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Equations et inclusions différentielles semi linéaires avec impulsions

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Dedication

This thesis is dedicated to

My parents, those who are waiting a long time to live this wonderful moment with their child, who grow up learning until he reach his Phd. They share with me every single moment between suffering and patient.

My brother, sisters who have supported me all the way since the beginning of my study.

To those who have been deprived from their right to study and to all those who believe in the richness of learning.

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Publications

1. H. Abdelli, J.R. Graef, H. Kadari, A. Ouahab and A. Oumansour, Existence of solutions to systems of second-order impulsive differential equations with integral boundary conditions on the half-line, *Dynamics of continuous, Discrete and Integral systems, Series A: Mathematical Analysis* **29**(2022), 91-109.

Contents

Introduction

Impulsive differential equations, that is, differential equations involving impulse effects,were considered for the first time in the 1960's by Mil'man and Myshkis [79, 80].

Impulsive equations consist of two parts:

- Differential equation that defines the continuous part of the solution;
- Impulsive part that defines the rapid change and the discontinuity of the solution.

The first part of impulsive equations, that is, described by differential equations,could consist of ordinary differential equations, integro-differential equations, functional differential equations, partial differential equations, fractional differential equations, etc

The second part of impulsive equations is called the impulsive condition. The points, at which the impulses occur, are called moments of impulses. The functions, that define the amount of the impulses are called impulsive functions. The time of action of the impulses, being small with respect to the whole duration of the studied process, can be negligibly small(instantaneous impulses), or the time could be an interval with a given length(non-instantaneous impulses). This leads to two basic types of impulsive equations:

- Instantaneous impulses: the duration of these changes is relatively short compared to the overall duration of the whole process. The model is given by impulsive differential equations (see, e.g monographs [58, 69, 97].
- Non-instantaneous impulses: an impulsive action, which starts at an arbitrary fixed point and remains active on a finite time interval. E.Hernandez and D.O'Regan [59] introduced this new class of abstract differential equations where the impulses are not instantaneous, and they investigated the existence of mild and classical solutions.

The development of the theory of impulsive differential equations gives an opportunity for some real-world processes and phenomena to be modeled more accurately.

There are many good monographs on the impulsive differential equations [97, 16, 17, 18, 19, 20]. Very recently impulsive semilinear differential equations and inclusions with nondensly defined were considered by Benchohra et al [26, 27], there are also many different studies in biology and medicine for which impulsive differential equations are good model (see for instant [11, 67, 68]), and for applications of the theory of impulsive differential equations see [40, 44, 72, 75, 96, 100, 106, 109, 111].

This thesis is devoted to the existence of solutions for different classes of initial values problems for semilinear differential equations with impulse effects.

This Thesis is arranged as follows:
In Chapter 1, we introduce notations, definitions, lemmas and fixed point theorems which are used throughout this thesis. In addition, this chapter contains background material on semigroup.
In chapter 2, we prove existence results for impulsive semilinear differential equations system.

$$\begin{cases} x'(t) = Ax(t) + f(t,x,y) \quad t \in [0,b], \ t \neq t_k, \\ y'(t) = Ay(t) + g(t,x,y) \quad t \in [0,b], \ t \neq t_k, \\ x(t_k^+) = x(t_k^-) + I_1(x(t_k), y(t_k)) \quad k = 1, \dots, m, \\ y(t_k^+) = y(t_k^-) + I_2(x(t_k), y(t_k)) \quad k = 1, \dots, m, \\ x(0) = a, \quad y(0) = b, \end{cases} \tag{0.0.1}$$

where $f, g : [0,b] \times E \times E \to E$ are given functions, A is an infinitesimal generator of a strongly continuous semigroup of bounded linear operators and $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$; $I_k : C([0,b], E) \times C([0,b], E) \to C([0,b], E)$, $k = 1, \dots, m$, are impulse functions and $x(t^+) = \lim_{s \to t^+} x(s)$, $x(t^-) = \lim_{s \to t^-} x(s)$.

We prove the uniqueness by the theorem of Perov.

Chapter 3 is concerned with the existence of solution for systems of second-order impulsive differential equations with integral boundary conditions :

$$\begin{cases} -u''(t) = f(t, u(t), v(t)), & t \in J, \ t \neq t_k; \\ -v''(t) = g(t, u(t), v(t)), & t \in J, \ t \neq t_k; \\ \Delta u(t_k) = J_{1,k}(u(t_k)), \quad -\Delta u'(t_k) = I_{1,k}(u(t_k)), & k = 1, 2, \cdots, \\ \Delta v(t_k) = J_{2,k}(v(t_k)), \quad -\Delta v'(t_k) = I_{2,k}(v'(t_k)), & k = 1, 2, \cdots, \\ u(0) = \displaystyle\int_0^\infty h_1(s) u(s) ds, \quad u'(\infty) = 0, & \\ v(0) = \displaystyle\int_0^\infty h_2(s) v(s) ds, \quad v'(\infty) = 0, & \end{cases} \tag{0.0.2}$$

where $J = [0, \infty)$, $f, g \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $0 < t_1 < t_2 < \cdots < t_k < \cdots$, $t_k \longrightarrow \infty$, $I_{i,k}, \ J_{i,k} \in C(\mathbb{R}, \mathbb{R}), \ for \ i = 1, 2$, $h_i \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\displaystyle\int_0^\infty h_i(s) ds \neq 1 \ for \ i = 1, 2$, $u'(\infty) = \lim_{t \longrightarrow \infty} u(t)$ and $v'(\infty) = \lim_{t \longrightarrow \infty} v(t)$, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$ and $\Delta v(t_k) = v(t_k^+) - v(t_k^-)$, where $u(t_k^+), v(t_k^+)$ and $u(t_k^-), v(t_k^-)$ represent the right-hand limit of $u(t), v(t)$ at $t = t_k$, respectively. $\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-)$ and $\Delta v'(t_k) = v'(t_k^+) - v'(t_k^-)$, where $u'(t_k^+), v'(t_k^+)$ and $u'(t_k^-), v'(t_k^-)$ represent the right-hand limit of $u'(t)$ $(v'(t))$ at $t = t_k$, respectively.

In Chapter 4, we are interested to study the following system :

$$\begin{cases} x'(t)+A_1x(t)=f(t,x(t),y(t)), & t\in[s^i_{(x,y)},t^{i+1}_{(x,y)}],\ H(x(s^i_{(x,y)}),y(s^i_{(x,y)}))=0,\\ x(t)=I(t^{i+1}_{(x,y)},t,x(t^{i+1}_{(x,y)}),y(t^{i+1}_{(x,y)}),x(t),y(t)), & t\in(t^{i+1}_{(x,y)},s^{i+1}_{(x,y)}],\ H(x(t^{i+1}_{(x,y)},y(t^{i+1}_{(x,y)})=1\\ y'(t)+A_2y(t)=g(t,x(t),y(t)), & t\in[s^i_{(x,y)},t^{i+1}_{(x,y)}],\ H(x(s^i_{(x,y)}),y(s^i_{(x,y)}))=0,\\ y(t)=\bar{I}(t^{i+1}_{(x,y)},t,x(t^{i+1}_{(x,y)}),y(t^{i+1}_{(x,y)}),x(t),y(t)), & t\in(t^{i+1}_{(x,y)},s^{i+1}_{(x,y)}],\ H(x(t^{i+1}_{(x,y)},y(t^{i+1}_{(x,y)})=1\\ x(0)=x_0,\quad y(0)=y_0,\quad H(x_0,y_0)=0, \end{cases} \tag{0.0.3}$$

$A_i : D(A_i)\subset E \longrightarrow E$ are infinitesimal generator of a strongly continuous semigroup of bounded linear operators $(T_i(t))_{t\geq 0}$ on a Banach space E, $H\in C(E\times E,[0,\infty)),(x_0,y_0)\in H^{-1}(0)$ and $f,g:[0,a]\times E\times E\longrightarrow E$ are given functions, $I\in C([0,\infty)\times[0,\infty)\times E\times E\times E\times E;E), \bar{I}\in C([0,\infty)\times[0,\infty)\times E\times E\times E\times E;E)$, $0=s^0_{(x,y)}<t^1_{(x,y)}<s^1_{(x,y)}<t^2_{(x,y)}\cdots<s^i_{(x,y)}<t^{i+1}_{(x,y)}\cdots\leq a$ are numbers depending on the state (x,y).

In Chapter 5, we establish the existence of solutions for the following boundary value problem.

$$\begin{cases} \Delta^3x(k)-f(k,x,\Delta x,y,\Delta y)=0, & k\in\mathbb{N}(0,b)\\ \Delta^3y(k)-z(k,x,\Delta x,y,\Delta y)=0, & k\in\mathbb{N}(0,b)\\ \alpha_0\Delta x(0)-\beta_0\Delta^2x(0)=0 & x(0)=0\\ \gamma_0\Delta x(b+1)+\delta_0\Delta^2x(b+1)=0, & \\ \bar{\alpha_0}\Delta y(0)-\bar{\beta_0}\Delta^2y(0)=0 & y(0)=0\\ \bar{\gamma_0}\Delta y(b+1)+\bar{\beta_0}\Delta^2y(b+1)=0, & \end{cases} \tag{0.0.4}$$

where $\beta_0,\delta_0,\bar{\beta_0},\bar{\delta_0}\in\mathbb{R}\setminus\{0\},\quad \alpha_0,\gamma_0,\bar{\alpha_0},\bar{\gamma}\in\mathbb{R},\quad f,z:\mathbb{N}(a,b)\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}^m$ are continuous functions.

Chapter 1

Preliminaries

In this chapter, we introduce notations, definitions, lemmas and fixed point theorems which are used throughout this thesis.

1.1 Generalized metric space

Definition 1.1.1. *[84, 85] Let X be a nonempty set. By a vector-valued metric on X we mean a map $d : X \times X \to \mathbb{R}^n$ with the following properties:*

(i) $d(u, v) \geq 0$ *for all* $u,\ v \in X$*.; if* $d(u, v) = 0$ *then u=v;*

(ii) $d(u, v) = d(v, u)$ *for all* $u,\ v \in X$*;*

(iii) $d(u, v) \leq d(u, w) + d(w, v)$ *for all* $u, v, w \in X$*.*

We call the pair (X, d) a generalized metric space for $(r_1, \ldots, r_n) \in \mathbb{R}^n_+$, we will denote by

$$B(x_0, r) = \{x \in X :\ d(x_0, x) < r\},$$

the open ball centered in x_0 with radius r. We mention that for generalized metric spaces, the notation of open subset, closed set, convergence, Cauchy sequence and completeness are similar to those in usual metric spaces. If, $x, y \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, by $x \leq y$ we mean $x_i \leq y_i$ for all $i = 1, \ldots, n$. Also $|x| = (|x_1|, \ldots, |x_n|)$ and $\max(x, y) = \max(\max(x_1, y_1), \ldots, \max(x_n, y_n))$. If $c \in \mathbb{R}$, then $x \leq c$ means $x_i \leq c$ for each $i = 1, \ldots, n$.

Definition 1.1.2. *Let (X, d) be a generalized metric space. A subset $A \subset X$ is called open if for any $x_0 \in A$, there exist $r \in \mathbb{R}^n_+$ with $r > 0$ such that $B(x_0, r) \subset A$, where $B(x_0, r) = \{x \in X : d(x_0, x) < r\}$ denote the open ball centered in x_0 with radius r, Any open ball is an open set and the collection of all open balls generates the generalized metric topology on X. Let*

$$\overline{B(x_0, r)} = \{x \in X : d(x_0, x) \leq r\}$$

the closed ball centered in x_0 with radius r.

Definition 1.1.3. *Let (X, d) be a generalized metric spaces, a sequence b_n in X is called the Cauchy sequence if for each $\epsilon > 0$ there exist $N \in \mathbb{N}$ such that for any $n, m \geq N : d(x_n, x_m) < \epsilon$*

Definition 1.1.4. *A generalized metric space (X, d) is called complete if each Cauchy sequence in X converges to a limit in X.*

Definition 1.1.5. *Let (X, d) be a generalized metric space, we say that a subset $Y \subset X$ is closed if $x_n \subset Y$ and $x_n \to x$ imply $x \in Y$.*

Definition 1.1.6. *Let (X, d) be a generalized metric space, A subset C of X is called compact if every open cover of C has a finite subcover.*
A subset C of X is sequentially compact if every sequence in C contains a convergent subsequence with limit in C.

Definition 1.1.7. *Let X, Y be two generalized metric spaces $K \subset X$ and $f : K \to Y$ be an open operator, then f is called :*

(P_1) Compact,if for any bounded subset $A \subset K$ then $f(A)$ is relatively compact or $\overline{f(A)}$ is compact.

(P_2) Completely continuous, if f is continuous and compact.

Definition 1.1.8. *A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc i.e. $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $det(M - \lambda I) = 0$, where I denote the unit matrix of $\mathcal{M}_{n\times n}(\mathbb{R})$.*

Theorem 1.1.1. *[94] Let $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$, the following assertions are equivalent:*

(a) M is convergent towards zero;

(b) $M^k \to 0$ as $k \to \infty$;

(c) The matrix $(I - M)$ is nonsingular and

$$(I - M)^{-1} = I + M + M^2 + \cdots + M^k + \cdots ,$$

(d) The matrix $(I - M)$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements.

Definition 1.1.9. *We say that a non-singular matrix $A = (a_{ij})_{1\leq i,j\leq n} \in M_{n\times n}(\mathbb{R})$ has the absolute value property if*

$$A^{-1}|A| \leq I,$$

where $|A| = (|a_{ij}|)_{1\leq i,j\leq n} \in M_{n\times n}(\mathbb{R}_+)$.

Some examples of matrices convergent to zero, $A \in M_{n\times n}(\mathbb{R})$, which also satisfies the property $(I-A)^{-1}|I-A| \leq I$ are:

1. $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ *and* $\max(a,b) < 1$

2. $A = \begin{pmatrix} a & -c \\ 0 & b \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ *and* $a+b<1,\ c<1$

3. $A = \begin{pmatrix} a & -a \\ b & -b \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ *and* $|a-b|<1, a>1,\ b>0$.

Lemma 1.1.2. *[104] Let*

$$Q = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

where $a, b, c, d \geq 0$ *and* $\det Q > 0$. *Then* Q^{-1} *is order preserving.*

Definition 1.1.10. *[84, 85] Let* (X, d) *be a generalized metric space. An operator* $N : X \to X$ *is called contractive associated with the above* d *on* X*, if there exists a convergent to zero matrix* M *such that* $d(T(x), T(y)) \leq Md(x, y)$ *for all* $x, y \in X$.

Let (X, d) be a generalized metric space with

$$d(x,y) := \begin{pmatrix} d_1(x,y) \\ \vdots \\ d_n(x,y) \end{pmatrix}.$$

Notice that d is a generalized metric on X if and only if $d_i,\ i = 1, 2, \ldots, n$ are metrics on X.

Definition 1.1.11. *Let* X *be a vector space metric on* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$*, we mean by a norm a map* $\|\cdot\| \longrightarrow \mathbb{R}^n$ *with the following properties:*

(i) $\|x\| \geq 0$ *for all* $x \in X$*; if* $\|x\| = 0$ *then* $x = (0, \cdots, 0)$*;*

(ii) $\|\lambda x\| = |\lambda|\|x\|$ *for all* $x \in X$ *and* $\lambda \in \mathbb{K}$*;*

(iii) $\|x+y\| \leq \|x\| + \|y\|$ *for all* $x,\ y \in X$.

The pair $(X, \|\cdot\|)$ is called a generalized normed space. If the generalized metric generated by $\|\cdot\|$ (i.e., $d(x, y) = \|x-y\|$) is complete then the space $(E, \|\cdot\|)$ is called a generalized Banach space.

Definition 1.1.12. *We say* $f : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is an* L^1*-Carathéodory function if*

1. $f(\cdot, x, y)$ *is measurable for any* $(x, y) \in \mathbb{R} \times \mathbb{R}$,

2. $f(t,\cdot,\cdot)$ *is a continuous for almost every* $t \in \mathbb{R}^+$.

3. *For each* $r_1,\ r_2 > 0$, *there exists* $\phi_{r_1,r_2} \in L^1([0,\infty))$ *such that*
$$|f(t,x,y)| \leq \Phi_{r_1,r_2}(t),$$
with $|x| \leq r_1$,$|y| \leq r_2$ *and almost every* $t \in [0,\infty)$.

1.2 Theory of fixed point

Theorem 1.2.1. *[41](**Schaefer's fixed point**)*
Let X *be a banach space and let* $N : E \to E$ *be a completely continuous map.If the set*
$$\Phi = \{x \in E : \lambda x = Nx \ \textit{for some} \ \lambda > 1\}$$
is bounded, the N *has a fixed point.*

Theorem 1.2.2. *[90](**Schauder's fixed point**) Let* M *be a closed bounded convex subset of a Banach space* E. *Assume that* $\Lambda : M \to M$ *is compact. Then* Λ *has at least one fixed point in* M.

Theorem 1.2.3. *[90, 93](**Perov's fixed point**) Let* (X,d) *be a complete generalized metric space and the mapping* $f : X \to X$ *with the property that there exists a matrix* $A \in M_{m,m}(\mathbb{R}_+)$ *such that* $d(f(x),f(y)) \leq Ad(x,y)$ *for all* $x,y \in X$*.If* A *is a matrix convergent to zero, then there exists a unique* $x^* \in X$ *such that* $x^* = f(x^*)$*,i.e; the mapping* f *has a unique fixed point.*

Theorem 1.2.4. *[51](**Krasnoselskii's fixed point**) Let* E *be a generalized Banach space. Suppose that* T *and* B *are two operators* $E \to E$ *such that*

($\mathcal{A}_1$) T *be a completely continuous operator.*

($\mathcal{A}_2$) B *be a continuous and* $M-$ *contraction operator.*

($\mathcal{A}_3$) *The matrix* $I - M$ *has the absolute property if*
$$M = \{x \in E \mid \lambda T(x) + \lambda B(\frac{x}{\lambda}) = x\}$$
is bounded for all $0 < \lambda < 1$. *then the equation*
$$x = T(x) + B(x), \ \ x \in X$$
has at least one solution.

Theorem 1.2.5. *[48] Let* X *be a generalized Banach space and* $N : X \to X$ *be a continuous compact mapping. Moreover, assume that the set*
$$\mathcal{K} = \{x \in X : x = \lambda N(x) \ \ \textit{for some} \ \lambda \in (0,1)\},$$
is bounded. Then N *has a fixed point.*

1.3 Semigroup of Linear Operators

Let X be a Banach space and $B(X)$ be the Banach space of bounded linear operators.

Definition 1.3.1. *A one-parameter family $S(t)$ for a bounded linear operators on a Banach space X is a C_0-semigroup (or strongly continuous) on X if*

(i) $S(t) \circ S(s) = S(t+s)$, *for* $t, s \geq 0$, *(semigroup property),*

(ii) $S(0) = I$, *(the identity on X);*

(iii) the map $t \to S(t)x$ is strongly continuous, for each $x \in X$, i.e;

$$\lim_{t \to 0} S(t)(x) = x, \; \forall x \in X.$$

Remark 1.3.1. *A semigroup of bounded linear operators $(S(t))_{t \geq 0}$ is uniformly continuous if*

$$\lim_{t \to 0} \|S(t) - I\| = 0.$$

We note that if a semigroup $S(t)$ is class (C_0), then we have the growth condition $\|S(t)\|_{B(X)} \leq M. \exp(\beta t)$, for $0 \leq t < \infty$, with some constants $M > 0$ and β.

If, in particular, $M = 1$ and $\beta = 0$, that is, $\|S(t)\|_{B(X)} \leq 1$, for $t \leq 0$, then the semigroup $S(t)$ is called a contraction semigroup (C_0).

Definition 1.3.2. *Let $S(t)$ be a semigroup of class (C_0) defined on X. The infinitesimal generator A of $S(t)$ is the linear operator defined by*

$$A(x) = \lim_{h \to 0} \frac{S(h)(x) - x}{h}, \text{ for } x \in D(A),$$

where $D(A) = \{x \in X \mid \lim_{h \to 0} \dfrac{S(h)(x) - x}{h} \text{ exists in } X\}$.

Let us recall the following property:

Theorem 1.3.1. *[89] If $S(t)$ is a C_0-semigroup, then there exist $\omega \geq 0$ and $M \geq 1$ such that*

$$\|S(t)\|_{B(X)} \leq M \exp(\omega t), \text{ for } 0 \leq t < \infty \tag{1.3.1}$$

Theorem 1.3.2. *If $(S(t))_{t \geq 0}$ is a C_0 semigroup then $t \to S(t)x$ is continuous, for each $x \in X$.*

Theorem 1.3.3. *Let $S(t)_{t \geq 0}$ be a C_0 semigroup and A be its infinitesimal generator. Then* (*a*) *For* $x \in X$,

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} S(s)x ds = S(t)x. \tag{1.3.2}$$

(*b*) *For* $x \in X$, $\int_0^t S(s)x ds \in D(A)$ *and*

$$A\left(\int_0^t S(s)x ds\right) = S(t)x - x. \tag{1.3.3}$$

(*c*) *For* $x \in D(A)$, $S(t) \in D(A)$ *and*

$$\frac{d}{dt}S(t)(x) = A(S(t)(x)) = S(t)(A(x)). \tag{1.3.4}$$

(*d*) *For* $x \in D(A)$

$$S(t)x - S(s)x = \int_s^t S(\tau)Ax d\tau = \int_s^t AS(\tau)x d\tau. \tag{1.3.5}$$

$$\lim_{t\to 0} S(t)(x) = x, \, \forall x \in X.$$

Corollary 1.3.4. *If* A *is the infinitesimal generator of a* C_0 *semigroup* $(S(t)_{t\geq 0})$ *then* $D(A)$ *the domain of* A, *is dense in* X *and* A *is closed linear operator.*

Theorem 1.3.5. *A linear operator* A *is the infinitesimal generator of a uniformly continuous semigroup if and only if* A *is a bounded linear operator.*

Theorem 1.3.6. *Let* $(T(t))_{t\geq 0}$ *and* $(S(t))_{t\geq 0}$ *be two* C_0 *semigroups on* X, *generated respectively by* A *and* B. *If* $A = B$ *then* $T(t) = S(t)$, $t \geq 0$.

