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Contribution à l'étude de l'existence de solutions des équations différentielles fractionnaires impulsives dans les espaces de Banach

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Ministry of Higher Education and Scientific Research
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Faculty of Exact Sciences
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THESIS

Presented to obtain the degree of

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Option: Differential Equations

by

Sadek HABANI

Contribution to the study of the existence of solutions for impulsive fractional differential equations in Banach spaces

Defended on .... in front of the evaluating committee composed of:

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Title : Contribution to the study of the existence of solutions for impulsive fractional differential equations in Banach spaces

Abstract : The objective of this thesis is to contribute to the development of the theory of the existence solutions of fractional differential equations, by studying three classes of equations in Banach spaces. The first and the second results concern the existence of weak solutions for certain impulsive differential equations of mixed type of fractional order and multi-order relative to the derivative in the sense of Caputo based on the theorem of the type-Krasnosel'skii fixed point combined with the technique of measures of weak non-compactness coupled with Henstock-Kurzweil-Pettis integrals. Then, we give sufficient conditions for a class of differential equations involving the fractional derivative of Caputo of order $\alpha \in (0;1]$ and the impulsive effect to prove that the set of weak solutions is nonempty and compact in $C_w(J;F)$ space using the Brouwer-Schauder-Tychonoff type theorem. Finally we conclude the results obtained by illustrative examples.

Keywords : Banach space, boundary value problem, Caputo fractional derivative, fixed point, Henstock-Kurzweil-Pettis integral, measure of weak noncompactness, impulses, weak solutions.

AMS Subject Classification : 26A33, 34A37, 34B37, 34G20

Titre : Contribution à l'étude de l'existence de solutions des équations différentielles fractionnaires impulsives dans les espaces de Banach

Résumé : L'objectif de cette thèse est de contribuer au développement de la théorie d'existence de solutions des équations différentielles fractionnaires, en étudiant trois types de classe d'équations dans des espaces de Banach. Les deux premiers résultats concernent l'existence de solutions faibles pour certaines équations différentielles impulsives de type mixte d'ordre fractionnaire et multi-ordre relatif à la dérivée au sens de Caputo en se basant sur le théorème du point fixe type-Krasnosel'skii et les techniques de mesure faible de non-compacité couplé aux intégrales Henstock-Kurzweil-Pettis. Ensuite on donne des conditions suffisantes pour une classe d'équations différentielles impliquant la dérivée fractionnaire de Caputo d'ordre $\alpha \in (0;1]$ et l'effet impulsif pour prouver que l'ensemble des solutions faibles est non vide et compact dans l'espace $C_w(J,F)$ en utilisant le théorème de type Brouwer-Schauder-Tychonoff. Enfin nous concluons les résultats obtenus par des exemples illustratifs.

Mots clés : Espace de Banach, problème limites, dérivée fractionnaire au sens de Caputo, point fixe, Intégrale de Henstock-Kurzweil-Pettis, mesure de non-compacité faible, impulsion, solutions faibles.

AMS Subject Classification : 26A33, 34A37, 34B37, 34G20

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List of publications

Papers in refereed journals/ Manuscripts under review

1. D. Seba, **S. Habani**, A. Benaissa, H. Rebai. The Henstock-Kurzweil-Pettis integral and multiorders fractional differential equations with impulses and multipoint fractional integral boundary conditions in Banach spaces. Math Meth Appl Sci. (2019); https://doi.org/10.1002/mma.6020

2. **S.Habani**, D. Seba and A. Benaissa, Existence result for impulsive fractional differential equations via weak Riemann integrals, Communications on Applied Nonlinear Analysis.Volume 27, Number 2, 27 - 44, (2020).

3. **S.Habani**, D. Seba and A. Benaissa, Weak solutions for impulsive fractional integral boundary value problems of mixed type via Henstock-Kurzweil-Pettis integrals, (submitted).

Contents

Introduction

The theory of impulsive differential equations has always received a great interest and a considerable attention for their controllability, stability and existence of solutions. They have been used to describe memory and hereditary properties of various materials and processes in many fields of engineering and scientific research: physics, mechanics, chemistry, electrical engineering, signal and image processing economics, biological and medical fields such as biology, bio technologies. It is used in medicine to observe blood flow phenomena. For more details on this theory and on its applications see for instance [15, 44, 53, 55] and references therein.

Indeed, they are used to describe the dynamics of many evolutionary processes subject to short-time perturbations which are considered to take place instantaneously in the form of impulses as they are performed discretely in their duration, compared with the total duration of the processes and phenomena, is negligible.

Among the different approaches investigations of impulsive differential equations, we are interested in the one based on the application of the classical methods of the theory of ordinary differential equations. In this direction, the works done by Mil'man and Myshkis [46], [47] and Myshkis and Samoilenko [49] were the first in which the general concepts of the theory of systems with pulse action are formulated from a new point of view and their basic specific features are investigated.

Nowadays, a lot of authors have more interest in this investigations due to their wide practical applications; see the monographs by Bainov and Simeonov [15], Lakshmikantham et al [44], and the joint work of Perestyuk et al [53], Samoilenko and Perestyuk [55].

