

Remark 1.6.2 *We have*

$$D^{\alpha,\eta} f = I^{1-\alpha,\eta} Df, \;\; 0 < \alpha < 1, \eta \geq 0.$$

1.7 Geometric control condition

Definition 1.7.1 *We say that the geometric control condition **GCC** holds if there exist* $x_0 \in \mathbb{R}^n$ *and a positive constant* $m > 0$ *such that:*

$$m \cdot \nu \leq 0 \text{ on } \Gamma_0, \text{ and } m \cdot \nu \geq 0 \text{ on } \Gamma_1,$$

with $m(x) = x - x_0$*.*

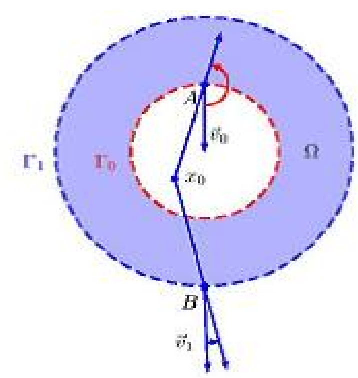

Figure 1.3: This model satisfies the usual geometric control condition

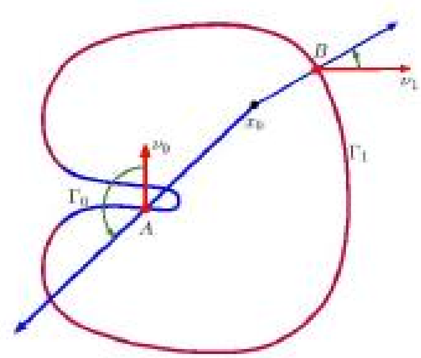

Figure 1.4: This model satisfies the GCC without $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$.

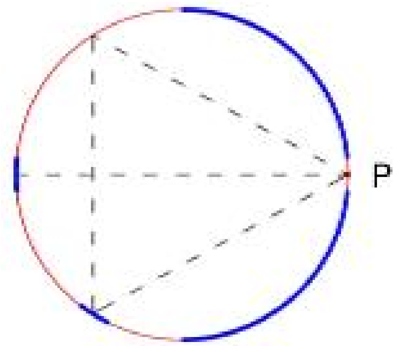

Figure 1.5: This model does not satisfy the usual geometric control condition

Chapter 2

ASYMPTOTIC STABILITY FOR THE LAMÉ SYSTEM WITH FRACTIONAL BOUNDARY DAMPING

2.1 Introduction

We consider the initial boundary value problem for the Lamé system given by:

$$(P1)\qquad \begin{cases} u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla(\text{div}\ \ u) = 0 & \text{in } \Omega\times(0,+\infty) \\ u = 0 & \text{in } \Gamma_0\times(0,+\infty) \\ \mu\dfrac{\partial u}{\partial \nu} + (\mu+\lambda)(\text{div}\ \ u)\nu = -\gamma\partial_t^{\alpha,\eta}u & \text{in } \Gamma_1\times(0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & \text{in } \Omega \end{cases}$$

where μ,λ are Lamé constants, γ is a positive constant, $u = (u_1, u_2, ..., u_n)^T$. Here Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ of class C^2 and ν is the unit outward normal to Γ. We assume that $\Gamma = \Gamma_0 \cup \Gamma_1$, where Γ_0 and Γ_1 are closed subsets of Γ with $\Gamma = \Gamma_0 \cap \Gamma_1 = \emptyset$. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α $(0 < \alpha < 1)$ with respect to the time variable (see **[12]**). It is defined as follows

$$\partial_t^{\alpha,\eta}w(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}e^{-\eta(t-s)}\frac{dw}{ds}(s)\,ds \qquad \eta \geq 0.$$

This problem has its origin in the mathematical description of memory-type elastic materials. It is well known that memory-type elastic materials exhibit nature damping, which is due to the special property of these materials to retain memory of their past history. From the mathematical point of view, these damping effects are modeled by integro-differential operators. Therefore, dynamics of memory-type elastic materials are very important and interesting as they have wide applications in natural sciences. From the physical point of view, the problem $(P1)$ describes the position $u(x,t)$ of the material particle $x = (x_1, x_2, \ldots, x_n)$ at time t, which is clamped in the portion Γ_0 of its boundary and its portion Γ_1 is supported by elastic bearings with fractional boundary responses, represented by the function $\partial_t^{\alpha,\eta} u$.

The problem of well-posedness and stability for the Lamé system has attracted a lot of attention in recent years, where diverse types of damping mechanisms have been introduced and many energy estimates have been obtained (polynomial, exponential or logarithmic decay).

Let us mention here some works concerning the stabilization of Lamé system of waves with different types of dampings.

In **[30]**, Lagnese obtained uniform stability estimates for linear homogeneous isotropic and bidimensional elasticity systems under a linear boundary damping. Komornik **[27]** proved the same estimates for the homogeneous isotropic system in 1-dimension and 2-dimension and under a linear boundary damping. Martinez **[36]** generalized the results of Komornik **[27]** to the case of elasticity systems of cubic crystals under a nonlinear boundary damping.

We should mention here that, to the best of our knowledge, there is no result concerning the Lame system with the presence of a fractional damping. In addition to being nonlocal, fractional derivatives involve singular and nonintegrable kernels ($t^{-\alpha}, 0 < \alpha < 1$). This makes the problem very delicate.

Noting that the case of the one-dimensional wave and plate equations with boundary fractional damping has treated in **[2]** and **[38]** where it is proven the strong stability and the lack of uniform stabilization.

Very recently, Benaissa and Rafa **[11]** (see also **[3]**) extended the result in **[38]** to higher-space dimension and boundary control of diffusive type and established a less precise decay estimate by adopting the multiplier method.

Our aim in this work is to prove that the stability of our system holds with fractional damping and to obtain an almost optimal polynomial decay.

This chapter is organized as follows. In Section **2**, we take advantage of the complete monotonicity of the power function integral kernel to represent it as a superposition of exponentials and derive what we call the "augmented model". In section **3**, we state a well-posedness result for problem $(P1)$. In section **4**, we show the lack of exponential stability by spectral analysis. Section **5** is devoted to results regarding strong asymptotic stability of solutions, and in Section **6**, we show an almost optimal polynomial energy decay rate depending on parameter α. We use a recent result of Borichev and Tomilov which relate resolvent bounds and decay rates.

2.2 Preliminaries

This section is concerned with the reformulation of the model ($P1$) into an augmented system. For that, we need the following claims.

Theorem 2.2.1 (see [38]) *Let μ be the function:*

$$\mu(\xi) = |\xi|^{(2\alpha-1)/2}, \quad -\infty < \xi < +\infty,\ 0 < \alpha < 1. \tag{2.1}$$

Then the relationship between the 'input' U and the 'output' O of the system

$$\partial_t \phi(\xi, t) + (\xi^2 + \eta)\phi(\xi, t) - U(t)\mu(\xi) = 0, \quad -\infty < \xi < +\infty, \eta \geq 0, t > 0, \tag{2.2}$$

$$\phi(\xi, 0) = 0, \tag{2.3}$$

$$O(t) = (\pi)^{-1} \sin(\alpha\pi) \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi, t)\, d\xi \tag{2.4}$$

is given by

$$O = I^{1-\alpha,\eta}U = D^{\alpha,\eta}U \tag{2.5}$$

where

$$[I^{\alpha,\eta} f](t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{-\eta(t-\tau)} f(\tau)\, d\tau.$$

Proof. From (2.2) and (2.3), we have

$$\phi(\xi, t) = \int_0^t \mu(\xi) e^{-(\xi^2+\eta)(t-\tau)} U(\tau) d\tau \tag{2.6}$$

Hence, by using (2.4), we get

$$O(t) = (\pi)^{-1} sin(\alpha\pi) e^{-\eta t} \int_0^t \left[2 \int_0^{+\infty} |\xi|^{2\alpha-1} e^{-\xi^2(t-s)} d\xi \right] e^{\eta\tau} U(\tau) d\tau \tag{2.7}$$

Thus,

$$\begin{array}{rcl} O(t) & = & (\pi)^{-1} sin(\alpha\pi) e^{-\eta t} \int_0^t \left[(t-s)^{-\alpha} \Gamma(\alpha) \right] e^{\eta\tau} U(\tau) d\tau \\ & = & (\pi)^{-1} sin(\alpha\pi) \int_0^t \left[(t-s)^{-\alpha} \Gamma(\alpha) \right] e^{-\eta(t-\tau)} U(\tau) d\tau \end{array} \tag{2.8}$$

Which completes the proof. Indeed, we know that $(\pi)^{-1} sin(\alpha\pi) = \dfrac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}$

Lemma 2.2.1 (see [2]) *If $\lambda \in D_\eta = \mathbb{C} \backslash]-\infty, -\eta]$ then*

$$\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda + \eta + \xi^2}\, d\xi = \frac{\pi}{\sin \alpha\pi} (\lambda + \eta)^{\alpha - 1}.$$

Proof. Let us set

$$f_{\tilde\lambda}(\xi) = \frac{\mu^2(\xi)}{\tilde\lambda + \eta + \xi^2}.$$

We have

$$\left|\frac{\mu^2(\xi)}{\tilde\lambda + \eta + \xi^2}\right| \le \begin{cases} \dfrac{\mu^2(\xi)}{Re\tilde\lambda + \eta + \xi^2} \text{ or} \\ \dfrac{\mu^2(\xi)}{|Im\tilde\lambda| + \eta + \xi^2} \end{cases}$$

Then the function $f_{\tilde\lambda}$ is integrable. Moreover

$$\left|\frac{\mu^2(\xi)}{\tilde\lambda + \eta + \xi^2}\right| \le \begin{cases} \dfrac{\mu^2(\xi)}{\eta_0 + \eta + \xi^2} & \text{for all } Re\tilde\lambda \ge \eta_0 > -\eta \\ \dfrac{\mu^2(\xi)}{\tilde\eta_0 + \xi^2} & \text{for all } |Im\tilde\lambda| \ge \tilde\eta_0 > 0 \end{cases}$$

From Theorem 1.16.1 in **[47]**, the function

$$f_{\tilde\lambda} : D \to \mathbb{C} \quad \text{is holomorphic.}$$

For a real number $\tilde\lambda > -\eta$, we have

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\tilde\lambda + \eta + \xi^2}\, d\xi = \int_{-\infty}^{+\infty} \frac{|\xi|^{2\alpha-1}}{\tilde\lambda + \eta + \xi^2}\, d\xi = \int_0^{+\infty} \frac{x^{\alpha-1}}{\tilde\lambda + \eta + x}\, dx \text{ (with } \xi^2 = x) \\
&= (\tilde\lambda + \eta)^{\alpha-1} \int_1^{+\infty} y^{-1}(y-1)^{\alpha-1}\, dy \text{ (with } y = x/(\tilde\lambda + \eta) + 1) \\
&= (\tilde\lambda + \eta)^{\alpha-1} \int_0^1 z^{-\alpha}(1-z)^{\alpha-1}\, dz \text{ (with } z = 1/y) \\
&= (\tilde\lambda + \eta)^{\alpha-1} B(1-\alpha, \alpha) = (\tilde\lambda + \eta)^{\alpha-1}\Gamma(1-\alpha)\Gamma(\alpha) = (\tilde\lambda + \eta)^{\alpha-1}\frac{\pi}{\sin \pi\alpha}.
\end{aligned}$$

Both holomorphic functions $f_{\tilde\lambda}$ and $\tilde\lambda \mapsto (\tilde\lambda + \eta)^{\alpha-1}\dfrac{\pi}{\sin \pi\alpha}$ coincide on the half line $]-\eta, +\infty[$, hence on D_η following the principe of isolated zeroes.

We are now in a position to reformulate system $(P1)$. Indeed, by using Theorem 2.2.1, system $(P1)$ may be recast into the augmented model:

$$(P'1) \quad \begin{cases} u_{tt} - \mu\Delta u - (\mu + \lambda)\nabla(\text{div } u) = 0 & \text{in } \Omega \times (0, +\infty) \\ \partial_t\phi(x, \xi, t) + (\xi^2 + \eta)\phi(x, \xi, t) - u(x, t)\mu(\xi) = 0 & \text{in } \Gamma_1 \times (-\infty, \infty) \times (0, +\infty) \\ u(x, t) = 0 & \text{on } \Gamma_0 \times (0, +\infty) \\ \mu\dfrac{\partial u}{\partial \nu} + (\mu + \lambda)(\text{div } u)\nu = -\zeta \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(x, \xi, t)\, d\xi & \text{in } \Gamma_1 \times (-\infty, \infty) \times (0, +\infty) \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) & \text{on } \Omega, \\ \phi(x, \xi, 0) = 0 & \text{on } \Gamma_1 \times (-\infty, \infty), \end{cases}$$

where $\zeta = \gamma(\pi)^{-1}\sin(\alpha\pi)$.

We define the energy of the solution by:

$$E(t) = \frac{1}{2}\sum_{j=1}^{n}\left(\|u_{jt}\|^2_{L^2(\Omega)} + \mu\|\nabla u_j\|^2_{L^2(\Omega)} + \zeta\int_{\Gamma_1}\int_{-\infty}^{+\infty}|\phi_j(x,\xi,t)|^2\,d\xi d\Gamma\right) + \frac{(\mu+\lambda)}{2}\|\text{div }u\|^2_{L^2(\Omega)} \tag{2.9}$$

Lemma 2.2.2 *Let (u,ϕ) be a regular solution of the problem $(P1)$. Then, the energy functional defined by (2.9) satisfies*

$$E'(t) = -\zeta\sum_{j=1}^{n}\int_{\Gamma_1}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi_j(x,\xi,t)|^2\,d\xi\,d\Gamma \le 0. \tag{2.10}$$

Proof. Multiplying the first equation in $(P1)$ by $\overline{u}_{jt}$, integrating over Ω and using integration by parts, we get

$$\frac{1}{2}\frac{d}{dt}\|u_{jt}\|_2^2 - \mu\Re\int_\Omega \Delta u_j\overline{u}_{jt}\,dx - (\mu+\lambda)\Re\int_\Omega\frac{\partial}{\partial x_j}(\text{div }u)\overline{u}_{jt}\,dx = 0.$$

Then

$$\frac{1}{2}\frac{d}{dt}\sum_{j=1}^{n}\left(\|u_{jt}\|^2_{L^2(\Omega)} + \mu\|\nabla u_j\|^2_{L^2(\Omega)}\right) + \frac{(\mu+\lambda)}{2}\|\text{div }u\|^2_{L^2(\Omega)} + \zeta\Re\sum_{j=1}^{n}\int_{\Gamma_1}\overline{u}_{jt}(x,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi_j(x,\xi,t)\,d\xi\,d\Gamma = 0 \tag{2.11}$$

Multiplying the second equation in $(P'1)$ by $\zeta\overline{\phi}_j$ and integrating over $\Gamma_1\times(-\infty,+\infty)$, to obtain:

$$\frac{\zeta}{2}\frac{d}{dt}\sum_{j=1}^{n}\|\phi_j\|^2_{L^2(\Gamma_1\times(-\infty,+\infty))} + \zeta\sum_{j=1}^{n}\int_{\Gamma_1}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi_j(x,\xi,t)|^2\,d\xi\,d\Gamma - \zeta\Re\sum_{j=1}^{n}\int_{\Gamma_1}u_{jt}(x,t)\int_{-\infty}^{+\infty}\mu(\xi)\overline{\phi}_j(x,\xi,t)\,d\xi\,d\Gamma = 0. \tag{2.12}$$

From (2.9), (2.11) and (2.12) we get

$$E'(t) = -\zeta\sum_{j=1}^{n}\int_{\Gamma_1}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi_j(x,\xi,t)|^2\,d\xi\,d\Gamma.$$

This completes the proof of the lemma.

2.3 Well-posedness

We rewrite the problem $(P'1)$ as a first-order system for $U = (u,v,\phi)^T$, where $v = u_t$. Then U satisfies

$$\begin{cases} U' = \mathcal{A}U, & t>0,\\ U(0) = (u_0,u_1,\phi_0). \end{cases} \tag{2.13}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ \phi \end{pmatrix} = \begin{pmatrix} v \\ \mu\Delta u + (\mu+\lambda)\nabla(\text{div } u) \\ -(\xi^2+\eta)\phi + v(x)\mu(\xi) \end{pmatrix}, \tag{2.14}$$

with domain

$$D(\mathcal{A}) = \{\mathcal{U} \in \mathcal{H} / \mathcal{A}\mathcal{U} \in \mathcal{H}\} \tag{2.15}$$

where the energy space $\mathcal{H}$ is defined as:

$$\mathcal{H} = \left(H^1_{\Gamma_0}(\Omega)\right)^n \times \left(L^2(\Omega)\right)^n \times \left(L^2(\Gamma_1 \times (-\infty, +\infty))\right)^n$$

equipped with the inner product

$$\begin{aligned} < U, \tilde{U} >_{\mathcal{H}} = & \sum_{j=1}^{n} \int_\Omega \left(v_j \bar{\tilde{v}}_j + \mu \nabla u_j \nabla \bar{\tilde{u}}_j\right) dx + (\mu+\lambda) \int_\Omega (\text{div } u)(\text{div } \bar{\tilde{u}})\, dx \\ & + \zeta \sum_{j=1}^{n} \int_{\Gamma_1} \int_{-\infty}^{+\infty} \phi_j(x,\xi,t) \bar{\tilde{\phi}}_j(x,\xi,t) d\xi d\Gamma. \end{aligned}$$

The domain of $\mathcal{A}$ is then

$$D(\mathcal{A}) = \left\{ \begin{array}{l} (u,v,\phi)^T \text{ in } \mathcal{H} : u \in \left(H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)\right)^n, \tilde{u} \in \left(H^1_{\Gamma_0}(\Omega)\right)^n, \\ -(\xi^2+\eta)\phi + v(x)\mu(\xi) \in \left(L^2(\Gamma_1 \times (-\infty,+\infty))\right)^n, \\ \mu \dfrac{\partial u}{\partial \nu} + (\mu+\lambda)(\text{div } u)\nu + \zeta \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(x,\xi)\, d\xi = 0, \text{ on } \Gamma_1 \\ |\xi|\phi \in \left(L^2(\Gamma_1 \times (-\infty,+\infty))\right)^n \end{array} \right\}. \tag{2.16}$$

Remark 2.3.1 *The condition $|\xi|\phi(\xi) \in (L^2(\Gamma_1 \times \mathbb{R}))^n$ is imposed to insure the existence of $-\bar{\zeta} \sum_{j=1}^{n} \int_{\Gamma_1} \int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi_j(x,\xi,t)|^2\, d\xi\, d\Gamma$ and $\mu(\xi)\phi(x,\xi) \in (L^1(\Gamma_1 \times \mathbb{R}))^n$.*

Our main result is giving by the following theorem.

Theorem 2.3.1 *The operator $\mathcal{A}$ defined by (2.14) and (2.16) generates a C_0-semigroup of contractions $e^{t\mathcal{A}}$ in the Hilbert space $\mathcal{H}$.*

Proof. To prove this result we shall use the Lumer-Phillips' theorem.
For any $U = (u,v,\phi) \in D(\mathcal{A})$, using (2.13), (2.10) and the fact that

$$E(t) = \frac{1}{2}\|U\|^2_{\mathcal{H}}, \tag{2.17}$$

we have

$$\Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\zeta \sum_{j=1}^{n} \int_{\Gamma_1} \int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi_j(x,\xi,t)|^2 \, d\xi \, d\Gamma \tag{2.18}$$

Then the operator $\mathcal{A}$ is dissipative.

Let $\tilde{\lambda} > 0$, we prose that the operator $(\tilde{\lambda} I - \mathcal{A})$ is a surjection.
In other words, we shall demonstrate that given any triplet $F = (F_1, F_2, F_3) \in \mathcal{H}$, there is an other triplet $U = (u, v, \phi) \in D(\mathcal{A})$ such that

$$(\tilde{\lambda} I - \mathcal{A})U = F. \tag{2.19}$$

Equation (2.19) is equivalent to

$$\begin{cases} \tilde{\lambda} u - v = F_1(x), \\ \tilde{\lambda} v - \mu\Delta u - (\mu+\lambda)\nabla(\text{div } u) = F_2(x), \\ \tilde{\lambda}\phi + (\xi^2+\eta)\phi - v(x)\mu(\xi) = F_3(x,\xi), \end{cases} \tag{2.20}$$

Suppose u is found with the appropriate regularity. Then, $(2.20)_1$ $(2.20)_2$ yield

$$v = \tilde{\lambda} u - F_1(x) \in \left(H^1_{\Gamma_0}(\Omega)\right)^n \tag{2.21}$$

and

$$\phi = \frac{F_3(x,\xi) + \mu(\xi) v(x)}{\xi^2 + \eta + \tilde{\lambda}}. \tag{2.22}$$

By using (2.20) and (2.21) it can easily be shown that u satisfies

$$\tilde{\lambda}^2 u - \mu\Delta u - (\mu+\lambda)\nabla(\text{div } u) = F_2(x) + \tilde{\lambda} F_1(x). \tag{2.23}$$

Solving system (2.23) is equivalent to finding $u \in \left(H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)\right)^n$ such that

$$\int_\Omega \left(\tilde{\lambda}^2 u_j \overline{w}_j - \mu\Delta u_j \overline{w}_j\right) dx - (\mu+\lambda)\int_\Omega \frac{\partial}{\partial x_j}(\text{div } u)\overline{w}_j dx = \int_\Omega (F_2^j(x) + \tilde{\lambda} F_1^j(x))\overline{w}_j dx, \tag{2.24}$$

For all $w \in \left(H^1_{\Gamma_0}(\Omega)\right)^n$. By using (2.24), the boundary condition $(2.16)_3$ and (2.22) the function u satisfying the following system

$$\begin{cases} \displaystyle\sum_{j=1}^{n} \int_\Omega \left(\tilde{\lambda}^2 u_j \overline{w}_j + \mu \nabla u_j \nabla \overline{w}_j \, dx\right) dx + (\mu+\lambda)\int_\Omega (\text{div } u)(\text{div } \overline{w}) \, dx + \tilde{\zeta} \sum_{j=1}^{n} \int_{\Gamma_1} v_j \overline{w}_j \, d\Gamma \\ \qquad = \displaystyle\sum_{j=1}^{n} \int_\Omega (F_2^j(x) + \tilde{\lambda} F_1^j(x))\overline{w}_j \, dx - \zeta \sum_{j=1}^{n} \int_{\Gamma_1} w_j \left(\int_{-\infty}^{\infty} \frac{\mu(\xi) F_3^j(x,\xi)}{\xi^2+\eta+\tilde{\lambda}} d\xi\right) d\Gamma. \end{cases} \tag{2.25}$$

where $\tilde{\zeta} = \zeta \displaystyle\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2+\eta+\tilde{\lambda}} \, d\xi$.

Using again (2.21), we deduce that

$$v(x) = \tilde{\lambda} u(x) - F_1(x), \quad \forall x \in \Gamma_1. \tag{2.26}$$

Inserting (2.26) into (2.25), we get

$$\begin{cases} \displaystyle\sum_{j=1}^{n}\int_{\Omega}\left(\tilde{\lambda}^2 u_j\overline{w}_j+\mu\nabla u_j\nabla\overline{w}_j\,dx\right)+(\mu+\lambda)\int_{\Omega}(\text{div }u)(\text{div }\overline{w})\,dx+\tilde{\zeta}\tilde{\lambda}\sum_{j=1}^{n}\int_{\Gamma_1}u_j\overline{w}_j\,d\Gamma \\ \quad =\displaystyle\sum_{j=1}^{n}\int_{\Omega}(F_2^j(x)+\tilde{\lambda}F_1^j(x))\overline{w}_j\,dx-\zeta\sum_{j=1}^{n}\int_{\Gamma_1}\overline{w}_j\left(\int_{-\infty}^{\infty}\frac{\mu(\xi)F_3^j(x,\xi)}{\xi^2+\eta+\tilde{\lambda}}\,d\xi\right)d\Gamma \\ \qquad +\tilde{\zeta}\displaystyle\sum_{j=1}^{n}\int_{\Gamma_1}F_1^j(x)\overline{w}_j\,d\Gamma. \end{cases} \tag{2.27}$$

Problem (2.27) is of the form

$$\mathcal{B}(u,w)=\mathcal{L}(w), \tag{2.28}$$

where $\mathcal{B}:\left[\left(H^1_{\Gamma_0}(\Omega)\right)^n\times\left(H^1_{\Gamma_0}(\Omega)\right)^n\right]\to\mathbb{C}$ is the sesquilinear form defined by

$$\mathcal{B}(u,w)=\sum_{j=1}^{n}\int_{\Omega}\left(\tilde{\lambda}^2 u_j\overline{w}_j+\mu\nabla u_j\nabla\overline{w}_j\,dx\right)+(\mu+\lambda)\int_{\Omega}(\text{div }u)(\text{div }\overline{w})\,dx+\tilde{\zeta}\tilde{\lambda}\sum_{j=1}^{n}\int_{\Gamma_1}u_j\overline{w}_j\,d\Gamma$$

and $\mathcal{L}:\left(H^1_{\Gamma_0}(\Omega)\right)^n\to\mathbb{C}$ is the antilinear form given by

$$\mathcal{L}(w)=\sum_{j=1}^{n}\int_{\Omega}(F_2^j(x)+\tilde{\lambda}F_1^j(x))\overline{w}_j\,dx-\zeta\sum_{j=1}^{n}\int_{\Gamma_1}\overline{w}_j\left(\int_{-\infty}^{\infty}\frac{\mu(\xi)F_3^j(x,\xi)}{\xi^2+\eta+\tilde{\lambda}}\,d\xi\right)d\Gamma$$
$$+\tilde{\zeta}\sum_{j=1}^{n}\int_{\Gamma_1}F_1^j(x)\overline{w}_j\,d\Gamma.$$

Clearly $\mathcal{L}$ is continuous form on $\left(H^1_{\Gamma_0}(\Omega)\right)^n$, while $\mathcal{B}$ is continuous and coercive form on $\left(H^1_{\Gamma_0}(\Omega)\right)^n$. Hence by Lax-Milgram Lemma, problem (2.28) has a unique solution $u\in\left(H^1_{\Gamma_0}(\Omega)\right)^n$.