For more details see [10, 13, 46, 47, 52, 56, 64]

Chapter 2

Impulsive Differential Equations

In this chapter we study the existence and uniqueness results to the coupled system of impulsive differential equations with initial conditions, the problem is in the form:

$$\begin{cases} x'(t) = Ax(t) + f_1(t, x(t), y(t)), \quad t \in [0, b] \;\; t \neq t_k \\ y'(t) = Ay(t) + g(t, x(t), y(t)), \quad t \in [0, b] \;\; t \neq t_k \\ x(t_k^+) - x(t_k^-) = I_{k1}(x(t_k), y(t_k)), \quad k = 1, \dots, m \\ y(t_k^+) - y(t_k^-) = I_{k2}(x(t_k), y(t_k)), \quad k = 1, \dots, m \\ x(0) = \bar{x}_0, \quad y(0) = \bar{y}_0, \end{cases} \tag{2.0.1}$$

where $f, g : [0, b] \times E \times E \to E$ are continuous functions, A is linear operator in Banach space $C([0, b], E)$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$; $\;I_{k1}, I_{k2} : C(E \times E, E)$, $k = 1, \dots, m$, are impulse functions and $x(t^-) = \lim_{s \to t^-} x(s)$, $x(t^+) = \lim_{s \to t^+} x(s)$ and $\bar{x}_0, \bar{y}_0 \in E$.

2.1 Existence and Uniqueness results

Let $J := [0, b]$. In order to define a solution for problem (2.0.1), consider the space $PC(J, E)$, where

$$PC(J, E) := \{y : J \to E, \quad y \in C(J \setminus \{t_k\}, E); k = 1, \dots, m, \\ y(t_k^-) \text{ and } y(t_k^+) \text{ exist and satisfy } y(t_k^-) = y(t_k^+)\}.$$

Endow end with the norm

$$\|y\|_{PC} = \sup\{\|y(t)\| : t \in J\}$$

$PC(J, E)$ is a Banach space.

Definition 2.1.1. *A function $(x,y) \in PC \times PC$ is a mild solution of (*2.0.1*) if*

$$\begin{cases} x(t) = T(t)\bar{x}_0 + \int_0^t T(t-s)f(s,x,y)ds + \sum\limits_{0<t_k<t} T(t-t_k)I_{k1}(x(t_k),y(t_k)), & t \in J \\ y(t) = T(t)\bar{y}_0 + \int_0^t T(t-s)g(s,x,y)ds + \sum\limits_{0<t_k<t} T(t-t_k)I_{k2}(x(t_k),y(t_k)), & t \in J. \end{cases} \tag{2.1.1}$$

Let us introduce the following hypotheses:

(P_1) There exist nonegative numbers a_i,b_i for each $i \in \{1,2\}$,such that

$$\begin{cases} |f(t,x,y) - f(t,\bar{x},\bar{y})| & \leq & a_1|x-\bar{x}| + b_1|y-\bar{y}| \\ |g(t,x,y) - g(t,\bar{x},\bar{y})| & \leq & a_2|x-\bar{x}| + b_2|y-\bar{y}| \end{cases}$$

for all $x, y, \bar{x}, \bar{y} \in E$.

Theorem 2.1.1. *Assume that (P1) is satisfied. Then the problem (*2.0.1*) has a unique solution.*

Proof. We give the prove of this theorem in several steps.

Step 1 Consider the problem (2.0.1) on $[0,t_1]$,

$$\begin{cases} x'(t) = Ax(t) + f(t,x,y), \quad t \in [0,t_1] \\ y'(t) = Ay(t) + g(t,x,y), \\ x(0) = \bar{x}_0 \quad, y(0) = \bar{y}_0. \end{cases} \tag{2.1.2}$$

We prove the uniqueness of the mild solution by the theorem of Perov,we consider the operator $\Lambda_1 : PC([0,t_1],E) \times PC([0,t_1],E) \to PC([0,t_1],E) \times PC([0,t_1],E)$ defined for $(x,y) \in PC([0,t_1],E) \times PC([0,t_1],E)$ by

$$\Lambda_1(x,y) = (N_1(x,y), N_2(x,y)), \tag{2.1.3}$$

where

$$N_1(x,y) = T(t)\bar{x}_0 + \int_0^t T(t-s)f(s,x,y)ds,$$

$$N_2(x,y) = T(t)\bar{y}_0 + \int_0^t T(t-s)g(s,x,y)ds,$$

we shall use theorem of Perov to prove that Λ_1 is a contraction. Indeed, let $(x,y),(\bar{x},\bar{y}) \in PC([0,t_1],E) \times PC([0,t_1],E)$.

$$\begin{aligned} |N_1(x(t),y(t)) - N_1(\bar{x}(t),\bar{y}(t))| &\leq \int_0^t |T(t-s)||f(s,x(s),y(s)) - f(s,\bar{x}(s),\bar{y}(s))|ds \\ &\leq M\int_0^t a_1|x(s)-\bar{x}(s)| + b_1|y(s)-\bar{y}(s)|ds \\ &\leq \tfrac{Ma_1}{\tau}\int_0^t \tau e^{s\tau}ds\|x-\bar{x}\| + \tfrac{Mb_1}{\tau}\int_0^t \tau e^{s\tau}ds\|y-\bar{y}\|. \end{aligned}$$

Thus, we have

$$\|N_1(x,y)-N_1(\bar{x},\bar{y})\|_1 \;\leq\; \tfrac{1}{\tau}\|x-\bar{x}\|_1+\tfrac{1}{\tau}\|y-\bar{y}\|_1.$$

where

$$\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|_1 = \sup_{t\in[0,t_1]} e^{-\tau t}\left\|\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}\right\|.$$

Similarly,

$$\|N_2(x,y)-N_2(\bar{x},\bar{y})\|_1 \;\leq\; \tfrac{1}{\tau}\|x-\bar{x}\|_1+\tfrac{1}{\tau}\|y-\bar{y}\|_1.$$

Hence,

$$\begin{aligned}\|\Lambda_1(x,y)-\Lambda_1(\bar{x},\bar{y})\|_1 &= \begin{pmatrix} \|N_1(x,y)-N_1(\bar{x},\bar{y})\|_1 \\ \|N_2(x,y)-N_2(\bar{x},\bar{y})\|_1 \end{pmatrix} \\ &\leq \tfrac{1}{\tau}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \|x-\bar{x}\|_1 \\ \|y-\bar{y}\|_1 \end{pmatrix}.\end{aligned}$$

Therefore,

$$\|\Lambda_1(x,y)-\Lambda_1(\bar{x},\bar{y})\|_1 \;\leq\; \bar{M}\begin{pmatrix} \|x-\bar{x}\|_1 \\ \|y-\bar{y}\|_1 \end{pmatrix}.$$

where,

$$\bar{M}=\frac{1}{\tau}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

For $\tau > 2$, $\bar{M}$ converge to 0, from Perov fixed point theorem 1.2.3, the mapping Λ_1 has a unique fixed $(x^0, y^0) \in PC([0,t_1],E)\times PC([0,t_1],E)$, which is the unique solution of the problem (2.1.2).

Step 2 Consider the system:

$$\begin{cases} x'(t) = Ax(t)+f(t,x,y), \quad t\in(t_1,t_2] \\ y'(t) = Ay(t)+g(t,x,y), \\ x(t_1^+) = x(t_1^-)+I_1(x(t_1),y(t_1)), \quad k=1 \\ y(t_1^+) = y(t_1^-)+\bar{I}_1(x(t_1),y(t_1)) \end{cases} \tag{2.1.4}$$

Consider the operator, $\Lambda_2 : PC((t_1,t_2],E)\times PC((t_1,t_2],E) \to PC((t_1,t_2],E)\times PC((t_1,t_2],E)$, defined for $(x,y)\in PC((t_1,t_2],E)\times PC((t_1,t_2],E)$ by

$$\Lambda_2(x,y) = (N_1(x,y), N_2(x,y)).$$

where,

$$N_1(x,y) = T(t-t_1)[x(t_1) + I_1(x(t_1), y(t_1))] + \int_0^t T(t-s)f(s,x(s),y(s))ds.$$

and

$$N_2(x,y) = T(t-t_1)[y(t_1) + \bar{I}_1(x(t_1), y(t_1))] + \int_0^t T(t-s)g(s,x(s),y(s))ds.$$

Let $(x,y),(\bar{x},\bar{y}) \in PC([t_1,t_2],E) \times PC([t_1,t_2],E)$,

$$\begin{aligned}
|N_1(x(t),y(t)) - N_1(\bar{x},\bar{y})| &\leq \int_{t_1}^t |T(t-s)||f(s,x(s),y(s)) - f(s,\bar{x}(s),\bar{y}(s))|ds \\
&\leq M\int_{t_1}^t a_1|x(s)-\bar{x}(s)| + b_1|y(s)-\bar{y}(s)|ds \\
&\leq \tfrac{Ma_1}{\tau}\int_{t_1}^t \tau e^{s\tau}ds\|x-\bar{x}\| + \tfrac{Mb_1}{\tau}\int_{t_1}^t \tau e^{s\tau}ds\|y-\bar{y}\| \\
&\leq \tfrac{Ma_1}{\tau}[e^{t\tau} - e^{t_1\tau}]\|x-\bar{x}\| + \tfrac{Mb_1}{\tau}[e^{t\tau} - e^{t_1\tau}]\|y-\bar{y}\| \\
&\leq [\tfrac{Ma_1}{\tau}\|x-\bar{x}\| + \tfrac{Mb_1}{\tau}\|y-\bar{y}\|]e^{t\tau} \\
&\quad -[\tfrac{Ma_1}{\tau}\|x-\bar{x}\| + \tfrac{Mb_1}{\tau}\|y-\bar{y}\|]e^{t_1\tau}.
\end{aligned}$$

Hence,

$$\begin{aligned}
|N_1(x(t),y(t)) - N_1(\bar{x}(t),\bar{y}(t))|e^{-t\tau} &\leq [\tfrac{Ma_1}{\tau}\|x-\bar{x}\| + \tfrac{Mb_1}{\tau}\|y-\bar{y}\|] \\
&\quad -[\tfrac{Ma_1}{\tau}\|x-\bar{x}\| + \tfrac{Mb_1}{\tau}\|y-\bar{y}\|]e^{t_1\tau}e^{-t\tau} \\
&\leq [1 - e^{t_1}e^{-t\tau}][\tfrac{Ma_1}{\tau}\|x-\bar{x}\| + \tfrac{Mb_1}{\tau}\|y-\bar{y}\|].
\end{aligned}$$

Then,

$$\|N_1(x,y) - N_1(\bar{x},\bar{y})\|_1 \leq \tfrac{1}{\tau}\|x-\bar{x}\|_1 + \tfrac{1}{\tau}\|y-\bar{y}\|_1.$$

where

$$\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|_1 = \sup_{t\in[t_1,t_2]} e^{-\tau t}\left\|\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}\right\|.$$

Similarly,

$$\begin{aligned}
\|N_2(x,y) - N_2(\bar{x},\bar{y})\|_1 &\leq [1 - e^{t_1}e^{-t\tau}][\tfrac{Ma_2}{\tau}]\|x-\bar{x}\| + [1 - e^{t_1}e^{-t\tau}][\tfrac{Mb_2}{\tau}]\|y-\bar{y}\| \\
&\leq \tfrac{1}{\tau}\|x-\bar{x}\|_1 + \tfrac{1}{\tau}\|y-\bar{y}\|_1.
\end{aligned}$$

Hence,

$$\begin{aligned}\|\Lambda_2(x,y)-\Lambda_2(\bar{x},\bar{y})\|_1 &= \begin{pmatrix} \|N_1(x,y)-N_1(\bar{x},\bar{y})\|_1 \\ \|N_2(x,y)-N_2(\bar{x},\bar{y})\|_1 \end{pmatrix} \\ &\leq \tfrac{1}{\tau}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \|x-\bar{x}\|_1 \\ \|y-\bar{y}\|_1 \end{pmatrix}.\end{aligned}$$

Therefore,

$$\|\Lambda_2(x,y)-\Lambda_2(\bar{x},\bar{y})\|_1 \leq \bar{M}\begin{pmatrix} \|x-\bar{x}\|_1 \\ \|y-\bar{y}\|_1 \end{pmatrix}.$$

where,

$$\bar{M}=\frac{1}{\tau}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

For $\tau > 2$, $\bar{M}$ converge to 0, then from Perov fixed point theorem 1.2.3, the mapping Λ_2 has a unique fixed $(x^1,y^1)\in PC((t_1,t_2],E)\times PC((t_1,t_2],E)$, which is the unique solution of the problem (2.1.4).

As a consequence, arguing inductively, the solution of problem (2.0.1) is given by

$$(x,y)(t)=\begin{cases} (x^0,y^0)(t), & t\in[0,t_1] \\ (x^1,y^1)(t), & t\in(t_1,t_2] \\ \vdots & \\ \vdots & \\ (x^m,y^m)(t), & t\in(t_m,b]. \end{cases}$$

□

For the next result we give the existence of mild solution of the problem without Lipsichiz conditions

Theorem 2.1.2. *Assume these conditions are satisfied,*

(A_1) *There exist continuous nondecreasing functions* $\psi_1:\mathbb{R}_+\to(0,\infty)$, $\psi_2:\mathbb{R}_+\to(0,\infty)$, *functions* $p_1\in L^1(J,\mathbb{R}_+)$, $p_2\in L^1(J,\mathbb{R}_+)$ *such that,*

$$\begin{cases} |f(t,x,y)| \leq p_1(t)\psi_1(|x|+|y|) \\ |g(t,x,y)| \leq p_2(t)\psi_2(|x|+|y|) \end{cases}$$

for almost all $t\in J$ *and all* $(x,y)\in E\times E$.

(A_2) *The linear operator $A : \mathcal{D}(A) \subset E \to E$ generates a compact strongly continuous semigroup $\{T(t) : t \geq 0\}$; i.e $T(t)$ is compact for any $t > 0$. Moreover, we denote $M = \sup_{0\leq t\leq b} \|T(t)\|$.*

Then the problem (2.0.1) *has at least one solution.*

Proof. Consider the operator $\Lambda : PC \times PC \to PC \times PC$ defined by

$$\Lambda(x, y) = (N_1(x, y), N_2(x, y)), \quad (x, y) \in PC \times PC,$$

where,

$$N_1(x(t), y(t)) = T(t)\bar{x}_0 + \int_0^t T(t-s)f(s, x, y)ds + \sum_{0<t_k<t} T(t-t_k)I_{k1}(x(t_k), y(t_k)), \quad t \in J$$

and

$$N_2(x(t), y(t)) = T(t)\bar{y}_0 + \int_0^t T(t-s)g(s, x, y)ds + \sum_{0<t_k<t} T(t-t_k)I_{k2}(x(t_k), y(t_k)), \quad t \in J.$$

We transform problem (2.0.1) into fixed point problem. It is clear that the fixed point of Λ are mild solution to (2.0.1).

The proof will be given in several steps.

Step 1 Λ is continuous.

Let (x_n, y_n) be a sequence in $PC(J, \mathbb{R}) \times PC(J, \mathbb{R})$ such that $(x_n, y_n) \to (x, y)$ it is enough to prove that $\Lambda(x_n, y_n) \to \Lambda(x, y)$. For all $t \in J$, we have

$$\begin{aligned} |N_1(x_n(t), y_n(t)) - N_1(x(t), y(t))| &\leq \int_0^b |T(t-s)||f(s, x_n(s), y_n(s)) - f(s, x(s), y(s))|ds \\ &+ M\sum_{k=1}^{n} |I_{k1}(x_n(t_k), y_n(t_k)) - I_{k1}(x(t_k), y(t_k))|. \end{aligned}$$

Since the function f, g are continuous, then, we have

$$\begin{aligned} \|N_1(x_n, y_n) - N_1(x, y)\| &\leq M \int_0^b \|f(\cdot, x_n(\cdot), y_n(\cdot)) - f(\cdot, x(\cdot), y(\cdot))\|ds \\ &+ M\sum_{k=1}^{n} |I_{k1}(x_n(t_k), y_n(t_k)) - I_{k1}(x(t_k), y(t_k))| \to 0 \,\text{as}\, n \to \infty. \end{aligned}$$

Similarly,

$$\begin{aligned} \|N_2(x_n, y_n) - N_2(x, y)\| &\leq M \int_0^b \|g(\cdot, x_n(\cdot), y_n(\cdot)) - g(\cdot, x(\cdot), y(\cdot))\|ds \\ &+ M\sum_{k=1}^{n} |I_{k2}(x_n(t_k), y_n(t_k)) - I_{k2}(x(t_k), y(t_k))| \to 0 \,\text{as}\, n \to \infty. \end{aligned}$$

Then Λ is continuous.

Step 2 Λ is bounded on bounded sets in $PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$.

Indeed, it is enough to show that for any $r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} > 0$, then there exists a positive constant $\ell = \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix}$ such that

$$(x,y) \in B_r = \{(x,y) \in PC \times PC : \|(x,y)\| \leq r\}, \text{ we have } \|\Lambda(x,y)\| \leq \ell.$$

Let $(x,y) \in B_r$. We have

$$\begin{aligned} \|\Lambda(x,y)\| &\leq \\ & (M\|\bar{x_0}\| + \textstyle\int_0^b \|f(s,x(s),y(s))\|ds + M\displaystyle\sum_{k=1}^{n} \|I_{k1}(x(t_k),y(t_k))\|, \\ & M\|\bar{y_0}\| + \textstyle\int_0^b \|g(s,x(s),y(s))\|ds + M\displaystyle\sum_{k=1}^{n} \|I_{k2}(x(t_k),y(t_k))\|), \end{aligned}$$

where,

$$\begin{aligned} \|N_1(x,y)\| &\leq M\|\bar{x_0}\| + \textstyle\int_0^b p(t)\psi(\|x\|_{PC} + \|y\|_{PC})dt + M\displaystyle\sum_{k=1}^{n} \|I_{k1}(x(t_k),y(t_k))\| \\ &\leq M\|\bar{x_0}\| + \textstyle\int_0^b p(t)\psi(\|x\|_{PC} + \|y\|_{PC})dt + M\displaystyle\sum_{k=1}^{n} \sup_{(x,y)\in \bar{B_r}} \|I_{k1}(x,y)\|, \\ &\leq M + \textstyle\int_0^b p(t)\psi(r_1 + r_2)dt \\ &\quad + M\displaystyle\sum_{k=1}^{n} \sup_{(x,y)\in \bar{B_r}} \|I_{k1}(x,y)\| := \ell_1. \end{aligned}$$

Similarly,

$$\begin{aligned} \|N_2(x,y)\| &\leq M\|\bar{y_0}\| + \textstyle\int_0^b p(t)\psi(\|x\|_{PC} + \|y\|_{PC})dt + M\displaystyle\sum_{k=1}^{n} \sup_{(x,y)\in \overline{B_r}} \|I_{k2}(x,y)\|, \\ &\leq M + \textstyle\int_0^b p(t)\psi(r_1 + r_2)dt \\ &\quad + M\displaystyle\sum_{k=1}^{n} \sup_{(x,y)\in \overline{B_r}} \|I_{k2}(x,y)\| := \ell_2. \end{aligned}$$

Step 3 Λ transforms every bounded set into an equicontinuous set in $PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$.

Let $\tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$, and let B_r be as in Step 2. Let $(x,y) \in B_r$. Then

1. If $\tau_1 \neq t_k$ or $(\tau_2 \neq t_k)$, $\forall k \in \{1,2,\dots,m\}$, we have

$$\begin{aligned}\|\Lambda(x,y)(\tau_2)-\Lambda(x,y)(\tau_1)\| &\leq \\ &(M\int_{\tau_1}^{\tau_2} p(s)\psi(r_1+r_2)ds \\ &+M\sum_{\tau_1<t_k<\tau_2}\sup_{(x,y)\in\bar{B}_r}\|I_{k1}(x,y)\| \\ &-M\int_{\tau_1}^{\tau_2} p(s)\psi(r_1+r_2)ds \\ &+M\sum_{\tau_1<t_k<\tau_2}\sup_{(x,y)\in\bar{B}_r}\|I_{k2}(x,y)\|) \to \begin{pmatrix}0\\0\end{pmatrix} \text{ as } \tau_1\to\tau_2.\end{aligned}$$

2. If $\tau_1 = t_i^-$, we consider $\delta_1 > 0$ such that $\{t_k, k\neq i\}\cap[t_i-\delta_1, t_i+\delta_1]=\emptyset$, so, for $0<h<\delta_1$, we have

$$\begin{aligned}&\|\Lambda(x,y)(t_i)-\Lambda(x,y)(t_i-h)\| \\ &\leq (M\int_{t_i-h}^{t_i} p(s)\psi(q_1+q_2)ds, M\int_{t_i-h}^{t_i} p(s)\psi(q_1+q_2)ds) \to \begin{pmatrix}0\\0\end{pmatrix} \text{ as } h\to 0.\end{aligned}$$

3. If $\tau_2 = t_i^+$, we consider $\delta_2 > 0$ such that $\{t_k, k\neq i\}\cap[t_i-\delta_2, t_i+\delta_2]=\emptyset$, so, for $0<h<\delta_2$, we have

$$\begin{aligned}&\|\Lambda(x,y)(t_i+h)-\Lambda(x,y)(t_i)\| \\ &\leq (M\int_{t_i}^{t_i+h} p(s)\psi(q_1+q_2)ds, M\int_{t_i}^{t_i+h} p(s)\psi(q_1+q_2)ds) \to \begin{pmatrix}0\\0\end{pmatrix} \text{ as } h\to 0.\end{aligned}$$

So by Steps 1,2 and 3, and by Arzela-Ascoli's theorem, Λ is completely continuous.

Step 4. A Priori Estimates.