The combination of the two theories created an effective domain of investigation see [7, 18, 20, 29, 31, 58]. It is to be noted that the method using the technique of measures of weak noncompactness coupled with Henstock-Kurzweil-Pettis integrals provides an effective mechanism to prove the existence of weak solutions for initial and boundary value problems for nonlinear differential and integral equations see [5, 24, 26, 46, 52, 57] and the references cited therein. On the other hand impulsive differential has grown and arise from the real world problems to describe the dynamics of processes in which sudden, discontinuous jumps occurs. These kind of processes are naturally seen in biology, physics, engineering. They undergo sudden changes in the course of their development; these changes are often of very short duration and are therefore produced instantaneously as impulses. As an example we can cite the effect of chemotherapy treatment on the dynamics cancer cells as well as the effects of earthquakes on the dynamics of human population, etc. see for instance [8, 18, 19, 35, 37, 60, 67, 69] and the references therein.

The modeling of such phenomena requires the use of models which involve explicitly and simultaneously the continuous evolution of the phenomenon as well as the changes instantly, Such models are said to be "impulsive"; they evolve with continuous process and governed by differential equations combined with equations of difference representing the impulsive effect experienced.

The theory of impulsive differential equations was initiated in 1960 by V. Milman and A. Myshkis and it has been developed mainly by V. Lakshmikantham from 1985. It has generated much more interest in recent decades; in particular, there has been appreciable development in the theory of impulsive differential equations with impulse of fixed moments.

In this thesis, we present the existence of weak solutions for initial value problems, for certain impulsive differential equations of mixed type of fractional order and multi-order relative to the derivative in the sense of Caputo and also Then we will give sufficient conditions for a class of differential equations with fractional order and impulses for the set of weak solutions is nonempty and compact in $C_w(J;F)$. Our results are based upon very fixed points theorems. This Thesis is arranged as follows:

In Chapter 1, we present some basic definitions, notations and some preliminary notions. In the first and the second section, we give some notations from the theory of Banach spaces and some properties of set valued maps. In the third section, we give definitions concerning fractional calculus theory. The last section is devoted to fixed points theory, here we give the main theorems that will be used in the following chapters.

In chapter2, we shall be concerned by boundary value problem of nonlinear impulsive differential equations of mixed type (sections 2.1 and 2.2). The section 2.1 will be concerned by multipoint fractional integral boundary value problem for an impulsive nonlinear differential equation involving multiorders fractional derivatives and deviating argument of the form:

$$\begin{cases} {}^cD^{r_k}_{t_k^+}x(t) = f(t,x(t),x(\theta(t))), \quad 1 < r_k \leq 2,\ k = 0,1,...,p,\ t \in J', \\ \\ \Delta x(t_k) = I_k(x(t_k)),\ \ \Delta x'(t_k) = I_k^*(x(t_k)),\ \ k = 1,2,...,p, \\ x(0) = \sum\limits_{k=0}^{p} \lambda_k \mathfrak{I}^{\beta_k}_{t_k^+}x(\eta_k),\ \ x'(0) = 0,\ t_k < \eta_k < t_{k+1}, \end{cases}$$

where ${}^cD^{r_k}_{t_k^+}$ is the Caputo fractional derivative of order r_k.

Then we deal with the section 2.2 where we are going to study integral boundary value problem of nonlinear impulsive differential equations of mixed type of the form :

$$\begin{cases} {}^cD^r x(t) = \psi(t, x(t)), \quad 1 < r \leq 2,\ t \in J', \\ \Delta x(t_k) = \varphi_k(x(t_k)),\ \ \Delta x'(t_k) = \varphi_k^*(x(t_k)),\ \ k = 0, 1, 2, ..., m, \\ \delta x(0) + \mu x'(0) = \int_0^T \sigma_1(x(s))ds,\ \ \delta x(T) + \mu x'(T) = \int_0^T \sigma_2(x(s))ds, \end{cases}$$

where ${}^cD^r$ is the Caputo fractional derivative of order r.
The purpose of this chapter is to use Henstock-Kurzweil-Pettis integrals and De Blasi measure of weak nonompactness to prove the existence of weak solutions for the problems (1.1) (section 2.1) and (1.2) (section 2.2). An example will be presented in the last part of each section illustrating the abstract theory.

In chapter3, we will be concerned by impulsive fractional differential equations. Section 3.1 we will study the existence result for a multipoint fractional integral boundary value problem of an impulsive nonlinear differential equation involving multiorders fractional derivatives and deviating argument. Our work is based on fixed point theorem of Brouwer-Schauder-Tychonoff type combined with the technique of measures of weak noncompactness. Section 3.2 is devoted to an illustrating example to show the applicability of the imposed conditions.

1

Preliminaries

1.1 Some notations and definitions

Let $\mathscr{F}^*$ be a Banach space, $\mathscr{F}^*$ denote the dual space of $\mathscr{F}$, $\mathscr{F}_w = (\mathscr{F}, w) = (\mathscr{F}, \sigma(\mathscr{F}, \mathscr{F}^*))$ the space $\mathscr{F}$ with its weak topology, $C(J = [0,T], \mathscr{F}_w) = (C(J,\mathscr{F}), w)$ is the space of all continuous functions from J to $\mathscr{F}$ with the usual supremum norm $\|x\|_\infty = \sup_{t\in J} \|x(t)\|$ and the weak topology, and the Banach space