In particular, setting $w\in(\mathcal{D}(\Omega))^n$ in (2.28), we get

$$(*)\qquad \tilde{\lambda}^2 u-\mu\Delta u-(\mu+\lambda)\nabla(\text{div}u)=F_2(x)+\tilde{\lambda}F_1(x)\text{ in }(\mathcal{D}(\Omega))^n.$$

As $F_2(x)+\tilde{\lambda}F_1(x)\in(L^2(\Omega))^n$, using $(*)$, we deduce that

$$(**)\qquad \tilde{\lambda}^2 u-\mu\Delta u-(\mu+\lambda)\nabla(\text{div }u)=F_2(x)+\tilde{\lambda}F_1(x)\text{ in }(L^2(\Omega))^n.$$

Due to the fact that $u\in\left(H^1_{\Gamma_0}(\Omega)\right)^n$, we get $\Delta u\in(L^2(\Omega))^n$ and we deduce that $u\in(H^2(\Omega))^n$.

Multiplying the conjugate of equalities $(**)$ by $w\in\left(H^1_{\Gamma_0}(\Omega)\right)^n$, integrating by parts on Ω, and comparing with (2.28) we get

$$-\int_{\Gamma_1}\overline{w}_j\Big(\mu\frac{\partial u_j}{\partial\nu}+(\mu+\lambda)(\text{div }u)\nu_j\Big)\,d\Gamma+\tilde{\zeta}\tilde{\lambda}\int_{\Gamma_1}u_j(x)\overline{w}_j\,d\Gamma$$
$$-\tilde{\zeta}\int_{\Gamma_1}F_1^j(x)\overline{w}_j\,d\Gamma+\zeta\int_{\Gamma_1}\overline{w}_j\left(\int_{-\infty}^{\infty}\frac{\mu(\xi)F_3^j(x,\xi)}{\xi^2+\eta+\tilde{\lambda}}\,d\xi\right)d\Gamma=0.$$

Consequently, defining $v_j(x) = \tilde{\lambda} u_j(x) - F_1^j(x), x \in \Gamma_1$ and ϕ_j as in (2.22), we deduce that

$$\mu \frac{\partial u}{\partial \nu} + (\mu + \lambda)(\operatorname{div} u)\nu + \zeta \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi = 0 \text{ on } \Gamma_1.$$

It follows that $v \in \left(H^1_{\Gamma_0}(\Omega)\right)^n$. Moreover from (2.22), we have

$$\begin{aligned}
\|\phi\|_{L^2(\Gamma_1 \times \mathbb{R})} &\leq \left\| \frac{F_3(x,\xi)}{\xi^2 + \eta + \tilde{\lambda}} \right\|_{L^2(\Gamma_1 \times \mathbb{R})} + \left\| \frac{\mu(\xi)}{\xi^2 + \eta + \tilde{\lambda}} \right\|_{L^2(\mathbb{R})} \|v\|_{L^2(\Gamma_1)} \\
&\leq \frac{1}{\tilde{\lambda}} \|F_3(x,\xi)\|_{L^2(\Gamma_1 \times \mathbb{R})} + \left[(1-\alpha)\frac{\pi}{\sin \alpha\pi}\right]^{1/2} (\tilde{\lambda} + \eta)^{(\alpha-2)/2} \|v\|_{L^2(\Gamma_1)}
\end{aligned}$$

and

$$\begin{aligned}
\||\xi|\phi\|_{L^2(\Gamma_1 \times \mathbb{R})} &\leq \left\| \frac{|\xi| F_3(x,\xi)}{\xi^2 + \eta + \tilde{\lambda}} \right\|_{L^2(\Gamma_1 \times \mathbb{R})} + \left\| \frac{|\xi|\mu(\xi)}{\xi^2 + \eta + \tilde{\lambda}} \right\|_{L^2(\mathbb{R})} \|v\|_{L^2(\Gamma_1)} \\
&\leq \frac{1}{\sqrt{2\tilde{\lambda}}} \|F_3(x,\xi)\|_{L^2(\Gamma_1 \times \mathbb{R})} + c\|v\|_{L^2(\Gamma_1)}.
\end{aligned}$$

Hence $\phi, |\xi|\phi \in (L^2(\Gamma_1 \times \mathbb{R}))^n$.
Therefore, the operator $(\tilde{\lambda} I - \mathcal{A})$ is surjective for any $\tilde{\lambda} > 0$.

Consequently, using Hille-Yosida theorem, we have the following well-posedness result:

Theorem 2.3.2 (Existence and uniqueness)

(1) If $U_0 \in D(\mathcal{A})$, then system (2.13) has a unique strong solution

$$U \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

(2) If $U_0 \in \mathcal{H}$, then system (2.13) has a unique weak solution

$$U \in C^0(\mathbb{R}_+, \mathcal{H}).$$

2.4 Lack of uniform stabilisation

In this section we shall prove that the system is not uniformly stable in general, since it is already the case for the unit disk $B(0,1)$ of $\mathbb{R}^2$ and $\Gamma_0 = \emptyset$ as shown below. This result is due to the fact that a subsequence of eigenvalues of $\mathcal{A}$ which is close to the imaginary axis.

Our main result is the following.

Theorem 2.4.1 *The semigroup generated by the operator $\mathcal{A}$ is not exponentially stable.*

Proof. We first compute the characteristic equation that gives the eigenvalues of $\mathcal{A}$. Let $\tilde{\lambda}$ be an eigenvalue of $\mathcal{A}$ with associated eigenvector $U=(u,v,\phi)^T$.
To solve $\mathcal{A}U=\tilde{\lambda}U$ is enough to solve

$$\begin{cases} \tilde{\lambda}u-v=0, \quad x\in B(0,1),\\ \tilde{\lambda}v-\mu\Delta u-(\mu+\lambda)\nabla(\mathrm{div}\ u)=0 \quad x\in B(0,1),\\ \tilde{\lambda}\phi+(\xi^2+\eta)\phi-v(x)\mu(\xi)=0, \quad x\in\partial B(0,1),\ \xi\in\mathbb{R}. \end{cases} \tag{2.29}$$

Next, by eliminating v from the above system we get the following system:

$$\begin{cases} \tilde{\lambda}^2u-\mu\Delta u-(\mu+\lambda)\nabla(\mathrm{div}\ u)=0 \quad x\in B(0,1),\\ \tilde{\lambda}\phi+(\xi^2+\eta)\phi-\tilde{\lambda}u(x)\mu(\xi)=0, \quad x\in\partial B(0,1),\ \xi\in\mathbb{R}. \end{cases} \tag{2.30}$$

with the following boundary condition

$$\mu\frac{\partial u}{\partial\nu}+(\mu+\lambda)(\mathrm{div}\ u)\nu=-\zeta\int_{-\infty}^{+\infty}\mu(\xi)\phi(x,\xi,t)\,dx. \tag{2.31}$$

As a consequence of (2.30), (div u) verifies the scalar equation

$$\tilde{\lambda}^2(\mathrm{div}\ u)-(2\mu+\lambda)\Delta(\mathrm{div}\ u)=0 \tag{2.32}$$

with the following boundary conditions (we use $(2.29)_1$ and (2.31))

$$\mu\frac{\partial u}{\partial\nu}+(\mu+\lambda)(\mathrm{div}\ u)\nu=-\gamma\tilde{\lambda}(\tilde{\lambda}+\eta)^{\alpha-1}u. \tag{2.33}$$

Then

$$\left(3\mu+2\lambda+\gamma\tilde{\lambda}(\tilde{\lambda}+\eta)^{\alpha-1}\right)(\mathrm{div}\ u)+(2\mu+\lambda)\frac{\partial}{\partial\nu}(\mathrm{div}\ u)=0. \tag{2.34}$$

From (2.32) and (2.34) we arrive at

$$\begin{cases} \tilde{\lambda}^2E-(2\mu+\lambda)\Delta E=0 & \text{in } B(0,1)\\ \left(3\mu+2\lambda+\gamma\tilde{\lambda}(\tilde{\lambda}+\eta)^{\alpha-1}\right)E+(2\mu+\lambda)\dfrac{\partial}{\partial\nu}E=0 & \text{in } \partial B(0,1), \end{cases} \tag{2.35}$$

where $E=(\mathrm{div}\ u)$.
We decompose $E(r,.)$ in Fourier series with respect to $1,\ \cos k\varphi,\ \sin k\varphi,\ k\in\mathbb{N}$:
$E(r,\varphi)=\sum_{k=0}^{\infty}\tilde{v}_k(r)\Phi_k(\varphi)$ with $\Phi_0=1,\ \Phi_{2k-1}(\varphi)=\cos k\varphi,\ \Phi_{2k}(\varphi)=\sin k\varphi,$
$k\in\mathbb{N}:\ \dfrac{\partial^2}{\partial\varphi^2}\Phi_k=\tau_k\Phi_k$ with $\tau_0=0,\ \tau_{2k}=\tau_{2k-1}=-k^2,\ k\in\mathbb{N}.$

Then the equation for E is equivalent to the system

$$\begin{cases} -(2\mu+\lambda)\tilde{v}_k''(r)+(\tilde{\lambda}^2+(2\mu+\lambda)\frac{k^2}{r})\tilde{v}_k(r)=0,\ r\in B(0,1),\\ (2\mu+\lambda)\tilde{v}_k'+\left(3\mu+2\lambda+\gamma\tilde{\lambda}(\tilde{\lambda}+\eta)^{\alpha-1}\right)\tilde{v}_k=0,\ r=1,k\in\mathbb{N}. \end{cases} \tag{2.36}$$

The theory of Bessel equations gives

$$\tilde{v}_k(r) = \left(\frac{2}{i\frac{\tilde{\lambda}}{\sqrt{2\mu+\lambda}}}\right)^k J_k\left(i\frac{\tilde{\lambda}}{\sqrt{2\mu+\lambda}}r,\right)$$

where J_k is the Bessel functions of the first kind of order k

$$J_k(s) = \sum_{j=1}^{\infty} \frac{(-1)^j}{\Gamma(k+j+1)}\left(\frac{s}{2}\right)^{k+2j}.$$

Therefore, using the second equation of (2.36), we find that if $\tilde{\lambda} \in \mathbb{C}^*$ satisfies

$$\frac{1}{\sqrt{2\mu+\lambda}} i\tilde{\lambda} J_k'\left(i\frac{\tilde{\lambda}}{\sqrt{2\mu+\lambda}}\right) + \left(\frac{3\mu+2\lambda}{2\mu+\lambda} + \frac{\gamma}{2\mu+\lambda}\tilde{\lambda}(\tilde{\lambda}+\eta)^{\alpha-1}\right) J_k\left(i\frac{\tilde{\lambda}}{\sqrt{2\mu+\lambda}}\right) = 0.$$

Then $\tilde{\lambda}$ is an eigenvalue of $\mathcal{A}$. Our goal is to find large eigenvalues which are close to the imaginary axis and to give their expansion.

Lemma 2.4.1 *There exists $N \in \mathbb{N}$ such that*

$$\{\lambda_m\}_{m\in\mathbf{Z}^*, |m|\geq N} \subset \sigma(\mathcal{A}), \tag{2.37}$$

where

$$\lambda_m = -i\sqrt{2\mu+\lambda}\left(m + \frac{k}{2} + \frac{1}{4}\right)\pi + \frac{\tilde{\alpha}}{m^{(1-\alpha)}} + \frac{\beta}{m^{(1-\alpha)}} + o\left(\frac{1}{m^{(1-\alpha)}}\right), m \geq N, \tilde{\alpha} \in i\mathbb{R}, \beta \in \mathbb{R}, \beta < 0.$$

$$\lambda_m = \overline{\lambda_{-m}} \text{ if } m \leq -N.$$

Moreover for all $|m| \geq N$, the eigenvalues λ_k are simple.

Proof. For clarity, the proof is divided into three steps:

Step 1. We have

$$\left(\beta + \varrho\tilde{\lambda}(\tilde{\lambda}+\eta)^{\alpha-1}\right) J_k\left(i\frac{\tilde{\lambda}}{\sqrt{2\mu+\lambda}}\right) + i\frac{\tilde{\lambda}}{\sqrt{2\mu+\lambda}} J_k'\left(i\frac{\tilde{\lambda}}{\sqrt{2\mu+\lambda}}\right) = 0, \tag{2.38}$$

where $\beta = (3\mu+2\lambda)/(2\mu+\lambda)$ and $\varrho = \gamma/(2\mu+\lambda)$. We know that

$$sJ_\nu'(s) = \nu J_\nu(s) - sJ_{\nu+1}(s). \tag{2.39}$$

Then (2.38) is equivalent to

$$\begin{aligned} f(\lambda) &= \left(k+\beta+\varrho\tilde{\lambda}(\tilde{\lambda}+\mu)^{\alpha-1}\right) J_k\left(i\frac{\tilde{\lambda}}{\sqrt{2\mu+\lambda}}\right) - i\frac{\tilde{\lambda}}{\sqrt{2\mu+\lambda}} J_{k+1}\left(i\frac{\tilde{\lambda}}{\sqrt{2\mu+\lambda}}\right) \\ &= -i\frac{\tilde{\lambda}}{\sqrt{2\mu+\lambda}}\left(J_{k+1}\left(i\frac{\tilde{\lambda}}{\sqrt{2\mu+\lambda}}\right) - \frac{\sqrt{2\mu+\lambda}}{i\tilde{\lambda}}\left(k+\beta+\varrho\tilde{\lambda}(\tilde{\lambda}+\mu)^{\alpha-1}\right) J_k\left(i\frac{\tilde{\lambda}}{\sqrt{2\mu+\lambda}}\right)\right) \\ &= 0. \end{aligned} \tag{2.40}$$

We set

$$\tilde{f}(\lambda) = J_{k+1}\left(i\frac{\tilde{\lambda}}{\sqrt{2\mu+\lambda}}\right) - \frac{\sqrt{2\mu+\lambda}}{i\tilde{\lambda}}\left(k+\beta+\varrho\tilde{\lambda}(\tilde{\lambda}+\mu)^{\alpha-1}\right) J_k\left(i\frac{\tilde{\lambda}}{\sqrt{2\mu+\lambda}}\right). \tag{2.41}$$

We will use the following classical asymptotic expansions of Bessels functions (see **[33]** p. 122, (5.11.6)): for all $\delta > 0$, the following development holds when $|\arg z| \le \pi - \delta$:

$$J_\nu(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cos\left(z - \nu\frac{\pi}{2} - \frac{\pi}{4}\right)\left(1 + O(\frac{1}{|z|^2})\right) - \left(\frac{2}{\pi z}\right)^{1/2} \sin\left(z - \nu\frac{\pi}{2} - \frac{\pi}{4}\right) O\left(\frac{1}{|z|^2}\right). \tag{2.42}$$

Then

$$\tilde{f}(\lambda) = \left(\frac{2}{\pi\tilde{z}}\right)^{1/2} \frac{e^{-iz}}{2i}\tilde{\tilde{f}}(\lambda), \tag{2.43}$$

where

$$z = i\frac{\tilde{\lambda}}{\sqrt{2\mu+\lambda}} - k\frac{\pi}{2} - \frac{\pi}{4}$$

$$\tilde{z} = i\frac{\tilde{\lambda}}{\sqrt{2\mu+\lambda}}$$

and

$$\begin{aligned} \tilde{\tilde{f}}(\lambda) &= (e^{2iz}-1) - \frac{\sqrt{2\mu+\lambda}\varrho}{\lambda^{1-\alpha}}(e^{2iz}+1) + o\left(\frac{1}{\lambda^{1-\alpha}}\right) \\ &= f_0(\lambda) + \frac{f_1(\lambda)}{\lambda^{1-\alpha}} + o\left(\frac{1}{\lambda^{1-\alpha}}\right), \end{aligned} \tag{2.44}$$

where

$$f_0(\lambda) = e^{2iz} - 1. \tag{2.45}$$

$$f_1(\lambda) = -\frac{\sqrt{2\mu+\lambda}\varrho}{\lambda^{1-\alpha}}(e^{2iz}+1). \tag{2.46}$$

Note that f_0 and f_1 remain bounded in the strip $-\alpha_0 \le \Re(\lambda) \le 0$.

Step 2. We look at the roots of f_0. From (2.45), f_0 has one family of roots that we denote λ_m^0.

$$f_0(\lambda) = 0 \Leftrightarrow e^{2iz} - 1 = 0$$

Hence

$$2i\left(i\frac{\tilde{\lambda}}{\sqrt{2\mu+\lambda}} - k\frac{\pi}{2} - \frac{\pi}{4}\right) = 2im\pi, \quad m \in \mathbf{Z},$$

i.e.,

$$\lambda_m^0 = -i\sqrt{2\mu+\lambda}\left(m + \frac{k}{2} + \frac{1}{4}\right)\pi, \quad m \in \mathbf{Z}.$$

Now with the help of Rouché's Theorem, we will show that the roots of $\tilde{f}$ are close to those of f_0. Let us start with the first family. Changing in (2.40) the unknown λ by $u = 2iz$ then (2.40) becomes

$$\tilde{f}(u) = (e^u - 1) + O\left(\frac{1}{u^{(1-\alpha)}}\right) = f_0(u) + O\left(\frac{1}{u^{(1-\alpha)}}\right)$$

The roots of f_0 are $u_m = -i\sqrt{2\mu+\lambda}\left(m+\frac{k}{2}+\frac{1}{4}\right)\pi, m \in \mathbf{Z}$, and setting:
$u = u_m + re^{it}, t \in [0, 2\pi]$, we can easily check that there exists a constant $C > 0$ independent of m such that $|e^u + 1| \geq Cr$ for r small enough. This allows to apply Rouché's Theorem.

Consequently, there exists a subsequence of roots of $\tilde{f}$ which tends to the roots u_m of f_0.
Equivalently, it means that there exists $N \in \mathbb{N}$ and a subsequence $\{\lambda_m\}_{|m|\geq N}$ of roots of $f(\lambda)$, such that $\lambda_m = \lambda_m^0 + o(1)$ which tends to the roots $-i\sqrt{2\mu+\lambda}\left(m+\frac{k}{2}+\frac{1}{4}\right)\pi$ of f_0.
Finally, for $|m| \geq N, \lambda_m$ is simple since λ_m^0 is.

Step 3. From Step 2, we can write

$$\lambda_m = -i\sqrt{2\mu+\lambda}\left(m+\frac{k}{2}+\frac{1}{4}\right)\pi + \varepsilon_m. \tag{2.47}$$

Using (2.47), we get

$$\begin{aligned} e^{2iz_m} &= e^{-\frac{2}{\sqrt{2\mu+\lambda}}\varepsilon_k} \\ &= 1 - \frac{2}{\sqrt{2\mu+\lambda}}\varepsilon_m + O(\varepsilon_m^2). \end{aligned} \tag{2.48}$$

Substituting (2.48) into (2.44), using that $\tilde{\tilde{f}}(\lambda_k) = 0$, we get:

$$\tilde{f}(\lambda_m) = -\frac{2}{\sqrt{2\mu+\lambda}}\varepsilon_m - \frac{2\sqrt{2\mu+\lambda}\varrho}{(-\sqrt{2\mu+\lambda}im\pi)^{1-\alpha}} + o(\varepsilon_m) + o\left(\frac{1}{m^{1-\alpha}}\right) = 0 \tag{2.49}$$

and hence

$$\begin{aligned} \varepsilon_m &= -\frac{(2\mu+\lambda)\varrho}{(-\sqrt{2\mu+\lambda}im\pi)^{1-\alpha}} + o\left(\frac{1}{m^{1-\alpha}}\right) \\ &= \begin{cases} -\sqrt{2\mu+\lambda}^{(\alpha-1)}\dfrac{\gamma}{(m\pi)^{1-\alpha}}\left(\cos(1-\alpha)\dfrac{\pi}{2} - i\sin(1-\alpha)\dfrac{\pi}{2}\right) + o\left(\dfrac{1}{m^{1-\alpha}}\right) & \text{for } m \succeq 0, \\ -\sqrt{2\mu+\lambda}^{(\alpha-1)}\dfrac{\gamma}{(-m\pi)^{1-\alpha}}\left(\cos(1-\alpha)\dfrac{\pi}{2} + i\sin(1-\alpha)\dfrac{\pi}{2}\right) + o\left(\dfrac{1}{m^{1-\alpha}}\right) & \text{for } m \preceq 0. \end{cases} \end{aligned} \tag{2.50}$$

From (2.50) we have in that case $|m|^{1-\alpha}\Re\lambda_m \sim \tilde{\beta}$, with

$$\tilde{\beta} = -\sqrt{2\mu+\lambda}^{(\alpha-1)}\frac{\gamma}{\pi^{1-\alpha}}\cos(1-\alpha)\frac{\pi}{2}.$$

The operator $\mathcal{A}$ has a non exponential decaying branche of eigenvalues. Thus the proof is complete.

2.5 Strong stability

One simple way to prove the strong stability of (2.13) is to use the following theorem due to Arendt-Batty and Lyubich-Vũ (see [**6**] and [**34**]).

Theorem 2.5.1 ([6]-[34]) *Let X be a reflexive Banach space and $(T(t))_{t\geq 0}$ be a C_0-semigroup generated by A on X. Assume that $(T(t))_{t\geq 0}$ is bounded and that no eigenvalues of A lie on the imaginary axis. If $\sigma(A)\cap iR$ is countable, then $(T(t))_{t\geq 0}$ is strongly stable.*

Our main result is the following theorem

Theorem 2.5.2 *The C_0-semigroup $e^{t\mathcal{A}}$ is strongly stable in $\mathcal{H}$; i.e, for all $U_0 \in \mathcal{H}$, the solution of (2.13) satisfies*

$$\lim_{t\to\infty} \|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} = 0.$$

For the proof of Theorem 2.5.2, we need the following two lemmas.

Lemma 2.5.1 *$\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.*

Proof.
Case 1. For $\tilde{\lambda} \neq 0$.
We will argue by contraction. Let us suppose that there $\tilde{\lambda} \in \mathbb{R}, \tilde{\lambda} \neq 0$ and $U \neq 0$, such that

$$\mathcal{A}U = i\tilde{\lambda}U. \tag{2.51}$$

Then, we get

$$\begin{cases} i\tilde{\lambda}u - v = 0, \quad x \in \Omega \\ i\tilde{\lambda}v - \mu\Delta u - (\mu+\lambda)\nabla(\text{div } u) = 0, \quad x \in \Omega \\ i\tilde{\lambda}\phi + (\xi^2+\eta)\phi - v(x)\mu(\xi) = 0, \quad x \in \Gamma_1. \end{cases} \tag{2.52}$$

Next, a straightforward computation gives

$$\Re(\mathcal{A}U, U) = -\zeta \sum_{i=1}^{n} \int_{\Gamma_1} \int_{-\infty}^{\infty} (\xi^2+\eta)|\phi_i(x,\xi,t)|^2 \, d\xi \, d\Gamma. \tag{2.53}$$

Then, from (2.53) we have

$$\phi = 0 \quad \text{on } \Gamma_1 \times (-\infty, \infty). \tag{2.54}$$

From $(2.52)_3$, we have

$$v(x) = 0, \quad \text{on } \Gamma_1. \tag{2.55}$$

Hence, from $(2.52)_1$ and $(2.16)_3$ we obtain

$$u(x) = 0 \quad \text{on } \Gamma \quad \text{and} \quad \mu\frac{\partial u}{\partial \nu}(x) + (\mu+\lambda)(\text{div } u)(x)\nu = 0 \quad \text{on } \Gamma_1. \tag{2.56}$$

Thus, by eliminating v, the system (2.52) implies that

$$\begin{cases} \tilde{\lambda}^2 u + \mu\Delta u + (\mu+\lambda)\nabla(\text{div } u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \\ \mu\dfrac{\partial u}{\partial \nu} + (\mu+\lambda)(\text{div } u)\nu = 0 & \text{on } \Gamma_1. \end{cases} \tag{2.57}$$

Therefore, using Holmgren's theorem, we deduce that $u = 0$ and consequently, $U = 0$.

Case 2. $\tilde{\lambda} = 0$. The system (2.52) becomes

$$\begin{cases} v = 0, \quad x \in \Omega \\ \mu\Delta u + (\mu+\lambda)\nabla(\text{div } u) = 0, \quad x \in \Omega \\ (\xi^2+\eta)\phi - v(x)\mu(\xi) = 0, \quad x \in \Gamma_1 \end{cases} \tag{2.58}$$

Multiplying the second equation of (2.58) by $\bar{u}$, integrating by parts over Ω and using the boundary conditions $u = 0$ on Γ_0, we get

$$\mu\sum_{j=1}^{n}\int_\Gamma \frac{\partial u_j}{\partial\nu}\bar{u}_j\,d\Gamma - \mu\sum_{j=1}^{n}\int_\Omega |\nabla u_j|^2\,dx + (\mu+\lambda)\sum_{j=1}^{n}\int_\Gamma (\text{div } u)\nu_j\bar{u}_j\,d\Gamma - (\mu+\lambda)\int_\Omega |(\text{div } u)|^2\,dx = 0$$

Now, from $(2.58)_3$ and $(2.16)_3$, we deduce that

$$-\mu\sum_{j=1}^{n}\int_\Omega |\nabla u_j|^2\,dx - (\mu+\lambda)\int_\Omega |(\text{div } u)|^2\,dx = 0.$$

Hence u is constant in the whole domain Ω. Therefore as Γ_0 is non empty, we have

$$u = 0, \text{ on } \Omega$$

It follows that $U = 0$.Consequently, $\mathcal{A}$ does not have purely imaginary eigenvalues.