Let $(x,y)\in PC(J,\mathbb{R})\times PC(J,\mathbb{R})$ such that $(x,y)=\lambda\Lambda(x,y)$, and $0<\lambda<1$. Then for all $t\in[0,t_1]$, we have

$$x(t) = \lambda x_0 + \lambda\int_0^t f(s,x(s),y(s))ds,$$

$$y(t) = \lambda y_0 + \lambda\int_0^t g(s,x(s),y(s))ds,$$

and so,

$$\begin{aligned}\|(x,y)(t)\| \leq\ & (\|x_0\| + \int_0^t p(s)\psi(\|x(s)\|+\|y(s)\|)ds, \|y_0\| \\ & + \int_0^t p(s)\psi(\|x(s)\|+\|y(s)\|)ds), \quad t\in[0,t_1].\end{aligned}$$

Consider the map $\vartheta = (\vartheta_1, \vartheta_2)$ such that

$$\vartheta_1 = \|x_0\| + \int_0^t p(s)\psi(\|x(s)\| + \|y(s)\|)ds, \quad t \in [0, t_1],$$

$$\vartheta_2 = \|y_0\| + \int_0^t p(s)\psi(\|x(s)\| + \|y(s)\|)ds, \quad t \in [0, t_1].$$

Then we have

$$\vartheta(0) = (\|x_0\|, \|y_0\|), \ \ \|(x, y)(t)\| \leq \vartheta(t), \quad t \in [0, t_1],$$

and

$$\vartheta_i(0) = p(\|x(t)\|, \|y(t)\|), \ \ \forall i = 1, 2, \quad t \in [0, t_1],$$

As ψ is a nondecreasing map, we have

$$\vartheta_i(t) \leq p(t)\psi(\vartheta_i(t)), \ \ \forall i = 1, 2 \ \ t \in [0, t_1],$$

which implies that for every $t \in [0, t_1]$,

$$\int_{\vartheta_i(0)}^{\vartheta_i(t)} \frac{du}{\psi(u)} \leq \int_0^{t_1} p(s)ds, \ \ \forall i = 1, 2.$$

The map $\Gamma_{i,0}(z) = \int_{\vartheta_i(0)}^{z} \frac{du}{\psi(u)}, \ \ \forall i = 1, 2$, is continuous and increasing. Then $\Gamma_{i,0}^{-1}$ exists and it is increasing, and we get

$$\vartheta_i(t) \leq \Gamma_{i,0}\left(\int_0^{t_1} p(s)ds\right) := M_{i,0}, \ \ \forall i = 1, 2.$$

As for all $t \in [0, t_1]$, $\|(x, y)(t)\| \leq \vartheta(t)$, and so,

$$\sup_{t\in[0,t_1]} \|(x, y)\| \leq \begin{pmatrix} M_{1,0} \\ M_{2,0} \end{pmatrix}.$$

Now, for $t \in (t_1, t_2]$, we have

$$\|x(t_1^+)\| \leq \|I_1(x(t_1), y(t_1))\| + \|x(t_1)\| \leq \sup_{(x,y)\in\overline{B_r}} \|I_1(x, y)\| + M_{1,0} := N_1,$$

$$\|y(t_1^+)\| \leq \|I_2(x(t_1), y(t_1))\| + \|y(t_1)\| \leq \sup_{(x,y)\in\overline{B_r}} \|I_2(x, y)\| + M_{2,0} := N_2,$$

where

$$q = \begin{pmatrix} M_{1,0} \\ M_{2,0} \end{pmatrix},$$

$$x(t) = \lambda(x(t_1) + I_1(x(t_1), y(t_1))) + \lambda \int_{t_1}^t f(s, x(s), y(s))ds,$$

$$y(t) = \lambda(x(t_1) + I_2(x(t_1), y(t_1))) + \lambda \int_{t_1}^{t} g(s, x(s), y(s))ds.$$

Then

$$\|x(t)\| \leq N_1 + \int_{t_1}^{t} p(s)\psi(\|x(s)\| + \|y(s)\|)ds, \quad t \in [t_1, t_2],$$

$$\|y(t)\| \leq N_2 + \int_{t_1}^{t} p(s)\psi(\|x(s)\| + \|y(s)\|)ds, \quad t \in [t_1, t_2],$$

Consider the map $W = (W_1, W_2)$ such that

$$W_1(t) = N_1 + \int_{t_1}^{t} p(s)\psi(\|x(s)\| + \|y(s)\|)ds, \ \ t \in [t_1, t_2],$$

$$W_2(t) = N_2 + \int_{t_1}^{t} p(s)\psi(\|x(s)\| + \|y(s)\|)ds, \ \ t \in [t_1, t_2].$$

So,

$$W(t_1^+) = (N_1, N_2), \quad \|(x, y)(t)\| \leq W(t), \ \ t \in [t_1, t_2],$$

and

$$W_i(t) = p(t)\psi(\|x(t)\| + \|y(t)\|), \ \ \forall i = 1, 2, \quad t \in [t_1, t_2].$$

Since ψ is nondecreasing, we get

$$W_i(t) \leq p(t)\psi(W_i(t)), \ \ \forall i = 1, 2, \quad t \in [t_1, t_2],$$

what implies that for every $t \in [t_1, t_2]$, we have

$$\int_{W_i(t_1^+)}^{W_i(t)} \frac{du}{\psi(u)} \leq \int_{t_1}^{t_2} p(s)ds, \quad i = 1, 2.$$

If we consider the map $\Gamma_{i,1}(z) = \int_{W_i(t_1^+)}^{z} \frac{ds}{\psi(s)}, \quad i = 1, 2$, we get

$$W_i(t) \leq \Gamma_{i,1}^{-1} \left(\int_{t_1}^{t_2} p(s)ds \right) := M_{i,1}, \quad i = 1, 2.$$

For all $t \in [t_1, t_2]$,

$$\|(x, y)(t)\| = \begin{pmatrix} \|x(t)\| \\ \|y(t)\| \end{pmatrix} \leq \begin{pmatrix} W_1(t) \\ W_2(t) \end{pmatrix},$$

so,

$$\sup_{t \in [t_1, t_2]} \|(x, y)(t)\| \leq \begin{pmatrix} M_{1,1} \\ M_{2,1} \end{pmatrix}.$$

We continue this process to the interval $(t_m, 1]$, and $(x, y) |_{(t_m,1]}$ is the solution of the problem $(x, y) = \lambda\Lambda(x, y)$ for $0 < \lambda < 1$. There exists $M_{i,m}, i = 1, 2$, such that

$$\sup_{t\in[t_m,b]} \|(x, y)(t)\| \leq \Gamma_{i,m}^{-1} \left(\int_{t_m}^{b} p(s)ds \right) := M_{i,m}.$$

As we choose (x, y) arbitrarily, for all solution of problem 2.0.1 we have

$$\|x(t, y)\| \leq \begin{pmatrix} \max_{k=0,1,\cdots,m}(M_{1,k}) \\ \max_{k=0,1,\cdots,m}(M_{2,k}) \end{pmatrix} := \begin{pmatrix} d_1^* \\ d_2^* \end{pmatrix}.$$

Thus, the set

$$\mathcal{K} = \{(x, y) \in PC \times PC : (x, y) = \Lambda(x, y), \lambda \in (0, 1)\}.$$

Since $\Lambda : PC \times PC \to PC \times PC$ is completely continuous and $\mathcal{K}$ is bounded, from Theorem 1.2.5, Λ has fixed point $(x, y) \in PC \times PC$ which is the solution of problem (2.0.1).

□

Chapter 3

System of impulsive differential equations on unbounded domain

The theory of boundary-value problems with integral boundary conditions for ordinary differential equations arises in different areas of applied mathematics and physics.

Boundary value problems with integral boundary conditions on the half-line for different classes of systems of differential equations have been intensively studied in the literature using a variety of methods (see [35, 36, 105, 108]).

This chapter is concerned with the existence of solution to systems of nonlinear second-order impulsive differential equations with integral boundary on the half-line of the forme

$$\begin{cases} -u''(t) = f(t, u(t), v(t)), & t \in J,\ t \neq t_k; \\ -v''(t) = g(t, u(t), v(t)), & t \in J,\ t \neq t_k; \\ \Delta u(t_k) = J_{1,k}(u(t_k)), \quad -\Delta u'(t_k) = I_{1,k}(u(t_k)), & k = 1, 2, \cdots, \\ \Delta v(t_k) = J_{2,k}(v(t_k)), \quad -\Delta v'(t_k) = I_{2,k}(v'(t_k)), & k = 1, 2, \cdots, \\ u(0) = \displaystyle\int_0^\infty h_1(s)u(s)ds, \quad u'(\infty) = 0, & \\ v(0) = \displaystyle\int_0^\infty h_2(s)v(s)ds, \quad v'(\infty) = 0, & \end{cases} \tag{3.0.1}$$

where $J = [0, \infty)$, $f, g \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $0 < t_1 < t_2 < \cdots < t_k < \cdots$, $t_k \longrightarrow \infty$, $I_{i,k},\ J_{i,k} \in C(\mathbb{R}, \mathbb{R}),\ for\ i = 1, 2,\ h_i \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $\displaystyle\int_0^\infty h_i(s)ds \neq 1\ for\ i = 1, 2$, $u'(\infty) = \lim_{t \longrightarrow \infty} u(t)$ and $v'(\infty) = \lim_{t \longrightarrow \infty} v(t)$, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$ and $\Delta v(t_k) = v(t_k^+) - v(t_k^-)$, where $u(t_k^+), v(t_k^+)$ and $u(t_k^-), v(t_k^-)$ represent the right-hand limit of $u(t), v(t)$ at $t = t_k$, respectively.Also $\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-)$ and $\Delta v'(t_k) = v'(t_k^+) - v'(t_k^-)$, where $u'(t_k^+), v'(t_k^+)$ and $u'(t_k^-), v'(t_k^-)$ represent the right-hand limit of $u'(t)$ and $v'(t)$ at $t = t_k$, respectively.

Since we are interested here in system of equations, we have opted for a vectorial approach based on the use of vector-valued norms, inverse-positive matric and a vectorial version of Krasnoselskii's fixed point theorem for sums of two operators [110].

Before setting the result of this section we consider the following spaces.

$$\begin{aligned} PC([0,\infty)) \;&= \{u : [0,+\infty) \to \mathbb{R} \mid u(t) \text{ is continuous at each } t \neq t_k, \\ &\text{left continuous at } t = t_k, u'(t_k^+) \text{ exists }, k = 1, 2, \cdots, \} \end{aligned}$$

and the space E defined by,

$$E = \{u \in PC([0,+\infty)), \sup_{t\in[0,+\infty)} \frac{\mid u(t) \mid}{1+t} < \infty\},$$

E is a Banach space, equipped with the norm:

$$\| u \|_E = \sup_{t\in[0,+\infty)} \frac{\mid u(t) \mid}{1+t} < \infty.$$

Then $E \times E$ is Banach space with the norm:

$$\| (u,v) \| = (\| u \|_E, \| v \|_E) \text{ for } (u,v) \in E \times E.$$

The following compactness criterion on unbounded domains is a simple extension of a compactness criterion in $C_b(\mathbb{R}^+,\mathbb{R})$ (see [15]).

Lemma 3.0.3. *We define the space $\eta = C_b(\mathbb{R}^+,\mathbb{R})$, let $N \subseteq \eta$, Then N is compact in η, if the following conditions hold:*

- (a) *N is uniformly bounded in η.*
- (b) *The functions from $\{y : \; y = \frac{x}{1+t}, \; x \in N\}$ belonging to N are almost equicontinuous on $\mathbb{R}^+$.*
- (c) *The functions from $\{y : \; y = \frac{x}{1+t}, \; x \in N\}$ are equiconvergent at $+\infty$, that is given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that $|y(t) - y(+\infty)| < \varepsilon$ for any $t \geq 0$ and $f \in N$ for all $t \geq T(\varepsilon)$ and $x \in N$.*

Our next lemma should come as no surprise. It shows that solutions of system (3.0.1) are equivalent to solutions of integral equations.

Lemma 3.0.4. *The vector $(u,v) \in PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$ is a solution of differential* *system* (3.0.1) *if and only if*

$$\begin{aligned}
u(t) &= \int_0^\infty H_1(t,s)f(s,u(s),v(s))ds + \sum_{k=1}^{\infty} H_1(t,t_k)I_{1.k}(u(t_k)) + \sum_{t_k<t} J_{1.k}(u(t_k)) \\
&+ \frac{\displaystyle\int_0^\infty h_1(s)\left(\sum_{t_k<s} J_{1.k}(u(t_k))\right)ds}{1-\displaystyle\int_0^\infty h_1(s)ds}, \\
v(t) &= \int_0^\infty H_2(t,s)g(s,u(s),v(s))ds + \sum_{k=1}^{\infty} H_2(t,t_k)I_{1.k}(u(t_k)) + \sum_{t_k<t} J_{2.k}(v(t_k)) \\
&+ \frac{\displaystyle\int_0^\infty h_2(s)\left(\sum_{t_k<s} J_{2.k}(v(t_k))\right)ds}{1-\displaystyle\int_0^\infty h_2(s)ds}.
\end{aligned}$$

where for $i = 1, 2$

$$H_i(t,s) = G(t,s) + \frac{1}{1-\displaystyle\int_0^\infty h_i(s)ds}\int_0^\infty G(\tau,s)h_i(\tau)d\tau$$

$$G(t,s) = \begin{cases} t, & 0 \le t \le s \le \infty, \\ s, & 0 \le s \le t \le \infty. \end{cases} \tag{3.0.2}$$

Proof. First we consider the following problem:

$$\begin{cases} -u''(t) = f(t,u(t),v(t)), & t \in J,\ t \neq t_k; \\ \Delta u(t_k) = J_{1,k}(u(t_k)), \quad -\Delta u'(t_k) = I_{1,k}(u(t_k)), & k = 1, 2, \cdots, \\ u(0) = \displaystyle\int_0^\infty h_1(s)u(s)ds, \quad u'(\infty) = 0. \end{cases} \tag{3.0.3}$$

Let u be a solution of problems (3.0.3), then by integration, we have

$$u'(t) = u'(0) - \int_0^t f(s,u(s),v(s))ds - \sum_{t_k<t} I_{1,k}(u(t_k)), \tag{3.0.4}$$

Taking limit for $t \longrightarrow \infty$,

$$u'(0) = \int_0^\infty f(s,u(s),v(s))ds + \sum_{k=1}^{\infty} I_{1,k}(u(t_k)),$$

Integrating (3.0.4), we can get

$$u(t) = u'(0)t+u(0)-\int_0^t (t-s)f(s,u(s),v(s))ds-\sum_{t_k<t} I_{1,k}(u(t_k))(t-t_k)+\sum_{t_k<t} J_{1,k}(u(t_k)).$$

Thus,

$$\begin{aligned} u(t) &= u(0)+\int_0^\infty tf(s,u(s),v(s))ds+\sum_{k=1}^{\infty} tI_{1,k}(u(t_k))-\int_0^t (t-s)f(s,u(s),v(s))ds \\ &\quad -\sum_{t_k<t} I_{1,k}(u(t_k))(t-t_k)+\sum_{t_k<t} J_{1,k}(u(t_k)). \end{aligned}$$

Thus,

$$\begin{aligned} u(t) &= u(0)+\int_0^\infty G(t,s)f(s,u(s),v(s))ds+\sum_{k=1}^{\infty} G(t,t_k)I_{1,k}(u(t_k)) \\ &\quad +\sum_{t_k<t} J_{1,k}(u(t_k)). \end{aligned}$$

Then,

$$\begin{aligned} u(t) &= \int_0^\infty h_1(s)u(s)ds+\int_0^\infty G(t,s)f(s,u(s),v(s))ds \\ &+ \sum_{k=1}^{\infty} G(t,t_k)I_{1,k}(u(t_k))+\sum_{t_k<t} J_{1,k}(u(t_k)). \end{aligned} \tag{3.0.5}$$

Thus,

$$\begin{aligned} \int_0^\infty h_1(s)u(s)ds &= \int_0^\infty h_1(s)\left(\int_0^\infty h_1(s)u(s)ds+\int_0^\infty G(s,\tau)f(\tau,u(\tau),v(\tau))d\tau\right)ds \\ &\quad +\int_0^\infty h_1(s)\left(\sum_{k=1}^{\infty} G(s,t_k)I_{1,k}(u(t_k))+\sum_{t_k<s} J_{1,k}(u(t_k))\right)ds. \end{aligned}$$

It follows that

$$\begin{aligned} \int_0^\infty h_1(s)u(s)ds &= \frac{1}{1-\int_0^\infty h_1(s)ds}\left(\int_0^\infty\int_0^\infty h_1(s)G(s,\tau)f(\tau,u(\tau),v(\tau))d\tau\right) \\ &\quad +\frac{1}{1-\int_0^\infty h_1(s)ds}\int_0^\infty h_1(s)\left(\sum_{k=1}^{\infty} G(s,t_k)I_{1,k}(u(t_k))\right)ds \\ &\quad +\frac{1}{1-\int_0^\infty h_1(s)ds}\int_0^\infty h_1(s)\left(\sum_{t_k<s} J_{1,k}(u(t_k))\right)ds. \end{aligned}$$

Substituting in (3.0.5) we have

$$\begin{aligned} u(t) &= \int_0^\infty G(t,s)f(s,u(s),v(s))ds + \frac{\int_0^\infty \int_0^\infty h_1(s)G(s,\tau)f(\tau,u(\tau),v(\tau))d\tau}{1-\int_0^\infty h_1(s)d\tau ds} \\ &+ \sum_{k=1}^\infty G(t,t_k)I_{1,k}(u(t_k)) + \frac{\int_0^\infty h_1(s)\left(\sum_{k=1}^\infty G(t,t_k)I_{1,k}(u(t_k))\right)ds}{1-\int_0^\infty h_1(s)ds} \\ &+ \sum_{t_k<t} J_{1,k}(u(t_k)) + \frac{\int_0^\infty h_1(s)\left(\sum_{t_k<t} J_{1,k}(u(t_k))\right)ds}{1-\int_0^\infty h_1(s)ds}. \end{aligned}$$

Hence,

$$\begin{aligned} u(t) &= \int_0^\infty H_1(t,s)f(s,u(s),v(s))ds + \sum_{k=1}^\infty H_1(t,t_k)I_{1.k}(u(t_k)) \\ &+ \sum_{t_k<t} J_{1.k}(u(t_k)) + \frac{\int_0^\infty h_1(s)\left(\sum_{t_k<s} J_{1.k}(u(t_k))\right)ds}{1-\int_0^\infty h_1(s)ds}, \end{aligned}$$

where,

$$H_1(t,s) = G(t,s) + \frac{1}{1-\int_0^\infty h_1(s)ds}\int_0^\infty G(\tau,s)h_1(\tau)d\tau.$$

Next, we consider the following problems

$$\begin{cases} -v''(t) = f(t,u(t),v(t)), & t\in J,\ t\neq t_k; \\ \Delta v(t_k) = J_{2,k}(v(t_k)),\quad \Delta v'(t_k) = -I_{2,k}(v(t_k)), & k=1,2,\cdots, \\ v(0) = \int_0^\infty h_2(s)v(s)ds,\quad v'(\infty)=0, \end{cases}$$

similarly, we have that

$$\begin{aligned} v(t) &= \int_0^\infty H_2(t,s)f(s,u(s),v(s))ds + \sum_{k=1}^{\infty} H_2(t,t_k)I_{1.k}(u(t_k)) + \sum_{t_k<t} J_{2.k}(u(t_k)) \\ &+ \frac{\int_0^\infty h_2(s)\left(\sum_{t_k<s} J_{2.k}(u(t_k))\right)ds}{1-\int_0^\infty h_2(s)ds}, \end{aligned}$$

where,

$$H_2(t,s) = G(t,s) + \frac{1}{1-\int_0^\infty h_2(s)ds}\int_0^\infty G(\tau,s)h_2(\tau)d\tau.$$

□

3.1 Existence results

In the section, we establish the existence of at least one solution of (3.0.1). We need following assumptions to obtain our result:

(H_1) f, g are L^1-Carathéodory functions.

(H_2) There exist nonnegative functions $P_i, \bar{P}_i \in L^1[0,+\infty)$ for $i = 1,2,3$ such that:

$$|f(t,u,v)| \leq P_1(t)|u| + P_2(t)|v| + P_3(t), for\ each\ t \in J, (u,v) \in \mathbb{R}^2,$$

and

$$|g(t,u,v)| \leq \bar{P}_1(t)|u| + \bar{P}_2(t)|u| + \bar{P}_3(t),\ \ for\ each\ t \in J,\ \ (u,v) \in \mathbb{R}^2.$$

(H_3) For all $u, v, \bar{u}, \bar{v} \in \mathbb{R}$, there exist nonnegative constants $a_{i,k}, b_{i,k} \geq 0, \imath = 1,2$ such that

$$\begin{cases} |I_{1,k}(u) - I_{1,k}(\bar{u})| \leq a_{1,k}|u-\bar{u}|, & \text{k=1,2,\dots} \\ |I_{2,k}(v) - I_{2,k}(\bar{v})| \leq a_{2,k}|v-\bar{v}|, & \text{k=1,2,\dots} \end{cases}$$

and

$$\begin{cases} |J_{1,k}(u) - J_{1,k}(\bar{u})| \leq b_{1,k}|u-\bar{u}|, & \text{k=1,2,\dots,m,\dots,} \\ |J_{2,k}(v) - J_{2,k}(\bar{v})| \leq b_{2,k}|v-\bar{v}|, & \text{k=1,2,\dots,m,\dots.} \end{cases}$$

(H_4) We define the constants N_i and C_i, $i = 1, 2, 3$, by

$$N_i = \left(1 + \frac{\|h_1\|_{L^1}}{h_1^*}\right) \int_0^\infty P_i(s)(1+s)ds < \infty,$$

$$C_i = \left(1 + \frac{\|h_2\|_{L^1}}{h_2^*}\right) \int_0^\infty \bar{P}_i(s)(1+s)ds < \infty, \ i = 1, 2;$$

$$N_3 = \left(1 + \frac{\|h_1\|_{L^1}}{h_1^*}\right) \left(\int_0^\infty P_3(s)ds + \sum_{k=1}^\infty |I_{1,k}(0)| + \sum_{k=1}^\infty |J_{1,k}(0)|\right) < \infty,$$

$$C_3 = \left(1 + \frac{\|h_2\|_{L^1}}{h_2^*}\right) \left(\int_0^\infty \bar{P}_3(s)ds + \sum_{k=1}^\infty |I_{2,k}(0)| + \sum_{k=1}^\infty |J_{2,k}(0)|\right) < \infty,$$

as well as the constants

$$K_i = \left(1 + \frac{\|h_i\|_{L^1}}{h_i^*}\right) \sum_{k=1}^\infty (a_{i,k} + b_{i,k})(1+t_k) < \infty, \ for \ i = 1, 2;$$

where,

$$h_1^* = \left|1 - \int_0^\infty h_1(s)ds\right|, \quad h_2^* = \left|1 - \int_0^\infty h_2(s)ds\right|.$$

Theorem 3.1.1. *Assume that (H_1)-(H_3) holds with $N_1 + K_1 < 1$ and $C_2 + K_2 < 1$. If*

$$\tilde{M} = \begin{pmatrix} 1 - N_1 - K_1 & -C_2 \\ -C_1 & 1 - C_2 - K_2 \end{pmatrix}.$$

and $\det \tilde{M} > 0$, *then problem* (3.0.1) *has at least one solution.*

Proof. Let $N : E \times E \longrightarrow E \times E$ be the operator defined by

$$N(u, v) = F(u, v) + B(u, v), \ (u, v) \in E \times E,$$

where,

$$F(u, v) = (F_1(u, v), F_2(u, v)); \ B(u, v) = (B_1(u, v), B_2(u, v)),$$

$$F_1(u(t), v(t)) = \int_0^\infty H_1(t, s) f(s, u(s), v(s))ds,$$

$$F_2(u(t), v(t)) = \int_0^\infty H_2(t, s) g(s, u(s), v(s))ds,$$

$$B_1(u(t),v(t)) = \sum_{k=1}^{\infty} H_1(t,t_k)I_{1.k}(u(t_k)) + \sum_{t_k<t} J_{1.k}(u(t_k)) + \frac{\int_0^\infty h_1(s)\left(\sum_{t_k<s} J_{1.k}(u(t_k))\right)ds}{1-\int_0^\infty h_1(s)ds},$$

and

$$B_2(u(t),v(t)) = \sum_{k=1}^{\infty} H_2(t,t_k)I_{2.k}(v(t_k)) + \sum_{t_k<t} J_{2.k}(v(t_k)) + \frac{\int_0^\infty h_2(s)\left(\sum_{t_k<s} J_{2.k}(v(t_k))\right)ds}{1-\int_0^\infty h_2(s)ds}.$$

In order to show that the condition in Theorem hold, we will proceed in several steps.