$$PC(J,\mathscr{F}) = \{x : J \to \mathscr{F} :\ x \in C(J_k, \mathscr{F}),\ x(t_k^+) \text{ and } x(t_k^-) \text{ exist, } k = 1,2,...,m\},$$

equipped with

$$\|x\|_{PC} = \sup_{t\in J} \|x(t)\|,$$

where $J_{k-1} = (t_{k-1}, t_k]$, $k = 1,2,...,m+1$ with $t_0 = 0$ and $t_{m+1} = T$.
We define the Banach space $\mathfrak{P} = PC^1(J,\mathscr{F}) = \{x \in PC(J,\mathscr{F}),\ x'(t_k^+),\ x'(t_k^-)$ exist and x' is left continuous at t_k, for , $k = 1,2,...,m\}$ with the norm

$$\|x\|_{\mathfrak{P}} = \sup_{t\in J}\{\|x(t)\|_{PC},\ \|x'(t)\|_{PC}\}.$$

Let us denote by $P_{wc}(\mathscr{F})$ the set of all weakly compact subsets of $\mathscr{F}$.

1.2 Absolutely Continuous Functions

Definition 1.2.1. *[50] A family $\mathscr{A}$ of functions Ψ is said to be uniformly $AC_*(\mathscr{F})$ i.e.uniformly absolutely continuous in the restricted sense on $\mathscr{F}$ if, for every $\varepsilon > 0$ there is $\eta > 0$ such that for every Ψ in $\mathscr{A}$ and every finite or infinite sequence of non-overlapping intervals $\{[c_i, d_i]\}$ with $c_i, d_i \in J$ and satisfying $\sum_i |d_i - c_i| < \eta$, we have $\sum_i U(\Psi, [c_i, d_i]) < \varepsilon$, where $U(\Psi, [c_i, d_i])$ denotes the oscillation of Ψ over $[c_i, d_i]$ (i.e. $U(\Psi, [c_i, d_i]) = \sup\{|\Psi(r) - \Psi(s)| :\ r, s \in [c_i, d_i]\}$).*

A family $\mathscr{A}$ of functions Ψ is said to be uniformly ACG_* on $[c,d]$, i.e.uniformly generalized absolutely continuous in the restricted sense on $[c,d]$, if $[c,d]$ is the union of a sequence of closed sets A_i such that on each A_i, the family $\mathscr{A}$ is uniformly $AC_*(A_i)$.

1.3 Banach Space-Valued Riemann Sum Type Integrals

1.3.1 Henstock-Kurzweil integral

Definition 1.3.1. *[25] A function $u : J \to \mathscr{F}$ is said to be Henstock-Kurzweil integrable on J if there exists an $\mathscr{L} \in \mathscr{F}$ such that, for every $\varepsilon > 0$, there exists $\delta(\xi) : J \to \mathbb{R}^+$ such that, for every δ-fine partition $D = \{(J_i, \xi_i)\}_{i=1}^n$, we have*

$$\left\| \sum_{i=1}^{n} u(\xi_i)u(J_i) - \mathscr{L} \right\| < \varepsilon,$$

$(HK)\int_0^T u(s)ds = \sum\limits_{i=1}^{n} u(\xi_i)u(J_i)$ denotes the Henstock-Kurzweil integral on J.

1.3.2 Henstock-Kurzweil-Pettis integral

Definition 1.3.2. *[25] A function $\psi : J \to \mathscr{F}$ is said to be Henstock-Kurzweil-Pettis integrable, shortly HKP-integrable on J, if there exists a function $f : J \to \mathscr{F}$ with the following properties:*

(i) *$\forall x^* \in \mathscr{F}^*$, $x^*\psi$ is HK-integrable on J;*

(ii) *$\forall t \in J$, $\forall x^* \in \mathscr{F}^*$, $x^* f(t) = (HK)\int_0^t x^*\psi(s)ds$.*

f is called a primitive of ψ and $f(t) = \int_0^t \psi(t)dt$ is the notation of the HKP-integral of ψ on the interval J.

Theorem 1.3.3. *[23] Let $\psi : J \to \mathscr{F}$ and assume that $\psi_n : J \to \mathscr{F}$, $n \in \mathbb{N}$, are HKP integrable on J. For each $n \in \mathbb{N}$, let Ψ_n be a primitive of ψ_n. If we assume that:*

(i) $\forall x^* \in \mathscr{F}^*$, $x^*(\psi_n(t)) \to x^*(\psi(t))$ *a.e. on J,*

(ii) *for each* $x^* \in \mathscr{F}^*$*, the family* $G = \{x^*\Psi_n : n = 1,2,...\}$ *is uniformly* ACG_* *on J (ie weakly uniformly* ACG_* *on J),*

(iii) *for each* $x^* \in \mathscr{F}^*$*, the set G is equicontinuous on J,*

then ψ *is HKP integrable on J and* $\int_0^t \psi_n(s)ds$ *tends weakly in* $\mathscr{F}$ *to* $\int_0^t \psi(s)ds$ *for each* $t \in J$.