The second condition of Theorem 2.5.1 will be satisfied if we show that $\sigma(\mathcal{A}) \cap \{i\mathbb{R}\}$ is at most a countable set. We have the following lemma.

Lemma 2.5.2 *We have*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \text{ if } \eta \neq 0,$$
$$i\mathbb{R}^* \subset \rho(\mathcal{A}) \text{ if } \eta = 0$$

where $\mathbb{R}^* = \mathbb{R} - \{0\}$.

Proof. We will prove that the operator $(i\tilde{\lambda}I - \mathcal{A})$ is surjective for $\tilde{\lambda} \neq 0$.
For this purpose, let $F = (F_1, F_2, F_3)^T \in \mathcal{H}$, we seek $X = (u, v, \phi)^T \in D(\mathcal{A})$ solution of the following equation

$$(i\tilde{\lambda}I - \mathcal{A})X = F. \tag{2.59}$$

Equivalently, we have

$$\begin{cases} i\tilde{\lambda}u - v = F_1(x), \\ i\tilde{\lambda}v - \mu\Delta u - (\mu+\lambda)\nabla(\text{div } u) = F_2(x), \\ i\tilde{\lambda}\phi + (\xi^2+\eta)\phi - v(x)\mu(\xi) = F_3(x,\xi). \end{cases} \tag{2.60}$$

From $(2.60)_1$ and $(2.60)_2$, we have

$$-\tilde{\lambda}^2 u - \mu\Delta u - (\mu+\lambda)\nabla(\text{div } u) = \Big(F_2(x) + i\tilde{\lambda}F_1(x)\Big) \tag{2.61}$$

with $u_{|\Gamma_0} = 0$. Solving system (2.61) is equivalent to finding $u \in \left(H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)\right)$ such that

$$\begin{cases} \displaystyle\sum_{j=1}^{n} \int_\Omega \left(-\tilde{\lambda}^2 u_j w_j + \mu \nabla u_j \nabla w_j\right)\, dx + \int_\Omega (\mu+\lambda)(\text{div } u)(\text{div } w)\, dx + i\tilde{\lambda}\tilde{\zeta} \sum_{j=1}^{n} \int_{\Gamma_1} u_j w_j \, d\Gamma \\ \displaystyle\quad = \sum_{j=1}^{n} \int_\Omega (F_2^j(x) + i\tilde{\lambda} F_1^j(x)) w_j \, dx - \zeta \sum_{j=1}^{n} \int_{\Gamma_1} w_i \left(\int_{-\infty}^{\infty} \frac{\mu(\xi) F_3^j(x,\xi)}{\xi^2+\eta+i\tilde{\lambda}}\, d\xi\right) d\Gamma \\ \displaystyle\qquad + \tilde{\zeta} \sum_{j=1}^{n} \int_{\Gamma_1} F_1^j(x) w_j \, d\Gamma, \end{cases} \tag{2.62}$$

For all $w \in \left(H^1_{\Gamma_0}(\Omega)\right)^n$, where $\tilde{\zeta} = \zeta \displaystyle\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2+\eta+i\tilde{\lambda}}\, d\xi$.

We can rewrite (2.62) as

$$-(L_{\tilde{\lambda}} u, w)_{((H^1_{\Gamma_0}(\Omega))^n, ((H^1_{\Gamma_0}(\Omega))')^n)} + a_{(H\Gamma_0{}^1(\Omega)}(u,w) = l(w) \tag{2.63}$$

with the bilinear form defined by

$$a_{(H^1_{\Gamma_0}(\Omega))}(u,w) = \mu \sum_{i=1}^{n} \int_\Omega \nabla u_i \nabla w_i \, dx + (\mu+\lambda) \int_\Omega (\text{div } u)(\text{div } w)\, dx + i\tilde{\lambda}\tilde{\zeta} \sum_{j=1}^{n} \int_{\Gamma_1} u_j w_j \, d\Gamma$$

and

$$(L_{\tilde{\lambda}} u, w)_{H^1_{\Gamma_0}} = \sum_{i=1}^{n} \int_\Omega \tilde{\lambda}^2 u_j \overline{w}_j \, dx.$$

Using the compactness embedding from $L^2(\Omega)$ into $H^{-1}(\Omega)$ and from $H^1_{\Gamma_0}(\Omega)$ into $L^2(\Omega)$ we deduce that the operator $L_{\tilde{\lambda}}$ is compact from $(L^2(\Omega))^n$ into $(L^2(\Omega))^n$. Consequently, by Fredholm alternative, proving the existence of u solution of (2.63) reduces to proving that there is not a nontrivial solution for (2.63) for $l \equiv 0$. Indeed if there exists $u \neq 0$, such that

$$(L_{\tilde{\lambda}} u, w)_{((H^1_{\Gamma_0}(\Omega))^n, ((H^1_{\Gamma_0}(\Omega))')^n)} = a_{(H^1_{\Gamma_0}(\Omega))}(u,w), \quad \forall w \in H^1_{\Gamma_0}(\Omega). \tag{2.64}$$

This means that $i\tilde{\lambda}$ is an eigenvalue of $\mathcal{A}$. Therefore from Lemma 2.5.1 we deduce that $u = 0$.

Now, if $\tilde{\lambda} = 0$ and $\eta \neq 0$, the system (2.60) is reduced to the following system

$$\begin{cases} v = -F_1(x), \\ \mu \Delta u + (\mu+\lambda)\nabla(\text{div } u) = -F_2(x), \\ (\xi^2+\eta)\phi - v(x)\mu(\xi) = F_3(x,\xi). \end{cases} \tag{2.65}$$

Solving system (2.65) is equivalent to finding $u \in \left(H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)\right)^n$ such that

$$\begin{cases} \displaystyle\sum_{j=1}^{n} \int_\Omega \mu \nabla u_j \nabla w_j dx + (\mu+\lambda) \int_\Omega (\text{div } u)(\text{div } w) dx = \sum_{j=1}^{n} \int_\Omega F_2^j w_j dx + \tilde{\zeta} \sum_{j=1}^{n} \int_{\Gamma_1} F_1^j w_j \, d\Gamma \\ \displaystyle\qquad - \zeta \sum_{j=1}^{n} \int_{\Gamma_1} w_j \left(\int_{-\infty}^{\infty} \frac{\mu(\xi) F_3^j(x,\xi)}{\xi^2+\eta}\, d\xi\right) d\Gamma \end{cases} \tag{2.66}$$

For all $w \in (H^1_{\Gamma_0}(\Omega))^n$.

Consequently, problem (2.66) is equivalent to the problem

$$\mathcal{B}(u,w) = \mathcal{L}(w), \tag{2.67}$$

where $\mathcal{B} : \left[\left(H^1_{\Gamma_0}(\Omega)\right)^n \times \left(H^1_{\Gamma_0}(\Omega)\right)^n\right] \to \mathbb{C}$, is the bilinear form defined by

$$\mathcal{B}(u,w) = \sum_{i=1}^{n} \int_\Omega \mu \nabla u_i \nabla w_i \, dx + (\mu+\lambda) \int_\Omega (\text{div } u)(\text{div } w) \, dx \tag{2.68}$$

and $\mathcal{L} : \left(H^1_{\Gamma_0}(\Omega)\right)^n \to \mathbb{C}$ is the linear form defined by

$$\mathcal{L}(w) = \sum_{j=1}^{n} \int_\Omega F_2^j w_j \, dx + \tilde{\zeta} \sum_{j=1}^{n} \int_{\Gamma_1} F_1^j w_j \, d\Gamma - \zeta \sum_{j=1}^{n} \int_{\Gamma_1} w_j \left(\int_{-\infty}^{\infty} \frac{\mu(\xi) F_3^j(x,\xi)}{\xi^2+\eta} \, d\xi \right) d\Gamma.$$

It is easy to verify that $\mathcal{B}$ is continuous and coercive, and $\mathcal{L}$ is continuous. So applying the Lax-Milgram theorem, we deduce that for all $w \in \left(H^1_{\Gamma_0}(\Omega)\right)^n$ problem (2.67) admits a unique solution $u \in \left(H^1_{\Gamma_0}(\Omega)\right)^n$. Applying the classical elliptic regularity, it follows from (2.66) that $u \in (H^2(\Omega))^n$. Therefore, the operator $\mathcal{A}$ is surjective.

Residual spectrum of $\mathcal{A}$

Lemma 2.5.3 *Let $\mathcal{A}$ be defined by (2.14). Then*

$$\mathcal{A}^* \begin{pmatrix} u \\ v \\ \phi \end{pmatrix} = \begin{pmatrix} -v \\ -\mu\Delta u - (\mu+\lambda)\nabla(\ div\ u) \\ -(\xi^2+\eta)\phi - v(x)\mu(\xi) \end{pmatrix} \tag{2.69}$$

with domain

$$D(\mathcal{A}^*) = \left\{ \begin{array}{l} (u,v,\phi)^T \ in\ \mathcal{H} : u \in \left(H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)\right)^n, v \in \left(H^1_{\Gamma_0}(\Omega)\right)^n, \\ -(\xi^2+\eta)\phi - v(x)\mu(\xi) \in \left(L^2(\Gamma_1 \times (-\infty,+\infty))\right)^n, \\ \mu\dfrac{\partial u}{\partial \nu} + (\mu+\lambda)(\ div\ u)\nu + \zeta \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi = 0,\ on \Gamma_1 \\ |\xi|\phi \in \left(L^2(\Gamma_1 \times (-\infty,+\infty))\right)^n \end{array} \right\}. \tag{2.70}$$

Proof. Let $U=(u,v,\phi)^T$ and $V=(\tilde{u},\tilde{v},\tilde{\phi})^T$. We have: $<\mathcal{A}U,V>_{\mathcal{H}}=<U,\mathcal{A}^*V>_{\mathcal{H}}$.

$$\begin{aligned}
<\mathcal{A}U,V>_{\mathcal{H}}=&\sum_{j=1}^{n}\int_{\Omega}(\mu\tilde{v}_j\Delta u_j+(\mu+\lambda)\tilde{v}_j\nabla(\operatorname{div} u)+\mu\nabla\tilde{u}_j\nabla u_j)\,dx+(\mu+\lambda)\int_{\Omega}(\operatorname{div} v)(\operatorname{div}\tilde{u})dx\\
&+\zeta\sum_{j=1}^{n}\int_{\Gamma_1}\int_{-\infty}^{+\infty}[-(\xi^2+\eta)\phi+v(x)\mu(\xi)]\tilde{\phi}\,d\xi\,d\Gamma\\
=&\,\mu\sum_{j=1}^{n}\int_{\Gamma_1}\tilde{v}_j\frac{\partial u}{\partial\nu}\,dx-\mu\sum_{j=1}^{n}\int_{\Omega}\nabla u_j\nabla\tilde{v}_j\,dx+(\mu+\lambda)\sum_{j=1}^{n}\int_{\Gamma_1}\tilde{v}_j(\operatorname{div} u)\nu\,dx\\
&-(\mu+\lambda)\int_{\Omega}(\operatorname{div} u)(\operatorname{div}\tilde{v})\,dx+\mu\sum_{j=1}^{n}\int_{\Gamma_1}v\frac{\partial\tilde{u}}{\partial\nu}\,d\Gamma-\mu\sum_{j=1}^{n}\int_{\Omega}v_j\Delta\tilde{u}_j\,dx\\
&+(\mu+\lambda)\sum_{j=1}^{n}\int_{\Gamma_1}v_j(\operatorname{div}\tilde{u})\nu\,d\Gamma-(\mu+\lambda)\sum_{j=1}^{n}\int_{\Omega}v_j\nabla(\operatorname{div}\tilde{u})\,dx\\
&-\zeta\int_{\Gamma_1}\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi\tilde{\phi}\,d\xi\,d\Gamma+\zeta\int_{\Gamma_1}v(x)\int_{-\infty}^{+\infty}\mu(\xi)\tilde{\phi}\,d\xi\,d\Gamma
\end{aligned}$$

If we set

$$\mu\frac{\partial\tilde{u}}{\partial\nu}=-\zeta\int_{-\infty}^{+\infty}\mu(\xi)\tilde{\phi}(x,t)\,d\xi-(\mu+\lambda)(\operatorname{div} u)\nu,$$

we get

$$\begin{aligned}
<\mathcal{A}U,V>_{\mathcal{H}}=&-\mu\sum_{j=1}^{n}\int_{\Omega}\nabla u_j\nabla\tilde{v}_j\,dx-(\mu+\lambda)\int_{\Omega}(\operatorname{div} u)(\operatorname{div}\tilde{v})\,dx-\mu\sum_{j=1}^{n}\int_{\Omega}v_j\Delta\tilde{u}_j\,dx\\
&-(\mu+\lambda)\sum_{j=1}^{n}\int_{\Omega}v_j\nabla(\operatorname{div}\tilde{u})\,dx-\zeta\int_{\Gamma_1}\int_{-\infty}^{+\infty}[(\xi^2+\eta)\tilde{\phi}+\tilde{v}(x)\mu(\xi)]\,d\xi\,d\Gamma
\end{aligned}$$

Theorem 2.5.3 *$\sigma_r(\mathcal{A})=\emptyset$, where $\sigma_r(\mathcal{A})$ denotes the set of residual spectrum of $\mathcal{A}$. It is defined as*

$$\sigma_r(\mathcal{A})=\{\tilde{\lambda}\in\mathbb{C}:ker(\lambda I-\mathcal{A})=0\ and\ Im(\tilde{\lambda}I-\mathcal{A})\ is\ not\ dense\ in\ \mathcal{H}\}.$$

Proof. Since $\tilde{\lambda}\in\sigma_r(\mathcal{A}),\overline{\lambda}\in\sigma_r(\mathcal{A}^*)$ the proof will be accomplished if we can show that $\sigma_r(\mathcal{A})=\sigma_r(\mathcal{A}^*)$. This is because obviously the eigenvalues of $\mathcal{A}$ are symmetric on the real axis. From (2.69), the eigenvalue problem $\mathcal{A}^*Z=\lambda Z$ for $\lambda\in\mathbb{C}$ and $0\neq Z=(u,v,\phi)\in D(\mathcal{A}^*)$ we have

$$\begin{cases}\tilde{\lambda}u+v=0,\\ \tilde{\lambda}v+\mu\Delta u+(\mu+\lambda)\nabla(\operatorname{div} u)=0,\\ \tilde{\lambda}\phi+(\xi^2+\eta)\phi+v(x)\mu(\xi)=0.\end{cases}\tag{2.71}$$

From $(2.71)_1$ and $(2.71)_2$, we find

$$\tilde{\lambda}^2u-\mu\Delta u-(\mu+\lambda)\nabla(\operatorname{div} u)=0.\tag{2.72}$$

As $v_{|\Gamma_1}=-\tilde{\lambda}u_{|\Gamma_1}$, we deduce from $(2.71)_3$ and $(2.70)_3$ that

$$\mu\frac{\partial u}{\partial\nu}+(\mu+\lambda)(\operatorname{div} u)\nu=-\zeta\tilde{\lambda}\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}\,d\xi u(x),\quad\forall x\in\Gamma_1\tag{2.73}$$

with the following conditions

$$u_{|\Gamma_1} = 0. \tag{2.74}$$

Hence $\mathcal{A}^*$ has the same eigenvalues with $\mathcal{A}$, and this completes The proof.

Remark 2.5.1 *When $\eta = 0$, then $\tilde{\lambda} = 0$ is in the continuous spectrum.*

Indeed, let $u_k \in H^1_{\Gamma_0}(\Omega)$ be an eigenfunction of the following problem

$$\begin{cases} -\mu\Delta u_k - (\mu+\lambda)\nabla(\text{ div } u_k) &= \omega_k^2 u_k \\ u_k &= 0 \text{ on } \Gamma_0 \\ \mu\dfrac{\partial u_k}{\partial \nu} + (\mu+\lambda)(\text{div } u_k)\nu &= 0 \end{cases}$$

such that

$$\|u_k\|^2_{H^1_{\Gamma_0}} = \int_{\Gamma_0} |\nabla u_k|^2 \, d\Gamma.$$

Now, we define the vector $F = (u_k, 0, 0) \in \mathcal{H}$. we suppose that there exists $U = (u, v, \phi) \in D(\mathcal{A})$ such that

$$-\mathcal{A}U = F.$$

It follows that

$$(EV) \qquad v = -u_k \text{ in } \Omega, \quad |\xi|^2\phi + \mu(\xi)v = 0 \text{ on } \Gamma_1$$

and

$$\begin{cases} -\mu\Delta u - (\mu+\lambda)\nabla(\text{ div } u) &= 0 \text{ on } \Omega \\ u &= 0 \text{ on } \Gamma_0 \\ \mu\dfrac{\partial u}{\partial \nu} + (\mu+\lambda)(\text{div } u)\nu &= 0 \text{ on } \Gamma_1 \end{cases}$$

From (EV), we deduce that $\phi(x,\xi) = |\xi|^{\frac{2\alpha-5}{2}} u_k|_{\Gamma_1}$. Clearly, for $\alpha \in (0,1)$, $\phi \notin L^{\Gamma_1 \times \mathbb{R}}$.

2.6 Polynomial stability (for $\eta \neq 0$)

In order to establish the polynomial energy decay rate, let us consider the usual geometrical control condition: there exists a point $x_0 \in \mathbb{R}^n$ such that

$$m \cdot \nu \leq 0 \text{ on } \Gamma_0, \qquad m \cdot \nu > 0 \text{ on } \Gamma_1, \tag{2.75}$$

where $m = x - x_0$.

Moreover, we use a recent result by Borichev and Tomilov **[13]**. Accordingly, if we consider a bounded C_0-semigroup $S_{\mathcal{A}}(t) = e^{\mathcal{A}t}$ on a Hilbert space. If

$$i\mathbb{R} \subset \rho(\mathcal{A}) \text{ and } \sup_{|\beta|\geq 1} \frac{1}{\beta^\delta} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < M$$

for some $\delta > 0$, then there exist c such that

$$\|e^{\mathcal{A}t}U_0\|^2 \leq \frac{c}{t^{\frac{2}{\delta}}} \|U_0\|^2_{D(\mathcal{A})}.$$

Our main result is as follows.

Theorem 2.6.1 *If $\varepsilon > 0$, then the semigroup $S_{\mathcal{A}}(t)_{t\geq 0}$ is polynomially stable and*

$$E(t) = \|S_{\mathcal{A}}(t)U_0\|^2_{\mathcal{H}} \leq \frac{1}{t^{\frac{2}{(1-\alpha)+\frac{\varepsilon}{2}}}}\|U_0\|^2_{D(\mathcal{A})}.$$

Proof. We will need to study the resolvent equation $(i\tilde{\lambda} - \mathcal{A})U = F$, for $\lambda \in \mathbb{R}$, namely

$$\begin{cases} i\tilde{\lambda}u - v = F_1, \\ i\tilde{\lambda}v - \mu\Delta u - (\mu+\lambda)\nabla(\text{ div } u) = F_2, \\ i\tilde{\lambda}\phi + (\xi^2+\eta)\phi - v(x)\mu(\xi) = F_3, \end{cases} \tag{2.76}$$

where $F = (F_1, F_2, F_3)^T$. Taking inner product in $\mathcal{H}$ with U and using (2.18) we get

$$|\Re\langle\mathcal{A}U, U\rangle| \leq \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{2.77}$$

This implies that

$$\zeta\int_{\Gamma_1}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi_j(x,\xi,t)|^2\,d\xi\,d\Gamma \leq \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{2.78}$$

Moreover, from the boundary condition $(P)_3$, we have

$$\begin{aligned} &\int_{\Gamma_1}\left|\mu\frac{\partial u_j}{\partial\nu} + (\mu+\lambda)(\text{div u})\nu_j\right|^2 d\Gamma \\ &\qquad\leq \zeta^2\int_{\Gamma_1}\left|\int_{-\infty}^{+\infty}\mu(\xi)\phi_j(x,\xi)\,d\xi\right|^2 d\Gamma \\ &\qquad\leq \zeta^2\left(\int_{-\infty}^{+\infty}(\xi^2+\eta)^{-1}|\mu(\xi)|^2\,d\xi\right)\int_{\Gamma_1}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi_j(x,\xi)|^2\,d\xi\,d\Gamma \\ &\qquad\leq c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \end{aligned} \tag{2.79}$$

From $(2.76)_3$, we obtain

$$v(x)\mu(\xi) = (i\tilde{\lambda}+\xi^2+\eta)\phi(x,\xi) - F_3(x,\xi), \quad \forall x \in \Gamma_1. \tag{2.80}$$

By multiplying (2.80) by $(i\tilde{\lambda}+\xi^2+\eta)^{-1}|\xi|^{\frac{1-\varepsilon}{2}}$, we get

$$(i\tilde{\lambda}+\xi^2+\eta)^{-1}v(x)|\xi|^{\frac{1-\varepsilon}{2}}\mu(\xi) = |\xi|^{\frac{1-\varepsilon}{2}}\phi - (i\tilde{\lambda}+\xi^2+\eta)^{-1}|\xi|^{\frac{1-\varepsilon}{2}}F_3(x,\xi), \quad x \in \Gamma_1. \tag{2.81}$$

Hence, by taking absolute values of both sides of (2.81), integrating over the interval $]-\infty,+\infty[$ with respect to the variable ξ, applying Cauchy-Schwartz inequality, we obtain

$$\frac{1}{\sqrt{2}}\mathcal{S}|v_j(x)| \leq \mathcal{U}\left(\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi_j(x,\xi)|^2\,d\xi\right)^{\frac{1}{2}} + \sqrt{2}\mathcal{V}\left(\int_{-\infty}^{+\infty}|F_3^j(x,\xi)|^2\,d\xi\right)^{\frac{1}{2}}, \tag{2.82}$$

where

$$\mathcal{S} = \int_{-\infty}^{+\infty} (|\tilde{\lambda}| + \xi^2 + \eta)^{-1} |\xi|^{\frac{1-\varepsilon}{2}} |\mu(\xi)| \, d\xi = \frac{\pi}{|\sin(\frac{\alpha+2}{2} - \frac{\varepsilon}{4})\pi|} \left(|\tilde{\lambda}| + \eta\right)^{\frac{\alpha+2}{2} - \frac{\varepsilon}{4}},$$

$$\mathcal{U} = \left(\int_{-\infty}^{+\infty} (\xi^2 + \eta)^{-1} |\xi|^{1-\varepsilon} \, d\xi \right)^{\frac{1}{2}},$$

$$\mathcal{V} = \left(\int_{-\infty}^{+\infty} (|\tilde{\lambda}| + \xi^2 + \eta)^{-2} |\xi|^{1-\varepsilon} \, d\xi \right)^{\frac{1}{2}} = \left(\frac{\varepsilon}{2} \frac{\pi}{\sin(\frac{2-\varepsilon}{2})\pi} \left(|\tilde{\lambda}| + \eta\right)^{-\frac{\varepsilon}{2} - 1} \right)^{\frac{1}{2}}.$$

Thus, by using again the inequality $2PQ \leq P^2 + Q^2, P \geq 0, Q \geq 0$, we get

$$\mathcal{S}^2 \int_{\Gamma_1} |v_j(x)|^2 \, d\Gamma \leq 4\mathcal{U}^2 \int_{\Gamma_1} \int_{-\infty}^{+\infty} (\xi^2 + \eta) |\phi_j|^2 \, d\xi \, d\Gamma + 8\mathcal{V}^2 \int_{\Gamma_1} \int_{-\infty}^{+\infty} |F_3^j(x, \xi)|^2 \, d\xi \, d\Gamma. \tag{2.83}$$

We conclude that

$$\int_{\Gamma_1} |v_j(x)|^2 \, d\Gamma \leq c |\tilde{\lambda}|^{1-\alpha+\frac{\varepsilon}{2}} \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}} + c |\tilde{\lambda}|^{1-\alpha} \|F\|_{\mathcal{H}}^2. \tag{2.84}$$

Let us introduce the following notation

$$\mathcal{I}_u(x) = \sum_{j=1}^{n} \left(|v_j(x)|^2 + \mu |\nabla u_j(x)|^2 \right) + (\mu + \lambda) |\text{div } u(x)|^2$$

and

$$\mathcal{E}_u = \int_{\Omega} \mathcal{I}_u(x) \, dx.$$

Lemma 2.6.1 *We have that*

$$\mathcal{E}_u \leq c \|F\|_{\mathcal{H}}^2 + c' \sum_{j=1}^{n} \int_{\Gamma_1} \left| \mu \frac{\partial u_j}{\partial \nu} + (\mu + \lambda)(div \ u) \nu_j \right|^2 d\Gamma + c'' \sum_{j=1}^{n} \int_{\Gamma_1} (m \cdot \nu) |v_j|^2 \, d\Gamma \tag{2.85}$$

for positive constants c, c' and c''.