Step 1 It is clear from its definition that B is a continuous operator. To show that B is an M-contraction, let $(u,v),(\bar{u},\bar{v}) \in E \times E$. From (H_3), we see that

$$\frac{|B_1(u(t),v(t)) - B_1(\bar{u}(t),\bar{v}(t))|}{1+t}$$

$$\leq \sum_{k=1}^{\infty} \frac{|H_1(t,t_k)|}{1+t}|I_{1,k}(u(t_k)) - I_{1,k}(\bar{u}(t_k))| + \sum_{t_k<t} |J_{1.k}(u(t_k)) - J_{1,k}(\bar{u}(t_k))|$$

$$+ \frac{\int_0^\infty h_1(s)\left(\sum_{t_k<s} |J_{1,k}(u(t_k)) - J_{1,k}(\bar{u}(t_k))|\right)ds}{\left|1-\int_0^\infty h_1(s)ds\right|}$$

$$\leq \sum_{k=1}^{\infty}\left(\frac{G_1(t,t_k)}{1+t} + \frac{1}{h_1^*}\int_0^\infty h_1(r)\frac{G_1(r,t_k)}{1+t}dr\right) a_{1,k}|u(t_k) - \bar{u}(t_k)| + \sum_{k=1}^{\infty} b_{1,k}|u(t_k) - \bar{u}(t_k)|$$

$$+ \frac{\int_0^\infty h_1(s)ds}{h_1^*}\sum_{k=1}^{\infty} b_{1,k}\left|u(t_k) - \bar{u}(t_k)\right|.$$

Thus,

$$\begin{aligned}\|B_1(u,v)-B_1(\bar{u},\bar{v})\|_E & \\ &\leq \left(1+\frac{\|h_1\|_{L^1}}{h_1^*}\right)\sum_{k=1}^{\infty}(a_{1,k}+b_{1,k})(1+t_k)\|u-\bar{u}\|_E := K_1\|u-\bar{u}\|_E.\end{aligned}$$

Similarly, we have

$$\|B_2(u,v)-B_2(\bar{u},\bar{v})\|_E \leq \left(1+\frac{\|h_2\|_{L^1}}{h_2^*}\right)\sum_{k=1}^{\infty}(a_{2,k}+b_{2,k})(1+t_k)\|v-\bar{v}\|_E := K_2\|u-\bar{u}\|_E.$$

Therefore,

$$\begin{bmatrix}\|B_1(u,v)-B_1(\bar{u},\bar{v})\|_E \\ \\ \|B_2(u,v)-B_2(\bar{u},\bar{v})\|_E\end{bmatrix} \leq M \begin{bmatrix}\|u-\bar{u}\|_E \\ \\ \|v-\bar{v}\|_E\end{bmatrix},$$

where,

$$M=\begin{pmatrix}K_1 & 0 \\ 0 & K_2\end{pmatrix}.$$

Since $K_1,\ K_2\in[0,1)$, M converge to zero, and this implies that B is a contraction operator.

Step 2 To show that the operator F is continuous, let $(u_n,v_n)\longrightarrow(u,v)$ as $n\longrightarrow\infty$. Then $u_n\longrightarrow u$ and $v_n\longrightarrow v$ as $n\longrightarrow\infty$.
We have

$$\frac{|F_1(u_n(t),v_n(t))-F_1(u(t),v(t))|}{1+t} \leq \int_0^\infty \frac{|H_1(t,s)|}{1+t}|f(s,u_n(s),v_n(s))-f(s,u(s),v(s))|ds$$

and so

$$\|F_1(u_n,v_n)-F_1(u,v)\|_E \leq \sup_{t\in[0,+\infty[}\int_0^\infty \frac{|H_1(t,s)|}{1+t}|f(s,u_n(s),v_n(s))-f(s,u(s),v(s))|ds$$

Since f is L^1-Carathéodory, by the lebesgue dominated convergence theorem,

$$\|F_1(u_n,v_n)-F_1(u,v)\|_E\longrightarrow 0,\ n\longrightarrow 0,$$

Similarly,

$$\|F_2(u_n,v_n)-F_2(u,v)\|_E\longrightarrow 0,\ n\longrightarrow 0.$$

so F is continuous.

Step 3 We want to show that F maps bounded sets into relatively compact sets, so let D be a bounded subset of E. Then there exists $q>0$ such that $\|u\|_E\leq q$ and

$\|v\|_E \le q$ for all $(u,v) \in D$.
Let $(u,v) \in D$. Then for each $t \in [0,+\infty[$, we have

$$\frac{|F_1(u(t),v(t))|}{1+t} \le \int_0^\infty \frac{|H_1(t,s)|}{1+t}|f(s,u(s),v(s))|ds$$

and similarly for F_2. Since f, g be are Carathédory function, there exists nonnegative function $\phi_{M_0,M_1} \in L^1[0,\infty[$ such that

$$|f(t,u(t),v(t))| \le \phi_{r_1,r_2}(t) \ and \ |g(t,u(t),v(t))| \le \phi_{r_1,r_2}(t) \ t \in \mathbb{R}.$$

Hence

$$\|F_1(u,v)\|_E \le \left(1 + \frac{\|h_1\|_{L^1}}{h_1^*}\right) \int_0^\infty \phi_{r_1,r_2}(s)ds.$$

Similarly, we have

$$\|F_2(u,v)\|_E \le \left(1 + \frac{\|h_2\|_{L^1}}{h_2^*}\right) \int_0^\infty \phi_{r_1,r_2}(s)ds.$$

Therefore, F maps bounded sets E into bounded sets in E.

Moreover, for any $T \in [0,+\infty[$ and $\tau_1, \tau_2 \in [0,T]$, $\tau_1 < \tau_2$,

$$\begin{aligned}\left|\frac{F_1(u(\tau_2),v(\tau_2))}{1+\tau_2} - \frac{F_1(u(\tau_1),v(\tau_1))}{1+\tau_1}\right| &\le \int_0^\infty \left|\frac{H_1(\tau_2,s)}{1+\tau_2} - \frac{H_1(\tau_1,s)}{1+\tau_1}\right| |f(s,u(s),v(s))|ds \\ &\le \int_0^\infty \left|\frac{H_1(\tau_2,s)}{1+\tau_2} - \frac{H_1(\tau_1,s)}{1+\tau_1}\right| \phi_{r_1,r_2}(s)ds \\ &\longrightarrow 0 \ as \ \tau_1 \longrightarrow \tau_2.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\left|\frac{F_2(u(\tau_2),v(\tau_2))}{1+\tau_2} - \frac{F_2(u(\tau_1),v(\tau_1))}{1+\tau_1}\right| &\le \int_0^\infty \left|\frac{H_2(\tau_2,s)}{1+\tau_2} - \frac{H_2(\tau_1,s)}{1+\tau_1}\right| \phi_{r_1,r_2}(s)ds \\ &\longrightarrow 0 \ as \ \tau_1 \longrightarrow \tau_2,\end{aligned}$$

Thus F is equicontinuous on any compact interval of $[0,\infty)$.

Step 4 Next, we wish to show that $N(D)$ is equiconvergent, i.e, for any $\varepsilon = (\varepsilon_1, \varepsilon_2) > 0$, there exists sufficiently large $T(\varepsilon) = \max(T_1(\varepsilon_1), T_2(\varepsilon_2))$ such that

$$\left|\frac{F(u(\tau_2),v(\tau_2))}{1+\tau_2} - \frac{F(u(\tau_1),v(\tau_1))}{1+\tau_1}\right| \le \varepsilon, \ \forall \ \tau_1, \tau_2 \ge T(\varepsilon), (u,v) \in E. \tag{3.1.1}$$

Since $\phi_{r_1,r_2} \in L^1[0,\infty)$ then $\displaystyle\int_0^\infty \frac{|H_i(t,s)|}{1+t}\phi_{r_1,r_2}(s)ds < \infty \ for \ i = 1,2$, so we can choose $T_1(\varepsilon), T_2(\varepsilon)$ such that

$$\int_0^\infty \frac{|H_i(t,s)|}{1+t}\phi_{r_1,r_2}(s)ds \le \frac{\varepsilon_i}{2}, \ for \ i = 1,2. \tag{3.1.2}$$

Then, for every $\tau_1, \tau_2 \geq T_1(\varepsilon_1)$, we have

$$\begin{aligned}
\left|\frac{F_1(u(\tau_2), v(\tau_2))}{1+\tau_2} - \frac{F_1(u(\tau_1), v(\tau_1))}{1+\tau_1}\right| &\leq \int_0^\infty \left|\frac{H_1(\tau_2, s)}{1+\tau_2} - \frac{H_1(\tau_1, s)}{1+\tau_1}\right| \phi_{r_1,r_2}(s)ds \\
&\leq \int_0^\infty \frac{|H_1(\tau_2, s)|}{1+\tau_2}\phi_{r_1,r_2}(s)ds \\
&\quad + \int_0^\infty \frac{|H_1(\tau_1, s)|}{1+\tau_1}\phi_{r_1,r_2}(s)ds \\
&\leq \frac{\varepsilon_1}{2} + \frac{\varepsilon_1}{2} = \varepsilon_1.
\end{aligned}$$

Similarly, for every $\tau_1, \tau_2 \geq T_2(\varepsilon_2)$, we have

$$\begin{aligned}
\left|\frac{F_2(u(\tau_2), v(\tau_2))}{1+\tau_2} - \frac{F_2(u(\tau_1), v(\tau_1))}{1+\tau_1}\right| &\leq \int_0^\infty \frac{|H_2(\tau_2, s)|}{1+\tau_2}\phi_{r_1,r_2}(s)ds \\
&\quad + \int_0^\infty \frac{|H_2(\tau_1, s)|}{1+\tau_1}\phi_{r_1,r_2}(s)ds \\
&\leq \frac{\varepsilon_2}{2} + \frac{\varepsilon_2}{2} = \varepsilon_2.
\end{aligned}$$

Hence for each $\tau_1, \tau_2 \geq \max(T_1(\varepsilon_1), T_2(\varepsilon_2))$

$$\left|\frac{F(u(\tau_2), v(\tau_2))}{1+\tau_2} - \frac{F(u(\tau_1), v(\tau_1))}{1+\tau_1}\right| \leq \varepsilon, \quad \forall (u, v) \in D.$$

That is (3.1.1), holds.

Step 5 To show that set $B = \left\{(u, v) \in E \times E : F(u, v) + \lambda B(\frac{u}{\lambda}, \frac{v}{\lambda}) = (u, v)\right\}$ is bounded for $0 < \lambda < 1$. Let $(u, v) \in B$, then

$$\begin{aligned}
u(t) &= \int_0^\infty H_1(t, s) f(s, u(s), v(s))ds + \lambda \sum_{k=1}^\infty H_1(t, t_k) I_{1.k}\left(\frac{u(t_k)}{\lambda}\right) + \lambda \sum_{t_k<t} J_{1.k}\left(\frac{u(t_k)}{\lambda}\right) \\
&\quad + \lambda \frac{\displaystyle\int_0^\infty h_1(s)\left(\sum_{t_k<s} J_{1.k}\left(\frac{u(t_k)}{\lambda}\right)\right) ds}{1 - \displaystyle\int_0^\infty h_1(s)ds},
\end{aligned}$$

$$\begin{aligned}
v(t) &= \int_0^\infty H_2(t, s) g(s, u(s), v(s))ds + \lambda \sum_{k=1}^\infty H_2(t, t_k) I_{2.k}\left(\frac{v(t_k)}{\lambda}\right) + \lambda \sum_{t_k<t} J_{2.k}\left(\frac{v(t_k)}{\lambda}\right) \\
&\quad + \lambda \frac{\displaystyle\int_0^\infty h_2(s)\left(\sum_{t_k<s} J_{2.k}\left(\frac{v(t_k)}{\lambda}\right)\right) ds}{1 - \displaystyle\int_0^\infty h_2(s)ds}.
\end{aligned}$$

Thus,

$$\begin{aligned}\frac{|u(t)|}{1+t} &\leq \left(1+\frac{\|h_1\|_{L^1}}{h_1^*}\right)\int_0^\infty P_1(s)(1+s)ds\|u\|_E + \left(1+\frac{\|h_1\|_{L^1}}{h_1^*}\right)\int_0^\infty P_2(s)(1+s)ds\|v\|_E \\ &+ \left(1+\frac{\|h_1\|_{L^1}}{h_1^*}\right)\int_0^\infty P_3(s)(1+s)ds + \left(1+\frac{\|h_1\|_{L^1}}{h_1^*}\right)\sum_{k=1}^\infty (a_{1,k}+b_{1,k})(1+t_k)\|u\|_E \\ &+ \left(1+\frac{\|h_1\|_{L^1}}{h_1^*}\right)\sum_{k=1}^\infty |I_{1,k}(0)| + \left(1+\frac{\|h_1\|_{L^1}}{h_1^*}\right)\sum_{k=1}^\infty |J_{1,k}(0)|,\end{aligned}$$

and

$$\begin{aligned}\frac{|v(t)|}{1+t} &\leq \left(1+\frac{\|h_2\|_{L^1}}{h_2^*}\right)\int_0^\infty \bar{P}_1(s)(1+s)ds\|u\|_E + \left(1+\frac{\|h_2\|_{L^1}}{h_2^*}\right)\int_0^\infty \bar{P}_2(s)(1+s)ds\|v\|_E \\ &+ \left(1+\frac{\|h_2\|_{L^1}}{h_2^*}\right)\int_0^\infty \bar{P}_3(s)(1+s)ds + \left(1+\frac{\|h_2\|_{L^1}}{h_2^*}\right)\sum_{k=1}^\infty (a_{2,k}+b_{2,k})(1+t_k)\|v\|_E \\ &+ \left(1+\frac{\|h_2\|_{L^1}}{h_2^*}\right)\sum_{k=1}^\infty |I_{2,k}(0)| + \left(1+\frac{\|h_2\|_{L^1}}{h_2^*}\right)\sum_{k=1}^\infty |J_{2,k}(0)|.\end{aligned}$$

This implies

$$\|u\|_E \leq N_1\|u\|_E + K_1\|u\|_E + N_2\|v\|_E + N_3.$$

and

$$\|v\|_E \leq C_1\|u\|_E + K_2\|v\|_E + C_2\|v\|_E + C_3.$$

Thus, we have

$$\begin{pmatrix} 1-N_1-K_1 & -C_2 \\ -C_1 & 1-C_2-K_2 \end{pmatrix}\begin{pmatrix} \|u\|_E \\ \|v\|_E \end{pmatrix} \leq \begin{pmatrix} N_3 \\ C_3 \end{pmatrix}.$$

Therefore,

$$\tilde{M}\begin{pmatrix} \|u\|_E \\ \|v\|_E \end{pmatrix} \leq \begin{pmatrix} N_3 \\ C_3 \end{pmatrix}. \tag{3.1.3}$$

Since $\tilde{M}$ satisfies the conditions of lemma (1.1.2) thus $(\tilde{M})^{-1}$ is order preserving. We apply $(\tilde{M})^{-1}$ to both sides of the inequality (3.1.3) to obtain

$$\begin{pmatrix} \|u\|_E \\ \|v\|_E \end{pmatrix} \leq (\tilde{M})^{-1}\begin{pmatrix} N_3 \\ C_3 \end{pmatrix}.$$

Thus by Theorem (1.2.4) there exists at least one fixed point for N which is solution of problem 3.0.1. □

Example 3.1.1. *Consider the problem:*

$$\begin{cases} -u'' = \dfrac{e^{-t}}{100}(1+u+v)^{\frac{2}{3}}, & t \in J,\ t \neq k, \\ -v'' = \dfrac{e^{-t}}{200}(1+u+v)^{\frac{1}{2}}, & t \in J,\ t \neq k, \\ \Delta u(k) = \dfrac{1}{8^k}\sqrt{u(k)}, & k = 1,2,\cdots, \\ -\Delta u'(k) = \dfrac{1}{10^k}\sqrt{u(k)}, & k = 1,2,\cdots, \\ \Delta v(k) = e^{-2k}\dfrac{v(k)}{(1+v(k))}, & k = 1,2,\cdots, \\ -\Delta v'(t_k) = e^{-3k}\ \dfrac{v(k)}{(1+v(k))}, & k = 1,2,\cdots, \\ u(0) = \displaystyle\int_0^\infty e^{-4s}u(s)ds, \quad u'(\infty) = 0, \\ v(0) = \displaystyle\int_0^\infty e^{-5s}v(s)ds, \quad v'(\infty) = 0. \end{cases} \tag{3.1.4}$$

$$f(t,u,v) = \frac{e^{-t}}{100}(1+u+v)^{\frac{2}{3}},$$

$$g(t,u,v) = \frac{e^{-t}}{200}(1+u+v)^{\frac{1}{2}},$$

$$J_{1,k}(u(t_k)) = \frac{1}{8^k}\sqrt{u(k)}\ k = 1,2,\cdots,$$

$$I_{1,k}(u(t_k)) = \frac{1}{10^k}\sqrt{u(k)}\ k = 1,2,\cdots,$$

$$J_{2,k}(v'(t_k)) = e^{-2k}\ \frac{v(k)}{(1+v(k))},\ k = 1,2,\cdots,$$

$$I_{2,k}(v'(t_k)) = e^{-3k}\ \frac{v(k)}{(1+v(k))},\ k = 1,2,\cdots,$$

$$h_1(s) = e^{-4s} \text{ and } h_2(s) = e^{-5s}.$$

Let $u,v \in [0,\infty[$ *et* $t \in J$
it is clear that $\displaystyle\int_0^\infty e^{-5s} = \frac{1}{5} \neq 1$ *and* $\displaystyle\int_0^\infty e^{-4s}ds = \frac{1}{4} \neq 1.$
By the inequality $(1+x+y)^\gamma \leq 1+\gamma x+\gamma y$, *for* $x \in \mathbb{R}^+$, $0 \leq \gamma \leq 1$, *we see that*

$$|f(t,u,v)| = \frac{e^{-t}}{100}\left(1+\frac{2}{3}|u|+\frac{2}{3}|v|\right),$$

and

$$|g(t,u,v)| = \frac{e^{-t}}{200}\left(1+\frac{1}{2}|u|+\frac{1}{2}|v|\right),$$

Hence the condition (H_2) *holds with* $P_i(t) = \dfrac{e^{-t}}{150}$ *and* $\bar{P}_i(t) = \dfrac{e^{-t}}{400}$ *for i=1,2 ,* $P_3(t) = \dfrac{e^{-t}}{100}$, $\bar{P}_3(t) = \dfrac{e^{-t}}{200}$.

Also for all $u, \bar{u}, v, \bar{v} \in \mathbb{R}^+$, *we have*

$$|I_{1,k}(u) - I_{1,k}(\bar{u})| \leq \frac{1}{10^k}|u - \bar{u}|, \ k = 1, 2, \cdots ,$$

and

$$|I_{2,k}(v) - I_{2,k}(\bar{v})| \leq e^{-3k}|v - \bar{v}|, \ k = 1, 2, \cdots ,$$

$$|J_{1,k}(u) - J_{1,k}(\bar{u})| \leq \frac{1}{8^k}|u - \bar{u}|, \ k = 1, 2, \cdots ,$$

and

$$|J_{2,k}(v) - J_{2,k}(\bar{v})| \leq e^{-2k}|v - \bar{v}|, \ k = 1, 2, \cdots ,$$

Thus (H_3) *holds with*

$a_{1,k} = \dfrac{1}{10^k}$, $b_{1,k} = \dfrac{1}{8^k}$, $a_{2,k} = e^{-3k}$, $b_{2,k} = e^{-2k}$, $k = 1, 2, \cdots,$

Then, we easily obtain:

$$N_i = \left(1 + \frac{\|h_1\|_{L^1}}{h_1^*}\right) \int_0^\infty P_i(s)(1+s)ds = \frac{2}{225} < \infty, \ i = 1, 2;$$

$$C_i = \left(1 + \frac{\|h_2\|_{L^1}}{h_2^*}\right) \int_0^\infty \bar{P}_i(s)(1+s)ds = \frac{1}{320} < \infty, \ i = 1, 2;$$

$$K_1 = \left(1 + \frac{\|h_1\|_{L^1}}{h_1^*}\right) \sum_{k=1}^{\infty}(a_{1,k} + b_{1,k})(1 + t_k) = \frac{1073}{1327} < \infty,$$

$$K_2 = \left(1 + \frac{\|h_2\|_{L^1}}{h_2^*}\right) \sum_{k=1}^{\infty}(a_{2,k} + b_{2,k})(1 + t_k) \simeq 0,41 < \infty,$$

$$N_3 = \left(1 + \frac{\|h_1\|_{L^1}}{h_1^*}\right) \left(\int_0^\infty P_3(s)ds + \sum_{k=1}^{\infty}|I_{1,k}(0)| + \sum_{k=1}^{\infty}|J_{1,k}(0)|\right) = \frac{1}{75} < \infty,$$

$$C_3 = \left(1 + \frac{\|h_2\|_{L^1}}{h_2^*}\right) \left(\int_0^\infty \bar{P}_3(s)ds + \sum_{k=1}^{\infty}|I_{2,k}(0)| + \sum_{k=1}^{\infty}|J_{2,k}(0)|\right) = \frac{1}{160} < \infty.$$

Thus $N_1 + K_1 \simeq 0,81 < 1$ *and* $C_2 + K_2 \simeq 0,41 < 1$

For this example

$$\tilde{M} \simeq \begin{pmatrix} 1 - 0,81 & -\frac{2}{225} \\ -\frac{1}{320} & 1 - 0,41 \end{pmatrix}.$$

$\det \tilde{M} \simeq 0,11 > 0$. *By Theorem 3.1.1, it follows that Problem has at least one solution.*

Chapter 4

Impulsive evolution equations without predefined time

Impulsive differential equations are studied by many authors, we can see the books [58, 69, 97, 1]. Hernandez and O'Regan [59] introduced a new class of abstract impulsive differential equations where the impulsive are non-instantaneous. In 2020 Hernandez [60], introduce new class of impulsive differential equation without predefined time of moments of impulses.
In this chapter, we are interested in studying the following system:

$$\begin{cases} x'(t) + A_1 x(t) = f(t, x(t), y(t)), & t \in [s^i_{(x,y)}, t^{i+1}_{(x,y)}],\ H(x(s^i_{(x,y)}), y(s^i_{(x,y)})) = 0, \\ x(t) = I(t^{i+1}_{(x,y)}, t, x(t^{i+1}_{(x,y)}), y(t^{i+1}_{(x,y)}), x(t), y(t)), & t \in (t^{i+1}_{(x,y)}, s^{i+1}_{(x,y)}],\ H(x(t^{i+1}_{(x,y)}), y(t^{i+1}_{(x,y)})) = 1 \\ y'(t) + A_2 y(t) = g(t, x(t), y(t)), & t \in [s^i_{(x,y)}, t^{i+1}_{(x,y)}],\ H(x(s^i_{(x,y)}), y(s^i_{(x,y)})) = 0, \\ y(t) = \bar{I}(t^{i+1}_{(x,y)}, t, x(t^{i+1}_{(x,y)}), y(t^{i+1}_{(x,y)}), x(t), y(t)), & t \in (t^{i+1}_{(x,y)}, s^{i+1}_{(x,y)}],\ H(x(t^{i+1}_{(x,y)}, y(t^{i+1}_{(x,y)}) = 1 \\ x(0) = x_0, \quad y(0) = y_0, \quad H(x_0, y_0) = 0, \end{cases} \tag{4.0.1}$$

$A_i : D(A_i) \subset E \longrightarrow E$ are infinitesimal generator of a strongly continuous semigroup of bounded linear operators $(T_i(t))_{t\geq 0}$ on a Banach space E, $H \in C(E \times E, [0, \infty)), (x_0, y_0) \in H^{-1}(0)$ and $f, g : [0, a] \times E \times E \longrightarrow E$ are given functions, $I \in C([0,\infty) \times [0,\infty) \times E \times E \times E \times E; E), \bar{I} \in C([0,\infty) \times [0,\infty) \times E \times E \times E \times E; E)$, $0 = s^0_{(x,y)} < t^1_{(x,y)} < s^1_{(x,y)} < t^2_{(x,y)} \cdots < s^i_{(x,y)} < t^{i+1}_{(x,y)} \cdots \leq a$ are numbers depending on the state (x, y).