1.4 Some properties of set-valued maps

Definition 1.4.1. *A function* $x(.) : J \to \mathscr{F}$ *is said to be:*

(i) weakly continuous at $a \in J$ *if for every* $x^* \in \mathscr{F}^*$ *the scalar function* $t \mapsto \langle x^*, x(t)\rangle$ *is continuous at a.*

(ii) weakly absolutely continuous (ω*AC) on J if for every* $x^* \in \mathscr{F}^*$ *the real valued function* $t \mapsto \langle x^*, x(t)\rangle$ *is AC on J.*

(iii) weakly differentiable at $a \in J$ *if there exists an element* $x'_\omega(a) \in \mathscr{F}$ *such that*

$$\lim_{h\to 0}\left\langle x^*, \frac{x(a+h)-x(a)}{h}\right\rangle = \left\langle x^*, x'_\omega(a)\right\rangle$$

for every $x^* \in \mathscr{F}^*$*. The element* $x'_\omega(a)$*, also denoted by* $\frac{d\omega}{dt}x(a)$*, is called the weak derivative of* $x(.)$ *at* $a \in J$*. Moreover* $x(.)$ *is said to be weakly differentiable on J, if it is weakly differentiable at each point* $t \in J$*. The vector valued function* $t \mapsto x'_\omega(t)$ *denotes the weak derivative of* $x(.)$*. If* $x(.) : J \to \mathscr{F}$ *is weakly differentiable on J, then the real function* $t \longmapsto \langle x^*, x(t)\rangle$ *is differentiable on J, and*

$$\frac{d}{dt}\langle x^*, x(t)\rangle = \left\langle x^*, x'_\omega(t)\right\rangle,\ t \in J,\ for\ every\ x^* \in \mathscr{F}^*.$$

Definition 1.4.2. *A function $\varphi : J \times \mathscr{F} \to \mathscr{F}$ is said to be weakly-weakly continuous at (t_0, x_0) if given $\varepsilon > 0$ and $x^* \in \mathscr{F}^*$, there exists $\delta > 0$ and a weakly open set U containing x_0 such that $|\langle x^*, \varphi(t,x) - \varphi_0(t,x)\rangle| < \varepsilon$ whenever $|t - t_0| < \delta$ and $x \in U$.*

Proposition 1.4.3. *If $\mathscr{F}$ is a weakly sequentially complete space and $x(.) : J \to \mathscr{F}$ is a function such that for every $x^* \in \mathscr{F}^*$ the real function $t \longmapsto \langle x^*, x(t)\rangle$ is differentiable on J, then $x(.)$ is weakly differentiable on on J.*

Definition 1.4.4. *[28] Let $\mathscr{F}$ be a Banach space, $\Omega_{\mathscr{F}}$ the bounded subsets of $\mathscr{F}$ and B_1 the unit ball of $\mathscr{F}$. The De Blasi measure of weak noncompactness is the map $\beta : \Omega_{\mathscr{F}} \longrightarrow [0, \infty)$ defined by*

$$\beta(A) = \inf\{\varepsilon > 0 : \textit{there exists } \Omega \in P_{wc}(\mathscr{F}) \textit{ such that } A \subset \varepsilon B_1 + \Omega\}.$$

Properties : The De Blasi measure of noncompactness satisfies some useful properties (for more details see [28]).

(a) $A \subset B \Rightarrow \beta(A) \subset \beta(B)$,

(b) $\beta(A) = 0 \Rightarrow A$ is relatively compact,

(c) $\beta(A \cup B) = \max(\beta(A), \beta(B))$,

(d) $\beta(\bar{A}^{\omega}) = \beta(A)$, ($\bar{A}^{\omega}$ denotes the weak closure of A),

(e) $\beta(A + B) \leq \beta(A) + \beta(B)$,

(f) $\beta(\lambda A) \leq |\lambda|\beta(A)$,

(g) $\beta(co(A)) \leq \beta(A)$

(h) $\beta(\bigcup_{|\lambda| \leq h} A) = h\beta(A)$.

(i) $\beta(A) \leq 2diam(A)$.

Lemma 1.4.5. *[38] Let $\mathscr{V} \subset C(J, \mathscr{F})$ be bounded and strongly equicontinuous and $u_0 \in C(J, \mathscr{F})$. Then*

- *The function $t \to \beta(\mathscr{V})$ is continuous on J,*
- *$\overline{co}\{\mathscr{V}, u_0\}$ is equicontinuous in $C(J,\mathscr{F})$,*
- *$\beta_c(\mathscr{V}) = \max_{t\in J} \beta(\mathscr{V}(t)) = \beta(\mathscr{V}(t))$,*

where $\beta_c(.)$ is the measure of weak noncompactness on $C(J,\mathscr{F})$.

Let us recall the definition of convex-power condensing operators.
For this, let $\mathscr{F}$ be a Banach space, D a nonempty closed convex subset of $\mathscr{F}$, $N : D \to D$ a mapping and $x_0 \in D$. For any $U \subset D$, we set

$$\begin{aligned} N^{(1,x_0)}(U) &= N(U), \\ N^{(n,x_0)}(U) &= N\left(\overline{co}N^{(n-1,x_0)}(U)\bigcup\{x_0\}\right), \text{ for } n = 2,3,... \end{aligned}$$

Definition 1.4.6. *Let $\mathscr{F}$ be a Banach space, D a nonempty closed convex subset of $\mathscr{F}$, and μ a measure of weak noncompactness on $\mathscr{F}$. Let $N : D \to D$ be a bounded mapping (that is, it takes bounded sets into bounded ones), $x_0 \in D$ and n_0 a positive integer. We say that N is a μ-convex-power condensing operator about x_0 and n_0 if, for any bounded set $U \subset D$ with $\mu(U) > 0$, we have $\mu(N^{(n_0,x_0)}(U)) < \mu(U)$.*

1.5 Fractional order operators and properties

This section deals with the definition and properties of various operators of fractional integration and fractional differentiation of arbitrary order.