Proof. Multiplying $(2.76)_2$ by $\bar{u}$,integrating on Ω we obtain

$$\begin{aligned} -\int_{\Omega} v_j (\overline{i\tilde{\lambda} u_j}) \, dx + \mu \int_{\Omega} |\nabla u_j|^2 \, dx + (\mu + \lambda) \int_{\Omega} (\text{div } u) \frac{\partial \bar{u}_j}{\partial x_j} \, dx - \mu \int_{\Gamma_1} \overline{u_j} \frac{\partial u_j}{\partial \nu} \, d\Gamma \\ -(\mu + \lambda) \int_{\Gamma_1} \nu_j \bar{u}_j (\text{div } u) \, d\Gamma = \int_{\Omega} \bar{u} F_2^j \, dx. \end{aligned} \tag{2.86}$$

From $(2.76)_1$, we have $i\tilde{\lambda} u_j = v_j + F_1^j$. Then

$$\begin{aligned} -\int_{\Omega} |v_j|^2 \, dx + \mu \int_{\Omega} |\nabla u_j|^2 \, dx + (\mu + \lambda) \int_{\Omega} (\text{div } u) \frac{\partial \bar{u}_j}{\partial x_j} \, dx - \mu \int_{\Gamma_1} \overline{u_j} \frac{\partial u_j}{\partial \nu} \, d\Gamma \\ -(\mu + \lambda) \int_{\Gamma_1} \nu_j \bar{u}_j (\text{div } u) \, d\Gamma = \int_{\Omega} \bar{u}_j F_2^j \, dx + \int_{\Omega} v_j \overline{F}_1^j \, dx. \end{aligned} \tag{2.87}$$

Hence

$$
\begin{aligned}
-\sum_{j=1}^{n}\int_{\Omega}|v_j|^2\,dx+\mu\sum_{j=1}^{n}\int_{\Omega}|\nabla u_j|^2\,dx+(\mu+\lambda)\int_{\Omega}|\text{div }u|^2\,dx-\mu\sum_{j=1}^{n}\int_{\Gamma_1}\overline{u_j}\frac{\partial u_j}{\partial \nu}\,d\Gamma\\
-(\mu+\lambda)\sum_{j=1}^{n}\int_{\Gamma_1}\nu_j\bar{u}_j(\text{div }u)\,d\Gamma=\sum_{j=1}^{n}\int_{\Omega}\bar{u}_jF_2^j\,dx+\sum_{j=1}^{n}\int_{\Omega}v_j\overline{F}_1^j\,dx.
\end{aligned}
\tag{2.88}
$$

Multiplying $(2.76)_2$ by $(2m\cdot\nabla\bar{u})$, integrating on Ω we obtain

$$
\begin{aligned}
-2\int_{\Omega}v_j(m\cdot\overline{i\tilde{\lambda}\nabla u_j})\,dx-2\mu\int_{\Omega}\Delta u_j(m\cdot\nabla\bar{u}_j)\,dx-2(\mu+\lambda)\int_{\Omega}\frac{\partial(\text{div }u)}{\partial x_j}(m\cdot\nabla\bar{u}_j)\,dx\\
=2\int_{\Omega}F_2^j(m\cdot\nabla\bar{u}_j)\,dx.
\end{aligned}
\tag{2.89}
$$

From $(2.76)_2$, we have $i\tilde{\lambda}\nabla u-\nabla v=\nabla F_1$, then

$$
\begin{aligned}
-2\int_{\Omega}v_j(m\cdot\nabla\bar{v}_j)\,dx-2\mu\int_{\Omega}\Delta u_j(m\cdot\nabla\bar{u}_j)\,dx-2(\mu+\lambda)\int_{\Omega}\frac{\partial(\text{div }u)}{\partial x_j}(m\cdot\nabla\bar{u}_j)\,dx\\
=2\int_{\Omega}F_2^j(m\cdot\nabla\bar{u}_j)\,dx+2\int_{\Omega}v_j(m\cdot\nabla\overline{F}_1^j)\,dx.
\end{aligned}
\tag{2.90}
$$

For $u\in H^2(\Omega)$, we have the following Rellich's identity

$$
\begin{aligned}
\int_{\Omega}\Delta u_j(m\cdot\nabla\bar{u}_j)\,dx=\int_{\Gamma}(m\cdot\nabla\bar{u}_j)\frac{\partial u_j}{\partial\nu}\,d\Gamma-\int_{\Omega}\nabla u_j\cdot\nabla(m\cdot\nabla\bar{u}_j)\,dx.\\
\int_{\Omega}\frac{\partial(\text{div }u)}{\partial x_j}(m\cdot\nabla\bar{u}_j)\,dx=\int_{\Gamma}(m\cdot\nabla\bar{u}_j)(\text{div }u)\nu_j\,d\Gamma-\int_{\Omega}(\text{div }u)\frac{\partial}{\partial x_j}(m\cdot\nabla\bar{u}_j)\,dx.
\end{aligned}
\tag{2.91}
$$

Moreover, using the following identity

$$
2\Re\nabla u_j\cdot\nabla(m\cdot\nabla\bar{u}_j)=2|\nabla u_j|^2+m\cdot\nabla(|\nabla u_j|^2)
$$

and integration by parts, we get

$$
\begin{aligned}
2\Re\int_{\Omega}\nabla u_j\cdot\nabla(m\cdot\nabla\bar{u}_j)\,dx=(2-n)\int_{\Omega}|\nabla u_j|^2\,dx+\int_{\Gamma}m\cdot\nu|\nabla u_j|^2\,d\Gamma\\
2\sum_{j=1}^{n}\Re\int_{\Omega}(\text{div }u)\frac{\partial}{\partial x_j}(m\cdot\nabla\bar{u}_j)\,dx=(2-n)\int_{\Omega}|\text{div }u|^2\,dx+\int_{\Gamma}m\cdot\nu|\text{div }u|^2\,d\Gamma.
\end{aligned}
\tag{2.92}
$$

From (2.90), (2.91) and (2.92), we get

$$
\begin{aligned}
n\sum_{j=1}^{n}\int_{\Omega}|v_j|^2\,dx+(2-n)\mu\sum_{j=1}^{n}\int_{\Omega}|\nabla u_j|^2\,dx+(2-n)(\mu+\lambda)\sum_{j=1}^{n}\int_{\Omega}|\text{div }u|^2\,dx\\
=2\sum_{j=1}^{n}\Re\int_{\Omega}F_2^j(m\cdot\nabla\bar{u}_j)\,dx+2\sum_{j=1}^{n}\Re\int_{\Omega}v_j(m\cdot\nabla\overline{F}_1^j)\,dx+\sum_{j=1}^{n}\int_{\Gamma_1}(m\cdot\nu)|v_j|^2\,d\Gamma\\
+2\mu\Re\sum_{j=1}^{n}\int_{\Gamma}(m\cdot\nabla\bar{u}_j)\frac{\partial u_j}{\partial\nu}\,d\Gamma-\mu\sum_{j=1}^{n}\int_{\Gamma}(m\cdot\nu)|\nabla u_j|^2\,d\Gamma\\
+2(\mu+\lambda)\Re\sum_{j=1}^{n}\int_{\Gamma}(m\cdot\nabla\bar{u}_j)(\text{div }u)\nu_j\,d\Gamma-(\mu+\lambda)\int_{\Gamma}(m\cdot\nu)|\text{div }u|^2\,d\Gamma.
\end{aligned}
\tag{2.93}
$$

Noting that $\nabla u_j = \dfrac{\partial u_j}{\partial \nu}\nu$ on Γ_0, it follows that

$$\begin{aligned}
& n\sum_{j=1}^{n}\int_\Omega |v_j|^2\,dx + (2-n)\mu\sum_{j=1}^{n}\int_\Omega |\nabla u_j|^2\,dx + (2-n)(\mu+\lambda)\sum_{j=1}^{n}\int_\Omega |\mathrm{div}\ u|^2\,dx \\
& = 2\sum_{j=1}^{n}\Re\int_\Omega F_2^j(m\cdot\nabla\bar{u}_j)\,dx + 2\sum_{j=1}^{n}\Re\int_\Omega v_j(m\cdot\nabla\overline{F}_1^j)\,dx + \sum_{j=1}^{n}\int_{\Gamma_1}(m\cdot\nu)|v_j|^2\,d\Gamma \\
& \quad +2\mu\Re\sum_{j=1}^{n}\int_{\Gamma_1}(m\cdot\nabla\bar{u}_j)\frac{\partial u_j}{\partial\nu}\,d\Gamma + \mu\sum_{j=1}^{n}\int_{\Gamma_0}(m\cdot\nu)|\nabla u_j|^2\,d\Gamma \\
& \quad +2(\mu+\lambda)\Re\sum_{j=1}^{n}\int_{\Gamma_1}(m\cdot\nabla\bar{u}_j)(\mathrm{div}\ u)\nu_j\,d\Gamma + (\mu+\lambda)\int_{\Gamma_0}(m\cdot\nu)|\mathrm{div}\ u|^2\,d\Gamma. \\
& \quad -\mu\sum_{j=1}^{n}\int_{\Gamma_1}(m\cdot\nu)|\nabla u_j|^2\,d\Gamma - (\mu+\lambda)\int_{\Gamma_1}(m\cdot\nu)|\mathrm{div}\ u|^2\,d\Gamma
\end{aligned}$$

Multiplying (2.87) by $(n-1)$ and summing the result relation with the above inequality, we get

$$\begin{aligned}
& \sum_{j=1}^{n}\int_\Omega |v_j|^2\,dx + \mu\sum_{j=1}^{n}\int_\Omega |\nabla u_j|^2\,dx + (\mu+\lambda)\int_\Omega |\mathrm{div}\ u|^2\,dx = 2\sum_{j=1}^{n}\Re\int_\Omega F_2^j(m\cdot\nabla\bar{u}_j)\,dx \\
& \quad +2\sum_{j=1}^{n}\Re\int_\Omega v_j(m\cdot\nabla\overline{F}_1^j)\,dx + (n-1)\left(\sum_{j=1}^{n}\int_\Omega \bar{u}_jF_2^j\,dx + \sum_{j=1}^{n}\int_\Omega v_j\overline{F}_1^j\,dx\right) \\
& \quad +\sum_{j=1}^{n}\int_{\Gamma_1}(m\cdot\nu)|v_j|^2\,d\Gamma + (\mu+\lambda)\int_{\Gamma_0}(m\cdot\nu)|\mathrm{div}\ u|^2\,d\Gamma + \mu\sum_{j=1}^{n}\int_{\Gamma_0}(m\cdot\nu)|\nabla u_j|^2\,d\Gamma \\
& \quad +2\Re\sum_{j=1}^{n}\int_{\Gamma_1}(m\cdot\nabla\bar{u}_j)\left(\mu\frac{\partial u_j}{\partial\nu} + (\mu+\lambda)(\mathrm{div}\ u)\nu_j\right)d\Gamma - \mu\sum_{j=1}^{n}\int_{\Gamma_1}(m\cdot\nu)|\nabla u_j|^2\,d\Gamma \\
& \quad -(\mu+\lambda)\int_{\Gamma_1}(m\cdot\nu)|\mathrm{div}\ u|^2\,d\Gamma + (n-1)\sum_{j=1}^{n}\int_{\Gamma_1}\bar{u}_j\left(\mu\frac{\partial u_j}{\partial\nu} + (\mu+\lambda)(\mathrm{div}\ u)\nu_j\right)d\Gamma
\end{aligned}$$

Since Γ_1 is compact and m, ν are sufficiently regular,
There exists $\delta > 0$ such that $m(x)\cdot\nu(x) \geq \delta > 0$, for all $x \in \Gamma_1$.
We deduce

$$\begin{aligned}
& \sum_{j=1}^{n}\int_\Omega |v_j|^2\,dx + \mu\sum_{j=1}^{n}\int_\Omega |\nabla u_j|^2\,dx + (\mu+\lambda)\int_\Omega |\mathrm{div}\ u|^2\,dx = 2\sum_{j=1}^{n}\Re\int_\Omega F_2^j(m\cdot\nabla\bar{u}_j)\,dx \\
& \quad +2\sum_{j=1}^{n}\Re\int_\Omega v_j(m\cdot\nabla\overline{F}_1^j)\,dx + (n-1)\left(\sum_{j=1}^{n}\int_\Omega \bar{u}_jF_2^j\,dx + \sum_{j=1}^{n}\int_\Omega v_j\overline{F}_1^j\,dx\right) \\
& \quad +\sum_{j=1}^{n}\int_{\Gamma_1}(m\cdot\nu)|v_j|^2\,d\Gamma + (\mu+\lambda)\int_{\Gamma_0}(m\cdot\nu)|\mathrm{div}\ u|^2\,d\Gamma + \mu\sum_{j=1}^{n}\int_{\Gamma_0}(m\cdot\nu)|\nabla u_j|^2\,d\Gamma \\
& \quad +2\Re\sum_{j=1}^{n}\int_{\Gamma_1}(m\cdot\nabla\bar{u}_j)\left(\mu\frac{\partial u_j}{\partial\nu} + (\mu+\lambda)(\mathrm{div}\ u)\nu_j\right)d\Gamma - \mu\delta\sum_{j=1}^{n}\int_{\Gamma_1}|\nabla u_j|^2\,d\Gamma \\
& \quad -(\mu+\lambda)\delta\int_{\Gamma_1}|\mathrm{div}\ u|^2\,d\Gamma + (n-1)\sum_{j=1}^{n}\int_{\Gamma_1}\bar{u}_j\left(\mu\frac{\partial u_j}{\partial\nu} + (\mu+\lambda)(\mathrm{div}\ u)\nu_j\right)d\Gamma
\end{aligned} \tag{2.94}$$

We can estimate

$$
\begin{aligned}
&2\int_{\Gamma_1}(m\cdot\nabla\bar{u_j})\left(\mu\frac{\partial u_j}{\partial\nu}+(\mu+\lambda)(\text{div } u)\nu_j\right)d\Gamma\\
&\qquad\le\frac{\delta\mu}{2}\int_{\Gamma_1}|\nabla u_j|^2\,d\Gamma+2\frac{\|m\|_\infty^2}{\delta\mu}\int_{\Gamma_1}\left|\mu\frac{\partial u_j}{\partial\nu}+(\mu+\lambda)(\text{div } u)\nu_j\right|^2 d\Gamma.
\end{aligned}
\tag{2.95}
$$

Moreover,

$$
\begin{aligned}
&(n-1)\int_{\Gamma_1}\bar{u_j}\left(\mu\frac{\partial u_j}{\partial\nu}+(\mu+\lambda)(\text{div } u)\nu_j\right)d\Gamma\\
&\qquad\le\frac{\varepsilon}{2}\int_{\Gamma_1}|u_j|^2\,d\Gamma+\frac{(n-1)^2}{2\varepsilon}\int_{\Gamma_1}\left|\left(\mu\frac{\partial u_j}{\partial\nu}+(\mu+\lambda)(\text{div } u)\nu_j\right)\right|^2 d\Gamma\\
&\qquad\le\frac{\varepsilon}{2}C(P)\int_{\Omega}|\nabla u_j|^2\,dx+\frac{(n-1)^2}{2\varepsilon}\int_{\Gamma_1}\left|\mu\frac{\partial u_j}{\partial\nu}+(\mu+\lambda)(\text{div } u)\nu_j\right|^2 d\Gamma,
\end{aligned}
\tag{2.96}
$$

where we have used trace inequality and Poincaré's theorem.

Remark 2.6.1 *In the above inequality $C(P)$ is the smallest positive constant such that*

$$\int_{\Gamma_1}|\vartheta|^2\,d\Gamma\le C(P)\int_{\Omega}|\nabla\vartheta|^2\,dx,\quad\forall\vartheta\in H^1_{\Gamma_0}(\Omega).$$

Indeed, we ca easily estimate

$$2\Re\int_{\Omega}F_2^j(m\cdot\nabla\bar{u_j})\,dx\le\frac{\varepsilon}{2}\int_{\Omega}|\nabla u_j|^2\,dx+\frac{2}{\varepsilon}\|m\|_\infty^2\|F_2^j\|^2_{L^2(\Omega)},\tag{2.97}$$

$$2\Re\int_{\Omega}v_j(m\cdot\nabla\overline{F}_1^j)\,dx\le\frac{\varepsilon}{2}\int_{\Omega}|v_j|^2\,dx+\frac{2}{\varepsilon}\|m\|_\infty^2\|\nabla F_1^j\|^2_{L^2(\Omega)},\tag{2.98}$$

$$
\begin{aligned}
(n-1)\int_{\Omega}F_2^j\bar{u}_j\,dx &\le \frac{\varepsilon}{2}\int_{\Omega}|u_j|^2\,dx+\frac{(n-1)^2}{2\varepsilon}\|F_2^j\|^2_{L^2(\Omega)}\\
&\le \frac{\varepsilon}{2}C(\Omega)\int_{\Omega}|\nabla u_j|^2\,dx+\frac{(n-1)^2}{2\varepsilon}\|F_2^j\|^2_{L^2(\Omega)},
\end{aligned}
\tag{2.99}
$$

$$(n-1)\int_{\Omega}v\overline{F}_1^j\,dx\le\frac{\varepsilon}{2}\int_{\Omega}|v_j|^2\,dx+\frac{(n-1)^2}{2\varepsilon}\|F_1^j\|^2_{L^2(\Omega)}.\tag{2.100}$$

Then

$$
\begin{aligned}
&(1-\varepsilon)\sum_{j=1}^n\int_{\Omega}|v_j|^2\,dx+\left(\mu-\frac{\varepsilon}{2}C(P)-\frac{\varepsilon}{2}-\frac{\varepsilon}{2}C(\Omega)\right)\sum_{j=1}^n\int_{\Omega}|\nabla u_j|^2\,dx+(\mu+\lambda)\int_{\Omega}|\text{div } u|^2\,dx\\
&\qquad\le\frac{2}{\varepsilon}\|m\|_\infty^2\sum_{j=1}^n\left(\|F_2^j\|^2_{L^2(\Omega)}+\|\nabla F_1^j\|^2_{L^2(\Omega)}\right)+\frac{(n-1)^2}{2\varepsilon}\sum_{j=1}^n\left(\|F_2^j\|^2_{L^2(\Omega)}+\|F_1^j\|^2_{L^2(\Omega)}\right)\\
&\qquad+\left(2\frac{\|m\|_\infty^2}{\delta\mu}+\frac{(n-1)^2}{2\varepsilon}\right)\int_{\Gamma_1}\left|\mu\frac{\partial u_j}{\partial\nu}+(\mu+\lambda)(\text{div } u)\nu_j\right|^2 d\Gamma+\sum_{j=1}^n\int_{\Gamma_1}(m\cdot\nu)|v_j|^2\,d\Gamma.
\end{aligned}
$$

Choosing ε small enough, we conclude (2.85).
For $\tilde{\lambda} \neq 0$, we obtain

$$\mathcal{E}_u \leq c\|F\|_{\mathcal{H}}^2 + c'\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c''|\tilde{\lambda}|^{1-\alpha+\frac{\varepsilon}{2}}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c'''|\tilde{\lambda}|^{1-\alpha}\|F\|_{\mathcal{H}}^2.$$

Since that

$$\int_{\Gamma_1}\int_{-\infty}^{+\infty} |\phi(x,\xi)|^2 \, d\xi \, d\Gamma \leq C \int_{\Gamma_1}\int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(x,\xi)|^2 \, d\xi \, d\Gamma$$

for $\tilde{\lambda} \neq 0$. If $|\tilde{\lambda}| > 1$ we get

$$\|U\|_{\mathcal{H}}^2 \leq |\tilde{\lambda}|^{2(1-\alpha)+\varepsilon}\|F\|_{\mathcal{H}}^2.$$

The conclusion then follows by applying Theorem (2.6.1).

Chapter 3

STABILITY RESULT OF THE LAMÉ SYSTEM WITH A DELAY TERM IN THE INTERNAL FRACTIONAL FEEDBACK

3.1 Introduction

In this chapter, we consider the initial boundary value problem for the Lamé system given by:

$$(P2)\qquad \begin{cases} u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla(\text{div } u) + a_1\partial_t^{\alpha,\eta}u(x,t-\tau) & \\ \qquad\qquad\qquad\qquad\qquad\qquad + a_2 u_t(x,t) = 0 & \text{in } \Omega\times(0,+\infty), \\ u = 0 & \text{in } \Gamma\times(0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau) & \text{in } \Omega\times(0,\tau), \end{cases}$$

where μ, λ are Lamé constants, $u = (u_1, u_2, ..., u_n)^T$. Here Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Moreover, $a_1 > 0$, $a_2 > 0$ and the constant $\tau > 0$ is the time delay. The notation $\partial_t^{\alpha,\eta}$ stands for the exponential fractional derivative operator of order α. It is defined by

$$\partial_t^{\alpha,\eta}w(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}e^{-\eta(t-s)}\frac{dw}{ds}(s)\,ds \qquad 0<\alpha<1, \qquad \eta>0.$$

Delay effects arise in many applications and pratical problems because, in most instances, physical, chemical, biological, thermal, and economic phenomena naturally depend not only on the present state but also on some past occurrences. In recent years, the control of PDEs with time delay effects has become an active area of research, see for example [**1**], [**46**], and references therein. In many cases it was shown that delay is a source of instability and even an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay unless additional conditions or control terms have been used.

The stability issue of systems with delay is, therefore, of theoretical and practical importance. In particular, consider the wave equation with homogeneous Dirichlet boundary condition

$$
(PW) \qquad \begin{cases} u''(x,t) - \Delta_x u(x,t) + \mu_1 u'(x,t) + \mu_2 u'(x,t-\tau) = 0 & \text{in } \Omega \times (0,+\infty), \\ u(x,t) = 0 & \text{on } \Gamma \times (0,+\infty), \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) & \text{in } \Omega, \\ u'(x,t-\tau) = f_0(x,t-\tau) & \text{in } \Omega \times (0,\tau). \end{cases}
$$

For instance in **[39]**, the authors studied the problem (PW). They determined suitable relations between μ_1 and μ_2, for which the stability or alternatively instability takes place. More precisely, they showed that the energy is exponentially stable if $\mu_2 < \mu_1$ and they also found a sequence of delays for which the corresponding solution of (P) will be instable if $\mu_2 > \mu_1$. The main approach used in **[39]** is an observability inequality obtained with a Carleman estimate.

Noting that the case of the wave equation with internal fractional feedback (without delay) have treated in **[25]** where it is proven global existence and uniqueness results. As far as we are concerned, this is the first work in the literature that takes into account the uniform decay rates for Lamé system with delay term in the internal fractional feedback.

The remainder of this chapter falls into five sections. In Section **2**, we show that the above system can be replaced by an augmented one obtained by coupling an equation with a suitable diffusion, and we study of energy functional associated to system. In section **3**, we state a well-posedness result for problem $(P2)$. In section **4**, we prove the strong asymptotic stability of solutions. In section **5** we show the exponential stability using the Gearhart-Huang-Pruss theorem.

3.2 Preliminaries

This section is concerned with the reformulation of the model $(P2)$ into an augmented system. For that, we need the following claims.

Theorem 3.2.1 (see [38]) *Let μ be the function:*

$$\mu(\xi) = |\xi|^{(2\alpha-1)/2}, \quad -\infty < \xi < +\infty,\ 0 < \alpha < 1. \tag{3.1}$$

Then the relationship between the 'input' U and the 'output' O of the system

$$\partial_t \phi(\xi,t) + (\xi^2+\eta)\phi(\xi,t) - U(t)\mu(\xi) = 0, \quad -\infty < \xi < +\infty, \eta \geq 0, t > 0, \tag{3.2}$$

$$\phi(\xi,0) = 0, \tag{3.3}$$

$$O(t) = (\pi)^{-1}\sin(\alpha\pi)\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi,t)\, d\xi \tag{3.4}$$

is given by

$$O = I^{1-\alpha,\eta}U = D^{\alpha,\eta}U, \tag{3.5}$$

where

$$[I^{\alpha,\eta}f](t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}e^{-\eta(t-\tau)}f(\tau)\,d\tau.$$

Proof. From (3.2) and (3.3), we have

$$\phi(\xi,t) = \int_0^t \mu(\xi)e^{-(\xi^2+\eta)(t-\tau)}U(\tau)d\tau. \tag{3.6}$$

Hence, by using (3.4), we get

$$O(t) = (\pi)^{-1}sin(\alpha\pi)e^{-\eta t}\int_0^t \left[2\int_0^{+\infty}|\xi|^{2\alpha-1}e^{-\xi^2(t-s)}d\xi\right]e^{\eta\tau}U(\tau)d\tau. \tag{3.7}$$

Thus,

$$\begin{aligned} O(t) &= (\pi)^{-1}sin(\alpha\pi)e^{-\eta t}\int_0^t \left[(t-s)^{-\alpha}\Gamma(\alpha)\right]e^{\eta\tau}U(\tau)d\tau \\ &= (\pi)^{-1}sin(\alpha\pi)\int_0^t \left[(t-s)^{-\alpha}\Gamma(\alpha)\right]e^{-\eta(t-\tau)}U(\tau)d\tau \end{aligned} \tag{3.8}$$

which completes the proof. Indeed, we know that $(\pi)^{-1}sin(\alpha\pi) = \dfrac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}$.

Lemma 3.2.1 (see [10]) *If* $\lambda \in D_\eta = \mathbb{C}\backslash]-\infty,-\eta]$ *then*

$$\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}\,d\xi = \frac{\pi}{\sin\alpha\pi}(\lambda+\eta)^{\alpha-1}.$$

We make the following hypotheses on the damping and the delay functions:

$$a_1\eta^{\alpha-1} < a_2. \tag{3.9}$$

We are now in a position to reformulate system $(P2)$. As in **[39]**, we introduce the new variable

$$z(x,\rho,t) = u_t(x,t-\rho\tau), \qquad x\in\Omega, \rho\in(0,1), t>0.$$

Then the above variable z satisfies

$$\tau z_t(x,\varrho,t) + z_\varrho(x,\varrho,t) = 0, \quad \varrho\in(0,1), \quad t>0.$$

Consequently, by using Theorem 3.2.1, the system $(P2)$ is equivalent to

$$
(P'2) \quad \begin{cases}
u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla(\text{div } u) & \\
\qquad\qquad + \zeta \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(x,\xi,t)\,d\xi + a_2 u_t(t) = 0 & \text{in } \Omega \times (0,+\infty), \\
\phi_t(x,\xi,t) + (\xi^2+\eta)\phi(x,\xi,t) - z(x,1,t)\mu(\xi) = 0 & \text{in } \Omega \times (-\infty,\infty) \times (0,+\infty), \\
\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 & \text{in } \Omega \times (0,1) \times (0,+\infty), \\
u(x,t) = 0 & \text{on } \Gamma \times (0,+\infty), \\
z(x,0,t) = u_t(x,t), & \text{in } \Omega \times (0,+\infty), \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{on } \Omega, \\
\phi(x,\xi,0) = 0 & \text{on } \Omega \times (-\infty,\infty), \\
z(x,\rho,0) = f_0(x,-\rho\tau) & \text{in } \Omega \times (0,1),
\end{cases}
$$

where $\zeta = (\pi)^{-1}\sin(\alpha\pi)a_1$.