4.1 Existence of mild solution

In this section, we study the existence of solution for (4.0.1). To begin, we introduce some concepts of solution.

Definition 4.1.1. *We said that the function* $(x, y) \in C([0, b]; X) \times C([0, b]; X)$*,* $0 < b < a$*, is a* N*-mild solution of* 4.0.1 *if* $x(0) = x_0, y(0) = y_0$ *and there exist numbers* $0 = s^0_{(x,y)} < t^1_{(x,y)} < s^1_{(x,y)} < t^2_{(x,y)} \dots < t^N_{(x,y)} < s^N_{(x,y)} \leq b$ *such that* $(x(\cdot), y(\cdot))$ *is continuous at* $t \neq t^i_{(x,y)}$*,* $H(x(s^i_{(x,y)}), y(s^i_{(x,y)})) = 0$ *for all* $i = 0, \dots, N,$*,*$H(x(t^j_{(x,y)}), y(t^j_{(x,y)})) = 1$ *for all* $j = 1, \dots, N$*,* $1 \geq H(x(s), y(s)) > 0$ *for all* $s \in [t^j_{(x,y)}, s^j_{(x,y)})$ *and* $j = 1, \dots, N$*,* $0 \leq H(x(s), y(s)) < 1$ *for all* $s \in [s^j_{(x,y)}, t^{j+1}_{(x,y)})$ *and* $j = 1, \dots, N - 1$*,* $(x(s), y(s)) = \psi(t^i_{(x,y)}, s, x(t^i_{(x,y)}), y(t^i_{(x,y)}), x(s); y(s))$ *for each* $s \in (t^i_{(x,y)}, s^i_{(x,y)}]$ *and* $i = 1, \dots, N$ *and*

(a) *for* $j = 1, \dots, N-1$*, the function* $(x, y)_{|_{[s^j_{(x,y)}, t^{j+1}_{(x,y)}]}}$ *is a mild solution of the problem*

$$\left\{\begin{array}{lcll} x_1'(t) &=& A_1x_1(t) + f(t, x_1(t), y_1(t)), & t \in [s^j_{(x,y)}, t^{j+1}_{(x,y)}] \\ x_1(s^j_{(x,y)}) &=& I(t^j_{(x,y)}, s^j_{(x,y)}, x_1(t^j_{(x,y)}), y_1(t^j_{(x,y)}), x_1(s^j_{(x,y)}), y_1(s^j_{(x,y)})), & \\ y_1'(t) &=& A_2y_1(t) + g(t, x_1(t), y_1(t)), & t \in [s^j_{(x,y)}, t^{j+1}_{(x,y)}] \\ y_1(s^j_{(x,y)}) &=& \bar{I}(t^j_{(x,y)}, s^j_{(x,y)}, x_1(t^j_{(x,y)}), y_1(t^j_{(x,y)}), x_1(s^j_{(x,y)}), y_1(s^j_{(x,y)})). & \end{array}\right. \quad (4.1.1)$$

(b) *the function* $(x, y)_{|_{[0, t^1_{(x,y)}]}}$ *is a mild solution of*

$$\left\{\begin{array}{lcll} x_1'(t) &=& A_1x_1(t) + f(t, x_1(t), y_1(t)), & t \in [0, t^1_{(x,y)}] \\ x_1(0) &=& x_0, & \\ y_1'(t) &=& A_2y_1(t) + g(t, x_1(t), y_1(t)), & t \in [0, t^1_{(x,y)}] \\ y_1(0) &=& y_0, & \end{array}\right. \quad (4.1.2)$$

(c) $s^N_{(x,y)} = b$ *or* $s^N_{(x,y)} < b$*,* $0 \leq H(x(s), y(s)) < 1$ *for* $s \in [s_(x, y)^N, b]$ *and* $(x, y)_{|_{[s^N_{(x,y)}, b]}}$ *is a mild solution of*

$$\left\{\begin{array}{lcll} x_1'(t) &=& A_1x_1(t) + f(t, x_1(t), y_1(t)), & t \in [s^N_{(x,y)}, b], \\ x_1(s^N_{(x,y)}) &=& I(t^N_{(x,y)}, s^N_{(x,y)}, x_1(t^N_{(x,y)}), y_1(t^N_{(x,y)}), x_1(s^N_{(x,y)}), y_1(s^N_{(x,y)})), & \\ y_1'(t) &=& A_2y_1(t) + g(t, x_1(t), y_1(t)), & t \in [s^N_{(x,y)}, b] \\ y_1(s^N_{(x,y)}) &=& \bar{I}(t^N_{(x,y)}, s^N_{(x,y)}, x_1(t^N_{(x,y)}), y_1(t^N_{(x,y)}), x_1(s^N_{(x,y)}), y_1(s^N_{(x,y)})), & \end{array}\right. \quad (4.1.3)$$

Definition 4.1.2. *We said that a function* $(x, y) \in C([0, b]; X) \times C([0, b]; X)$ *is a* $N\psi$*-mild solution of* (4.0.1) *on* $[0, b]$ *if* $x(0) = x_0, y(0) = y_0$ *and there exist numbers* $0 = s^0_{(x,y)} < t^1_{(x,y)} < s^1_{(x,y)} < t^2_{(x,y)} \dots < s^{N-1}_{(x,y)} < t^N_{(x,y)} \leq b$ *such that* $x(.), y(.)$ *is continuous at* $t \neq t^i$*,* $H(x(s^i_{(x,y)}), y(s^i_{(x,y)})) = 0$ *for all* $i = 0, \dots, N - 1$*,*$H(x(t^j_{(x,y)})) = 1$ *for all* $j = 1, \dots, N$*, the function* $(x, y)_{|_{[0, s^N_{(x,y)}]}}$ *is a* $N - 1$*-mild solution of* (4.0.1) *on* $[0, s^{N-1}_{(x,y)}]$ *and*

(a) *$t^N_{(x,y)} = b$ or $t^N_{(x,y)} < b$, $1 \geq H(x(s), y(s)) > 0$ for $s \in [t^N_{(x,y)}, b]$ and*

$$x_1(s) = I(t^N_{(x,y)}, s, x_1(t^N_{(x,y)}), y_1(t^N_{(x,y)}), x(s), y_1(s))$$

$$y_1(s) = \bar{I}(t^N_{(x,y)}, s, x_1(t^N_{(x,y)}), y_1(t^N_{(x,y)}), x(s), y_1(s)).$$

To prove our results we include the following conditions. Next,∂B denotes the boundary of a set B.

(H_1) There exist $L_i, \overline{L}_i \in \mathbb{R}_+,\ i = 1, 2$ such that

$$\|f(t, x, y) - f(t, \bar{x}, \bar{y})\| \leq L_1\|x - \bar{x}\| + L_2\|y - \bar{y}\|, \quad x, y, \bar{x}, \bar{y} \in X$$

and

$$\|g(t, x, y) - g(t, \bar{x}, \bar{y})\| \leq \bar{L}_1\|x - \bar{x}\| + \bar{L}_2\|y - \bar{y}\|, \quad x, y, \bar{x}, \bar{y} \in X.$$

(H_2) $I \in C([0, \infty)\times[0, \infty)\times E\times E\times E\times E; E)$, $\bar{I} \in C([0, \infty)\times[0, \infty)\times E\times E\times E\times E; E)$ and

$$L_I = \sup\{[I(t, s, x(t), y(t), x(s), y(s))]_{C_{Lip(E;E)}} : t \in I_a, s \in I_{t,a}, x, y \in \delta H^{(-1)}\} < \infty,$$

$$L_{\bar{I}} = \sup\{[\bar{I}(t, s, x(t), y(t), x(s), y(s))]_{C_{Lip(E;E)}} : t \in I_a, s \in I_{t,a}, x, y \in \delta H^{(-1)}\} < \infty,$$

where $I_a = [0, a],\ I_{t,a} = [t, a]$.

(H_3) There exist q_1, q_2, q_3, q_4, such that,

$$|I(t, s, e, p, x, y) - I(t, s, e, p, \bar{x}, \bar{y})| \leq q_1|x - \bar{x}| + q_2|y - \bar{y}|,$$

and

$$|\bar{I}(t, s, e, p, x, y) - \bar{I}(t, s, e, p, \bar{x}, \bar{y})| \leq q_3|x - \bar{x}| + q_4|y - \bar{y}|.$$

Now, we can establish and prove our result on the existence of solution.

Theorem 4.1.1. *Assume that the conditions (H_1),(H_2) and (H_3) are satisfied and $(x_0, y_0) \in H^{-1}(0)$, where,*

$$K = \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix}.$$

If K is converge to 0, then there exists a unique N-mild solution, or a unique $N(I\bar{I})$-mild solution of $(4.0.1)$ on $[0, a]$, or $0 < b_{x_0} < a$, $0 < b_{y_0} < a$.

Proof.

Step 1 We consider the following problem

$$\begin{cases} x(t) = A_1x(t) + f(t, x(t), y(t)), & t \in [0, a] \\ y(t) = A_2y(t) + g(t, x(t), y(t)), & t \in [0, a] \\ x(0) = x_0, \quad y(0) = y_0, \end{cases} \tag{4.1.4}$$

It is clear that the mild solutions of (4.1.4) are the fixed point of the following operator $N : C([0, a], E) \times C([0, a], E) \to C([0, a], E) \times C([0, a], E)$ defined by

$$N(x, y) = (N_1(x, y), N_2(x, y)),$$

where,

$$N_1(x, y) = T_1(t)x_0 + \int_0^t T_1(t - s)f(s, x, y)ds, \quad (x, y) \in C([0, a], E) \times C([0, a], E)$$

and

$$N_2(x, y) = T_2(t)y_0 + \int_0^t T_2(t - s)g(s, x, y)ds, \quad (x, y) \in C([0, a], E) \times C([0, a], E).$$

Indeed, let $(x, y), (\bar{x}, \bar{y}) \in C([0, a], E) \times C([0, a], E)$,

$$\begin{aligned} |N_1(x, y) - N_1(\bar{x}, \bar{y})| &\leq \int_0^t |T(t - s)||f(s, x, y) - f(s, \bar{x}, \bar{y})|ds \\ &\leq M \int_0^t a_1|x - \bar{x}| + b_1|y - \bar{y}|ds \\ &\leq \tfrac{Ma_1}{\tau} \int_0^t \tau e^{s\tau}ds\|x - \bar{x}\| + \tfrac{Mb_1}{\tau} \int_0^t \tau e^{s\tau}ds\|y - \bar{y}\|. \end{aligned}$$

Thus,

$$\|N_1(x, y) - N_1(\bar{x}, \bar{y})\|_1 \leq \tfrac{1}{\tau}\|x - \bar{x}\|_1 + \tfrac{1}{\tau}\|y - \bar{y}\|_1,$$

where,

$$\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_1 = \sup_{t\in[0,a]} e^{-\tau t} \left\| \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \right\|.$$

Similarly,

$$|N_2(x, y) - N_2(\bar{x}, \bar{y})| \leq \tfrac{Ma_2}{\tau} \int_0^t \tau e^{s\tau}ds\|x - \bar{x}\| + \tfrac{Mb_2}{\tau} \int_0^t \tau e^{s\tau}ds\|y - \bar{y}\|.$$

Then,

$$\|N_2(x, y) - N_2(\bar{x}, \bar{y})\|_1 \leq \tfrac{1}{\tau}\|x - \bar{x}\|_1 + \tfrac{1}{\tau}\|y - \bar{y}\|_1.$$

Hence,

$$\|N(x,y)-N(\bar{x},\bar{y})\|_1 = \begin{pmatrix} \|N_1(x,y)-N_1(\bar{x},\bar{y})\|_1 \\ \|N_2(x,y)-N_2(\bar{x},\bar{y})\|_1 \end{pmatrix}$$

$$\leq \tfrac{1}{\tau}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \|x-\bar{x}\|_1 \\ \|y-\bar{y}\|_1 \end{pmatrix}.$$

Therefore,

$$\|N(x,y)-N(\bar{x},\bar{y})\|_1 \leq \bar{M}\begin{pmatrix} \|x-\bar{x}\|_1 \\ \|y-\bar{y}\|_1 \end{pmatrix},$$

where

$$\bar{M}=\frac{1}{\tau}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \tau > 2.$$

From Perov fixed point theorem 1.2.3, the mapping N has a unique fixed $(x,y) \in C([0,a],E)\times C([0,a],E)$ which is a unique solution of problem (4.1.4), we denote this solution by (x_1,y_1).

If $H(x_1(t),y_1(t)) \neq 1$ for all $t \in [0,a]$, hence (x_1,y_1) is a mild solution of (4.0.1) on $[0,a]$.

If $H((x_1(t),y_1(t)) \neq 1$ for all $t \in [0,a)$ and $H((x_1(a),y_1(a)) = 1$ then (x_1,y_1) is a 1ψ-mild solution of (4.0.1) on $[0,a]$.

Step 2 Otherwise $0 < t^1 = \inf\{s \in [0,a] : H(x(s),y(s)) = 1\} < a$ and $H(x(s),y(s)) = 1$. In this case, we consider the map $\theta : C([t^1,a],E)\times C([t^1,a],E) \to C([t^1,a],E)\times C([t^1,a],E)$ defined by

$$\theta(v_1,v_2) = (\theta_1(v_1,v_2),\theta_2(v_1,v_2)),$$

where,

$$\theta_1(v_1,v_2) = I(t^1,t,e,p,v_1,v_2), \quad (v_1,v_2) \in C([t^1,a],E)\times C([t^1,a],E).$$

and

$$\theta_2(v_1,v_2) = \bar{I}(t^1,t,e,p,v_1,v_2), \quad (v_1,v_2) \in C([t^1,a],E)\times C([t^1,a],E).$$

Indeed, let $(v_1,v_2),(\bar{v_1},\bar{v_2}) \in C([t^1,a],E)\times C([t^1,a],E)$. From assumption H_3, it is easy to see that $\theta(\cdot)$ is a contraction and has a fixed point $(v_1,v_2) \in C([t^1,a],X)\times C([t^1,a],X)$.

$$\begin{aligned} |\theta_1(v_1,v_2)-\theta_1(\bar{v_1},\bar{v_2})| &= |I(t,s,e,p,v_1,v_2)-I(t,s,e,p,\bar{v_1},\bar{v_2})| \\ &\leq q_1|v_1-\bar{v_1}| + q_2|v_2-\bar{v_2}| \\ e^{t\tau}e^{-t\tau}|\theta_1(v_1,v_2)-\theta_1(\bar{v_1},\bar{v_2})| &\leq q_1e^{t\tau}e^{-t\tau}|v_1-\bar{v_1}| + q_2e^{t\tau}e^{-t\tau}|v_2-\bar{v_2}|. \end{aligned}$$

Then,

$$\|\theta_1(v_1,v_2)-\theta_1(\bar{v_1},\bar{v_2})\|_2 \leq q_1\|v_1-\bar{v_1}\|_2 + q_2\|v_2-\bar{v_2}\|_2,$$

where,

$$\left\|\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right\|_2 = \sup_{t\in[t_1,a]} e^{\tau t}e^{-\tau t}\left\|\begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}\right\|.$$

Similarly,

$$\|\theta_2(v_1,v_2)-\theta_2(\bar{v_1},\bar{v_2})\|_2 \leq q_3\|v_1-\bar{v_1}\|_2 + q_4\|v_2-\bar{v_2}\|_2.$$

Hence,

$$\|\theta(v_1,v_2)-\theta(\bar{v_1},\bar{v_2})\|_2 = \begin{pmatrix} \|\theta_1(v_1,v_2)-\theta_1(\bar{v_1},\bar{v_2})\|_2 \\ \|\theta_2(v_1,v_2)-\theta_2(\bar{v_1},\bar{v_2})\|_2 \end{pmatrix} \leq \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix}\begin{pmatrix} \|v_1-\bar{v_1}\|_2 \\ \|v_2-\bar{v_2}\|_2 \end{pmatrix}.$$

Therefore,

$$\|\theta(v_1,v_2)-\theta(\bar{v_1},\bar{v_2})\|_2 \leq K\begin{pmatrix} \|v_1-\bar{v_1}\|_2 \\ \|v_2-\bar{v_2}\|_2 \end{pmatrix}.$$

we define now the function $u^1 : [0,a]\times[0,a] \to E$ given by $u^1(t) = (x_1(t), y_1(t))$ for $t \in [0,t^1]$ and $u^1(t) = (v_1(t), v_2(t))$ for $t \in [t^1,a]$. It obvious that $u^1_{|_{[0,t^1]}}$ is a mild solution of (4.1.4) on $[0,t^1]$ and

$$u^1(s) = \begin{cases} I(t^1,s,x_1(t^1),y_1(t^1),v_1(s),v_2(s)), & \text{for all } s\in[t^1,a] \\ \bar{I}(t^1,s,x_1(t^1),y_1(t^1),v_1(s),v_2(s)), \end{cases}$$

if $H(u^1(s)) \neq 0$ for all $s \in [t^1,a]$, then $u^1(.)$ is the unique $1(I\bar{I})$-mild solution of (4.0.1) on $[0,a]$. if $H(u^1(s)) \neq 0$ for all $s \in [t^1,a]$ and $H(u^1(a)) = 0$, then $u^1(.)$ is the unique 1-mild solution of (4.0.1) on $[0,a]$.

Step 3 We assume now that $t^1 < s^1 = \inf\{s \in [t^1,a] : H(x(s),y(s)) = 0\} < a$. obviously, $x(s^1) = 0, y(s^1) = 0$. Similar to the above, we know that there exist a unique mild solution $(\sigma_1,\sigma_2) \in C([s^1,a],X)\times C([s^1,a],X)$ of

$$\begin{cases} \sigma_1'(t) = A_1\sigma_1(t) + f(t,\sigma_1(t),\sigma_2(t)), \quad t\in[s_{(x,y)},a] \\ \sigma_2'(t) = A_2\sigma_2(t) + g(t,\sigma_1(t),\sigma_2(t)) \\ \sigma_1(s^1) = I(t^1,s^1,x(t^1),y(t^1),x(s^1),y(s^1)) \\ \sigma_2(s^1) = \bar{I}(t^1,s^1,x(t^1),y(t^1),x(s^1),y(s^1)). \end{cases} \tag{4.1.5}$$

It is clear that the mild solutions of (4.1.5) are the fixed point of the following operator $\Theta : C([s^1,a],E)\times C([s^1,a],E) \to C([s^1,a],E)\times C([s^1,a],E)$ defined by

$$\Theta(\sigma_1,\sigma_2) = (\Theta_1(\sigma_1,\sigma_2), \Theta_2(\sigma_1,\sigma_2)),$$

where,

$$\begin{aligned}\Theta_1(\sigma_1(t),\sigma_2(t)) = & \int_{s^1}^t T(t-s)f(s,\sigma_1(s),\sigma_2(s))ds \\ & +T(t-s^1)\sigma_1(s^1), \quad (\sigma_1,\sigma_2)\in C([s^1,a],E)\times C([s^1,a],E).\end{aligned}$$

and

$$\begin{array}{rcl}\Theta_2(\sigma_1(t),\sigma_2(t)) & = & \int_{s^1}^{t} T(t-s)g(s,\sigma_1(s),\sigma_2(s))ds \\ & & +T(t-s^1)\sigma_2(s^1), \quad (\sigma_1,\sigma_2)\in C([s^1,a],E)\times C([s^1,a],E).\end{array}$$

Indeed, let $(\sigma_1,\sigma_2),(\bar{\sigma_1},\bar{\sigma_2})\in C([s^1,a],E)\times C([s^1,a],E)\to C([s^1,a],E)\times C([s^1,a],E)$,

$$\begin{array}{rcl}|\Theta_1(\sigma_1,\sigma_2)-\Theta_1(\bar{\sigma_1},\bar{\sigma_2})| & \leq & |T(t_s^1)|I(t^1,s^1,e,p,x,y)-I(t^1,s^1,e,p,\bar{x},\bar{y})| \\ & & +\int_{s^1}^{t}|T(t-s)||f(s,\sigma_1(s),\sigma_2(s))-f(s,\bar{\sigma_1},\bar{\sigma_2})|ds \\ & \leq & M\int_{s^1}^{t} L_1|\sigma_1-\bar{\sigma_1}|+L_2|\sigma_2-\bar{\sigma_2}|ds \\ & \leq & \frac{ML_1}{\tau}\int_{s^1}^{t}\tau e^{s\tau}ds\|\sigma_1-\bar{\sigma_1}\|+\frac{ML_2}{\tau}\int_{s_1}^{t}\tau e^{s\tau}ds\|\sigma_2-\bar{\sigma_2}\| \\ & \leq & \frac{ML_1}{\tau}(e^{t\tau}-e^{s_1\tau})\|\sigma_1-\bar{\sigma_1}\|+\frac{ML_2}{\tau}(e^{t\tau}-e^{s_1\tau}))\|\sigma_2-\bar{\sigma_2}\| \\ & \leq & (\frac{ML_1}{\tau})e^{t\tau}\|\sigma_1-\bar{\sigma_1}\|+(\frac{ML_2}{\tau})e^{t\tau}\|\sigma_2-\bar{\sigma_2}\| \\ & \leq & e^{t\tau}(\frac{1}{\tau}\|\sigma_1-\bar{\sigma_1}\|+\frac{1}{\tau}\|\sigma_2-\bar{\sigma_2}\|),\end{array}$$

then

$$\|\Theta_1(\sigma_1,\sigma_2)-\Theta_1(\bar{\sigma_1},\bar{\sigma_2})\|_2\leq\frac{1}{\tau}\|\sigma_1-\bar{\sigma_1}\|_2+\frac{1}{\tau}\|\sigma_2-\bar{\sigma_2}\|_2,$$

where,

$$\left\|\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix}\right\|_2=\sup_{t\in[s^1,a]} e^{-\tau t}\left\|\begin{pmatrix}\sigma_1(t)\\ \sigma_2(t)\end{pmatrix}\right\|.$$