1.5.1 Special Functions of the Fractional Calculus

Here, we give some informations on the Euler's gamma, Beta and the Mittag-Leffler functions which play the most important role in the theory of the differentiation of arbitrary order.

•The Gamma Function

One of the basic function of the fractional calculus is Euler's gamma function $\Gamma(z)$, which generalizes the factorial $n!$ and allows n to take also non-integer and even complex values.

Definition 1.5.1. *(Gamma Function) The gamma function $\Gamma(z)$ is defined by the integral*

$$\Gamma(z) = \int_0^{+\infty} e^t t^{z-1} dt \tag{1.1}$$

which converges in the right half of the complex plane $Re(z) > 0$.

One of the basic properties of the gamma function is that it satisfies the following function equation

$$\Gamma(z+1) = z\Gamma(z)$$

•The Beta Function

Definition 1.5.2. *The function $(x,y) \to \beta(x,y)$, where $Re(x) > 0, Re(y) > 0$, defined by*

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt,$$

is called the Beta function.

There is relation between Gamma and Beta functions given in the relation :

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

It should also be mentioned that the Beta function is symmetric, i.e.,

$$\beta(x,y) = \beta(y,x).$$

•The Mittag-Leffler Function

The exponential function e^z is very important in the theory of integer-order differential equations. Its one parameter generalization, called the Mittag-Leffler function, is the function which is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$$

For $\alpha = 1$, we obtain

$$E_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = e^z$$

The generalizations of this function, so called functions of the Mittag-Leffler type, play an important role in the theory of fractional differential equations

Definition 1.5.3. *A two-parameter Mittag-Leffler function defined by*

$$E_{(\alpha,\beta)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \alpha > 0,\ \beta > 0 \tag{1.2}$$

For special choices of the values of the parameters α, β we obtain:

$$\begin{aligned} E_{(1,1)}(z) &= E_1(z) = e^z, \\ E_{(2,1)}(z^2) &= ch(z), \\ E_{(1,2)}(z) &= \frac{e^z - 1}{z}, \text{ etc.} \end{aligned}$$

1.5.2 Fractional-Order Integration

1.5.3 Grünwald-Letnikov fractional derivative

The main idea of the fractional derivative of Grünwald-Letnikov is to give a generalization from the classical definition of integer order derivation of a function to arbitrary real orders.

The first derivative (of order 1) of a function f at point t is given by:

$$f'(t) = \lim_{h\to 0} \frac{f(t+h)-f(t)}{h} = \lim_{h\to 0} \frac{f(t)-f(t-h)}{h} \tag{1.3}$$

By successive derivation of the function f; we obtain a generalization of formula (1.3) to the order n ($n \in \mathbb{N}^*$) of the form:

$$f^n(t) = \lim_{h\to 0} \frac{1}{h^n} \sum_{p=0}^{p=n} (-1)^p \binom{n}{p} f(t-ph) \tag{1.4}$$

If the integer n is positive, the formula (1.4) represents the derivative of order n, and if n is negative represents the integral repeated n times.

From the fundamental property $\Gamma(n+1) = n!, \forall n \in \mathbb{N}$, so in the case where n is negative or null can be written

$$(-1)^p \binom{n}{p} = \frac{-n(-n+1)...(-n+p-1)}{p!}$$

Definition 1.5.4. *If f is a continuous function over the interval $[\mu, x]$ the derivative fractionnaries of order α and order $(-\alpha)$ in the sense of Grünwald-Letnikov of the function f are defined respectively by:*

$$\begin{aligned} {}_{\mu}^{G}D^{\alpha} f(t) &= \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{p=0}^{\infty} \frac{\Gamma(p-\alpha)}{\Gamma(p+1)\Gamma(-\alpha)} f(t-ph) \\ &= \frac{1}{\Gamma(-\alpha)} \int_{\mu}^{t} (t-s)^{-\alpha-1} f(s) ds \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} {}_{\mu}^{G}D^{-\alpha} f(t) &= \lim_{h\to 0} \frac{1}{h^{-\alpha}} \sum_{p=0}^{\infty} \frac{\Gamma(p+\alpha)}{\Gamma(p+1)\Gamma(\alpha)} f(t-ph) \\ &= \frac{1}{\Gamma(\alpha)} \int_{\mu}^{t} (t-s)^{\alpha-1} f(s) ds \end{aligned} \tag{1.6}$$

Fractional derivative of a constant

Proposition 1.5.5. *The fractional derivative of Grünwald-Letnikov of order of a constant function $f(x) = \lambda$ is defined by:*

$$ {}^{G}_{\mu}D^{\alpha} f(t) = \frac{\lambda}{\Gamma(1-\alpha)}(t-\mu)^{-\alpha} \tag{1.7}$$

Composition with the order derivatives entire

Proposition 1.5.6.