We define the energy of the solution by:

$$
\begin{aligned}
E(t) = \frac{1}{2}\sum_{j=1}^{n}\Big(\|u_{jt}\|^2_{L^2(\Omega)} + \mu\|\nabla u_j\|^2_{L^2(\Omega)} + \zeta\int_\Omega\int_{-\infty}^{+\infty}|\phi_j(x,\xi,t)|^2\,d\xi dx\Big) \\
+\frac{\nu}{2}\sum_{j=1}^{n}\int_\Omega\int_0^1 |z_j(x,\rho,t)|^2\,d\rho dx + \frac{(\mu+\lambda)}{2}\|\text{div } u\|^2_{L^2(\Omega)}.
\end{aligned} \tag{3.10}
$$

where ν is a positive constant verifying

$$
\tau\zeta\left(\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta}\,d\xi\right) < \nu < \tau\left(2a_2 - \zeta\left(\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta}\,d\xi\right)\right). \tag{3.11}
$$

Remark 3.2.1 *Using Lemma 3.2.1, the condition (3.11) means that*

$$
\tau a_1\eta^{\alpha-1} < \nu < \tau(2a_2 - a_1\eta^{\alpha-1}).
$$

Lemma 3.2.2 *Let (u,ϕ,z) be a regular solution of the problem $(P'2)$. Then, the energy functional defined by (3.10) satisfies*

$$
E'(t) \le -C\sum_{j=1}^{n}\int_\Omega\left(u_t^2 + z^2(x,1,t)\right)dx. \tag{3.12}
$$

Proof. Multiplying the first equation in $(P2)$ by $\overline{u}_{jt}$, integrating over Ω and using integration by parts, we get

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_{jt}\|_2^2 - \mu\Re\int_\Omega \Delta u_j\overline{u}_{jt}\,dx - (\mu+\lambda)\Re\int_\Omega\frac{\partial}{\partial x_j}(\text{div } u)\overline{u}_{jt}\,dx \\
+\zeta\int_\Omega\overline{u}_{jt}\int_{-\infty}^{+\infty}\mu(\xi)\phi_j(x,\xi,t)\,d\xi\,dx + a_2\int_\Omega u_{jt}^2(t) = 0.
\end{aligned}
$$

Then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\sum_{j=1}^{n}\left(\|u_{jt}\|^2_{L^2(\Omega)}+\mu\|\nabla u_j\|^2_{L^2(\Omega)}\right)+\frac{(\mu+\lambda)}{2}\|\text{div }u\|^2_{L^2(\Omega)}+a_2\sum_{j=1}^{n}\|u_{jt}\|^2_{L^2}\\ +\zeta\Re\sum_{j=1}^{n}\int_\Omega \overline{u}_{jt}\int_{-\infty}^{+\infty}\mu(\xi)\phi(x,\xi,t)\,d\xi\,dx=0.\end{aligned} \tag{3.13}$$

Multiplying the second equation in $(P'2)$ by $\zeta\overline{\phi}_j$ and integrating over $\Omega\times(-\infty,+\infty)$, we obtain:

$$\begin{aligned}\frac{\zeta}{2}\frac{d}{dt}\sum_{j=1}^{n}\|\phi_j\|^2_{L^2(\Omega\times(-\infty,+\infty))}+\zeta\sum_{j=1}^{n}\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi_j(x,\xi,t)|^2\,d\xi\,dx\\ -\zeta\Re\sum_{j=1}^{n}\int_\Omega z_j(x,1,t)\int_{-\infty}^{+\infty}\mu(\xi)\overline{\phi}_j(x,\xi,t)\,d\xi\,dx=0.\end{aligned} \tag{3.14}$$

Multiplying the third equation in $(P'2)$ by $\nu\overline{z}_j$ and integrating over $\Omega\times(0,1)$, we get:

$$\frac{1}{2}\frac{d}{dt}\sum_{j=1}^{n}\|z_j\|^2_{L^2(\Omega\times(0,1))}+\frac{\tau^{-1}}{2}\sum_{j=1}^{n}\int_\Omega\left(z_j^2(x,1,t)-u_{jt}^2(x,t)\right)\,dx=0. \tag{3.15}$$

From $(3.10),(3.13)$ and (3.15) we get

$$\begin{aligned}E'(t)=-a_2\sum_{j=1}^{n}\|u_{jt}\|^2_{L^2}-\zeta\sum_{j=1}^{n}\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi_j(x,\xi,t)|^2\,d\xi\,dx\\ -\zeta\Re\sum_{j=1}^{n}\int_\Omega\overline{u}_{jt}\int_{-\infty}^{+\infty}\mu(\xi)\phi_j(x,\xi,t)\,d\xi\,dx+\zeta\Re\sum_{j=1}^{n}\int_\Omega z_j(x,1,t)\int_{-\infty}^{+\infty}\mu(\xi)\overline{\phi}_j(x,\xi,t)\,d\xi\,dx\\ +\frac{\nu\tau^{-1}}{2}\sum_{j=1}^{n}\int_\Omega u_t^2(x,t)\,dx-\frac{\nu\tau^{-1}}{2}\sum_{j=1}^{n}\int_\Omega z_j^2(x,1,t)\,dx\end{aligned}. \tag{3.16}$$

Moreover, we have

$$\left|\int_{-\infty}^{+\infty}\mu(\xi)\phi_j(x,\xi,t)\,d\xi\right|\le\left(\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta}\,d\xi\right)^{\frac{1}{2}}\left(\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi_j(x,\xi,t)|^2\,d\xi\right)^{\frac{1}{2}}.$$

Then

$$\begin{aligned}&\left|\int_\Omega z_j(x,1,t)\int_{-\infty}^{+\infty}\mu(\xi)\overline{\phi}_j(x,\xi,t)\,d\xi\,dx\right|\\ &\le\left(\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta}\,d\xi\right)^{\frac{1}{2}}\|z_j(x,1,t)\|_{L^2(\Omega)}\left(\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi_j(x,\xi,t)|^2\,dx\,d\xi\right)^{\frac{1}{2}},\end{aligned}$$

$$\begin{aligned}&\left|\int_\Omega \overline{u}_{jt}(x,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi_j(x,\xi,t)\,d\xi\,dx\right|\\ &\le\left(\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta}\,d\xi\right)^{\frac{1}{2}}\|u_j(x,t)\|_{L^2(\Omega)}\left(\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi_j(x,\xi,t)|^2\,dx\,d\xi\right)^{\frac{1}{2}}.\end{aligned}$$

Applying the Cauchy-Schwarz inequality we obtain

$$E'(t) \leq \left(-a_2 + \frac{\zeta I}{2} + \frac{\nu\tau^{-1}}{2}\right)\sum_{j=1}^{n}\int_\Omega u_{jt}^2(x,t)\,dx + \left(\frac{\zeta I}{2} - \frac{\nu\tau^{-1}}{2}\right)\sum_{j=1}^{n}\int_\Omega z_j^2(x,1,t)\,dx \quad ,$$

where $I = \displaystyle\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2+\eta}\,d\xi$, which implies

$$E'(t) \leq -C\sum_{j=1}^{n}\int_\Omega \left(u_{jt}^2(x,t) + z_j^2(x,1,t)\right)\,dx$$

with

$$C = \min\left\{\left(a_2 - \frac{\zeta I}{2} - \frac{\nu\tau^{-1}}{2}\right), \left(-\frac{\zeta I}{2} + \frac{\nu\tau^{-1}}{2}\right)\right\}.$$

Since ν is chosen satisfying assumption (3.11), the constant C is positive. This completes the proof of the lemma.

3.3 Well-posedness

In this section, we give the existence and uniqueness result for system $(P'2)$ using the semigroup theory. Let us denote $U = (u, v, \phi, z)^T$, where $v = u_t$. The system $(P'2)$ can be rewrite as follows:

$$\begin{cases} U' = \mathcal{A}U, & t > 0, \\ U(0) = (u_0, u_1, \phi_0, f_0), \end{cases} \tag{3.17}$$

where $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is the linear operator defined by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ \phi \\ z \end{pmatrix} = \begin{pmatrix} v \\ \mu\Delta u + (\mu+\lambda)\nabla(\text{div } u) - \zeta\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi(x,\xi)\,d\xi - a_2 v \\ -(\xi^2+\eta)\phi + z(x,1)\mu(\xi) \\ -\tau^{-1}z_\rho(x,\rho) \end{pmatrix} \tag{3.18}$$

and $\mathcal{H}$ is the energy space given by

$$\mathcal{H} = \left(H_0^1(\Omega)\right)^n \times \left(L^2(\Omega)\right)^n \times \left(L^2(\Omega\times(-\infty,+\infty))\right)^n \times \left(L^2(\Omega\times(0,1))\right)^n$$

For any $U = (u, v, \phi, z)^T \in \mathcal{H}, \tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\phi}, \tilde{z})^T \in \mathcal{H}$, we equip $\mathcal{H}$ with the inner product defined by

$$< U, \tilde{U} >_{\mathcal{H}} = \sum_{j=1}^{n}\int_\Omega \left(v_j\bar{\tilde{v}}_j + \mu\nabla u_j\nabla\bar{\tilde{u}}_j\right)dx + (\mu+\lambda)\int_\Omega(\text{div } u)(\text{div } \bar{\tilde{u}})\,dx$$
$$+\zeta\sum_{j=1}^{n}\int_\Omega\int_{-\infty}^{+\infty}\phi_j(x,\xi)\bar{\tilde{\phi}}_j(x,\xi)d\xi dx + \nu\sum_{j=1}^{n}\int_\Omega\int_0^1 z(x,\rho)\bar{\tilde{z}}_j(x,\rho)d\rho dx.$$

The domain of $\mathcal{A}$ is given by

$$D(\mathcal{A})=\left\{\begin{array}{l}(u,v,\phi,z)^T \text{ in } \mathcal{H}: u\in\left(H^2(\Omega)\cap H_0^1(\Omega)\right)^n, v\in\left(H^1(\Omega)\right)^n,\\ -(\xi^2+\eta)\phi+z(x,1)\mu(\xi)\in\left(L^2(\Omega\times(-\infty,+\infty))\right)^n,\\ z\in\left(L^2\left(\Omega;H^1(0,1)\right)\right)^n,\\ |\xi|\phi\in\left(L^2(\Omega\times(-\infty,+\infty))\right)^n, v=z(.,0) \text{ in } \Omega\end{array}\right\}. \tag{3.19}$$

We show that the operator $\mathcal{A}$ generates a C_0 semigroup in $\mathcal{H}$. We prove that $\mathcal{A}$ is a maximal dissipative operator. For this we need the following two Lemmas.

Lemma 3.3.1 *The operator $\mathcal{A}$ is dissipative and satisfies for any $U\in D(\mathcal{A})$,*

$$\Re\langle\mathcal{A}U,U\rangle_{\mathcal{H}}\leq -C\sum_{j=1}^{n}\int_{\Omega}\left(v^2+z^2(x,1)\right).dx \tag{3.20}$$

Proof. For any $U=(u,v,\phi,z)\in D(\mathcal{A})$, using (3.17), (3.12) and the fact that

$$E(t)=\frac{1}{2}\|U\|_{\mathcal{H}}^2, \tag{3.21}$$

estimate (3.20) easily follows.

Lemma 3.3.2 *The operator $(\tilde{\lambda}I-\mathcal{A})$ is surjective for $\tilde{\lambda}>0$.*

Proof. For any $F=(F_1,F_2,F_3,F_4)^T\in\mathcal{H}$, where $F_i=(f_i^1,f_i^2,...,f_i^n)^T$, we show that there exists $U\in D(\mathcal{A})$ satisfying

$$(\tilde{\lambda}I-\mathcal{A})U=F. \tag{3.22}$$

Equation (3.22) is equivalent to

$$\begin{cases}\tilde{\lambda}u-v=F_1(x),\\ \tilde{\lambda}v-\mu\Delta u-(\mu+\lambda)\nabla(\operatorname{div} u)+\zeta\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi(x,\xi)\,d\xi+a_2v=F_2(x),\\ \tilde{\lambda}\phi+(\xi^2+\eta)\phi-z(x,1)\mu(\xi)=F_3(x,\xi),\\ \lambda z(x,\rho)+\tau^{-1}z_\rho(x,\rho)=F_4(x,\rho).\end{cases} \tag{3.23}$$

Suppose u is found with the appropriate regularity. Then, $(3.23)_1$ $(3.23)_3$ yield

$$v=\tilde{\lambda}u-F_1(x)\in\left(H^1(\Omega)\right)^n \tag{3.24}$$

and

$$\phi=\frac{F_3(x,\xi)+\mu(\xi)z(x,1)}{\xi^2+\eta+\tilde{\lambda}}. \tag{3.25}$$

We note that the last equation in (3.23) with $z(x,0)=v(x)$ has a unique solution given by

$$z(x,\rho)=v(x)e^{-\tilde{\lambda}\rho\tau}+\tau e^{-\tilde{\lambda}\rho\tau}\int_0^\rho F_4(x,\sigma)e^{\tilde{\lambda}\sigma\tau}d\sigma. \tag{3.26}$$

Inserting (3.24) in (3.26), we get

$$z(x,\rho)=\tilde{\lambda}u(x)e^{-\tilde{\lambda}\rho\tau}-F_1(x)e^{-\tilde{\lambda}\rho\tau}+\tau e^{-\tilde{\lambda}\rho\tau}\int_0^\rho F_4(x,\sigma)e^{\tilde{\lambda}\sigma\tau}d\sigma,\qquad x\in\Omega,\ \rho\in(0,1). \tag{3.27}$$

In particular,

$$z(x,1)=\tilde{\lambda}u(x)e^{-\tilde{\lambda}\tau}+z_0(x),\qquad x\in\Omega \tag{3.28}$$

with $z_0\in L^2(\Omega)$ defined by

$$z_0(x)=-F_1(x)e^{-\tilde{\lambda}\tau}+\tau e^{-\tilde{\lambda}\tau}\int_0^1 F_4(x,\sigma)e^{\tilde{\lambda}\sigma\tau}d\sigma,\ x\in\Omega. \tag{3.29}$$

Inserting (3.24) in $(3.23)_2$, we get

$$\begin{aligned}(\tilde{\lambda}^2+\tilde{\lambda}a_2)u-\mu\Delta u-(\mu+\lambda)\nabla(\text{div } u)+\zeta\int_{-\infty}^{+\infty}\mu(\xi)\phi(x,\xi)\,d\xi\\ =F_2(x)+(\tilde{\lambda}+a_2)F_1(x).\end{aligned} \tag{3.30}$$

Solving system (3.30) is equivalent to finding $u\in\left(H^2(\Omega)\cap H_0^1(\Omega)\right)^n$ such that

$$\begin{aligned}&\sum_{j=1}^n\int_\Omega\left((\tilde{\lambda}^2+\tilde{\lambda}a_2)u_j\overline{w}_j-\mu\Delta u_j\overline{w}_j\right)dx-(\mu+\lambda)\int_\Omega\frac{\partial}{\partial x_j}(\text{div } u)\overline{w}_jdx\\ &+\zeta\sum_{j=1}^n\int_\Omega\overline{w}_j\int_{-\infty}^{+\infty}\mu(\xi)\phi_j(x,\xi)\,d\xi dx=\sum_{j=1}^n\int_\Omega(F_2^j(x)+(\tilde{\lambda}+a_2)F_1^j(x))\overline{w}_jdx\end{aligned} \tag{3.31}$$

for all $w\in\left(H_0^1(\Omega)\right)^n$. Inserting (3.25) in (3.31), the function u satisfies the following system

$$\left\{\begin{aligned}&\sum_{j=1}^n\int_\Omega\left((\tilde{\lambda}^2+\tilde{\lambda}a_2)u_j\overline{w}_j+\mu\nabla u_j\nabla\overline{w}_j\,dx\right)\,dx+(\mu+\lambda)\int_\Omega(\text{div } u)(\text{div }\overline{w})\,dx\\ &\quad+\theta\sum_{j=1}^n\int_\Omega\overline{w}_jz(x,1)\,d\xi\,dx=\sum_{j=1}^n\int_\Omega(F_2^j(x)+(\tilde{\lambda}+a_2)F_1^j(x))\overline{w}_j\,dx\\ &\qquad\qquad-\zeta\sum_{j=1}^n\int_\Omega\overline{w}_j\left(\int_{-\infty}^\infty\frac{\mu(\xi)F_3^j(x,\xi)}{\xi^2+\eta+\tilde{\lambda}}\,d\xi\right)\,dx,\end{aligned}\right. \tag{3.32}$$

where $\theta=\zeta\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta+\tilde{\lambda}}\,d\xi$. Inserting (3.28) into (3.32), we get

$$\left\{\begin{aligned}&\sum_{j=1}^n\int_\Omega\left((\tilde{\lambda}^2+\tilde{\lambda}a_2)u_j\overline{w}_j+\mu\nabla u_j\nabla\overline{w}_j\,dx\right)+(\mu+\lambda)\int_\Omega(\text{div } u)(\text{div }\overline{w})\,dx\\ &\quad+\lambda\theta\sum_{j=1}^n\int_\Omega u_j\overline{w}_je^{-\lambda\tau}\,dx=\sum_{j=1}^n\int_\Omega\left(F_2^j(x)+(\tilde{\lambda}+a_2)F_1^j(x)\right)\overline{w}_j\,dx\\ &\quad-\zeta\sum_{j=1}^n\int_\Omega\overline{w}_j\left(\int_{-\infty}^\infty\frac{\mu(\xi)F_3^j(x,\xi)}{\xi^2+\eta+\tilde{\lambda}}\,d\xi\right)\,dx-\theta\sum_{j=1}^n\int_\Omega\overline{w}_jz_0(x)\,dx.\end{aligned}\right. \tag{3.33}$$

Problem (3.33) is of the form

$$\mathcal{B}(u,w)=\mathcal{L}(w), \tag{3.34}$$

where $\mathcal{B}:\left[(H_0^1(\Omega))^n\times(H_0^1(\Omega))^n\right]\to\mathbb{C}$ is the sesquilinear form defined by

$$\begin{aligned}\mathcal{B}(u,w)=\sum_{j=1}^n\int_\Omega\left((\tilde\lambda^2+\tilde\lambda a_2)u_j\overline{w}_j+\mu\nabla u_j\nabla\overline{w}_j\,dx\right)+(\mu+\lambda)\int_\Omega(\text{div } u)(\text{div }\overline{w})\,dx\\ +\lambda\theta\sum_{j=1}^n\int_\Omega u_j\overline{w}_je^{-\lambda\tau}\,dx\end{aligned}$$

and $\mathcal{L}:(H_0^1(\Omega))^n\to\mathbb{C}$ is the antilinear functional given by

$$\begin{aligned}\mathcal{L}(w)=\sum_{j=1}^n\int_\Omega\left(F_2^j(x)+(\tilde\lambda+a_2)F_1^j(x)\right)\overline{w}_j\,dx-\zeta\sum_{j=1}^n\int_\Omega\overline{w}_j\left(\int_{-\infty}^{\infty}\frac{\mu(\xi)F_3^j(x,\xi)}{\xi^2+\eta+\tilde\lambda}\,d\xi\right)dx\\ -\theta\sum_{j=1}^n\int_\Omega\overline{w}_jz_0(x)\,dx.\end{aligned}$$

It is easy to verify that $\mathcal{B}$ is continuous and coercive, and $\mathcal{L}$ is continuous. Consequently, by the Lax-Milgram theorem, we deduce that for all $w\in(H_0^1(\Omega))^n$, the system (3.34) has a unique solution $u\in(H_0^1(\Omega))^n$. By the regularity theory for the linear elliptic equations, it follows that $u\in(H^2(\Omega))^n$. Therefore, the operator $(\tilde\lambda I-\mathcal{A})$ is surjective for any $\tilde\lambda>0$.

Consequently, using Hille-Yosida theorem, we have the following well-posedness result:

Theorem 3.3.1 (Existence and uniqueness)

(1) If $U_0\in D(\mathcal{A})$, then system (3.17) has a unique strong solution

$$U\in C^0(\mathbb{R}_+,D(\mathcal{A}))\cap C^1(\mathbb{R}_+,\mathcal{H}).$$

(2) If $U_0\in\mathcal{H}$, then system (3.17) has a unique weak solution

$$U\in C^0(\mathbb{R}_+,\mathcal{H}).$$

3.4 Strong stability

One simple way to prove the strong stability of (3.17) is to use the following theorem due to Arendt-Batty and Lyubich-Vũ (see [**6**] and [**34**]).

Theorem 3.4.1 ([6]-[34]) *Let X be a reflexive Banach space and $(T(t))_{t\geq 0}$ be a C_0-semigroup generated by A on X. Assume that $(T(t))_{t\geq 0}$ is bounded and that no eigenvalues of A lie on the imaginary axis. If $\sigma(A)\cap iR$ is countable, then $(T(t))_{t\geq 0}$ is strongly stable.*

Our main result is the following theorem

Theorem 3.4.2 *The C_0-semigroup $e^{t\mathcal{A}}$ is strongly stable in $\mathcal{H}$; i.e, for all $U_0 \in \mathcal{H}$, the solution of (3.17) satisfies*

$$\lim_{t\to\infty} \|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} = 0.$$

For the proof of Theorem 3.4.2, we need the following two lemmas.

Lemma 3.4.1 *$\mathcal{A}$ does not have eigenvalues on $i\mathrm{I\!R}$.*

Proof. We will argue by contraction. Let us suppose that there $\tilde\lambda \in \mathrm{I\!R}, \tilde\lambda \neq 0$ and $U \neq 0$, such that

$$\mathcal{A}U = i\tilde\lambda U. \tag{3.35}$$

Then, we get

$$\begin{cases} i\tilde\lambda u - v = 0, \\ i\tilde\lambda v - \mu\Delta u - (\mu+\lambda)\nabla(\mathrm{div}\ u) + \zeta \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(x,\xi)\,d\xi + a_2 v = 0, \\ i\tilde\lambda\phi + (\xi^2+\eta)\phi - z(x,1)\mu(\xi) = 0, \\ i\tilde\lambda z(x,\rho) + \tau^{-1} z_\rho(x,\rho) = 0. \end{cases} \tag{3.36}$$

Then, from (3.20) we have

$$v = 0, \quad z(x,1) = 0. \tag{3.37}$$

Hence, from $(3.36)_3$ and (3.37) we obtain

$$u \equiv 0, \quad \phi \equiv 0. \tag{3.38}$$

Note that $(3.36)_4$ gives us $z = ve^{-i\tilde\lambda\rho\tau} = 0$ as the unique solution of $(3.36)_4$. Hence $U \equiv 0$.

Now if $\tilde\lambda = 0$, inserting $(3.36)_1$ into $(3.36)_2$, we deduce that

$$\begin{cases} -\mu\Delta u - (\mu+\lambda)\nabla(\mathrm{div}\ u) = 0, \\ u = 0 \text{ in } \Gamma. \end{cases} \tag{3.39}$$

Multiplying by $\overline{u}$, integrating over Ω we have

$$\|\nabla u\|^2_{L^2(\Omega)} + \|\mathrm{div}\ u\|^2_{L^2(\Omega)} = 0. \tag{3.40}$$

Hence $u = 0$. Then $U \equiv 0$.

Lemma 3.4.2 *We have*

$$i\mathbb{R} \subset \rho(\mathcal{A}).$$

Proof. To prove this, we need the following generalization of the Lax-Milgram Lemma.

Lemma 3.4.3 (see [18])

Let V and H be Hilbert spaces such that the embedding $V \subset H$ is compact and dense. Suppose that $a_V : V \times V \to \mathbb{C}$ and $a_H : H \times H \to \mathbb{C}$ are two bounded sesquilinear forms such that a_V is V-coercive and $G : V \to \mathbb{C}$ is a continuous conjugate linear form. The equation

$$a_H(u,v) + a_V(u,v) = G(v), \qquad \forall v \in V$$

has either a unique solution $u \in V$ for all $G \in V'$ or has a nontrivial solution for $G = 0$.