Similarly,

$$\begin{array}{rcl}|\Theta_2(\sigma_1,\sigma_2)-\Theta_2(\bar{\sigma_1},\bar{\sigma_2})| & \leq & |T(t_s^1)|\bar{I}(t^1,s^1,e,p,x,y)-\bar{I}(t^1,s^1,e,p,\bar{x},\bar{y})| \\ & & +\int_{s^1}^{t}|T(t-s)||g(s,\sigma_1(s),\sigma_2(s))-g(s,\bar{\sigma_1},\bar{\sigma_2})|ds \\ & \leq & M\int_{s^1}^{t} L_3|\sigma_1-\bar{\sigma_1}|+L_4|\sigma_2-\bar{\sigma_2}|ds \\ & \leq & \frac{ML_3}{\tau}\int_{s^1}^{t}\tau e^{s\tau}ds\|\sigma_1-\bar{\sigma_1}\|+\frac{ML_4}{\tau}\int_{s_1}^{t}\tau e^{s\tau}ds\|\sigma_2-\bar{\sigma_2}\| \\ & \leq & \frac{ML_3}{\tau}(e^{t\tau}-e^{s_1\tau})\|\sigma_1-\bar{\sigma_1}\|+\frac{ML_4}{\tau}(e^{t\tau}-e^{s_1\tau}))\|\sigma_2-\bar{\sigma_2}\| \\ & \leq & e^{t\tau}(\frac{1}{\tau}\|\sigma_1-\bar{\sigma_1}\|+\frac{1}{\tau}\|\sigma_2-\bar{\sigma_2}\|).\end{array}$$

Therefore,

$$\|\Theta_2(\sigma_1,\sigma_2)-\Theta_2(\bar{\sigma_1},\bar{\sigma_2})\|_2\leq\frac{1}{\tau}\|\sigma_1-\bar{\sigma_1}\|_2+\frac{1}{\tau}\|\sigma_2-\bar{\sigma_2}\|_2.$$

Hence,

$$\|\Theta(\sigma_1,\sigma_2)-\Theta(\bar{\sigma_1},\bar{\sigma_2})\|_2 = \begin{pmatrix} \|\Theta_1(\sigma_1,\sigma_2)-\Theta_1(\bar{\sigma_1},\bar{\sigma_2})\|_2 \\ \|\Theta_2(\sigma_1,\sigma_2)-\Theta_2(\bar{\sigma_1},\bar{\sigma_2})\|_2 \end{pmatrix}$$

$$\leq \begin{pmatrix} \frac{1}{\tau} & \frac{1}{\tau} \\ \\ \frac{1}{\tau} & \frac{1}{\tau} \end{pmatrix} \begin{pmatrix} \|\sigma_1-\bar{\sigma_1}\|_2 \\ \|\sigma_2-\bar{\sigma_2}\|_2 \end{pmatrix}.$$

Therefore,

$$\|\Theta(\sigma_1,\sigma_2)-\Theta(\bar{\sigma_1},\bar{\sigma_2})\|_2 \leq \frac{1}{\tau}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \|\sigma_1-\bar{\sigma_1}\|_2 \\ \|\sigma_2-\bar{\sigma_2}\|_2 \end{pmatrix}.$$

From Perov fixed point theorem 1.2.3, the mapping Θ has a unique fixed $(\sigma_1,\sigma_2) \in C([s_1,a],E)\times C([s_1,a],E)$ which is a unique solution of problem (4.1.5), we denote this solution by (x_3,y_3). and we define $(x_3,y_3):[0,a]\times[0,a]\to E$ by

$$(x_3,y_3) = \begin{cases} (x_2,y_2), & \text{for } t\in[0,s^1_{(x,y)}] \\ \\ (\sigma_1,\sigma_2), & \text{for } t\in[s^1,a] \end{cases}$$

if $H(x_3(s),y_3(s))<1$ for all $s_\in[s^1,a]$ then $(x_3(.),y_3(.))$ is the unique 1-mild solution on $[0,a]$.
if $H(x_3(s),y_3(s))<1$ for all $s_\in[s^1,a]$ and $H(x_3(a),y_3(a))=1$ then $(x_3(.),y_3(.))$ is the unique 2-mild solution on $[0,a]$.

Step 4 Otherwise, we have that $s^1<t^2=\inf\{s\in[s^1,a]:H(u(s))=1\}<a$ and arguing as in the first part of the proof, we have that there exists a unique fixed point $(\omega_1,\omega_2)\in C([t^2,a];E]\times C([t^2,a];E]$ of the map $\Lambda: C([t^2,a],E)\times C([t^2,a],E)\to C([t^2,a],E)\times C([t^2,a],E)$ defined by

$$\Lambda(\omega_1,\omega_2) = (\Lambda_1(\omega_1,\omega_2),\Lambda_2(\omega_1,\omega_2)),$$

where,

$$\Lambda_1(\omega_1,\omega_2) = I(t^1,t,e,p,\omega_1,\omega_2), \quad (\omega_1,\omega_2)\in C([t^2,a],E)\times C([t^2,a],E)$$

and

$$\Lambda_2(\omega_1,\omega_2) = \bar{I}(t^1,t,e,p,\omega_1,\omega_2), \quad (\omega_1,\omega_2)\in C([t^2,a],E)\times C([t^2,a],E).$$

We continue this process for the following problem

$$(x(t), y(t)) = \begin{cases} (x_1(t), y_1(t)) \text{ for all } t \in [0, t^1] \\ (x_2(t), y_2(t)) \text{ for all } t \in [t^1, s^1) \\ (x_3(t), y_3(t)) \text{ for all } t \in [s^1, t^2) \\ \vdots \\ (x_m(t), y_m(t)) \text{ for all } t \in [t^m, s^m). \end{cases}$$

□

Chapter 5

Difference equations

Since the publication of the landmark paper of Hertman [55] in the year 1978 difference equations has become a major field of research. In fact during this period several books, e,g Agarwal [2] Agarwal and Wang [4], Ahlbrant and Peterson [12] , Elydi[43], Kelley and Peterson[63], Kocic and Ladas[65], Lakshmikantham and Trigiante[70], Mickens Sharkovsky, Mdistrenko and Romanenko[98], and hundreds of research articles on the theory, methods and applications of difference equations have been written. In this chapter we will follow Agarwal and O'Regan [3, 5, 6, 7, 8, 9] and establish the existence of solutions following boundary value problem.

$$\begin{cases} \Delta^3 x(k) - f(k,x,\Delta x,y,\Delta y) = 0, & k \in \mathbb{N}(0,b) \\ \Delta^3 y(k) - z(k,x,\Delta x,y,\Delta y) = 0, & k \in \mathbb{N}(0,b) \\ \alpha_0 \Delta x(0) - \beta_0 \Delta^2 x(0) = 0 & x(0) = 0 \\ \gamma_0 \Delta x(b+1) + \delta_0 \Delta^2 x(b+1) = 0, & \\ \bar{\alpha_0} \Delta y(0) - \bar{\beta_0} \Delta^2 y(0) = 0 & y(0) = 0 \\ \bar{\gamma_0} \Delta y(b+1) + \bar{\beta_0} \Delta^2 y(b+1) = 0, & \end{cases} \tag{5.0.1}$$

where $\beta_0, \delta_0, \bar{\beta_0}, \bar{\delta_0} \in \mathbb{R} \setminus \{0\}$, $\alpha_0, \gamma_0, \bar{\alpha_0}, \bar{\gamma} \in \mathbb{R}$, $f, z : \mathbb{N}(0,b) \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ are continuous functions.

Throughout this chapter, we shall use some of the following notations: $\mathbb{N} = 0, 1, \ldots$ the set of natural numbers including zero, $\mathbb{N}(a) = a, a+1, \ldots$ where $a \in \mathbb{N}, \mathbb{N}(a, b-1) = a, a+1, \ldots, b-l$ where $a < b-1 < \infty$ and $a, b \in \mathbb{N}$. Any one of these three sets will be denoted by $\bar{\mathbb{N}}$. the *forward* difference operators Δ is defined as $\Delta f(k) = f(k+1) - f(k)$. The higher order differences for a positive integer m are defined as $\Delta^m f(k) = \Delta[\Delta^{m-1} f(k)]$. Let I be the identity operator, i.e. $If(k) = f(k)$, then obviously $\Delta = E - I$ and for a positive integer m we may deduce the relation

$$E^m f(k) = (I + \Delta)^m f(k) = \sum_{i=0}^{m} \binom{m}{i} \Delta^i f(k), \quad \Delta^0 = I. \tag{5.0.2}$$

5.1 Existence and uniqueness result

In this section we will study the existence and uniqueness solutions.

Lemma 5.1.1. *A function $x \in C(\mathbb{N}(0,b),\mathbb{R}^m)$ is solution of problem*

$$\begin{cases} \Delta^3 x(k) = -h(k), & k \in \mathbb{N}(0,b) \\ \alpha_0 \Delta x(0) - \beta_0 \Delta^2 x(0) = 0 \\ \gamma_0 \Delta x(b+1) + \delta_0 \Delta^2 x(b+1) = 0, \end{cases} \tag{5.1.1}$$

where $h \in C(\mathbb{N}(0,b),\mathbb{R}^m)$, and $(\delta_0\alpha_0)^2-(\gamma_0\beta_0)^2-(b+1)\gamma_0^2\beta_0\alpha_0-(b+1)^2(\alpha_0\gamma_0)^2 \neq 0$ if and only if

$$x(k) = \sum_{l=0}^{b} g(k,i)h(i), \quad k \in \mathbb{N}(0,b),$$

where

$$g(k,i) = \begin{cases} \frac{(\beta_0^2+k\alpha_0\beta_0+k^2\alpha_0^2)(\delta_0^2-\gamma_0^2(b-i)^2)}{(\delta_0\alpha_0)^2-(\gamma_0\beta_0)^2-(b+1)\gamma_0^2\beta_0\alpha_0-(b+1)^2(\alpha_0\gamma_0)^2} - (k-i-1)^2 & i \in \{0,\cdots,k-1\} \\ \\ \frac{(\beta_0^2+k\alpha_0\beta_0+k^2\alpha_0^2)(\delta_0^2-\gamma_0^2(b-i)^2)}{(\delta_0\alpha_0)^2-(\gamma_0\beta_0)^2-(b+1)\gamma_0^2\beta_0\alpha_0-(b+1)^2(\alpha_0\gamma_0)^2} & i \in \{k,\cdots,b\}. \end{cases}$$

Proof. Let $x \in C(\mathbb{N}(0,b),\mathbb{R}^m)$ be a solution of problem (5.1.1), then

$$\Delta^3 x(k) = -h(k) \Rightarrow \Delta^2(k+1) - \Delta^2(k) = -h(k).$$

Then,

$$\begin{array}{lll} i=0, & \Delta^2 x(1) - \Delta^2 x(0) & = & -h(0) \\ i=1, & \Delta^2 x(2) - \Delta^2 x(1) & = & -h(1) \\ \vdots & & & \\ i=k-1, & \Delta^2 x(k) - \Delta^2 x(k-1) & = & -h(k-1). \end{array}$$

By summing the above equations, we get

$$\Delta^2 x(k) = \Delta^2 x(0) - \sum_{i=0}^{k-1} h(i). \tag{5.1.2}$$

Thus,

$$\begin{array}{llll} i=0, & \Delta x(1) - \Delta x(0) & = & \Delta^2 x(0) - 0 \\ i=1, & \Delta x(2) - \Delta x(1) & = & \Delta^2 x(0) - h(0) \\ \vdots & & & \\ i=k-1, & \Delta^2 x(k) - \Delta^2 x(k-1) & = & \Delta^2 x(0) - h(0) - h(1) \cdots - h(k-1). \end{array}$$

Hence,

$$\Delta x(k) = \Delta x(0) + k\Delta^2 x(0) - \sum_{i=0}^{k-1} (k-i-1)h(i). \tag{5.1.3}$$

Thus,

$$\begin{array}{lll} i=0, & x(1)-x(0) & = \Delta x(0)+k\Delta^2 x(0)-0 \\ i=1, & x(2)-x(1) & = \Delta x(0)+k\Delta^2 x(0)-(k-1)h(0) \\ \vdots & & \\ i=k-1, & x(k)-x(k-1) & = \Delta x(0)+k\Delta^2 x(0)-(k-1)h(0) \\ & & \quad -(k-2)h(1)-\ldots-h(k-2). \end{array}$$

Hence,

$$x(k)=x(0)+k\Delta x(0)+k^2\Delta^2 x(0)-\sum_{i=0}^{k-1}(k-i-1)^2 h(i). \tag{5.1.4}$$

From (5.1.2) and (5.1.4), we have

$$x(b+1)=x(0)+(b+1)\Delta x(0)+(b+1)^2\Delta^2 x(0)-\sum_{i=0}^{b}(b-i)^2 h(i). \tag{5.1.5}$$

and

$$\Delta^2 x(b+1)=\Delta^2 x(0)-\sum_{i=0}^{b}h(i). \tag{5.1.6}$$

Since

$$\alpha_0\Delta x(0)-\beta_0\Delta^2 x(0)=0 \quad ; \quad \gamma_0\Delta x(b+1)+\delta_0\Delta^2 x(b+1)=0$$

$$\alpha_0 x(0)-\beta_0\Delta x(0)=0 \quad ; \quad \gamma_0 x(b+1)+\delta_0\Delta x(b+1)=0.$$

Then,

$$\Delta^2 x(0)=\left(\frac{\alpha_0}{\beta_0}\right)^2 x(0) \quad ; \quad \Delta^2 x(b+1)=\left(-\frac{\gamma_0}{\delta_0}\right)\Delta x(b+1)$$

$$\Delta x(0)=\left(\frac{\alpha_0}{\beta_0}\right) x(0) \quad ; \quad \Delta x(b+1)=\left(-\frac{\gamma_0}{\delta_0}\right) x(b+1).$$

On the other hand, from (5.1.5) and (5.1.6), we have

$$\begin{aligned} x(0) \;=\; & \frac{(\beta_0\delta_0)^2}{(\delta_0\alpha_0)^2-(\gamma_0\beta_0)^2-(b+1)\gamma_0^2\beta_0\alpha_0-(b+1)^2(\alpha_0\gamma_0)^2}\textstyle\sum_{i=0}^{b}h(i) \\ & -\frac{(\beta_0\gamma_0)^2}{(\delta_0\alpha_0)^2-(\gamma_0\beta_0)^2-(b+1)\gamma_0^2\beta_0\alpha_0-(b+1)^2(\alpha_0\gamma_0)^2}\textstyle\sum_{i=0}^{b}(b-i)^2 h(i) \end{aligned}$$

and

$$x(k)=x(0)\left[\frac{\beta_0^2+k\alpha_0\beta_0+k^2\alpha_0^2}{\beta_0^2}\right]-\sum_{i=0}^{k-1}(k-i-1)^2 h(i).$$

By above relation, we get

$$\begin{aligned} x(k) &= \frac{\delta_0^2(\beta_0^2+k\alpha_0\beta_0+k^2\alpha_0^2)}{(\delta_0\alpha_0)^2-(\gamma_0\beta_0)^2-(b+1)\gamma_0^2\beta_0\alpha_0-(b+1)^2(\alpha_0\gamma_0)^2}\sum_{i=0}^{b}h(i) \\ &\quad -\frac{\gamma_0^2(\beta_0^2+k\alpha_0\beta_0+k^2\alpha_0^2)}{(\delta_0\alpha_0)^2-(\gamma_0\beta_0)^2-(b+1)\gamma_0^2\beta_0\alpha_0-(b+1)^2(\alpha_0\gamma_0)^2}\sum_{i=0}^{b}(b-i)^2h(i) \\ &\quad -\sum_{i=0}^{k-1}(k-i-1)^2h(i) \\ &= \frac{\beta_0^2+k\alpha_0\beta_0+k^2\alpha_0^2}{(\delta_0\alpha_0)^2-(\gamma_0\beta_0)^2-(b+1)\gamma_0^2\beta_0\alpha_0-(b+1)^2(\alpha_0\gamma_0)^2}\sum_{i=k}^{b}(\delta_0^2-\gamma_0^2(b-i)^2)h(i) \\ &\quad +\frac{\beta_0^2+k\alpha_0\beta_0+k^2\alpha_0^2}{(\delta_0\alpha_0)^2-(\gamma_0\beta_0)^2-(b+1)\gamma_0^2\beta_0\alpha_0-(b+1)^2(\alpha_0\gamma_0)^2}\sum_{i=0}^{k-1}(\delta_0^2-\gamma_0^2(b-i)^2)h(i) \\ &\quad -\sum_{i=0}^{k-1}(k-i-1)^2h(i). \end{aligned}$$

This implies that

$$x(k)=\sum_{l=0}^{b}g(k,i)h(i), \quad k\in\mathbb{N}(0,b).$$

□

Let us introduce the following hypothesis:

(H_1) the exist nonnegative numbers $a_i,\bar{a}_i$ and $b_i,\bar{b}_i$ for each $i=1,2$

$$\begin{aligned}\|f_1(k,x_1,x_2,y_1,y_2)-f_1(k,\bar{x}_1,\bar{x}_2,\bar{y}_1,\bar{y}_2)\| &\le a_1\|x_1-\bar{x}_1\|+a_2\|x_2-\bar{x}_2\| \\ &\quad +b_1\|y_1-\bar{y}_1\|+b_2\|y_2-\bar{y}_2\|\end{aligned}$$

and

$$\begin{aligned}\|f_2(k,x_1,x_2,y_1,y_2)-f_2(k,\bar{x}_1,\bar{x}_2,\bar{y}_1,\bar{y}_2)\| &\le \bar{a}_1\|x_1-\bar{x}_1\|+\bar{a}_2\|x_2-\bar{x}_2\|+\bar{b}_1\|y_1-\bar{y}_1\| \\ &\quad +\bar{b}_2\|y_2-\bar{y}_2\|\end{aligned}$$

for all $x_1,x_2,y_1,y_2,\bar{x}_1,\bar{x}_2,\bar{y}_1,\bar{y}_2\in\mathbb{R}^m$.

Theorem 5.1.2. *Assume that (H_1) is satisfied and the matrix*

$$M=2(b+1)g_*\begin{pmatrix}(a_1+a_2) & (b_1+b_2)\\ (\bar{a}_1+\bar{a}_2) & (\bar{b}_1+\bar{b}_2)\end{pmatrix},$$

where,

$$g_*=\sup\{|g(i,j)|:(i,j)\in\mathbb{N}(0,b)\times\mathbb{N}(0,b)\}.$$

Hence,

$$g_*=\begin{cases}\frac{\|\beta_0^2+b\alpha_0\beta_0+b^2\alpha_0^2\|\|\delta_0^2-\gamma_0^2(b)^2\|}{\|(\delta_0\alpha_0)^2+(\gamma_0\beta_0)^2-(b+1)\gamma_0^2\beta_0\alpha_0-(b+1)^2(\alpha_0\gamma_0)^2\|}+\|(b-1)^2\|, & i\in\{0,\cdots,k-1\}\\[2ex] \frac{\|\beta_0^2+b\alpha_0\beta_0+b^2\alpha_0^2\|\|\delta_0^2-\gamma_0^2(b)^2\|}{\|(\delta_0\alpha_0)^2-(\gamma_0\beta_0)^2-(b+1)\gamma_0^2\beta_0\alpha_0-(b+1)^2(\alpha_0\gamma_0)^2\|}, & i\in\{k,\cdots,b\}.\end{cases}$$

If M converges to zero, then the problem (5.1.1) has unique solution.