$$\frac{d^n}{dx^n}({}^{G}_{\mu}D^{\alpha} f(t)) = {}^{G}_{\mu} D^{n+\alpha} f(t) \tag{1.8}$$

and

$${}^{G}_{\mu}D^{\alpha}\left(\frac{d^n}{dx^n} f(t)\right) = {}^{G}_{\mu} D^{n+\alpha} f(t) - \sum_{p=0}^{n-1} \frac{f^{(p)}(\mu)(t-\mu)^{p-m-n}}{\Gamma(p-m-n+1)} \tag{1.9}$$

1.5.4 Riemann Liouville fractional derivative

We begin this section by stating Cauchy formula for repeated integration

Theorem 1.5.7. ***(Cauchy formula for repeated integration)***. *Let f be some continuous function on the interval $[a,b]$. The n-th repeated integral of f based at a,*

$$f^{(-n)}(t) = \int_a^t \int_a^{s_1} \ldots \int_a^{s_{n-1}} f(s_n) ds_n ds_{n-1} \ldots ds_2 ds_1,$$

is given by single integration

$$f^{(-n)}(t) = \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1} f(s) ds$$

From this formula the definition of fractional integral is constructed, so we can take an integral of any real order. Replacing $(n-1)!$ by $\Gamma(n)$ and the power n in the integrand with some $\alpha \in \mathbb{R}_+$, we have Riemann-Liouville fractional integral.

Definition 1.5.8. *(Riemann-Liouville fractional derivative). Let $\alpha > 0$ and $n \in \mathbb{N}$ such that $n-1 < \alpha < n$, and $a < t < b$. Left hand and Right hand Riemann-Liouville fractional derivative is defined as:*

$$D^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_a^t (t-s)^{n-\alpha-1} f(s) ds, \tag{1.10}$$

$$D^{\alpha}_{b^-} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_t^b (t-s)^{n-\alpha-1} f(s) ds, \tag{1.11}$$

respectively.

If $\alpha \in \mathbb{N}$ then these definitions acts as like classical derivative of order α.

Remark 1.5.9. *In general Riemann-Liouville fractional derivative of order α of the function $f(t)$ with $a < t < b$ is defined as*

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_a^t (t-s)^{n-\alpha-1} f(s) ds \tag{1.12}$$

If $0 < \alpha < 1$, we obtain

$$D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(s)}{(t-s)^{\alpha}} ds \tag{1.13}$$

Definition 1.5.10. *(Riemann-Liouville fractional integral). Let $\alpha > 0$ and $n-1 < \alpha < n, n \in \mathbb{N}$, and a<t<b. Left hand and Right hand Riemann-Liouville fractional integral is defined as:*

$$I^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \tag{1.14}$$

$$I^{\alpha}_{b^-} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (t-s)^{\alpha-1} f(s) ds, \tag{1.15}$$

respectively.

Remark 1.5.11. *In general Riemann-Liouville fractional integral of order $\alpha \in \mathbb{R}$ of the* *function $f(t)$ with $a < t < b$ is defined as*

$$I^{\alpha} f(t) = D^{(-\alpha)} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds \tag{1.16}$$

Example 1.5.12. . ***Constant function***

Let us find the fractional integral of a constant function C using the remark 1.5.11

$$\begin{aligned} I^{\alpha} C = D^{-\alpha} C &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} C ds \\ &= \frac{C}{\Gamma(\alpha+1)} (t-a)^{\alpha} \end{aligned}$$

Example 1.5.13. . ***The Power Function***

The Power function is very important in Mathematics since many functions can be derived from an infinite power series. we will use the Riemann- Liouville fractional integral given in remark 1.5.11 to compute the integral of order $\alpha > 0$ of the power function $(t-a)^{\delta}$. Plugging this into the equation it gives

$$D^{-\alpha} (t-a)^{\delta} = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} (t-a)^{\delta} ds$$

If we make the substitution $s = (t-a)x + a$, it follows that $ds = (t-a)dx$ and the new interval of integration is $[0,1]$, we can rewrite the last expression as

$$\begin{aligned} D^{-\alpha} (t-a)^{\delta} &= \frac{(t-a)^{\delta+\alpha}}{\Gamma(\alpha)} \int_0^1 (1-x)^{\alpha-1} x^{\delta} ds \\ &= \frac{(t-a)^{\delta+\alpha}}{\Gamma(\alpha)} B(\alpha, \delta+1) \\ &= \frac{\Gamma(\delta+1)}{\Gamma(\delta+\alpha+1)} (t-a)^{\delta+\alpha} \end{aligned}$$

1.5.5 Caputo Fractional Derivative

Applied problems require definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions, which contain $f(0), f'(0), f''(0)$, etc. Unfortunately, the Riemann- Liouville approach leads to initial conditions containing the limit values of the Riemann-Liouville derivative at the lower terminal

$$\lim_{t\to 0} D^{\alpha-1} f(t), \lim_{t\to 0} D^{\alpha-2} f(t), ..., \lim_{t\to 0} D^{\alpha-n} f(t)$$

and there is no physical interpretation for such types of initial conditions.
A certain solution to this conflict was proposed by the so-called Caputo fractional derivative definition

Definition 1.5.14. *(Caputo fractional derivative). Let $\alpha > 0$ and $n-1 < \alpha < n, n \in \mathbb{N}$, and a<t<b. Left hand and Right hand Caputo is defined as:*

$$^{c}D^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \quad (1.17)$$

$$^{c}D^{\alpha}_{b^-} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \quad (1.18)$$

respectively.