We will prove that the operator $(i\tilde{\lambda} I - \mathcal{A})$ is surjective for $\tilde{\lambda} \neq 0$. For this purpose, let $F = (F_1, F_2, F_3, F_4)^T \in \mathcal{H}$, we seek $U = (u, v, \phi, z)^T \in D(\mathcal{A})$ of solution of the following equation

$$(i\tilde{\lambda} I - \mathcal{A})U = F. \tag{3.41}$$

Equivalently, we have

$$\begin{cases} i\tilde{\lambda} u - v = F_1, \\ i\tilde{\lambda} v - \mu \Delta u - (\mu + \lambda)\nabla(\text{div } u) + \zeta \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(x,\xi)\, d\xi + a_2 v = F_2, \\ i\tilde{\lambda}\phi + (\xi^2 + \eta)\phi - z(x,1)\mu(\xi) = F_3, \\ i\tilde{\lambda} z(x,\rho) + \tau^{-1} z_\rho(x,\rho) = F_4. \end{cases} \tag{3.42}$$

From $(3.42)_1$ and $(3.42)_2$, we have

$$-\tilde{\lambda}^2 u - \mu\Delta u - (\mu+\lambda)\nabla(\text{div } u) + \zeta \int_{-\infty}^{+\infty} \mu(\xi)\phi(x,\xi)\, d\xi + a_2 v(t) = (F_2 + i\tilde{\lambda} F_1) \tag{3.43}$$

with $u_{|\Gamma} = 0$. Solving system (3.43) is equivalent to finding $u \in (H^2 \cap H_0^1(\Omega))^n$ such that

$$\begin{cases} \displaystyle\sum_{j=1}^{n} \int_\Omega \Big((-\tilde{\lambda}^2 + i\tilde{\lambda} a_2) u_j \overline{w}_j + \mu \nabla u_j \nabla \overline{w}_j \, dx \Big) + (\mu+\lambda) \int_\Omega (\text{div } u)(\text{div } \overline{w})\, dx \\ \qquad + i\tilde{\lambda}\theta \displaystyle\sum_{j=1}^{n} \int_\Omega u_j \overline{w}_j e^{-\tilde{\lambda}\tau}\, dx = \sum_{j=1}^{n} \int_\Omega \Big(F_2^j(x) + (i\tilde{\lambda} + a_2) F_1^j(x) \Big) \overline{w}_j \, dx \\ \qquad\quad -\zeta \displaystyle\sum_{j=1}^{n} \int_\Omega \overline{w}_j \left(\int_{-\infty}^{\infty} \frac{\mu(\xi) F_3^j(x,\xi)}{\xi^2 + \eta + i\tilde{\lambda}}\, d\xi \right) dx - \theta \sum_{j=1}^{n} \int_\Omega \overline{w}_j z_0(x)\, dx \end{cases} \tag{3.44}$$

for all $w \in (H_0^1(\Omega))^n$. We can rewrite (3.44) as

$$-(L_{\tilde{\lambda}} u, w)_{((L^2(\Omega))^n, ((L^2(\Omega))')^n)} + a_{(H_0^1(\Omega))^n}(u, w) = l(w) \tag{3.45}$$

with the sesquilinear form defined by

$$a_{(H_0^1(\Omega))^n}(u,w)=\mu\sum_{j=1}^n\int_\Omega \nabla u_j\nabla\overline{w}_j\,dx+i\tilde{\lambda}a_2\sum_{j=1}^n\int_\Omega u_j\;\overline{w}_j\,dx+i\tilde{\lambda}\theta\sum_{j=1}^n\int_\Omega u_j\overline{w}_je^{-\tilde{\lambda}\tau}\,dx$$

and

$$(L_{\tilde{\lambda}}u,w)_{((L^2(\Omega))^n,((L^2(\Omega))')^n)}=\sum_{j=1}^n\int_\Omega \tilde{\lambda}^2u_j\overline{w}_j\,dx.$$

Using the compactness of the embedding from $L^2(\Omega)$ into $H^{-1}(\Omega)$ and from $H_0^1(\Omega)$ into $L^2(\Omega)$ we deduce that the operator $L_{\tilde{\lambda}}$ is compact from $(L^2(\Omega))^n$ into $(L^2(\Omega))^n$. Consequently, by the Fredholm alternative, proving the existence of a solution u of (3.45) reduces to proving that there is not a nontrivial solution for (3.45) for $l\equiv 0$. Indeed if there exists $u\neq 0$, such that

$$(L_\lambda u,w)_{((H_0^1(\Omega))^n,((H_0^1(\Omega))')^n)}=a_{(H_0^1(\Omega))^n}(u,w)\quad\forall w\in(H_0^1(\Omega))^n,\tag{3.46}$$

then $i\tilde{\lambda}$ is an eigenvalue of $\mathcal{A}$. Therefore from Lemma 3.4.1 we deduce that $u=0$.

Now, if $\tilde{\lambda}=0$, the system (3.42) is reduced to the following system

$$\begin{cases} v=-F_1,\\ -\mu\Delta u-(\mu+\lambda)\nabla(\text{div } u)+\zeta\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi(x,\xi)\,d\xi+a_2v=F_2,\\ (\xi^2+\eta)\phi-z(x,1)\mu(\xi)=F_3,\\ \tau^{-1}z_\rho(x,\rho)=F_4.\end{cases}\tag{3.47}$$

Solving system (3.47) is equivalent to finding $u\in(H^2\cap H_0^1(\Omega))^n$ such that

$$\begin{gathered}\mu\sum_{j=1}^n\int_\Omega\nabla u_j\nabla\overline{w}_j\,dx+(\mu+\lambda)\int_\Omega(\text{div } u)(\text{div }\overline{w})\,dx=\sum_{j=1}^n\int_\Omega F_2^j\overline{w}_j\,dx\\ +\left(\zeta\int_{-\infty}^{\infty}\frac{\mu^2(\xi)}{\xi^2+\eta}\,d\xi+a_2\right)\sum_{j=1}^n\int_\Omega F_1^j\overline{w}_j\,dx-\tau\zeta\int_{-\infty}^{\infty}\frac{\mu^2(\xi)}{\xi^2+\eta}\,d\xi\sum_{j=1}^n\int_\Omega\int_0^1F_4^j(x,s)\,ds\overline{w}_j\,dx\\ -\zeta\sum_{j=1}^n\int_\Omega\overline{w}_j\int_{-\infty}^{\infty}\frac{\mu(\xi)F_3^j(x,\xi)}{\xi^2+\eta}\,d\xi\,dx.\end{gathered}\tag{3.48}$$

for all $w\in(H_0^1(\Omega))^n$.

Consequently, problem (3.48) is equivalent to the problem

$$\mathcal{B}(u,w)=\mathcal{L}(w),\tag{3.49}$$

where $\mathcal{B}:[(H_0^1(\Omega))^n\times(H_0^1(\Omega))^n]\rightarrow\mathbb{C}$ is the sesquilinear form defined by

$$\mathcal{B}(u,w)=\mu\sum_{j=1}^n\int_\Omega\nabla u_j\nabla\overline{w}_j\,dx+(\mu+\lambda)\int_\Omega(\text{div } u)(\text{div }\overline{w})\,dx\tag{3.50}$$

and $\mathcal{L} : (H_0^1(\Omega))^n \to \mathbb{C}$ is the antilinear form defined by

$$\begin{aligned}\mathcal{L}(w) &= \sum_{j=1}^n \int_\Omega F_2^j \overline{w}_j \, dx + \left(\zeta \int_{-\infty}^{\infty} \frac{\mu^2(\xi)}{\xi^2+\eta} \, d\xi + a_2\right) \sum_{j=1}^n \int_\Omega F_1^j \overline{w}_j \, dx \\ &-\tau\zeta \int_{-\infty}^{\infty} \frac{\mu^2(\xi)}{\xi^2+\eta} \, d\xi \sum_{j=1}^n \int_\Omega \int_0^1 F_4^j(x,s) \, ds \overline{w}_j \, dx - \zeta \sum_{j=1}^n \int_\Omega \overline{w}_j \int_{-\infty}^{\infty} \frac{\mu(\xi) F_3^j(x,\xi)}{\xi^2+\eta} \, d\xi \, dx.\end{aligned} \tag{3.51}$$

It is easy to verify that $\mathcal{B}$ is continuous and coercive, and $\mathcal{L}$ is continuous. So applying the Lax-Milgram theorem, we deduce that for all $w \in (H_0^1(\Omega))^n$ problem (3.49) admits a unique solution $u \in (H_0^1(\Omega))^n$. Applying the classical elliptic regularity, it follows from (3.48) that $u \in (H^2(\Omega))^n$. Therefore, the operator $\mathcal{A}$ is surjective.

3.5 Exponential stability

The necessary and suficient conditions for the exponential stability of the C_0- semigroup of contractions on a Hilbert space were obtained by Gearhart [**21**] and Huang [**26**] independently, see also Pruss [**43**]. We will use the following result due to Gearhart.

Theorem 3.5.1 ([43]- [26]) *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on Hilbert space $\mathcal{H}$. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{i\beta : \beta \in \mathbb{R}\} \equiv i\mathbb{R} \tag{3.52}$$

and

$$\overline{\lim_{|\beta|\to\infty}} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{3.53}$$

Our main result is as follows.

Theorem 3.5.2 *The semigroup $S_{\mathcal{A}}(t)_{t\geq 0}$ generated by $\mathcal{A}$ is exponentially stable.*

Proof. We will need to study the resolvent equation $(i\tilde{\lambda} - \mathcal{A})U = F$, for $\lambda \in \mathbb{R}$, namely

$$\begin{cases} i\tilde{\lambda}u - v = F_1, \\ i\tilde{\lambda}v - \mu\Delta u - (\mu+\lambda)\nabla(\text{div } u) + \zeta \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(x,\xi) \, d\xi + a_2 v = F_2, \\ i\tilde{\lambda}\phi + (\xi^2+\eta)\phi - z(x,1)\mu(\xi) = F_3, \\ i\tilde{\lambda}z(x,\rho) + \tau^{-1} z_\rho(x,\rho) = F_4, \end{cases} \tag{3.54}$$

where $F = (F_1, F_2, F_3, F_4)^T$. Taking inner product in $\mathcal{H}$ with U and using (3.20) we get

$$|Re\langle \mathcal{A}U, U\rangle| \leq \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}}. \tag{3.55}$$

This implies that

$$\sum_{j=1}^n \int_\Omega v_j^2(x) \, dx, \quad \sum_{j=1}^n \int_\Omega z_j^2(x,1) \, dx \leq C\|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}}. \tag{3.56}$$

From $(3.54)_3$, we obtain

$$\phi = \frac{z(x,1)\mu(\xi) + F_3}{i\tilde{\lambda} + \xi^2 + \eta}. \tag{3.57}$$

Then

$$\begin{aligned} \|\phi\|_{L^2(\Omega\times(-\infty,+\infty))} &\leq \left\| \frac{\mu(\xi)}{i\tilde{\lambda} + \xi^2 + \eta} \right\|_{L^2(-\infty,+\infty)} \|z(x,1)\|_{L^2(\Omega)} + \left\| \frac{F_3}{i\tilde{\lambda} + \xi^2 + \eta} \right\|_{L^2(\Omega\times(-\infty,+\infty))} \\ &\leq \left(2(1-\alpha)\frac{\pi}{\sin\alpha\pi}(|\tilde{\lambda}| + \eta)^{\alpha-2} \right)^{\frac{1}{2}} \|z(x,1)\|_{L^2(\Omega)} + \frac{\sqrt{2}}{|\tilde{\lambda}| + \eta} \|F_3\|_{L^2(\Omega\times(-\infty,+\infty))}. \end{aligned} \tag{3.58}$$

Similarly, we have

$$\begin{aligned} \|\xi\phi\|_{L^2(\Omega\times(-\infty,+\infty))} &\leq \left\| \frac{\xi\mu(\xi)}{i\tilde{\lambda} + \xi^2 + \eta} \right\|_{L^2(-\infty,+\infty)} \|z(x,1)\|_{L^2(\Omega)} + \left\| \frac{\xi F_3}{i\tilde{\lambda} + \xi^2 + \eta} \right\|_{L^2(\Omega\times(-\infty,+\infty))} \\ &\leq \left(2\alpha\frac{\pi}{\sin\alpha\pi}(|\tilde{\lambda}| + \eta)^{\alpha-1} \right)^{\frac{1}{2}} \|z(x,1)\|_{L^2(\Omega)} + \frac{\sqrt{2}}{\sqrt{|\tilde{\lambda}| + \eta}} \|F_3\|_{L^2(\Omega\times(-\infty,+\infty))}. \end{aligned} \tag{3.59}$$

Let us introduce the following notation

$$\mathcal{I}_u(x) = \sum_{j=1}^{n} \left(|v_j(x)|^2 + \mu|\nabla u_j(x)|^2 \right) + (\mu+\lambda)|\text{div } u(x)|^2$$

and

$$\mathcal{E}_u = \int_\Omega \mathcal{I}_u(x)\, dx.$$

Lemma 3.5.1 *We have that*

$$\mathcal{E}_u \leq c\|F\|^2_{\mathcal{H}} + c'\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{3.60}$$

for positive constants c and c'.

Proof. Multiplying the equation $(3.54)_2$ by $\bar{u}$, integrating on Ω we obtain

$$\begin{gathered} -\int_\Omega v_j(\overline{i\tilde{\lambda}u_j})\, dx + \mu\int_\Omega |\nabla u_j|^2\, dx + (\mu+\lambda)\int_\Omega (\text{div } u)\frac{\partial \bar{u}_j}{\partial x_j}\, dx + \zeta\int_\Omega \bar{u}_j(\int_{-\infty}^{+\infty} \mu(\xi)\phi_j(x,\xi)\, d\xi)\, dx \\ +a_2\int_\Omega \bar{u}_j v_j\, dx = \int_\Omega \bar{u}F_2^j\, dx. \end{gathered} \tag{3.61}$$

From $(3.54)_1$, we have $i\tilde{\lambda}u_j = v_j + F_1^j$. Then

$$\begin{gathered} -\int_\Omega |v_j|^2\, dx + \mu\int_\Omega |\nabla u_j|^2\, dx + (\mu+\lambda)\int_\Omega (\text{div } u)\frac{\partial \bar{u}_j}{\partial x_j}\, dx + \zeta\int_\Omega \bar{u}_j(\int_{-\infty}^{+\infty} \mu(\xi)\phi_j(x,\xi)\, d\xi)\, dx \\ +a_2\int_\Omega \bar{u}_j v_j\, dx = \int_\Omega \bar{u}_j F_2^j\, dx + \int_\Omega v_j\overline{F}_1^j\, dx. \end{gathered} \tag{3.62}$$

Hence

$$-\sum_{j=1}^{n}\int_{\Omega}|v_j|^2\,dx+\mu\sum_{j=1}^{n}\int_{\Omega}|\nabla u_j|^2\,dx+(\mu+\lambda)\int_{\Omega}|\mathrm{div}\ u|^2\,dx+\zeta\sum_{j=1}^{n}\int_{\Omega}\bar{u_j}(\int_{-\infty}^{+\infty}\mu(\xi)\phi_j(x,\xi)\,d\xi)\,dx$$
$$+a_2\sum_{j=1}^{n}\int_{\Omega}\bar{u_j}v_j\,dx=\sum_{j=1}^{n}\int_{\Omega}\bar{u_j}F_2^j\,dx+\sum_{j=1}^{n}\int_{\Omega}v_j\overline{F}_1^j\,dx. \tag{3.63}$$

We can estimate

$$\begin{aligned}
&\left|\int_{\Omega}\bar{u_j}\left(\int_{-\infty}^{+\infty}\mu(\xi)\phi_j(x,\xi)\,d\xi\right)\,dx\right|\\
&\le\|u_j\|_{L^2(\Omega)}\left(\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta}\,d\xi\right)^{\frac{1}{2}}\left(\int_{\Omega}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi_j(x,\xi)|^2\,d\xi\,dx\right)^{\frac{1}{2}}\\
&\le\frac{\varepsilon}{2}\left(\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta}\,d\xi\right)\|u_j\|^2_{L^2(\Omega)}+\frac{1}{2\varepsilon}\int_{\Omega}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi_j(x,\xi)|^2\,d\xi\,dx\\
&\le\frac{\varepsilon}{2}C(\Omega)\left(\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta}\,d\xi\right)\|\nabla u_j\|^2_{L^2(\Omega)}+\frac{1}{2\varepsilon}\int_{\Omega}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi_j(x,\xi)|^2\,d\xi\,dx,
\end{aligned}$$

$$\begin{aligned}
\left|\int_{\Omega}\bar{u_j}v_j\,dx\right| &\le \|u_j\|_{L^2(\Omega)}\|v_j\|_{L^2(\Omega)}\\
&\le \frac{\varepsilon}{2}C(\Omega)\|\nabla u_j\|^2_{L^2(\Omega)}+\frac{1}{2\varepsilon}\|v_j\|^2_{L^2(\Omega)},
\end{aligned}$$

$$\left|\int_{\Omega}\bar{u_j}F_2^j\,dx\right|\le\frac{\varepsilon}{2}C(\Omega)\|\nabla u_j\|^2_{L^2(\Omega)}+\frac{1}{2\varepsilon}\|F_2^j\|^2_{L^2(\Omega)},$$

$$\left|\int_{\Omega}v_j\overline{F}_1^j\,dx\right|\le\frac{\varepsilon}{2}\|v_j\|^2_{L^2(\Omega)}+\frac{1}{2\varepsilon}\|F_1^j\|^2_{L^2(\Omega)}.$$

Choosing ε small enough, we conclude (3.60). Moreover, the equation $(3.54)_4$ has a unique solution

$$\begin{aligned}
z(x,\rho) &= e^{-i\tau\tilde{\lambda}\rho}z(x,0)+\tau e^{-i\tau\tilde{\lambda}\rho}\int_0^{\rho}e^{-i\tau\tilde{\lambda}\sigma}F_4(x,\sigma)\,d\sigma\\
&= e^{-i\tau\tilde{\lambda}\rho}v(x)+\tau e^{-i\tau\tilde{\lambda}\rho}\int_0^{\rho}e^{-i\tau\tilde{\lambda}\sigma}F_4(x,\sigma)\,d\sigma.
\end{aligned}$$

Then

$$\|z(x,\rho)\|_{L^2(\Omega\times(0,1))}\le\|v(x)\|_{L^2(\Omega)}+\tau\|F_4(x,\rho)\|_{L^2(\Omega\times(0,1))}. \tag{3.64}$$

Finally, (3.58), (3.60) and (3.64) imply that

$$\|U\|_{\mathcal{H}}\le C$$

for a positive constant C. The conclusion then follows by applying Theorem 3.5.1.

Remark 3.5.1 *We can extend the results of this chapter to more general measure density instead of (3.1), that is μ is an even nonnegative measurable function such that*

$$\int_{-\infty}^{\infty}\frac{\mu(\xi)^2}{1+\xi^2}\,d\xi<\infty. \tag{3.65}$$

Chapter 4

EXPONENTIAL DECAY FOR THE LAMÉ SYSTEM WITH FRACTIONAL TIME DELAY AND BOUNDARY FEEDBACK

4.1 Introduction

This chapter is devoted to the study of well-posedness and boundary stabilization of the Lamé system in an bounded domain Ω of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ of class C^2. We assume that $\Gamma = \Gamma_0 \cup \Gamma_1$, where Γ_0 and Γ_1 are closed subsets of Γ with $\Gamma_0 \cap \Gamma_1 = \emptyset$.

The system is given by :

$$(P3) \qquad \begin{cases} u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla(\text{div } u) + a_1\partial_t^{\alpha,\eta}u(x,t-\tau) = 0 & \text{in } \Omega \times (0,+\infty), \\ u = 0 & \text{in } \Gamma_0 \times (0,+\infty), \\ \mu\dfrac{\partial u}{\partial \nu} + (\mu+\lambda)(\text{div } u)\nu = -a_2 u_t(x,t) & \text{in } \Gamma_1 \times (0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau) & \text{in } \Omega \times (0,\tau). \end{cases}$$

where μ, λ are Lamé constants, $u = (u_1, u_2, ..., u_n)^T$. Moreover, $a_1 > 0$, $a_2 > 0$ and the constant $\tau > 0$ is the time delay. ν stands for the unit normal vector of $\partial\Omega$ pointing towards the exterior of Ω and $\dfrac{\partial u}{\partial \nu}$ is the normal derivative. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative (see **[12]**) of order α with respect to the time variable and is defined by

$$\partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s),\, ds \qquad 0 < \alpha < 1, \qquad \eta \geq 0.$$

One very active area of mathematical control theory has been the investigation of the delay effect in the stabilization of hyperbolic systems. It is well known that an arbitrarily small delay can have a destabilizing effect to systems that are asymptotically stable in the absence of delay (see [**5**], [**16**], [**17**], [**22**], [**41**], [**39**]).

In particular, the following boundary stabilization problem for the N-dimensional wave equation with interior delay was studied In [**5**],

$$(PA) \qquad \begin{cases} u_{tt}(x,t) - \Delta u(x,t) - a u_t(x,t-\tau) = 0 & x \in \Omega, \ \ t > 0, \\ u = 0 & x \in \Gamma_0, \ \ t > 0, \\ \dfrac{\partial u}{\partial \nu} = -k u_t(x,t) & x \in \Gamma_1, \ \ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & x \in \Omega, \\ u_t(x,t) = g(x,t), & x \in \Omega, \ \ t \in (-\tau, 0) \end{cases}$$

where the authors showed an exponential stability result under the usual Lions geometric condition on the domain Ω, providing that the delay coefficient a is sufficiently small. However, if the damping factor is larger than the delay factor then one can show exponential stability for the wave equation.

In the absence of the delay in system (PA), that is for $\tau = 0$, a large amount of literature is available on this model, addressing problems of the existence, uniqueness and asymptotic behavior in time when some damping effects are considered, such as: frictional damping, viscoelastic damping and thermal dissipation. Furthermore, in the case of absence of both of the delay and danping, that is for $a = 0$ and $k = 0$, the asymptotic stability of (PA) has been shown in [**18**] using the well-known Arendt-Batty- Lyubic-Vu Theorem. This is the best we can obtain since it is possible to have eigenvalues arbitrarily close to the imaginary axis, see for instance [**19**].

Moreover, the result in [**5**] was extended to the Timoshenko system in [**44**] (see also [**20**]), where the authors studied a Timoshenko beam system given by two coupled hyperbolic equations, with delay terms in the first and second equation and two boundary controls, they proved the exponential decay of the total energy.

To our best knowledge the Lamé system with internal fractional time delay terms is not considered previously. Motivated by the above research, we will consider the Lamé with internal fractional time delays and boundary feedbacks (P). The main objectives of the present work are to establish the global well-posedness and exponential stability of system (P).

The idea in this work is that a damping with time delay does not destroy the stability if there is another boundary dissipative damping in which contrasts appropriately with the previous one. (i.e., by giving the control in the feedback form $-a_2 u_t(x,t) \ \ x \in \Gamma_1, \ \ t > 0$). We will show that system $(P3)$ is exponentially stable for a_1 sufficiently small.

This chapter is organized as follows. In Section **2**, we take advantage of the complete monotonicity of the power function integral kernel to represent it as a superposition of exponentials and derive what we call the "augmented model", while in Section **3**, we deal with the well-posedness result of the problem using the semigroup theory. Lastly, in Section **4**, we obtain exponential stability results by constructing an appropriate Lyapunov functional as in [**5**].

4.2 Preliminaries

This section is concerned with the reformulation of the model $(P3)$ into an augmented system. For that, we need the following claims.

Theorem 4.2.1 (see [38]) *Let μ be the function:*

$$\mu(\xi) = |\xi|^{(2\alpha-1)/2}, \quad -\infty < \xi < +\infty,\ 0 < \alpha < 1. \tag{4.1}$$

Then the relationship between the 'input' U and the 'output' O of the system

$$\partial_t\phi(\xi,t) + (\xi^2+\eta)\phi(\xi,t) - U(t)\mu(\xi) = 0, \quad -\infty < \xi < +\infty, \eta \geq 0, t > 0, \tag{4.2}$$

$$\phi(\xi,0) = 0, \tag{4.3}$$

$$O(t) = (\pi)^{-1}\sin(\alpha\pi)\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi,t)\,d\xi \tag{4.4}$$

is given by

$$O = I^{1-\alpha,\eta}U = D^{\alpha,\eta}U, \tag{4.5}$$

where

$$[I^{\alpha,\eta}f](t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}e^{-\eta(t-\tau)}f(\tau)\,d\tau.$$

Proof. From (4.2) and (4.3), we have

$$\phi(\xi,t) = \int_0^t \mu(\xi)e^{-(\xi^2+\eta)(t-\tau)}U(\tau)d\tau. \tag{4.6}$$

Hence, by using (4.4), we get

$$O(t) = (\pi)^{-1}sin(\alpha\pi)e^{-\eta t}\int_0^t\left[2\int_0^{+\infty}|\xi|^{2\alpha-1}e^{-\xi^2(t-s)}d\xi\right]e^{\eta\tau}U(\tau)d\tau. \tag{4.7}$$

Thus,

$$\begin{aligned} O(t) &= (\pi)^{-1}sin(\alpha\pi)e^{-\eta t}\int_0^t\left[(t-s)^{-\alpha}\Gamma(\alpha)\right]e^{\eta\tau}U(\tau)d\tau \\ &= (\pi)^{-1}sin(\alpha\pi)e^{-\eta t}\int_0^t\left[(t-s)^{-\alpha}\Gamma(\alpha)\right]e^{-\eta(t-\tau)}U(\tau)d\tau \end{aligned} \tag{4.8}$$

which completes the proof. Indeed, we know that $(\pi)^{-1}\sin(\alpha\pi) = \dfrac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}$.

Lemma 4.2.1 (see [2]) *If $\lambda > 0$ then*

$$\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda + \xi^2}\, d\xi = \frac{\pi}{\sin \alpha\pi}(\lambda + \eta)^{\alpha - 1}.$$

We are now in a position to reformulate system $(P3)$. Indeed, by using Theorem 4.2.1, system $(P3)$ may be recast into the augmented model:

$$(P'3)\quad \begin{cases} u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla(\text{div } u) & \\ \qquad\qquad + \zeta \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(x,\xi,t)\, d\xi = 0 & \text{in } \Omega \times (0,+\infty), \\ \partial_t \phi(x,\xi,t) + (\xi^2 + \eta)\phi(x,\xi,t) - z(x,1,t)\mu(\xi) = 0 & \text{in } \Omega \times (-\infty,\infty) \times (0,+\infty), \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 & \text{in } \Omega \times (0,1) \times (0,+\infty), \\ u(x,t) = 0 & \text{on } \Gamma_0 \times (0,+\infty), \\ \mu \dfrac{\partial u}{\partial \nu} + (\mu+\lambda)(\text{div } u)\nu = -a_2 u_t(x,t) & \text{in } \Gamma_1 \times (-\infty,\infty) \times (0,+\infty), \\ z(x,0,t) = u_t(x,t), & \text{in } \Omega \times (0,+\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{on } \Omega, \\ \phi(x,\xi,0) = 0 & \text{on } \Omega \times (-\infty,\infty), \\ z(x,\rho,0) = f_0(x,-\rho\tau) & \text{in } \Omega \times (0,1), \end{cases}$$

where $\zeta = a_1(\pi)^{-1}\sin(\alpha\pi)$.