Proof. Consider the operator $N : C(\mathbb{N}(0,b),\mathbb{R}^m)\times C(\mathbb{N}(0,b),\mathbb{R}^m)\to C(\mathbb{N}(0,b),\mathbb{R}^m)$ defined for $(x,y)\in C(\mathbb{N}(0,b),\mathbb{R}^m)\times C(\mathbb{N}(0,b),\mathbb{R}^m)$ by

$$N(x,y)=(N_1(x,y),N_2(x,y)),$$

where,

$$\begin{aligned} N_1(x(k),y(k)) &= \textstyle\sum_{l=0}^{b} g(k,l)f_1(k,x,\Delta x,y,\Delta y). \\ N_2(x(k),y(k)) &= \textstyle\sum_{l=0}^{b} g(k,l)f_2(k,x,\Delta x,y,\Delta y). \end{aligned} \tag{5.1.7}$$

Let $x,\bar{x},y,\bar{y}\in C(\mathbb{N}(0,b),\mathbb{R}^m)$, then

$$|N_1(x,y)-N_1(\bar{x},\bar{y})| \leq \textstyle\sum_{l=0}^{b}|g(k,l)||f_1(l,x,\Delta x,y,\Delta y)-f_1(l,\bar{x},\Delta\bar{x},\bar{y},\Delta\bar{y})|.$$

and

$$|N_2(x,y)-N_2(\bar{x},\bar{y})| \leq \textstyle\sum_{l=0}^{b}|g(k,l)||f_2(l,x,\Delta x,y,\Delta y)-f_2(l,\bar{x},\Delta\bar{x},\bar{y},\Delta\bar{y})|.$$

Thus,

$$\begin{aligned} \|N_1(x,y)-N_1(\bar{x},\bar{y})\|_\infty &\leq (b+1)\sup_{(k,l)\in\mathbb{N}(0,b)\times\mathbb{N}(0,b)}|g(k,l)|[a_1\|x-\bar{x}\|_\infty \\ &\quad +a_2\|\Delta x-\Delta\bar{x}\|_\infty+b_1\|y-\bar{y}\|_\infty+b_2\|\Delta y-\Delta\bar{y}\|_\infty] \\ &\leq (b+1)g_*[(a_1+a_2)\|x-\bar{x}\|_*+(b_1+b_2)\|y-\bar{y}\|_*]. \end{aligned}$$

Similarly

$$\begin{aligned} \|N_2(x,y)-N_2(\bar{x},\bar{y})\|_\infty &\leq (b+1)\sup_{(k,l)\in\mathbb{N}(0,b)\times\mathbb{N}(0,b)}|g(k,l)|[\bar{a_1}\|x-\bar{x}\|_\infty \\ &\quad +\bar{a_2}\|\Delta x-\Delta\bar{x}\|_\infty+\bar{b_1}\|y-\bar{y}\|_\infty+\bar{b_2}\|\Delta y-\Delta\bar{y}\|_\infty] \\ &\leq (b+1)g_*[(\bar{a_1}+\bar{a_2})\|x-\bar{x}\|_*+(\bar{b_1}+\bar{b_2})\|y-\bar{y}\|_*]. \end{aligned}$$

$$\Delta N(x,y)=(\Delta N_1(x,y),\Delta N_2(x,y)),$$

where,

$$\Delta N_1(x,y)=N_1(x(k+1),y(k+1))-N_1(x(k),y(k)).$$

$$\Delta N_2(x,y)=N_2(x(k+1),y(k+1))-N_2(x(k),y(k)).$$

Then,

$$\begin{aligned} |\Delta N_1(x,y)-\Delta N_1(\bar{x},\bar{y})| &\leq |N_1(x(k+1),y(k+1))-N_1(\bar{x}(k+1),\bar{y}(k+1))| \\ &\quad +|N_1(x,y)-N_1(\bar{x},\bar{y})| \\ &\leq \textstyle\sum_{l=0}^{b}|g(k,l+1)||f_1(l+1,x,\Delta x,y,\Delta y)-f_1(l+1,\bar{x},\Delta\bar{x},\bar{y},\Delta\bar{y})| \\ &\quad +\textstyle\sum_{l=0}^{b}|g(k,l)||f_1(l,x,\Delta x,y,\Delta y)-f_1(k,\bar{x},\Delta\bar{x},\bar{y},\Delta\bar{y})|. \end{aligned}$$

Similarly we have

$$\begin{aligned}|\Delta N_2(x,y)-\Delta N_2(\bar{x},\bar{y})| &\le \textstyle\sum_{l=0}^{b}|g(k,l+1)||f_2(l+1,x,\Delta x,y,\Delta y)-f_2(l+1,\bar{x},\Delta\bar{x},\bar{y},\Delta\bar{y})|\\ &\quad+\textstyle\sum_{l=0}^{b}|g(k,l)||f_2(l,x,\Delta x,y,\Delta y)-f_2(k,\bar{x},\Delta\bar{x},\bar{y},\Delta\bar{y})|.\end{aligned}$$

Thus,

$$\|\Delta N_1(x,y)-\Delta N_1(\bar{x},\bar{y})\|_\infty \le (b+1)g_*[(a_1+a_2)\|x-\bar{x}\|_*+(b_1+b_2)\|y-\bar{y}\|_*].$$

Similarly,

$$\|\Delta N_2(x,y)-\Delta N_2(\bar{x},\bar{y})\|_\infty \le (b+1)g_*[(\bar{a}_1+\bar{a}_2)\|x-\bar{x}\|_*+(\bar{b}_1+\bar{b}_2)\|y-\bar{y}\|_*].$$

Hence,

$$\begin{aligned}\|\Delta N_1(x,y)-\Delta N_1(\bar{x},\bar{y})\|_\infty+\|N_1(x,y)-N_1(\bar{x},\bar{y})\|_\infty &\le 2(b+1)g_*[(a_1+a_2)\|x-\bar{x}\|_*\\ &\quad+(b_1+b_2)\|y-\bar{y}\|_*]\\ \|N_1(x,y)-N_1(\bar{x},\bar{y})\|_* &\le 2(b+1)g_*[(a_1+a_2)\|x-\bar{x}\|_*\\ &\quad+(b_1+b_2)\|y-\bar{y}\|_*].\end{aligned}$$

Similarly we have

$$\begin{aligned}\|\Delta N_2(x,y)-\Delta N_2(\bar{x},\bar{y})\|_\infty+\|N_2(x,y)-N_2(\bar{x},\bar{y})\|_\infty &\le 2(b+1)g_*[(\bar{a}_1+\bar{a}_2)\|x-\bar{x}\|_*\\ &\quad+(\bar{b}_1+\bar{b}_2)\|y-\bar{y}\|_*]\\ \|N_2(x,y)-N_2(\bar{x},\bar{y})\|_* &\le 2(b+1)g_*[(\bar{a}_1+\bar{a}_2)\|x-\bar{x}\|_*\\ &\quad+(\bar{b}_1,\bar{b}_2)\|y-\bar{y}\|_*].\end{aligned}$$

Then,

$$\begin{pmatrix}\|N_1(x,y)-N_1(\bar{x},\bar{y}\|_*\\ \|N_2(x,y)-N_2(\bar{x},\bar{y}\|_*\end{pmatrix} \le 2(b+1)g_*\begin{pmatrix}a_1+a_2 & b_1+b_2\\ \bar{a}_1+\bar{a}_2 & \bar{b}_1+\bar{b}_2\end{pmatrix}\begin{pmatrix}\|x-\bar{x}\|_*\\ \|y-\bar{y}\|_*\end{pmatrix}.$$

For all, $(x,y),(\bar{x},\bar{y})\in C(\mathbb{N}(0,b),\mathbb{R}^n)\times C(\mathbb{N}(0,b),\mathbb{R}^n)$.

From Perov fixed point theorem, the mapping N has a unique fixed $(x,y)\in C(\mathbb{N}(0,b),\mathbb{R}^n)\times C(\mathbb{N}(0,b),\mathbb{R}^n)$ which is unique solution of problem (5.1.1). □

Theorem 5.1.3. *Let Ω be a closed subset of $C(\mathbb{N}(0,b),E)$ .If Ω is uniformly bounded and the set $\{y(k) : y\in\Omega\}$ is relatively compact for each $k\in\mathbb{N}(0,b)$, then Ω is compact.*

Theorem 5.1.4. *Let $f,p:\mathbb{N}(0,b)\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ are continuous functions. Assume that condition*

(H_2) There exist $d_1, d_2 \in C(\mathbb{N}(0,b),\mathbb{R}_+)$ and α , $\beta \in (0,1)$ such that

$$|f(k,x_1,x_2,y_1,y_2) \leq d_1(k)(|x_1|+|x_1|+|y_1|+|y_2|)^{\alpha},$$

$$k \in \mathbb{N}(0,b), \;\; (x_1,x_2,y_1,y_2) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m,$$

and

$$|p(k,x_1,x_2,y_1,y_2) \leq d_2(k)(|x_1|+|x_1|+|y_1|+|y_2|)^{\beta},$$

$$k \in \mathbb{N}(0,b), \;\; (x_1,x_2,y_1,y_2) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m,$$

holds. Then the problem (5.1.1) *has at least one solution.*

Proof. Clearly, the fixed point of N are solutions to (5.1.1), where N is defined in (5.1.7). In order to apply theorem (5.1.3), we first show that N is completely continuous.

The proof will be given in several steps.

Step 1. $N = (N_1, N_2)$ is continuous.

let (x_m, y_m) be a sequence such that $(x_m, y_m) \to (x,y) \in C(\mathbb{N}(0,b),\mathbb{R}_+) \times C(\mathbb{N}(0,b),\mathbb{R}_+)$ as $m \to \infty$. Then

$$\begin{aligned} |N_1(x_m(k),y_m(k)) - N_1(x(k),y(k))| &= |\textstyle\sum_{l=0}^{b} g(k,l)f(l,x_m,\Delta x_m,y_m,\Delta y_m) \\ &\quad - \textstyle\sum_{l=0}^{b} g(k,l)f(l,x,\Delta x,y,\Delta y)| \\ &\leq g_* \textstyle\sum_{l=0}^{b} |f(l,x_m,\Delta x_m,y_m,\Delta y_m) \\ &\quad - f(l,x,\Delta x,y,\Delta y)|. \end{aligned}$$

Similarly,

$$\begin{aligned} |N_2(x_m(k),y_m(k)) - N_2(x(k),y(k))| &= |\textstyle\sum_{l=0}^{b} g(k,l)p(l,x_m,\Delta x_m,y_m,\Delta y_m) \\ &\quad - \textstyle\sum_{l=0}^{b} g(k,l)p(l,x,\Delta x,y,\Delta y)| \\ &\leq g_* \textstyle\sum_{l=0}^{b} |p(l,x_m,\Delta x_m,y_m,\Delta y_m) - p(l,x,\Delta x,y,\Delta y)| \end{aligned}$$

Since f, p are continuous functions, we get

$$\|N_1(x_m,y_m) - N_1(x,y)\|_\infty \leq g_* \textstyle\sum_{l=0}^{b} |f(k,x_m,\Delta x_m,y_m,\Delta y_m) - f(k,x,\Delta x,y,\Delta y)| \to 0 \quad \text{as } m \to \infty,$$

and

$$\|N_2(x_m,y_m) - N_2(x,y)\|_\infty \leq g_* \textstyle\sum_{l=0}^{b} |f(k,x_m,\Delta x_m,y_m,\Delta y_m) - f(k,x,\Delta x,y,\Delta y)| \to 0 \quad \text{as } m \to \infty.$$

In other side

$$\Delta N = (\Delta N_1, \Delta N_2),$$

then

$$\begin{aligned}|\Delta N_1(x_m(k),y_m(k))-\Delta N_1(x(k),y(k))| &= |(N_1(x_m(k+1),y_m(k+1))-N_1(x_m(k),y_m(k)))\\ &- (N_1(x(k+1),y(k+1))-N_1(x(k),y(k)))|\\ &\leq\\ &\quad |N_1(x_m(k+1),y_m(k+1))-N_1(x(k+1),y(k+1))|\\ &+ |N_1(x_m(k),y_m(k))-N_1(x(k),y(k))|.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\Delta N_1(x_m,y_m)-\Delta N_1(x,y)\|_\infty \leq\ & g_*[\textstyle\sum_{l=0}^{b}|f(l+1,x_m(l+1),\Delta x_m(l+1),y_m(l+1),\\ &\Delta y_m(l+1))\\ &-f(l+1,x(l+1),\Delta x(l+1),y(l+1),\Delta y(l+1))|\\ &+\textstyle\sum_{l=0}^{b}|f(l,x_m(l),\Delta x_m(l),y_m(l),\Delta y_m(l))\\ &-f(l,x(l),\Delta x(l),y(l),\Delta y(l))|\,]\longrightarrow 0\\ &\text{ as } m\longrightarrow\infty.\end{aligned}$$

Similarly,

$$\begin{aligned}\|\Delta N_2(x_m,y_m)-\Delta N_2(x,y)\|_\infty \leq\ & g_*[\textstyle\sum_{l=0}^{b}|p(l+1,x_m(l+1),\Delta x_m(l+1),y_m(l+1),\\ &\Delta y_m(l+1))\\ &-p(l+1,x(l+1),\Delta x(l+1),y(l+1),\Delta y(l+1))|\\ &+\textstyle\sum_{l=0}^{b}|p(l,x_m(l),\Delta x_m(l),y_m(l),\Delta y_m(l))\\ &-p(l,x(l),\Delta x(l),y(l),\Delta y(l))|\,]\longrightarrow 0\\ &\text{ as } m\longrightarrow\infty.\end{aligned}$$

Hence

$$\begin{aligned}\|\Delta N_1(x_m,y_m)-\Delta N_1(x,y)\|_\infty &+ \|N_1(x_m,y_m)-N_1(x,y)\|_\infty\\ &\leq g_*[\textstyle\sum_{l=0}^{b}|f(l+1,x_m(l+1),\Delta x_m(l+1),y_m(l+1),\\ &\quad\Delta y_m(l+1))\\ &\quad-f(l+1,x(l+1),\Delta x(l+1),y(l+1),\Delta y(l+1))|\\ &\quad+2\textstyle\sum_{l=0}^{b}|f(l,x_m(l),\Delta x_m(l),y_m(l),\Delta y_m(l))\\ &\quad-f(l,x(l),\Delta x(l),y(l),\Delta y(l))|\,].\longrightarrow 0\\ &\quad\text{ as } m\longrightarrow\infty.\end{aligned}$$

Similarly

$$\begin{aligned}\|\Delta N_2(x_m,y_m)-\Delta N_2(x,y)\|_\infty &+ \|N_2(x_m,y_m)-N_2(x,y)\|_\infty\\ &\leq g_*[\textstyle\sum_{l=0}^{b}|p(l+1,x_m(l+1),\Delta x_m(l+1),y_m(l+1),\\ &\quad\Delta y_m(l+1))\\ &\quad-p(l+1,x(l+1),\Delta x(l+1),y(l+1),\Delta y(l+1))|\\ &\quad+2\textstyle\sum_{l=0}^{b}|p(l,x_m(l),\Delta x_m(l),y_m(l),\Delta y_m(l))\\ &\quad-p(l,x(l),\Delta x(l),y(l),\Delta y(l))|\,].\longrightarrow 0\\ &\quad\text{ as } m\longrightarrow\infty.\end{aligned}$$

Step 2. N maps bounded sets into bounded sets in $C(\mathbb{N}(0,b),\mathbb{R}^n)\times C(\mathbb{N}(0,b),\mathbb{R}^n)$. Indeed, it is enough to show that for any $q,\bar{q}\geq 0$ there exists a positive constants $l,\bar{l}$ such that for each

$$(x,y)\in B_q=\{(x,y)\in C(\mathbb{N}(0,b),\mathbb{R}^n)\times C(\mathbb{N}(0,b),\mathbb{R}^n):\|x\|_\infty\leq q,\|y\|_\infty\leq q\}$$

and

$$(x,y)\in B_{\bar{q}}=\{(x,y)\in C(\mathbb{N}(0,b),\mathbb{R}^n)\times C(\mathbb{N}(0,b),\mathbb{R}^n):\|\Delta x\|_\infty\leq \bar{q},\|\Delta y\|_\infty\leq \bar{q}\}$$

we have

$$\|N(x,y)\|_\infty\leq l=(l_1,l_2),$$
$$\|\Delta N(x,y)\|_\infty\leq \bar{l}=(\bar{l}_1,\bar{l}_2).$$

Then for each $k\in\mathbb{N}(0,b)$, we get

$$\begin{aligned}|N_1(x(k),y(k)| &= |\textstyle\sum_{l=0}^b g(k,l)f(l,x(l),\Delta x(l),y(l),\Delta y(l))|\\ &\leq g_*\textstyle\sum_{l=0}^b|f(l,x(l),\Delta x(l),y(l),\Delta y(l))|\\ &\leq g_*\textstyle\sum_{k=0}^b c_1(k)(|x|+|\Delta x|+|y|+|\Delta y|)^\alpha.\end{aligned}$$

Therefore,

$$\begin{aligned}\|N_1(x,y)\|_\infty &\leq g_*(\|x\|_*+\|y\|_*)^\alpha\textstyle\sum_{k=0}^b c_1(k)\\ &\leq q_*^\alpha g_*\textstyle\sum_{k=0}^b c_1(k):=l_1.\end{aligned}$$

Similarly,

$$\|N_2(x,y)\|_\infty\leq q_*^\beta g_*\sum_{k=0}^b c_2(k):=l_2.$$

$$\begin{aligned}|\Delta N_1(x(k),y(k))| &\leq |\textstyle\sum_{l=0}^b g(k,l+1)f(l+1,x(l+1),\Delta x(l+1),y(l+1),\Delta y(l+1))|\\ &\quad+|\textstyle\sum_{l=0}^b g(k,l)f(l,x(l),\Delta x(l),y(l),\Delta y(l))|\\ &\leq\\ &\quad g_*[\textstyle\sum_{k=0}^b c_1(k+1)(|x(k+1)|+|\Delta x(k+1)|+|y(k+1)|+|\Delta y(k+1)|)^\alpha\\ &\quad+\textstyle\sum_{k=0}^b c_1(k)(|x(k)|+|\Delta x(k)|+|y(k)|+|\Delta y(k)|)^\alpha],\end{aligned}$$

Therefore,

$$\begin{aligned}\|\Delta N_1(x,y)\|_\infty &\leq g_*[\textstyle\sum_{k=0}^b c_1(k+1)(|x|_*+|y|_*)^\alpha+\sum_{k=0}^b c_1(k)(|x|_*+|y|_*)^\alpha]\\ &\leq q_*^\alpha g_*\textstyle\sum_{k=0}^b c_*(k):=\bar{l}_1.\end{aligned}$$

Similarly,

$$\begin{aligned}\|\Delta N_2(x,y)\|_\infty &\leq g_*[\textstyle\sum_{k=0}^b c_1(k+1)(|x|_*+|y|_*)^\beta+\sum_{k=0}^b c_1(k)(|x|_*+|y|)^\beta]\\ &\leq q_*^\beta g_*\textstyle\sum_{k=0}^b c_*(k):=\bar{l}_2.\end{aligned}$$

On the other side,

$$\begin{array}{rcl}\|\Delta N_1(x,y)\|_\infty + \|N_1(x,y)\|_\infty & \leq & q_*^\alpha g_* \sum_{k=0}^b c_*(k) + q_*^\alpha g_* \sum_{k=0}^b c_1(k) \\ \|N_1(x,y)\|_* & \leq & l_1^*.\end{array}$$

and

$$\begin{array}{rcl}\|\Delta N_2(x,y)\|_\infty + \|N_2(x,y)\|_\infty & \leq & q_*^\beta g_* \sum_{k=0}^b c_*(k) + q_*^\beta g_* \sum_{k=0}^b c_1(k) \\ \|N_2(x,y)\|_* & \leq & l_2^*.\end{array}$$

Moreover, for each $k \in \mathbb{N}(0,b)$, we have $\{N_1(x(k),y(k)) : (x,y) \in B_q\}$ $\{N_2(x(k),y(k)) : (x,y) \in B_q\}$ and $\{\Delta N_1(x(k),y(k)) : (x,y) \in B_{\bar{q}}\}$ $\{\Delta N_2(x(k),y(k)) : (x,y) \in B_{\bar{q}}\}$ are relatively compact in $\mathbb{R}^n$. Then from a consequence of Theorem 5.1.3 we conclude that N is compact. As a consequence of Steps 1 to 2, the operator $N : C(\mathbb{N}(0,b),\mathbb{R}) \to C(\mathbb{N}(0,b),\mathbb{R})$ is completely continuous.

Step 3. It remains to show that

$$\mathcal{A} = \{(x,y) \in C(\mathbb{N}(0,b),\mathbb{R}^n) \times C(\mathbb{N}(0,b),\mathbb{R}^n) : (x,y) = \lambda N(x,y), \lambda \in (0,1)\}.$$

is bounded.

Let $(x,y) \in \mathcal{A}$. Then $x = \lambda N_1(x,y)$ and $y = \lambda N_2(x,y)$ for some $0 < \lambda < 1$. Thus, for $k \in \mathbb{N}(0,b)$, we have

$$\begin{array}{rcl}|x(k)| & \leq & \sum_{l=0}^b |g(k,l)f(l,x,\Delta x,y,\Delta y)| \\ & \leq & g_* \sum_{l=0}^b |f(l,x,\Delta x,y,\Delta y)| \\ & \leq & g_* \sum_{k=0}^b c_1(k)(|x| + |\Delta x| + |y| + |\Delta y|)^\alpha,\end{array}$$

therefore,

$$\|x\|_\infty \leq g_* \sum_{k=0}^b c_1(k)(\|x\|_\infty + \|\Delta x\|_\infty + \|y\|_\infty + \|\Delta y\|_\infty)^\alpha.$$

and

$$\begin{array}{rcl}|y(k)| & \leq & \sum_{l=0}^b |g(k,l)p(l,x,\Delta x,y,\Delta y)| \\ & \leq & g_* \sum_{l=0}^b |p(l,x,\Delta x,y,\Delta y)| \\ & \leq & g_* \sum_{k=0}^b c_2(k)(|x| + |\Delta x| + |y| + |\Delta y|)^\beta,\end{array}$$

therefore,

$$\|y\|_\infty \leq g_* \sum_{k=0}^b c_2(k)(\|x\|_\infty + \|\Delta x\|_\infty + \|y\|_\infty + \|\Delta y\|_\infty)^\beta.$$

Then we have,

$$\begin{array}{rcl}\|\Delta x\|_\infty & \leq & g_*[\sum_{k=0}^b c_1(k+1)(\|x\|_\infty + \|\Delta x\|_\infty + \|y\|_\infty + \|\Delta y\|_\infty)^\alpha \\ & & + \sum_{k=0}^b c_1(k)(\|x\|_\infty + \|\Delta x\|_\infty + \|y\|_\infty + \|\Delta y\|_\infty)^\alpha] \\ & \leq & g_* \sum_{k=0}^b c_1^*(k)(\|x\|_\infty + \|\Delta x\|_\infty + \|y\|_\infty + \|\Delta y\|_\infty)^\alpha.\end{array}$$

and

$$\begin{aligned}\|\Delta y\|_\infty &\leq g_*[\textstyle\sum_{k=0}^{b} c_2(k+1)(\|x\|_\infty+\|\Delta x\|_\infty+\|y\|_\infty+\|\Delta y\|_\infty)^\beta \\ &\quad +\textstyle\sum_{k=0}^{b} c_2(k)(\|x\|_\infty+\|\Delta x\|_\infty+\|y\|_\infty+\|\Delta y\|_\infty)^\beta] \\ &\leq g_*\textstyle\sum_{k=0}^{b} c_2^*(k)(\|x\|_\infty+\|\Delta x\|_\infty+\|y\|_\infty+\|\Delta y\|_\infty)^\beta.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\Delta x\|_\infty+\|x\|_\infty &\leq g_*[\textstyle\sum_{k=0}^{b} c_1^*(k)(\|x\|_\infty+\|\Delta x\|_\infty+\|y\|_\infty+\|\Delta y\|_\infty)^\alpha \\ &\quad +\textstyle\sum_{k=0}^{b} c_1(k)(\|x\|_\infty+\|\Delta x\|_\infty+\|y\|_\infty+\|\Delta y\|_\infty)^\alpha].\end{aligned} \tag{5.1.8}$$

and

$$\begin{aligned}\|\Delta y\|_\infty+\|y\|_\infty &\leq g_*[\textstyle\sum_{k=0}^{b} c_2^*(k)(\|x\|_\infty+\|\Delta x\|_\infty+\|y\|_\infty+\|\Delta y\|_\infty)^\beta \\ &\quad +\textstyle\sum_{k=0}^{b} c_2(k)(\|x\|_\infty+\|\Delta x\|_\infty+\|y\|_\infty+\|\Delta y\|_\infty)^\beta].\end{aligned} \tag{5.1.9}$$

Then by summing (5.1.8) and (5.1.9).
we get,

$$\begin{aligned}(\|x\|_\infty+\|\Delta x\|_\infty+\|y\|_\infty+\|\Delta y\|_\infty) &\leq g_*[\textstyle\sum_{k=0}^{b} c_1(k)(\|x\|_\infty+\|\Delta x\|_\infty+\|y\|_\infty+\|\Delta y\|_\infty)^\alpha \\ &\quad +\textstyle\sum_{k=0}^{b} c_2(k)(\|x\|_\infty+\|\Delta x\|_\infty+\|y\|_\infty+\|\Delta y\|_\infty)^\beta] \\ &\leq \\ &\quad g_*\textstyle\sum_{k=0}^{b} c(k)(\|x\|_\infty+\|\Delta x\|_\infty+\|y\|_\infty+\|\Delta y\|_\infty)^{\max(\alpha,\beta)}.\end{aligned}$$

If $(\|x\|_\infty+\|\Delta x\|_\infty+\|y\|_\infty+\|\Delta y\|_\infty)>1$ then since $0\leq \max(\alpha,\beta)<1$, we have

$$(\|x\|_\infty+\|\Delta x\|_\infty+\|y\|_\infty+\|\Delta y\|_\infty)^{1-\max(\alpha,\beta)} \leq g_*\textstyle\sum_{k=0}^{b} c(k).$$

Hence,

$$(\|x\|_\infty+\|\Delta x\|_\infty+\|y\|_\infty+\|\Delta y\|_\infty) \leq \left(g_*\textstyle\sum_{k=0}^{b} c(k)\right)^{\frac{1}{1-\max(\alpha,\beta)}} := M.$$

Therefore,

$$(\|x\|_\infty+\|\Delta x\|_\infty+\|y\|_\infty+\|\Delta y\|_\infty) \leq \max(1,M) := \overline{M}.$$

Let

$$U := \{(x,y)\in C(\mathbb{N}(0,b),\mathbb{R}^n)\times C(\mathbb{N}(0,b),\mathbb{R}^n) : (\|x\|_\infty,\|\Delta x\|_\infty,\|y\|_\infty,\|\Delta y\|_\infty)<\overline{M}+1\},$$

and consider the operator $N:\overline{U}\to C(\mathbb{N}(a,b-1),\mathbb{R}^n)\times C(\mathbb{N}(a,b-1),\mathbb{R}^n)$. From the choice of U, there is no $y\in\partial U$ such that $y\in\gamma N(y)$ for some $\gamma\in(0,1)$. As a consequence of the version of the nonlinear alternative of Leray-Schauder in generalized Banach space, N has a fixed point $(x;y)$ in U, solution of Problem (5.1.1). □

Conclusion and Perspectives

In this thesis, we have considered the problem of the existence of solutions for different classes of initial and boundary value problems for differential equations. Existence results was given for some classes by the nonlinear alternative of Leray-Schauder,Perov and Krasnoselskii fixed point theorem . We plan to look forward for the qualitative theory of these equations, in order to prove its importance in different scientific areas.