Remark 1.5.15. *1. In general Caputo fractional derivative of order $\alpha \in \mathbb{R}$ of the function $f(t)$ with $a < t < b$ is defined as*

$$^{c}D^{\alpha}_{t} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \quad (1.19)$$

where $n-1 < \alpha < n$.

2. This definition is of course more restrictive than Riemann-Liouville definition, since it requires the absolute integrability of the derivative of order n. The main advantage of

Caputo's approach is that the initial conditions for fractional differential equation with Caputo derivative take the same form of integer-order.
In particular, according to this definition, the relevant property is that the fractional derivative of a constant is still zero.

Definition 1.5.16. *[40] For a function $h:[d,\infty)\longrightarrow\mathscr{F}$, the Caputo fractional-order derivative of h, is defined by*

$$^{c}D^{\alpha}_{d^{+}}h(t)=\frac{1}{\Gamma(n-\alpha)}\int_{d}^{t}\frac{h^{(n)}(s)}{(t-s)^{1-n+\alpha}}ds,$$

where $n=[\alpha]+1$ and $[\alpha]$ denote the integer part of α.

Definition 1.5.17. *(Caputo fractional integral). Let $\alpha>0$ and $n-1<\alpha<n, n\in\mathbb{N}$, and a<t<b. Left hand and Right hand Riemann-Liouville fractional integral is defined as:*

$$^{c}I^{\alpha}_{a^{+}}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}(t-s)^{n-\alpha-1}f^{(n)}(s)ds, \tag{1.20}$$

$$^{c}I^{\alpha}_{b^{-}}f(t)=\frac{(-1)^{n}}{\Gamma(n-\alpha)}\int_{t}^{b}(t-s)^{n-\alpha-1}f^{(n)}(s)ds, \tag{1.21}$$

respectively.

Lemma 1.5.18. *[71] Let $\alpha>0$, then*

(i) The differential equation $^{c}D^{\alpha}_{d^{+}}h(t)=0$ has solutions $h(t)=c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1}$, $c_i\in\mathscr{F}$, $i=0,1,\ldots,n-1$, $n=[\alpha]+1$.

(ii) $\mathfrak{I}^{\alpha}_{d^{+}}(^{c}D^{\alpha}_{d^{+}})h(t)=h(t)+c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1}$ for some $c_i\in\mathscr{F}$, $i=0,1,...,n-1$, $n=[\alpha]+1$.

Example 1.5.19. *we will use the Caputo fractional derivative given in remark 1.5.15 to compute the integral of order $\alpha>0$ of the power function $f(t)=(t-a)^{\delta}$. Plugging this* *into the equation gives*

$$
\begin{aligned}
{}^cD_t^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds \\
&= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} \frac{\Gamma(\delta+1)}{\Gamma(\delta-n+1)} (s-a)^{\delta-n} ds
\end{aligned}
$$

put $s=(t-a)u+a$, $u\in[0,1]$ *and* $ds=(t-a)du$

$$
\begin{aligned}
{}^cD_t^{\alpha}(t-a)^{\delta} &= \frac{\Gamma(\delta+1)}{\Gamma(n-\alpha)\Gamma(\delta-n+1)}(t-a)^{\delta-a}\int_0^1 (1-u)^{n-\alpha-1}u^{\delta-n}ds \\
&= \frac{\Gamma(\delta+1)}{\Gamma(n-\alpha)\Gamma(\delta-n+1)}(t-a)^{\delta-a}B(\delta-n+1,n-\alpha) \\
&= \frac{\Gamma(\delta+1)}{\Gamma(n-\alpha)\Gamma(\delta-n+1)}(t-a)^{\delta-a}\frac{\Gamma(\delta-n+1)\Gamma(n-\alpha)}{\Gamma(\delta-\alpha+1)} \\
&= \frac{\Gamma(\delta+1)}{\Gamma(\delta-\alpha+1)}(t-a)^{\delta-a}
\end{aligned}
$$

1.6 Impulsive system

1.6.1 Description of an impulsive system

An impulse differential system is generally defined by a differential equation ordinary subject to a difference equation which represents the impulsive condition. A such system is given by

$$
\begin{cases}
u'(t) = f(t,u(t)), & \text{if} h(t,u)\neq 0 \\
\Delta u(t) = I(t,u(t)), & \text{if} h(t,u)=0,
\end{cases}
\tag{1.22}
$$

The problem (1.22) describes an evolutionary process governed by the differential equation when $h(t,u(t))\neq 0$. For h(t,u(t))=0 i.e. $t=t_i$, go from position $(t_i,u(t_i))$ to position $(t_i^+,u(t_i^+))$ with a quantity equal to $I(t,u(t))$, this time jump is called instant impulse.

In order to simplify the study of the system (1.22), we focus on a particular case of the relation $h(t,u)=0$ where we give an infinity of uncountable functions $\tau_k : J\to\mathbb{R}^+$ Such as $\tau_k < \tau_k < ... < \tau_k < ...$ and $\lim_{k\to\infty}\tau_k(u) = \infty$, for $u \in J$, thus the system (1.22) becomes

$$\begin{cases} u'(t) = f(t,u(t)), & t \neq \tau_k(u(t)) \\ \Delta u(t) = I_k(t,\tau_k(u(t))), & t = \tau_k(u(t)) \end{cases} \tag{1.23}$$

As the functions τ_k depend on the state u, we will say that the impulsive system (1.23) is a impulse system with variable impulse times.