We define the energy of the solution by:

$$\begin{aligned} E(t) = \frac{1}{2}\sum_{j=1}^{n}\Big(&\|u_{jt}\|^2_{L^2(\Omega)} + \mu\|\nabla u_j\|^2_{L^2(\Omega)} + \overline{\zeta}\int_\Omega \int_{-\infty}^{+\infty} |\phi_j(x,\xi,t)|^2\, d\xi dx\Big) \\ &+ \frac{\nu}{2}\sum_{j=1}^{n}\int_\Omega \int_0^1 |z_j(x,\rho,t)|^2\, d\rho dx + \frac{(\mu+\lambda)}{2}\|\text{div } u\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.9}$$

where $\overline{\zeta} = \dfrac{\nu}{2\tau I}$, $I = \displaystyle\int_0^\infty \frac{\mu^2(\xi)}{\xi^2 + \eta}\, d\xi$ and ν is a strictly positive real number.

In order to establish the exponential energy decay rate, let us consider the usual geometrical control condition: there exists a point $x_0 \in \mathbb{R}^n$ such that

$$m \cdot \nu \le 0 \ \text{ on } \ \Gamma_0 \qquad m \cdot \nu > 0 \ \text{ on } \ \Gamma_0,$$

where $m = x - x_0$

The main result of this chapter is the following.

Theorem 4.2.2 *For any $a_2 > 0$ there exist positive constants a_0, C_1, C_2 such that*

$$E(t) \leq C_1 e^{-C_2 t} E(0), \tag{4.10}$$

for any regular solution of problem $(P3)$ with $0 \leq a_1 < a_0$. The constants a_0, C_1, C_2 are independent of the initial data but they depend on a_2 and on the geometry of Ω.

4.3 Well-posedness

In this section, we give the existence and uniqueness result for system $(P'3)$ using the semigroup theory. Let us denote $U = (u, v, \phi, z)^T$, where $v = u_t$. The system $(P'3)$ can be rewrite as follows:

$$\begin{cases} U' = \mathcal{A}U, & t > 0, \\ U(0) = (u_0, u_1, \phi_0, f_0), \end{cases} \tag{4.11}$$

where $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is the linear operator defined by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ \phi \\ z \end{pmatrix} = \begin{pmatrix} v \\ \mu\Delta u + (\mu+\lambda)\nabla(\text{div } u) - \zeta\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(x,\xi)\, d\xi \\ -(\xi^2+\eta)\phi + z(x,1)\mu(\xi) \\ -\tau^{-1} z_\rho(x,\rho) \end{pmatrix} \tag{4.12}$$

and $\mathcal{H}$ is the energy space given by

$$\mathcal{H} = \left(H^1_{\Gamma_0}(\Omega)\right)^n \times \left(L^2(\Omega)\right)^n \times \left(L^2(\Omega \times (-\infty, +\infty))\right)^n \times \left(L^2(\Omega \times (0,1))\right)^n.$$

where

$$H^1_{\Gamma_0}(\Omega) = \left\{ u \in H^1(\Omega) :\, u|\Gamma_0 = 0 \right\}.$$

For any $U = (u, v, \phi, z)^T \in \mathcal{H}, \tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\phi}, \tilde{z})^T \in \mathcal{H}$, we equip $\mathcal{H}$ with the inner product defined by

$$\begin{aligned} < U, \tilde{U} >_{\mathcal{H}} = & \sum_{j=1}^{n} \int_\Omega (v_j \tilde{v}_j + \mu \nabla u_j \nabla \tilde{u}_j)\, dx + (\mu+\lambda) \int_\Omega (\text{div } u)(\text{div } \tilde{u})\, dx \\ & + \bar{\zeta} \sum_{j=1}^{n} \int_\Omega \int_{-\infty}^{+\infty} \phi_j(x,\xi) \tilde{\phi}_j(x,\xi) d\xi dx + \nu \sum_{j=1}^{n} \int_\Omega \int_0^1 z(x,\rho) \tilde{z}_j(x,\rho) d\rho dx. \end{aligned}$$

The domain of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \left\{ \begin{array}{l} (u, v, \phi, z)^T \text{ in } \mathcal{H} : u \in \left(H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)\right)^n, v \in \left(H^1(\Omega)\right)^n, \\ -(\xi^2+\eta)\phi + z(x,1,t)\mu(\xi) \in \left(L^2(\Omega \times (-\infty, +\infty))\right)^n, \\ z \in \left(L^2\left(\Omega; H^1(0,1)\right)\right)^n, \\ \mu \dfrac{\partial u}{\partial \nu} + (\mu+\lambda)(\text{div } u)\nu + a_2 v = 0 \text{ on } \Gamma_1, \\ |\xi|\phi \in \left(L^2(\Omega \times (-\infty, +\infty))\right)^n, v = z(.,0) \text{ in } \Omega \end{array} \right\}. \tag{4.13}$$

Remark 4.3.1 *The condition* $|\xi|\phi(\xi) \in (L^2(\Omega \times \mathbb{R}))^n$ *is imposed to insure the existence of* $-\overline{\zeta}\sum_{j=1}^{n}\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi_j(x,\xi,t)|^2\,d\xi\,dx$ *and* $\mu(\xi)\phi(x,\xi) \in (L^1(\Omega \times \mathbb{R}))^n$.

We show that there exists a positive constant c such that $(\mathcal{A}-cI)$ is dissipative. Let $U=(u,v,\phi,z)^T \in D(\mathcal{A})$, then

$$\begin{aligned}
\langle \mathcal{A}U,U\rangle_{\mathcal{H}} = & -a_2\sum_{j=1}^{n}\|v_j\|^2_{L^2(\Gamma_1)} - \zeta\sum_{j=1}^{n}\int_\Omega v_j\int_{-\infty}^{+\infty}\mu(\xi)\phi_j(x,\xi)\,d\xi\,dx \\
& +\overline{\zeta}\sum_{j=1}^{n}\int_\Omega z_j(x,1)\int_{-\infty}^{+\infty}\mu(\xi)\phi_j(x,\xi)\,d\xi\,dx - \overline{\zeta}\sum_{j=1}^{n}\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi_j(x,\xi,t)|^2\,d\xi\,dx \\
& +\frac{\nu}{2\tau}\sum_{j=1}^{n}\|v_j\|^2_{L^2(\Omega)} - \frac{\nu}{2\tau}\sum_{j=1}^{n}\|z_j(x,1)\|^2_{L^2(\Omega)} \\
\leq & -a_2\sum_{j=1}^{n}\|v_j\|^2_{L^2(\Gamma_1)} + \left(\frac{\zeta^2 I}{2\overline{\zeta}}+\frac{\nu}{2\tau}\right)\sum_{j=1}^{n}\|v_j\|^2_{L^2(\Omega)} \\
\leq & \left(\frac{\zeta^2 I}{2\overline{\zeta}}+\frac{\nu}{2\tau}\right)\sum_{j=1}^{n}\|v_j\|^2_{L^2(\Omega)}
\end{aligned}$$

This shows that $(\mathcal{A}-cI)$ is dissipative.

In the sequel, we claim that the operator $\mathcal{A}$ has the property $R(\tilde{\lambda}I-\mathcal{A})=\mathcal{H}$ for fixed $\tilde{\lambda}>0$. Indeed, let $F=(F_1,F_2,F_3,F_4)^T \in \mathcal{H}$, where $F_i=(f_i^1,f_i^2,...,f_i^n)^T$, we must solve the problem $(\tilde{\lambda}I-\mathcal{A})U=F$. for some $U=(u,v,\phi,z)^T \in D(\mathcal{A})$. The equation becomes the system

$$\begin{cases}
\tilde{\lambda}u - v = F_1(x), \\
\tilde{\lambda}v - \mu\Delta u - (\mu+\lambda)\nabla(\operatorname{div}\, u) + \zeta\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi(x,\xi)\,d\xi = F_2(x), \\
\tilde{\lambda}\phi + (\xi^2+\eta)\phi - z(x,1)\mu(\xi) = F_3(x,\xi), \\
\lambda z(x,\rho) + \tau^{-1}z_\rho(x,\rho) = F_4(x,\rho).
\end{cases} \tag{4.14}$$

Suppose u is found with the appropriate regularity. Then, $(4.14)_1$ and $(4.14)_3$ yield

$$v = \tilde{\lambda}u - F_1(x) \in \left(H^1_{\Gamma_0}(\Omega)\right)^n \tag{4.15}$$

and

$$\phi = \frac{F_3(x,\xi)+\mu(\xi)z(x,1)}{\xi^2+\eta+\tilde{\lambda}}. \tag{4.16}$$

We note that the last equation in (4.14) with $z(x,0)=v(x)$ has a unique solution given by

$$z(x,\rho) = v(x)e^{-\tilde{\lambda}\rho\tau} + \tau e^{-\tilde{\lambda}\rho\tau}\int_0^\rho F_4(x,r)e^{\tilde{\lambda}r\tau}dr. \tag{4.17}$$

Inserting (4.15) in (4.17), we get

$$z(x,\rho) = \tilde{\lambda}u(x)e^{-\tilde{\lambda}\rho\tau} - F_1(x)e^{-\tilde{\lambda}\rho\tau} + \tau e^{-\tilde{\lambda}\rho\tau}\int_0^\rho F_4(x,r)e^{\tilde{\lambda}r\tau}dr, \qquad x \in \Omega, \rho \in (0,1). \tag{4.18}$$

In particular,

$$z(x,1) = \tilde{\lambda}u(x)e^{-\tilde{\lambda}\tau} + z_0(x), \quad x \in \Omega \tag{4.19}$$

where for $x \in \Omega$

$$z_0(x) = -F_1(x)e^{-\tilde{\lambda}\tau} + \tau e^{-\tilde{\lambda}\tau}\int_0^1 F_4(x,r)e^{\tilde{\lambda}r\tau}dr. \tag{4.20}$$

In light of the above results, the function u satisfies the following equation

$$\tilde{\lambda}^2 u - \mu\Delta u - (\mu+\lambda)\nabla(\text{div } u) + \zeta\int_{-\infty}^{+\infty}\mu(\xi)\phi(x,\xi)\,d\xi = F_2(x) + \tilde{\lambda}F_1(x). \tag{4.21}$$

Then for any $w \in (H^1_{\Gamma_0}(\Omega))^n$, it follows from problem (4.21) that such that

$$\begin{aligned}\int_\Omega\left(\tilde{\lambda}^2 u_j w_j - \mu\Delta u_j w_j\right)dx - (\mu+\lambda)\int_\Omega\frac{\partial}{\partial x_j}(\text{div } u)w_j dx + \zeta\sum_{j=1}^n\int_\Omega w_j\int_{-\infty}^{+\infty}\mu(\xi)\phi_j(x,\xi)\,d\xi dx\\ = \int_\Omega(F_2^j(x) + \tilde{\lambda}F_1^j(x))w_j dx,\end{aligned} \tag{4.22}$$

By using integration by parts, the boundary condition $(4.13)_4$ and (4.16), we infer that

$$\begin{cases}\displaystyle\sum_{j=1}^n\int_\Omega\left(\tilde{\lambda}^2 u_j w_j + \mu\nabla u_j\nabla w_j\right)dx + (\mu+\lambda)\int_\Omega(\text{div } u)(\text{div } w)\,dx + \tilde{\lambda}\theta\sum_{j=1}^n\int_\Omega u_j w_j e^{-\tilde{\lambda}\tau}\,dx\\ \displaystyle\quad +\tilde{\lambda}a_2\sum_{j=1}^n\int_{\Gamma_1}u_j w_j\,d\Gamma = \sum_{j=1}^n\int_\Omega\left(F_2^j(x) + \tilde{\lambda}F_1^j(x)\right)w_j\,dx + a_2\sum_{j=1}^n\int_{\Gamma_1}F_1^j(x)w_j\,d\Gamma\\ \displaystyle\quad -\zeta\sum_{j=1}^n\int_\Omega w_j\left(\int_{-\infty}^{\infty}\frac{\mu(\xi)F_3^j(x,\xi)}{\xi^2+\eta+\tilde{\lambda}}\,d\xi\right)dx - \theta\sum_{j=1}^n\int_\Omega w_j z_0(x)\,dx.\end{cases} \tag{4.23}$$

where $\theta = \zeta\displaystyle\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta+\tilde{\lambda}}\,d\xi$.

Problem (4.23) is of the form

$$\mathcal{B}(u,w) = \mathcal{L}(w), \tag{4.24}$$

where $\mathcal{B} : \left[(H_0^1(\Omega))^n \times (H_0^1(\Omega))^n\right] \to \mathbb{C}$ is the sesquilinear form defined by

$$\begin{aligned}\mathcal{B}(u,w) = \sum_{j=1}^n\int_\Omega\left(\tilde{\lambda}^2 u_j w_j + \mu\nabla u_j\nabla w_j\right)dx + (\mu+\lambda)\int_\Omega(\text{div } u)(\text{div } w)\,dx\\ +\tilde{\lambda}\theta\sum_{j=1}^n\int_\Omega u_j w_j e^{-\tilde{\lambda}\tau}\,dx + \tilde{\lambda}a_2\sum_{j=1}^n\int_{\Gamma_1}u_j w_j\,d\Gamma\end{aligned}$$

and $\mathcal{L} : (H_0^1(\Omega))^n \to \mathbb{C}$ is the antilinear functional given by

$$\begin{aligned}\mathcal{L}(w) = \sum_{j=1}^n\int_\Omega\left(F_2^j(x) + \tilde{\lambda}F_1^j(x)\right)w_j\,dx - \zeta\sum_{j=1}^n\int_\Omega w_j\left(\int_{-\infty}^{\infty}\frac{\mu(\xi)F_3^j(x,\xi)}{\xi^2+\eta+\tilde{\lambda}}\,d\xi\right)dx\\ -\theta\sum_{j=1}^n\int_\Omega w_j z_0(x)\,dx + a_2\sum_{j=1}^n\int_{\Gamma_1}F_1^j(x)w_j\,d\Gamma.\end{aligned}$$

It is easy to verify that $\mathcal{B}$ is continuous and coercive, and $\mathcal{L}$ is continuous. Consequently, by the Lax-Milgram theorem, we conclude that for all $w \in (H_0^1(\Omega))^n$, the system (4.24) has a unique solution $u \in (H_0^1(\Omega))^n$. By the regularity theory for the linear elliptic equations, it follows that $u \in (H^2(\Omega))^n$. Therefore, the operator $(\tilde{\lambda} I - \mathcal{A})$ is surjective for any $\tilde{\lambda} > 0$.

Consequently, using Hille-Yosida theorem, we have the following existence result:

Theorem 4.3.1 (Existence and uniqueness)

(1) If $U_0 \in D(\mathcal{A})$, then system (4.11) has a unique strong solution

$$U \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

(2) If $U_0 \in \mathcal{H}$, then system (4.11) has a unique weak solution

$$U \in C^0(\mathbb{R}_+, \mathcal{H}).$$

4.4 Proof of Theorem 4.2.2

The proof will be divided into the following several technique propositions.

Proposition 4.4.1 *For any solution of problem $(P3)$ the following estimate holds:*

$$\begin{aligned} E'(t) \quad \leq \quad & -a_2 \sum_{j=1}^{n} \int_{\Gamma_1} |u_{jt}(x,t)|^2 \, d\Gamma + \frac{\zeta I + \nu\tau^{-1}}{2} \sum_{j=1}^{n} \int_{\Omega} u_{jt}^2(x,t)\, dx \\ & + \frac{\zeta I - \nu\tau^{-1}}{2I} \sum_{j=1}^{n} \int_{\Omega} \int_{-\infty}^{+\infty} (\xi^2 + \eta) |\phi_j(x,\xi,t)|^2 \, d\xi \, dx \end{aligned} \tag{4.25}$$

Proof. Multiplying the first equation in $(P3)$ by $\overline{u}_{jt}$, integrating over Ω and using integration by parts, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u_{jt}\|_2^2 - \mu \Re \int_{\Omega} \Delta u_j u_{jt}\, dx - (\mu+\lambda)\Re \int_{\Omega} \frac{\partial}{\partial x_j}(\text{div } u) u_{jt}\, dx \\ &+\zeta \int_{\Omega} u_{jt} \int_{-\infty}^{+\infty} \mu(\xi)\phi_j(x,\xi,t)\, d\xi\, dx = 0. \end{aligned}$$

Then

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\sum_{j=1}^{n}\left(\|u_{jt}\|_{L^2(\Omega)}^2 + \mu\|\nabla u_j\|_{L^2(\Omega)}^2\right) + \frac{(\mu+\lambda)}{2}\|\text{div } u\|_{L^2(\Omega)}^2 + a_2\sum_{j=1}^{n}\|u_{jt}\|_{L^2(\Gamma_1)}^2 \\ &+\zeta \Re \sum_{j=1}^{n} \int_{\Omega} u_{jt} \int_{-\infty}^{+\infty} \mu(\xi)\phi_j(x,\xi,t)\, d\xi\, dx = 0. \end{aligned} \tag{4.26}$$

Multiplying the second equation in $(P'3)$ by $\overline{\zeta}\phi_j$ and integrating over $\Omega\times(-\infty,+\infty)$, we obtain:

$$\begin{aligned}\frac{\overline{\zeta}}{2}\frac{d}{dt}\sum_{j=1}^{n}\|\phi_j\|^2_{L^2(\Omega\times(-\infty,+\infty))}+\overline{\zeta}\sum_{j=1}^{n}\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi_j(x,\xi,t)|^2\,d\xi\,dx&\\ -\overline{\zeta}\Re\sum_{j=1}^{n}\int_\Omega z_j(x,1,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi_j(x,\xi,t)\,d\xi\,dx=0.&\end{aligned}\tag{4.27}$$

Multiplying the third equation in $(P'3)$ by $\nu\overline{z}_j$ and integrating over $\Omega\times(0,1)$, we get:

$$\frac{\nu}{2}\frac{d}{dt}\sum_{j=1}^{n}\|z_j\|^2_{L^2(\Omega\times(0,1))}+\frac{\nu\tau^{-1}}{2}\sum_{j=1}^{n}\int_\Omega\left(z_j^2(x,1,t)-u_{jt}^2(x,t)\right)\,dx=0.\tag{4.28}$$

From $(4.9),(4.26)$ and (4.28) we get

$$\begin{aligned}E'(t)=&-a_2\sum_{j=1}^{n}\|u_{jt}\|^2_{L^2(\Gamma_1)}-\overline{\zeta}\sum_{j=1}^{n}\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi_j(x,\xi,t)|^2\,d\xi\,dx\\ &-\zeta\Re\sum_{j=1}^{n}\int_\Omega u_{jt}\int_{-\infty}^{+\infty}\mu(\xi)\phi_j(x,\xi,t)\,d\xi\,dx+\overline{\zeta}\Re\sum_{j=1}^{n}\int_\Omega z_j(x,1,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi_j(x,\xi,t)\,d\xi\,dx\\ &+\frac{\nu\tau^{-1}}{2}\sum_{j=1}^{n}\int_\Omega u_{jt}^2(x,t)\,dx-\frac{\nu\tau^{-1}}{2}\sum_{j=1}^{n}\int_\Omega z_j^2(x,1,t)\,dx.\end{aligned}\tag{4.29}$$

Moreover, we have

$$\left|\int_{-\infty}^{+\infty}\mu(\xi)\phi_j(x,\xi,t)\,d\xi\right|\leq\left(\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta}\,d\xi\right)^{\frac{1}{2}}\left(\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi_j(x,\xi,t)|^2\,d\xi\right)^{\frac{1}{2}}.$$

Then

$$\begin{aligned}&\left|\int_\Omega z_j(x,1,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi_j(x,\xi,t)\,d\xi\,dx\right|\\ &\leq\left(\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta}\,d\xi\right)^{\frac{1}{2}}\|z_j(x,1,t)\|_{L^2(\Omega)}\left(\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi_j(x,\xi,t)|^2\,d\xi\,dx\right)^{\frac{1}{2}},\end{aligned}$$

$$\begin{aligned}&\left|\int_\Omega u_{jt}(x,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi_j(x,\xi,t)\,d\xi\,dx\right|\\ &\leq\left(\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta}\,d\xi\right)^{\frac{1}{2}}\|u_j(x,t)\|_{L^2(\Omega)}\left(\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi_j(x,\xi,t)|^2\,d\xi\,dx\right)^{\frac{1}{2}}.\end{aligned}$$

Applying the Cauchy-Schwarz inequality we obtain (4.25).

Proposition 4.4.2 *For any regular solution of problem $(P3)$ and for every $\varepsilon > 0$, we have*

$$\begin{aligned}
&\sum_{j=1}^{n} \frac{d}{dt}\left\{\int_\Omega [2m\cdot\nabla u_j + (n-1)u_j]\, u_{jt}\, dx\right\} \\
&\qquad \le -\sum_{j=1}^{n}\int_\Omega \left(|u_{jt}|^2 + (\mu - \frac{\varepsilon}{2}C(P))|\nabla u_j|^2\right) dx - (\mu+\lambda)\int_\Omega |div\ u|^2\, dx \\
&\qquad\quad -\zeta\sum_{j=1}^{n}\int_\Omega [2m\cdot\nabla u_j + (n-1)u_j]\left[\int_{-\infty}^{+\infty}\mu(\xi)\phi_j(x,\xi)\, d\xi\right] dx \\
&\qquad\quad +\sum_{j=1}^{n}\int_{\Gamma_1}\left(\left(\|m\|_\infty + \frac{(n-1)^2}{2\varepsilon}a_2^2 + 2\frac{\|m\|_\infty^2}{\delta\mu}a_2^2\right)|u_{jt}|^2 - \left(\mu\delta - \frac{\delta\mu}{2}\right)|\nabla u_j|^2\right) d\Gamma \\
&\qquad\quad -(\mu+\lambda)\delta\int_{\Gamma_1}|div\ u|^2\, d\Gamma
\end{aligned}$$

where $C(P)$ is a sort of Poincaré constant, which is a positive constant depending on Ω and independent of the solution u.