Bibliography

[1] R.P. Agarwal, S. Hristova, D. O'Regan, *Non-Instantaneous Impulses in Differential Equations*, Springer International Publishing AG 2017.

[2] R.P. Agarwal,*Difference Equations and Inequalities*, Marcel Dekker, NewYork,1992.

[3] R.P. Agarwal and D. O'Regan, Boundary value problems for discrete equations, *Applied Math. Letters* **10(4)** (1997), 83–89.

[4] R.P. Agarwal and Patricia.J.Y. Wong, *Advanced Topics in Difference Equations*, Kluwer, Dordrecht, 1997.

[5] R.P. Agarwal and D. O'Regan, Discrete systems on infinite intervals, *Computers Math. Applic.* **35(9)** (1998), 97–105.

[6] R.P. Agarwal and D. O'Regan, A fixed point approach for nonlinear discrete boundary value problems,*Computers. Math. Applic.* **36** (1998),115–121.

[7] R.P. Agarwal and D. O'Regan, Singular discrete boundary value problems,*Applied Math. Letters.* **12** (1999),113–119.

[8] R.P. Agarwal and D. O'Regan, Nonpositone discrete boundary value problems,*Nonlinear Anal.* **39** (2000),207–215

[9] R.P. Agarwal and D. O'Regan, Difference equations in abstract spaces, *J. Austr. Math. Soc. (Series A)* **64** (1998),277–284.

[10] R. P. Agarwal, M. Meehan and D. O'Regan,*Fixed Point Theory and Applications*, Cambridge Tracts in Mathematics,**141**. Cambridge University Press, Cambridge, 2001.

[11] Z. Agur, L. Cojocaru, G. Mazaur, R.M. Anderson and Y.L. Danon, Pulse mass measles vaccination across age cohorts, *Proc. Nat. Acad. Sci. USA.* **90**(24), (1993), 11698-11702.

[12] C.D. Ahlbrandt and A.C. Peterson, *Discrete Hamiltonian Systems: Difference Equations, Continued Fractions and Riccati Equations*, Kluwer, Boston, 1996.

[13] J. Andres and L. Górniewicz, *Topological Fixed Point Principles for Boundary Value Problems*, Kluwer Academic Publishers, Dordrecht, 2003.

[14] J.P. Aubin, H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston, 2009.

[15] C. Avramescu, Sur l'existence des solutions convergentes des systèmes d'équations différentielles non linéaires, *Ann. Mat. Pura Appl.* **81**, (1969), 147–168.

[16] D.D. Bainov and P.S. Simeonov, *Oscillation Theory of Impulsive Differential Equations*, International Publications, Orlando, Fla, USA, 1998.

[17] D.D. Bainov and P.S. Simeonov, *Impulsive Differential Equations: Asymptotic Properties of the Solutions* , vol.**28** of Series on Advances in Mathematics for Applied Sciences, World Scientific, Singapore, 1995.

[18] D.D. Bainov and V. Covachev, *Impulsive Differential Equations with a Small Parameter*, vol. **24** of Series on Advances in Mathematics for Applied Sciences, World Scientific, Siingapore, 1994.

[19] D.D. Bainov and P.S. Simeonov, *Impulsive Differential Equations: Periodic Solutions and Applications* , Longman Scientific and Technical. Essex, England, 1993.

[20] D.D. Bainov, V. Lakshmikantham, and P.S. Simeonov, *Theory of Impulsive Differential Equations* , vol.**6** of Series in Modern Applied Mathematics, World Scientific, Singapore, 1989.

[21] D.D. Bainov and A.Dishliev, Population dynamics control in regard to minimizing the time necessary for the regeneration of a biomass taken away from the population, *Comp. Rend. Bulg. Scie.*, **42**, (1998), 29–32.

[22] D.D. Bainov and P.S. Simeonov, *Systems with Impulse Effect Stability Theory and Applications* , Ellis Horwood Limited Chichester, 1989.

[23] A.V. Balakrishnan, Semigroups of operators: theory and applications. Proceedings of the international conference in Newport Beach, CA, USA, December 14-18, 1998. *Prog. Nonlinear Differ. Equ. Appl.*, **42** Basel: Birkhäuser, 2000.

[24] A.V. Balakrishnan, Fractional powers of closed operators and the semi-groups generated, *Pac. J. Math.* **10** (1960), 419–437.

[25] M. Benchohra, J. Henderson and S.K. Ntouyas, *Impulsive Differential Equations and Inclusions*, *Hindawi Publishing Corporation,* Vol. **2**, New York, 2006.

[26] M. Benchohra, E. Gatsori, J. Henderson and S.K. Ntouyas, Nondensely defined evolution mpulsive differential inclusions with nonlocal conditions, *J. Math. Anal. Appl.* **286**, (2003), 307–325.

[27] M. Benchohra, L. Gòrniewicz, S.K. Ntouyas and A. Ouahab, Existence results for nondensely defined mpulsive semilinear functional differential equations. Nonlinear analysis and applications: to V. Lakshmikantham on his 80th birthday. Vol. 1,2, 289–300, Kluwer Acad. Publ., Dordrecht, 2003.

[28] A. T. Bharuch-Reid, *Random Integral Equations*, New York, Academic Press, 1972.

[29] O. Bolojan, Implicit first order differential systems with nonlocal conditions, *Electron. J. Qual. Theory Differ. Equ.***69** (2014),1–13.

[30] O. Bolojan-Nica, G. Infante and R. Precup, Existence results for systems with coupled nonlocal initial conditions, *Nonlinear Anal.* **94** (2014), 231–242.

[31] C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Mathematics, Springer-Verlag, Berlin–Heidelberg–New York, **580**, 1977.

[32] Y.K. Chang, W.T. Li and J.J. Nieto, Controllability of evolution differential inclusions in Banach spaces, *Nonlinear Anal.* **67**(2) (2007), 623–632.

[33] X. Chen, X. Zhang, Existence of positive solutions for singular impulsive differential equations with integral boundary conditions on an infinite interval in Banach spaces,*Electron. J. Qual. Theory Differ. Equ.* **29** (2011), 1-18.

[34] M. Choicy, J.F. Guegan and P. Rohani, Dynamics of infectious diseases and pulse vaccinations: tasting apart the embedded resonance effects, *Phys. D.* **223**, (2006), 26–35.

[35] S. Djebali, K. Mebarki, System of singular second-order differential equations with integral condition on the positive half-line, *Electron. J. Qual. Theory Differ. Equ.* **50** (2013), 1–42.

[36] S. Djebali, K. Mebarki, Semi-positone Sturm-Liouville differential systems on unbounded intervals, *Acta Math. Univ. Comenianae.* **2** (2016), 231–259.

[37] A. Dishliev and D.D. Bainov, Dependence upon initial conditions and perameters of solutions of impulsive differential equations with variable structure, *International Journal of theoretical Physics* , **29**, (1990), 655–676.

[38] G. Dore and S. Yakubov, Semigroup Estimates and Noncoercive Boundary Value Problems,*Semigroup Forum*, **60** (2000), 93–121.

[39] A.D'Onofrio, Stability properties of pulse Vaccination strategy in SEIR epidemic model, *Math. Biosci.* **179**, (2002), 57–72.

[40] A.D'Onofrio, On pulse vaccination strategy in the SIR epidemic model with vertical transmission, *Appl. Math. Lett.*, **18**, (2005), 729–732.

[41] J. Dugundji and A. Granas, *Fixed Point Theory*, Spribg-Verlag, New York, 2003.

[42] K.J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations,* Graduate Texts in Mathematics **194**, Springer-Verlag, New York, 2000.

[43] S. Elaydi, *An Introduction to Difference Equations*, Springer-Verlag, NewYork, 1996.

[44] S. Gao, J.J. Nieto and A. Toress, Analysis of a delayed epidemic model with pulse vaccination and saturation incidence, *Vaccine* **24** (2006) 6037–6045.

[45] R.K. George, A.K. Nandakumaran and A. Arapostathis, A note on controllability of impulsive systems, *J. Math. Anal. Appl.*, **241**, (2000), 276–283.

[46] L. Górniewicz, *Topological Fixed Point Theory of Multivalued Mappings*, Mathematics and its Applications, **495**, Kluwer Academic Publishers, Dordrecht, 1999.

[47] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[48] J. R. Graef, J. Henderson and A. Ouahab, *Topological Methods for Differential Equations and Inclusions*, Monographs and Research Notes in Mathematics, CRC Press, 2019.

[49] J.R. Graef, J. Henderson and A. Ouahab, *Impulsive differential inclusions. A fixed point approach.* De Gruyter Series in Nonlinear Analysis and Applications **20**. de Gruyter, Berlin, 2013.

[50] J. R. Graef, H. Kadari, A. Ouahab and A. Oumansour, Existence results for systems of second-order impulsive differential equations. *Acta Math. Univ. Comenianae.* **88**(1), (2019), 51–66.

[51] J.R. Graef, J. Henderson and A. Ouahab, Some Multivalued Versions of a Krasnoselskii Type Fixed Point Theorem. *J. Fixed Point Theory Appl.* **19** (2017), 1059–1082.

[52] D.J. Guo, The fixed point and eigenvalue of a class of concave and convex operator, *Chinese Sci. Bull*, **15** (1985), 1132-1135 (in Chinese).

[53] D. Guo, Second order impulsive integro-differential equations on unbounded domains in Banach spaces, *Nonlinear Anal. Theory, Methods Appl.* **35**(4), (1999), 413-423.

[54] M. Guo, X. Xue. R. Li, Controllability of impulsive evolution inclusions with nonlocal conditions, *J. Optim. Theory Appl.* **120** (2004) 355–374.

[55] P. Hartman, Difference equations: Disconjugacy, principal solutions, Green's functions,complete monotonicity, *trans. Amer. Math. Soc.* **246** (1978),1-30.

[56] S. Heikkilä and V. Lakshmikantham, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*, Monographs and Textbooks in Pure and Applied Mathematics, **181**, Marcel Dekker, New York, 1994.

[57] J. Henderson, A. Ouahab and M. Slimani, Existence results of system of semilinear discrete equations, *Int. J. Difference Equ.* **12**, (2017), 235–253.

[58] S.Hristova, *Qualitative Investigations and Approximate Methods for Impulsive Equations*, Nova Science, New York, 2009.

[59] E. Hernandez, D. O'Regan, On a new class of abstract impulsive differential equations, *Proc. Am. Math. Soc.* **141**(5), (2013), 1641-1649.

[60] E. Hernandez, Abstract impulsive differential equations without predefined time impulses, *J. Math. Anal.Appl.* **491**(1), (2020), 124288.

[61] B.S. Jensen, *The Dynamic Systems of Basic Economic Growth Models*, Kluwer, Dordrecht, 1994.

[62] G. Jiang and Q. Lu, Impulsive state feedback control of a predator-prey model, *J. Comput. Appl. Math.* **200**, (2007), 193–207.

[63] W.G. Kelley and A.C. Peterson, *Difference Equations: An Introduction with Applications*, Academic Press, New York, 1991.

[64] W.A. Kirk and B. Sims, *Handbook of Metric Fixed Point Theory*, Springer-Science + Business Media, B.V, Dordrecht, 2001.

[65] V.L. Kocic and G. Ladas,*Global Behavior of Nonlinear Difference Equations of Higher Order with Applications,* Kluwer, Dordrecht, 1993.

[66] S.G. Krein, *Linear Differential Equation in Banach Spaces*, Translations of mathematical monographs, American mathematical society, 1972.

[67] E. Kruger-Thiemr, Formal theory of drug dosage regiment. I. *J. Theoret. Biol.* **13**, (1966), 212–235.

[68] E. Kruger-Thiemr, Formal theory of drug dosage regiment. II. *J. Theoret. Biol.* **23**, (1969), 169–190.

[69] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, *Theory of imulsive Differential Equations.* World Scientific, Singapore, 1989.

[70] V. Lakshmikantham and D. Trigiante, *Difference Equations: Numerical Methods and Applications,* Academic Press, New York, 1986.

[71] A. Lasota and Z. Opial, An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations, *Bull. Acad. Pol. Sci. Ser. Sci. Math.Astronom. Phys.* **13**, (1965), 781-786.

[72] W. Li, H. Huo, Global attractivity of positive periodic solutions for an impulsive delay of respiratory dynamics, *J. Comput. Appl. Math.* **174**, (2005), 227–238.

[73] J. Li and J. J. Nieto, Existence of positive solutions for multipoint boundary value problem on the half-line with impulses, *Boundary Value Problems*, vol. 2009, Article ID 834158, 12 pp, 2009.

[74] B. Liu, Controllability of impulsive neutral functional differential inclusions with infinite delay, *Nonlinear Anal.* **60**, (2005), 1533–1552.

[75] X.Y. Lou and B.T. Cui, Global asymptotic stability of delay BAM neutral network with impulses, *Chaos Solitons Fractals.* **29**, (2006), 1023–1031.

[76] S. Lubkin, C_0*-Semigroups and Applications*, North-Hollnd Mathmatics studies 2003.

[77] J. Musielak, *Introduction to Functional Analysis*, PWN, Warszawa, 1976 (in Polish).

[78] J.D. Murray, *Mathematical Biology*, Springer, Berlin, 1989.

[79] V.D. Mil'man and A. D. Myshkis, On the stability of motion in the presence of impulses, *Sib.Math.Zh [Siberian Math.J]* **1**, (1960), 233-237.

[80] V.D. Mil'man, A.D.Myshkis, Random impulses in linear dynamical system, in Approximate Methods for solving Differential Equations, *Publishing House of the Academy of Sciences of Ukrainian SSR, Kiev,pp* (1963), 64-81 (in Russian).

[81] R.E. Mickens, Difference Equations: Theory and Applications Van Nostrand Reinhold, New York, 2nd edition, 1990.

[82] R.E. Mickens, *Nonstandard Finite Difference Models of Differential Equations*, World Scientific, Singapore, 1994.

[83] S. Nenov, Impulsive controllability and optimization problem in population dynamics, *Nonl. Anal.*, **36**, (1999), 881–890.

[84] O. Nica, R. Precup, On the nonlocal initial value problem for first order differential systems. *Stud. Univ. Babes -Bolyai Math.* **56**(3) (2011), 125–137.

[85] O. Nica, Nonlocal initial value problems for first order differential systems. *Fixed Point Theory.* **13** (2012), 603-612.

[86] O. Nica, Initial-value problems for first-order differential systems with general nonlocal conditions, *Electron. J. Differential Equations*, **2012**(74) (2012), 1–15.

[87] J. J. Nieto, A. Ouahab and M.A. Slimani, Existence and boundedness of solutions for systems of difference equations with infinite delay, *Glas. Mat.*, **53**(73) (2018), 123–141.

[88] A. Ouahab, Some Perov's and Krasnoles'skii type fixed point results and application, *Communications in Applied Analysis*, **19** (2015), 623-642.

[89] A. Pazy, Semigroups of Linear operators and Applications to Partial Differential Equations. New York (NY): Springer-Verlag, 1983.

[90] A.I.Perov, On the Cauchy problem for a system of ordinary differential equations, *Pribliz. Met. Rechen. Differ. Uravn.* **2**, (1964), 115-134(in Russian).

[91] R. Precup, The role of matrices that are convergent to zero in the study of semilinear operator systems, *Math. Comp. Modelling* **49** (2009), 703-708.

[92] R. Precup and A. Viorel, Existence results for systems of nonlinear evolution equations, *Int. J. Pure Appl. Math. IJPAM* **47** (2008), 199–206.

[93] R. Precup, *Methods in Nonlinear Integral Equations*, Kluwer Dordrecht, 2000.

[94] I. A. Rus, The theory of a metrical fixed point theorem: theoretical and applicative relevances, *Fixed Point Theory.* **9** (2008), 541-559.

[95] I. Rachunkov and J. Tomecek, *State-Dependent Impulses. Boundary Value Problems on Compact Interval.* Atlantis Briefs in Differential Equations, **6.** Atlantis Press, Paris, 2015.

[96] S.H. Saker, Oscillation and global attractivity of impulsive periodic delay respiratory dynamics model, *Chinese Ann. Math. Ser.* **B26**, (2005), 511–522.

[97] A.M.Samoilenko, N.A.Perestyuk, Impulsive Differential Equations (translated from the Russsian). World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises, Vol.**14**. World Scientific.River Edge,NJ, 1995.

[98] A.N. Sharkovsky, Y.L. Maistrenko and E.Y. Romanenko, *Difference Equations and Their Applications.* Kluwer, Dordrecht, 1993

[99] R.Shi and L.Chen, Stage-structured impulsive SI model for pest management, *Discrect. Dyn. Nat. Soc. Hindawi.* **2007** (2017), 1–11.

[100] S. Tang, L. Chen, Density-dependent birth rate, birth pulses and their population dynamic consequences, *J. Math. Biol.* **44** (2002) 185–199.

[101] A. Viorel, *Contributions to the Study of Nonlinear Evolution Equations*, Ph.D. thesis, Babeş-Bolyai University Cluj-Napoca, Department of Mathematics, 2011.

[102] I.I. Vrabie, C_0*-semigroups and applications*, North-Holland Mathematics Studies, 191. North-Holland Publishing Co., Amsterdam, 2003.

[103] R.S. Varga, *Matrix iterative analysis.* Second revised and expanded edition. Springer Series in Computational Mathematics, **27**. Springer-Verlag, Berlin, 2000.

[104] J.R.L. Webb, G. Infante, Positive solutions of nonlocal boundary value problems: a unified approach, *J. London Math. Soc.* **74** (2006), 673-693.

[105] S.L. Xi, M. Jia and H.P. Ji , Positive solutions of boundary value problem for systems of second-order differential equations with integral boundary condition on the half line, *Elec. J. Qual. Theo. Diff. Equ.* **31** (2009), 1–13.

[106] J. Yan, A. Zhao, J.J. Neito, Existence and global attractivity of positive periodic solution of periodic single-species impulsive Lotka-Volterra systems, *Math. Comput. Modelling* **40** (2004) 509–518.

[107] K. Yosida, Functional Analysis, 6^{th} edn. Springer-Verlag, Berlin, 1980.

[108] X. Q. Zhang, *Positive solutions of singular multipoint boundary value problem for systems of nonlinear second-order differential equations on infnite intervals in Banach spaces*, *Bound. Value Probl.* **2009** (2009), 1–22.

[109] W. Zhang. and M.Fan, Periodicity in a generalized ecological competition system governed by impulsive differential equations with delay, *Math. Comput. Modelling* **39** (2004) 479–493.

[110] M. Zhang, Y.Liu, *Existence of solutions for implicit fractional differential systems with coupled nonlocal conditions*, *Advances in Analysis.* **2**(1)(2017), 1–9.

[111] X. Zhang, Z. Shuai, K. Wang, Optimal impulsive hervasting policy for single population, *Nonlinear Anal.* RWA **4** (2003) 639–651.

[112] X. M. Zhang, X. Z. Yang, M. Q. Feng, Minimal Nonnegative Solution of Nonlinear Impulsive Differential Equations on Infinite Interval, *Boundary Value Problems,Volume* **2011**, Article ID 684542, 15pp.

Abstract

The objective of this thesis is to establish existence and uniqueness results for several types of Semi-linear differential equation, we mentions here:

- Impulsive Differential Equations.
- Difference equations.

Our results are based upon very recently fixed point theorems.

Résumé

L’objectif de notre thèse est d'établir des résultats d'existence et d'unicité

pour des équations différentiels semi linaire avec impulsion et les équations aux différences.

Toute cette étude a été faite dans les espaces de Banach.

Nos résultats sont basés sur des théorèmes récents de point fixe.

ملخص

الهدف من هذه الرسالة مناقشة " وجود و وحدانية " حلول من المسائل ذات معادلات تفاضلية نصف خطية وذلك وفق شروط ابتدائية ومجالات محددة.

النتائج اعتمدت على نظرية النقطة الصامدة .