Proof. Differentiating and integrating over Ω we obtain

$$\begin{aligned}
&\frac{d}{dt}\left\{\int_\Omega [2m\cdot\nabla u_j + (n-1)u_j]\, u_{jt}\, dx\right\} = \int_\Omega [2m\cdot\nabla u_{jt} + (n-1)u_{jt}]\, u_{jt}\, dx \\
&+\int_\Omega [2m\cdot\nabla u_j + (n-1)u_j]\left[\mu\Delta u_j + (\mu+\lambda)\frac{\partial}{\partial x_j}(\text{div } u) - \zeta\int_{-\infty}^{+\infty}\mu(\xi)\phi_j(x,\xi)\, d\xi\right] dx
\end{aligned}$$

For $u \in H^2(\Omega)$, we have the following Rellich's identity

$$\begin{aligned}
&\int_\Omega \Delta u_j(m\cdot\nabla u_j)\, dx = \int_\Gamma (m\cdot\nabla u_j)\frac{\partial u_j}{\partial\nu}\, d\Gamma - \int_\Omega \nabla u_j\cdot\nabla(m\cdot\nabla u_j)\, dx. \\
&\int_\Omega \frac{\partial(\text{div } u)}{\partial x_j}(m\cdot\nabla u_j)\, dx = \int_\Gamma (m\cdot\nabla u_j)(\text{div } u)\nu_j\, d\Gamma - \int_\Omega (\text{div } u)\frac{\partial}{\partial x_j}(m\cdot\nabla u_j)\, dx.
\end{aligned} \tag{4.30}$$

Hence

$$\begin{aligned}
\frac{d}{dt}\left\{\int_\Omega [2m\cdot\nabla u_j + (n-1)u_j]\, u_{jt}\, dx\right\} &= \int_\Omega [2m\cdot\nabla u_{jt} + (n-1)u_{jt}]\, u_{jt}\, dx \\
&-\zeta\int_\Omega [2m\cdot\nabla u_j + (n-1)u_j]\left[\int_{-\infty}^{+\infty}\mu(\xi)\phi_j(x,\xi)\, d\xi\right] dx \\
&2\mu\int_\Gamma (m\cdot\nabla u_j)\frac{\partial u_j}{\partial\nu}\, d\Gamma - 2\mu\int_\Omega \nabla u_j\cdot\nabla(m\cdot\nabla u_j)\, dx \\
&+2(\mu+\lambda)\int_\Gamma (m\cdot\nabla u_j)(\text{div } u)\nu_j\, d\Gamma - 2(\mu+\lambda)\int_\Omega (\text{div } u)\frac{\partial}{\partial x_j}(m\cdot\nabla u_j)\, dx \\
&-(n-1)\mu\int_\Omega |\nabla u_j|^2\, dx + (n-1)\mu\int_{\Gamma_1} u_j\frac{\partial u_j}{\partial\nu}\, d\Gamma \\
&-(n-1)(\mu+\lambda)\int_\Omega \frac{\partial u_j}{\partial x_j}(\text{div } u)\, dx + (n-1)(\mu+\lambda)\int_\Gamma u_j(\text{div } u)\nu_j\, d\Gamma
\end{aligned} \tag{4.31}$$

Moreover, using the following identity

$$2\nabla u_j \cdot \nabla(m\cdot\nabla u_j) = 2|\nabla u_j|^2 + m\cdot\nabla(|\nabla u_j|^2)$$

and integration by parts, we get

$$\begin{aligned}
&2\int_\Omega \nabla u_j\cdot\nabla(m\cdot\nabla u_j)\,dx = (2-n)\int_\Omega |\nabla u_j|^2\,dx + \int_\Gamma m\cdot\nu|\nabla u_j|^2\,d\Gamma\\
&2\sum_{j=1}^n\int_\Omega (\text{div } u)\frac{\partial}{\partial x_j}(m\cdot\nabla u_j)\,dx = (2-n)\int_\Omega |\text{div } u|^2\,dx + \int_\Gamma m\cdot\nu|\text{div } u|^2\,d\Gamma.
\end{aligned}\tag{4.32}$$

Substituting (4.32) into (4.31), we get

$$\begin{aligned}
\sum_{j=1}^n \frac{d}{dt}\left\{\int_\Omega [2m\cdot\nabla u_j + (n-1)u_j]\,u_{jt}\,dx\right\} &= \sum_{j=1}^n\int_\Omega [2m\cdot\nabla u_{jt} + (n-1)u_{jt}]\,u_{jt}\,dx\\
&-\zeta\sum_{j=1}^n\int_\Omega [2m\cdot\nabla u_j + (n-1)u_j]\left[\int_{-\infty}^{+\infty}\mu(\xi)\phi_j(x,\xi)\,d\xi\right]dx\\
&+2\mu\sum_{j=1}^n\int_\Gamma (m\cdot\nabla u_j)\frac{\partial u_j}{\partial\nu}\,d\Gamma - \mu\sum_{j=1}^n\int_\Gamma m\cdot\nu|\nabla u_j|^2\,d\Gamma\\
&-(\mu+\lambda)\int_\Gamma m\cdot\nu|\text{div } u|^2\,d\Gamma + 2(\mu+\lambda)\sum_{j=1}^n\int_\Gamma (m\cdot\nabla u_j)(\text{div } u)\nu_j\,d\Gamma\\
&-\mu\sum_{j=1}^n\int_\Omega |\nabla u_j|^2\,dx - (\mu+\lambda)\int_\Omega |\text{div } u|^2\,dx + (n-1)\mu\sum_{j=1}^n\int_{\Gamma_1} u_j\frac{\partial u_j}{\partial\nu}\,d\Gamma\\
&+(n-1)(\mu+\lambda)\sum_{j=1}^n\int_\Gamma u_j(\text{div } u)\nu_j\,d\Gamma
\end{aligned}\tag{4.33}$$

Noting that $\nabla u_j = \dfrac{\partial u_j}{\partial\nu}\nu$ on Γ_0, it follows that

$$\begin{aligned}
\sum_{j=1}^n \frac{d}{dt}\left\{\int_\Omega [2m\cdot\nabla u_j + (n-1)u_j]\,u_{jt}\,dx\right\} &= -\sum_{j=1}^n\int_\Omega |u_{jt}|^2\,dx + \sum_{j=1}^n\int_{\Gamma_1}(m\cdot\nu)|u_{jt}|^2\,d\Gamma\\
&-\zeta\sum_{j=1}^n\int_\Omega [2m\cdot\nabla u_j + (n-1)u_j]\left[\int_{-\infty}^{+\infty}\mu(\xi)\phi_j(x,\xi)\,d\xi\right]dx\\
&+\mu\sum_{j=1}^n\int_{\Gamma_0} m\cdot\nu|\nabla u_j|^2\,d\Gamma + (\mu+\lambda)\int_{\Gamma_0} m\cdot\nu|\text{div } u|^2\,d\Gamma\\
&+2\sum_{j=1}^n\int_{\Gamma_1}(m\cdot\nabla u_j)\left(\mu\frac{\partial u_j}{\partial\nu} + (\mu+\lambda)(\text{div } u)\nu_j\right)d\Gamma - \mu\sum_{j=1}^n\int_{\Gamma_1} m\cdot\nu|\nabla u_j|^2\,d\Gamma\\
&-(\mu+\lambda)\int_{\Gamma_1} m\cdot\nu|\text{div } u|^2\,d\Gamma - \mu\sum_{j=1}^n\int_\Omega |\nabla u_j|^2\,dx - (\mu+\lambda)\int_\Omega |\text{div } u|^2\,dx\\
&+(n-1)\sum_{j=1}^n\int_{\Gamma_1} u_j\left(\mu\frac{\partial u_j}{\partial\nu} + (\mu+\lambda)(\text{div } u)\nu_j\right)d\Gamma
\end{aligned}\tag{4.34}$$

Since Γ_1 is compact and m, ν are sufficiently regular,
there exists $\delta > 0$ such that $m(x)\cdot\nu(x) \geq \delta > 0$, for all $x \in \Gamma_1$. From (4.34) we deduce

$$\begin{aligned}
&\sum_{j=1}^{n} \frac{d}{dt}\left\{\int_\Omega [2m\cdot\nabla u_j + (n-1)u_j]\, u_{jt}\, dx\right\} \\
&\qquad \leq -\sum_{j=1}^{n}\int_\Omega |u_{jt}|^2\, dx - \mu\sum_{j=1}^{n}\int_\Omega |\nabla u_j|^2\, dx - (\mu+\lambda)\int_\Omega |\text{div } u|^2\, dx \\
&\qquad -\zeta\sum_{j=1}^{n}\int_\Omega [2m\cdot\nabla u_j + (n-1)u_j]\left[\int_{-\infty}^{+\infty}\mu(\xi)\phi_j(x,\xi)\, d\xi\right] dx \\
&\qquad +\|m\|_\infty\sum_{j=1}^{n}\int_{\Gamma_1}|u_{jt}|^2\, d\Gamma - \mu\delta\sum_{j=1}^{n}\int_{\Gamma_1}|\nabla u_j|^2\, d\Gamma - (\mu+\lambda)\delta\int_{\Gamma_1}|\text{div } u|^2\, d\Gamma \\
&\qquad +2\sum_{j=1}^{n}\int_{\Gamma_1}(m\cdot\nabla u_j)\left(\mu\frac{\partial u_j}{\partial\nu} + (\mu+\lambda)(\text{div } u)\nu_j\right) d\Gamma \\
&\qquad +(n-1)\sum_{j=1}^{n}\int_{\Gamma_1} u_j\left(\mu\frac{\partial u_j}{\partial\nu} + (\mu+\lambda)(\text{div } u)\nu_j\right) d\Gamma
\end{aligned} \tag{4.35}$$

where we have used also $m(x)\cdot\nu(x) < 0$ on Γ_0. We can estimate

$$\begin{aligned}
&2\int_{\Gamma_1}(m\cdot\nabla u_j)\left(\mu\frac{\partial u_j}{\partial\nu} + (\mu+\lambda)(\text{div } u)\nu_j\right) d\Gamma \\
&\qquad \leq \frac{\delta\mu}{2}\int_{\Gamma_1}|\nabla u_j|^2\, d\Gamma + 2\frac{\|m\|_\infty^2}{\delta\mu}\int_{\Gamma_1}\left|\mu\frac{\partial u_j}{\partial\nu} + (\mu+\lambda)(\text{div } u)\nu_j\right|^2 d\Gamma \\
&\qquad \leq \frac{\delta\mu}{2}\int_{\Gamma_1}|\nabla u_j|^2\, d\Gamma + 2\frac{\|m\|_\infty^2}{\delta\mu}a_2^2\int_{\Gamma_1}|u_{jt}|^2\, d\Gamma
\end{aligned} \tag{4.36}$$

Moreover,

$$\begin{aligned}
&(n-1)\int_{\Gamma_1}\bar{u_j}\left(\mu\frac{\partial u_j}{\partial\nu} + (\mu+\lambda)(\text{div } u)\nu_j\right) d\Gamma \\
&\qquad \leq \frac{\varepsilon}{2}\int_{\Gamma_1}|u_j|^2\, d\Gamma + \frac{(n-1)^2}{2\varepsilon}\int_{\Gamma_1}\left|\left(\mu\frac{\partial u_j}{\partial\nu} + (\mu+\lambda)(\text{div } u)\nu_j\right)\right|^2 d\Gamma \\
&\qquad \leq \frac{\varepsilon}{2}C(P)\int_\Omega|\nabla u_j|^2\, dx + \frac{(n-1)^2}{2\varepsilon}\int_{\Gamma_1}\left|\mu\frac{\partial u_j}{\partial\nu} + (\mu+\lambda)(\text{div } u)\nu_j\right|^2 d\Gamma, \\
&\qquad \leq \frac{\varepsilon}{2}C(P)\int_\Omega|\nabla u_j|^2\, dx + \frac{(n-1)^2}{2\varepsilon}a_2^2\int_{\Gamma_1}|u_{jt}|^2\, d\Gamma,
\end{aligned} \tag{4.37}$$

where we have used trace inequality and Poincaré's theorem.

Remark 4.4.1 *In the above inequality $C(P)$ is the smallest positive constant such that*

$$\int_{\Gamma_1}|\vartheta|^2\, d\Gamma \leq C(P)\int_\Omega|\nabla\vartheta|^2\, dx, \quad \forall\vartheta\in H^1_{\Gamma_0}(\Omega).$$

Then by using the Young inequality and the Sobolev-Poincaré inequality, we can easily get the following corollary.

Corollary 4.4.1 *For any regular solution of* $(P3)$

$$\sum_{j=1}^{n}\frac{d}{dt}\left\{\int_\Omega [2m\cdot\nabla u_j+(n-1)u_j]\,u_{jt}\,dx\right\}$$
$$\le -\sum_{j=1}^{n}\int_\Omega |u_{jt}|^2\,dx-\left(\mu-\frac{\varepsilon}{2}C(P)-\zeta\|m\|_\infty^2 I-\frac{\zeta}{2}I(n-1)^2C(\Omega)\right)\sum_{j=1}^{n}\int_\Omega |\nabla u_j|^2\,dx$$
$$-(\mu+\lambda)\int_\Omega |div\ u|^2\,dx+\left(\|m\|_\infty+2\frac{\|m\|_\infty^2}{\delta\mu}a_2^2+\frac{(n-1)^2}{2\varepsilon}a_2^2\right)\sum_{j=1}^{n}\int_{\Gamma_1}|u_{jt}|^2\,d\Gamma$$
$$+\frac{3}{2}\zeta\sum_{j=1}^{n}\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi_j(x,\xi,t)|^2\,d\xi\,dx$$
$$-\frac{\mu\delta}{2}\sum_{j=1}^{n}\int_{\Gamma_1}|\nabla u_j|^2\,d\Gamma-(\mu+\lambda)\delta\int_{\Gamma_1}|div\ u|^2\,d\Gamma$$

Now, let us introduce the functional

$$\mathcal{S}(t)=\sum_{j=1}^{n}\int_\Omega\int_0^1 e^{-\tau\rho}|z_j(x,\rho,t)|^2 d\,\rho\,dx$$

We can easily estimate

$$\begin{aligned}
\mathcal{S}'(t) &= 2\int_\Omega\int_0^1 e^{-\tau\rho}z_t(x,\rho,t)z(x,\rho,t)d\rho\,dx\\
&= -\frac{2}{\tau}\int_\Omega\int_0^1 e^{-\tau\rho}z_\rho(x,\rho,t)z(x,\rho,t)d\rho\,dx\\
&= -\frac{1}{\tau}\int_\Omega\int_0^1 e^{-\tau\rho}\frac{d}{d\rho}|z(x,\rho,t)|^2d\rho\,dx\\
&= -\frac{1}{\tau}\int_\Omega e^{-\tau}|z(x,1,t)|^2\,dx+\frac{1}{\tau}\int_\Omega|u_t|^2\,dx-\int_\Omega\int_0^1 e^{-\tau\rho}|z(x,\rho,t)|^2d\,\rho\,dx\\
&\le \frac{1}{\tau}\int_\Omega|u_t|^2\,dx-\frac{1}{\tau}e^{-\tau}\int_\Omega|z(x,1,t)|^2\,dx-e^{-\tau}\int_\Omega\int_0^1|z(x,\rho,t)|^2d\,\rho\,dx
\end{aligned}$$

Let us introduce the Lyapunov functional

$$\mathcal{E}(t)=E(t)+\gamma_1\sum_{j=1}^{n}\int_\Omega[2m\cdot\nabla u_j+(n-1)u_j]u_{jt}\,dx+\gamma_2\mathcal{S}(t).$$

where γ_1,γ_2 are suitable positive small constants that will be precised later on. Note that $\mathcal{E}(t)$ is equivalent to the energy $E(t)$ if γ_1 is small enough. In particular, there exists a positive constant C_1 and suitable positive constants α_1,α_2 such that

$$\alpha_1E(t)\le\mathcal{E}(t)\le\alpha_2E(t),\quad \forall 0<\gamma_1,\quad \gamma_1\le C_1. \tag{4.38}$$

Proposition 4.4.3 *For every $a_2 > 0$ there exist a_0, c_1, c_2 such that for any solution of problem (P) with $0 \leq a_1 < a_0$ we have*

$$\mathcal{E}(t) \leq c_1 e^{-c_2 t}, \quad t > 0. \tag{4.39}$$

The constants a_0, c_1, c_2 are independent of the initial data but they depend on a_2 and on the geometry of Ω.

Proof. Differentiating the Lyapunov functional $\mathcal{E}$ and using the propositions above we deduce

$$\begin{aligned}
\mathcal{E}'(t) \quad \leq \quad & \left(\frac{\zeta I + \nu\tau^{-1}}{2} - \gamma_1 + \frac{\gamma_2}{\tau}\right) \sum_{j=1}^{n} \int_\Omega |u_{jt}|^2 \, dx \\
& - \gamma_2 e^{-\tau} \sum_{j=1}^{n} \int_\Omega \int_0^1 |z_j(x,\rho,t)|^2 d\rho \, dx - \frac{\gamma_2}{\tau} e^{-\tau} \sum_{j=1}^{n} \int_\Omega |z_j(x,1,t)|^2 \, dx \\
& + \left(\frac{\zeta I - \nu\tau^{-1}}{2I} + \frac{3}{2}\zeta\gamma_1\right) \sum_{j=1}^{n} \int_\Omega \int_{-\infty}^{+\infty} (\xi^2 + \eta) |\phi_j(x,\xi,t)|^2 \, d\xi \, dx \\
& - \gamma_1 \left(\mu - \frac{\varepsilon}{2} C(P) - \zeta \|m\|_\infty^2 I - \frac{\zeta}{2} I (n-1)^2 C(\Omega)\right) \sum_{j=1}^{n} \int_\Omega |\nabla u_j|^2 \, dx \\
& + \left(\gamma_1 \|m\|_\infty + \gamma_1 a_2^2 \left(2\frac{\|m\|_\infty^2}{\delta\mu} + \frac{(n-1)^2}{2\varepsilon}\right) - a_2\right) \sum_{j=1}^{n} \int_{\Gamma_1} |u_{jt}|^2 \, d\Gamma
\end{aligned} \tag{4.40}$$

For a fixed $a_2 > 0$ we want to choose $\varepsilon, \gamma_1, \gamma_2 < C_1$ and a_1 sufficiently small in order to obtain

$$\mathcal{E}'(t) \leq -cE(t). \tag{4.41}$$

Applying the second inequality of (4.38) estimate (4.39) easily follows. To show that (4.40) implies (4.41) we simply need that

$$\begin{aligned}
& \frac{\zeta I + \nu\tau^{-1}}{2} - \gamma_1 + \frac{\gamma_2}{\tau} < 0, \\
& \frac{\zeta I - \nu\tau^{-1}}{2I} + \frac{3}{2}\zeta\gamma_1 < 0, \\
& \mu - \frac{\varepsilon}{2} C(P) - \zeta \|m\|_\infty^2 I - \frac{\zeta}{2} I (n-1)^2 C(\Omega) > 0, \\
& \gamma_1 \|m\|_\infty + \gamma_1 a_2^2 \left(2\frac{\|m\|_\infty^2}{\delta\mu} + \frac{(n-1)^2}{2\varepsilon}\right) - a_2 < 0.
\end{aligned}$$

For any $a_2 > 0$ this last condition is satisfied for γ_1 sufficiently small. It then remains to the first and third conditions. For the first one, we need to assume that $\gamma_1 > \gamma_2/\tau\ 2$, while for the third equation we need to fix ε small enough such that

$$\mu - \frac{\varepsilon}{2} C(P) > 0.$$

Then we now fix γ_1, γ_2 and ε and fulfilling the above requirements and look at the first equation to the third equation as conditions on a_1 and ν.

Conclusion

The summary provided below considers some problems for future research works that arise from this dissertation.

The second chapter of this thesis, was devoted to study of the stabilization of the Lamé system with fractional damping. The fractional velocity feedbacks considered has provided a weaker damping than the velocity feedbacks. Therefore, no exponential decay was expected. As for an interesting open problem, is to prove that the results obtained in this chapter hold for the Lamé beam system with two boundary control conditions of fractional derivative type.

The study done in the third and the forth chapters can be approached from a different angle under suitable conditions on the delay terms. Indeed, as to hyperbolic equations, like ordinary differential equations, can have solutions that do not exist globally, these lasts are said to "blow-up" in finite time. Therefore, the interpretation of the blow-up of the solutions may indicate a real phenomenon, or a failure of the physical model, this leads to physical problems often posing more difficulties regarding the case of the instability of the system.

Although, the last chapter has dealt with the Lyapunov's stability which is widely used to control various systems and became nowadays, an indispensable tool for the study of all systems. whether they are finite or infinite, linear or nonlinear, time-invariant or time varying, continuous or discrete. Consequently, reproducing the result obtained in the forth chapter, with a time-varying delay of fractional type would be very interesting.

Bibliography

[1] C. Abdallah, P. Dorato, J. Benitez-Read & R. Byrne, *Delayed Positive Feedback Can Stabilize Oscillatory System,* ACC, San Francisco, (1993), 3106-3107.

[2] Z. Achouri, N. Amroun & A. Benaissa, *The Euler-Bernoulli beam equation with boundary dissipation of fractional derivative type,* Mathematical Methods in the Applied Sciences **40**, (2017)-11, 3837-3854.

[3] M. Akil, A. Wehbe, *Stabilization of multidimensional wave equation with locally boundary fractional dissipation law under geometric conditions,* Math. Control Relat. Fields **9** (2019)-1, 97-116.

[4] F. Alabau-Boussouira & V. Komornik, *Boundary observability, controllability and stabilization of linear elastodynamic systems,* SIAM J. Control and Optim, **37** (1999), 521-542.

[5] K. Ammari, S. Nicaise, & C. Pignotti, *Feedback boundary stabilization of wave equations with interior delay,* Syst. Control Lett, (2010)-59, 623-628.

[6] W. Arendt & C. J. K. Batty, *Tauberian theorems and stability of one-parameter semigroups,* Trans. Am. Math. Soc, **306** (1988), pp. 837-852.

[7] R. L. Bagley & P. J. Torvik, *A theoretical basis for the application of fractional calculus to viscoelasticity,* J. Rheology. **27** (1983), 201-210.

[8] R. L. Bagley & P. J. Torvik, *On the appearance of the fractional derivative in the behavior of real material,* J. Appl. Mech. **51** (1983), 294-298.

[9] C. Bardos, G. Lebeau, & J. Rauch, *Sharp sufficient conditions for the observation, control and stabilisation of waves from the boundary,* SIAM J. Control Optim, **30** (1998)-5, 1024-1065, 92-2.

[10] A. Benaissa & S. Gaouar, *Asymptotic stability for the Lamé system with fractional boundary damping,* Comput. Math. Appl, **77** (2019)-5, 1331-1346.

[11] A. Benaissa & S. Rafaa, Well-posedness and energy decay of solutions to a wave equation with a general boundary control of diffusive type, Math. Nachr. **292** (2019)-8, 1644-1673.

[12] E. Blanc, G. Chiavassa & B. Lombard, *Biot-JKD model: Simulation of 1D transient poroelastic waves with fractional derivatives,* J. Comput. Phys, **237** (2013), 1-20.

[13] A. Borichev & Y. Tomilov, *Optimal polynomial decay of functions and operator semigroups,* Math. Ann. **347** (2010)-2, 455-478.

[14] H. Brézis, *Operateurs Maximaux Monotones et semi-groupes de contractions dans les espaces de Hilbert,* Notas de Matemàtica (50), Universidade Federal do Rio de Janeiro and University of Rochester, North-Holland, Amsterdam, (1973).

[15] J. U. Choi & R. C. Maccamy, *Fractional order Volterra equations with applications to elasticity,* J. Math. Anal. Appl, **139** (1989), 448-464.

[16] R. Datko, *Not all feedback stabilised hyperbolic systems are robust with respect to small time delays in their feedback*, SIAM J. Control Optim, **26** (1988), 697-713.

[17] R. Datko, J. Lagnese & P. Polis, *An example on the effect of time delays in boundary feedback stabilization of wave equations,* SIAM J. Control Optim, **24** (1985), 152-156.

[18] W. Desch, E. Fašangová, J. Milota & G. Propst, *Stabilization through viscoelastic boundary damping: a semigroup approach,* Semigroup Forum, **80** (2010), 405-415.

[19] W. Desch, E. Fašangová, J. Milota & G. Propst, *Spectrum of a viscoelastic boundary damping model,* J. Integral Equations Appl, **23** (2011)-4, 521-539.

[20] B. Feng, *Well-posedness and exponential decay for laminated Timoshenko beams with time delays and boundary feedbacks,* Math. Methods Appl. Sci. **41** (2018)-3, 1162-1174.

[21] L. Gearhart, *Spectral theory for contraction semigroups on Hilbert spaces,* Trans. Amer. Math. Soc. **236** (1978), 385-394.

[22] S. Gerbi, & B. Said-Houari, *Existence and exponential stability for a damped wave equation with dynamic boundary conditions and a delay term,* Appl. Math. Comput, **218** (2012)-24, 11900-11910.

[23] A. Guesmia, *On the decay estimates for elasticity systems with some localized dissipations,* Asymptotic Analysis, **22** (2000), 1-13.

[24] A. Guesmia, *Contributions à la contrôLabilité exacte et la stabilisation des systèmes d'évolution,* Ph.D. Thesis, Louis Pasteur University, France, (2000).

[25] H. Haddar, J.R Li & D. Matignon, *Efficient solution of a wave equation with fractional-order dissipative terms,* J. Comput. Appl. Math, **234** (2010)-6, 2003-2010.

[26] F. Huang, *Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces,* Ann. Differ. Equ, **1** (1985), 43-55.

[27] V. Komornik, *Boundary stabilization of linear elasticity systems,* Lecture Notes in Pure and Appl. Math, **174** (1996), 135-146.

[28] J. E. Lagnese, *Boundary stabilization of linear elastodynamic systems,* SIAM J. Control and Optim, **21** (1983), 968-984.

[29] J. Lagnese, *Decay of solutions of wave equations in a bounded region with boundary dissipation,* J. Differential Equation, **83** (1983), 50(2) :163-182.

[30] J. E. Lagnese, *Uniform asymptotic energy estimates for solution of the equation of dynamic plane elasticity with nonlinear dissipation at the boundary,* Nonlinear Anal. T. M. A, **16** (1991), 35-54.

[31] I. Lasiecka & R. Triggiani. *Uniform stabilization of the wave equation with dirichlet or neumann feedback control without geometrical conditions,* Appl. Math. Optim, **92** (1992), 25(2) :189-224.

[32] P. D. Lax, *Functional Analysis,* John Wiley & Sons, New York, (2002).

[33] N. N. Lebedev, *Special Functions and their Applications,* Dover Publications, New York, (1972).

[34] I. Lyubich Yu & V. Q. Phng, *Asymptotic stability of linear differential equations in Banach spaces,* Studia Mathematica, **88** (1988)-1, 37-42.

[35] F. Mainardi & E. Bonetti, *The applications of real order derivatives in linear viscoelasticity,* Rheol. Acta **26** (1988), 64-67.

[36] P. Martinez, *Stabilisation de Systèmes Distribués Semi Linéaires : Domaines Presques Etoilés et Inégalités Intégrales généralisées,,* Ph. D. Thesis, Louis Pasteur University, France, (1998).

[37] B. Mbodje & G. Montseny, *Boundary fractional derivative control of the wave equation,* IEEE Transactions on Automatic Control, **40** (1995), 368-382.

[38] B. Mbodje, *Wave energy decay under fractional derivative controls,* IMA Journal of Mathematical Control and Information, **23** (2006), 237-257.

[39] S. Nicaise and C. Pignotti, *Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks,* SIAM J. Control Optim, **45** (2006)-5, 1561-1585.

[40] I. Podlubny, *Fractional differential equations,* Mathematics in Science and Engineering, Academic Press, **198** (1999).

[41] G. Propst & J. Prüss, *On wave equations with boundary dissipation of memory type,* J. Integral Equations App, **8** (1996)-1, 99-123.

[42] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations,* volume 44 of Applied Math. Sciences, 2nd ed. Springer-Verlag, New York, (1983).

[43] J. Prüss, *On the spectrum of C_0-semigroups,* Transactions of the American Mathematical Society, **284** (1984)-2, 847-857.

[44] B. Said-Houari & A. Soufyane, *Stability result of the Timoshenko system with delay and boundary feedback,* IMA J Math Control Info, (2012), 29:383-398.

[45] S.G. Samko, A.A. Kilbas & O.I. Marichev, *Fractional integrals and derivatives,* Amsterdam, Gordon and Breach (1993) [Engl. Trans. from the Russian (1987)].

[46] I.H. Suh & Z. Bien, *Use of time delay action in the controller design,* IEEE Trans. Autom. Control, **25** (1980) 600-603.

[47] C. Wagschal, *Fonctions holomorphes - Equations différentielles : Exercices corrigés,* Herman, Paris, (2003).

[48] G.N. Watson, *A Treatise on the Theory of Bessel Functions,* Second edition, Cambridge University Press, London (1962).

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