If τ_k is constant, then the system (1.23) is said to be an impulsive system with fixed impulse time. We can simply describe an impulsive differential system with fixed moments by

$$\begin{cases} u'(t) = f(t,u(t)), & t \neq t_k \\ \Delta u(t) = I_k(u(t_k^-))), & t = t_k, \quad k = 0,1,2,... \end{cases} \tag{1.24}$$

In this case the problem is subject to a finite (or infinite) number of impulses which take place at fixed times given by an increasing sequence $0 < t_0 < t_1 < ... < t_k < ...$ having no accumulation points, namely $\lim\limits_{k\to\infty} t_k$=$+\infty$. Note that this type of system is the most currents in the natural case.

As the solutions of an impulsive differential equation are piecewise continuous, then the appropriate functional space for such solutions is the space of continuous functions in pieces

$$\mathscr{PC}([a,b];\mathscr{F}) = \left\{ \begin{array}{l} u:[a,b]\to\mathscr{F}, u\in\mathscr{C}((t_k,t_{k+1}];\mathscr{F}), k=0,1,...,m, u(t_k^-) \text{ et } u(t_k^+) \\ \text{exists with } u(t_k^-) = u(t_k), k=1,2,...,m \end{array} \right\}$$

where $\mathscr{F}$ is a Banach space

1.6.2 Applications of impulse systems

Its known that the theory of impulsive system shows a natural cadre in mathematics modelisation for a lot of real phenomens. For example in chemestry, in medecine mathematics models based on IDE are used to optimise therapy existed before, for example the model studied by Panetta describe the dynamic of normal and cancer cells which are in interaction with impulsive effect of chemical therapy The models which describe the growing of normal and cancer cells that are in interaction are given by:

$$\begin{cases} \frac{dx}{dt} = r_1 x\left(1 - \frac{x}{k_1} - \delta_1 y\right) \\ \frac{dy}{dt} = r_2 y\left(1 - \frac{y}{k_2} - \delta_2 y\right) \end{cases} \tag{1.25}$$

x, y represent biomass of normal and cancer cells respectively.

r_1, r_2 shows the growth rate of normal and cancer cells respectively.

k_1, k_2 represent the interaction's parameters between normal and cancer cells.

$\delta_i, i \in 1,2$ describe different interaction between normal and cancer cells for example δ_1 describes negative effect of tumor on of normal cells, δ_2 describes negative effect of normal cells on tumor. Finally if $\delta_i = 0$, we suppose that there isn't an interaction between the two typed of cells. For each injection of chemical therapy, it acts on the two types cells, the system (1.25) is subject of a perturbation of this form

$$\begin{cases} x(t_n^+) = e^{-\alpha_1 D} x(t_n^-) \\ y(t_n^+) = e^{-\alpha_2 D} y(t_n^-) \end{cases} \tag{1.26}$$

$e^{-\alpha_1 D}, e^{-\alpha_2 D}$ are the fractions of surviving cells both of normal and cancer respectively, after the injection of one dose of medicine, where α_i are given positive constants

$x(t_n^-), y(t_n^-)$ represent the biomass of normal and cancer cells just before the medicine injection

$x(t_n^+), y(t_n^+)$ represent the biomass of normal and cancer cells just after the medicine injection

System analysis (1.25)-(1.26) has shown that if the period $\tau(\tau = t_{i+1} - t_i)$. Between each treatment (impulsion) exceeds a certain threshold andif the dose D is less than the volume

$$\frac{\tau r_2(1-\delta_2 k_1}{\alpha_2 - \frac{\alpha_1\delta_2 k_1 r_2}{r_1}}$$

Than cancer cells may be replenished. If the dose exceeds the amount

$$\frac{-1}{\alpha_1} ln\left(a + e^{-r_1\tau}(1-a)\right),$$

the treatment may destroy normal cells and eventually kill the patient.
The chemotherapy protocol is a very useful tool in mathematics especially on the resolution of Differential equations and Integrals Indeed these theorems provide sufficient conditions for which a given function admits a fixed point, thus we ensure the existence of the solution of a given problem.

1.7 Some fixed point theorems

Theorem 1.7.1. ***(Krasnosel'skii-type)*** *[38] Let W be a subset of a Banach space $\mathscr{X}$. Suppose that W is nonempty, closed, convex and bounded, $G : W \to W$ is weakly sequentially continuous and there exist a vector $x_0 \in W$ and an integer n_0 for which G is power-convex condensing. Then G has at least one fixed point in W.*

Theorem 1.7.2. ***(Brouwer-Schauder-Tychonoff type)****[11] Let F be a Banach space and V be a non-empty subset of $C(J,F)$. Also assume that V is a closed convex subset of $C_w(J,F)$, $N : V \to V$ is continuous with respect to the weak uniform convergence topology, $N(V)$ is bounded and N is β-condensing. In addition, suppose the family $N(V)$ is strongly equicontinuous. Then the set of fixed points of N is non-empty and compact in $C_w(J,F)$.*

2

BVP of nonlinear impulsive differential equations of mixed type

2.1 BVP of multi-order $r_k \in (1, 2]$

2.1.1 Introduction

we consider the following multipoint fractional integral boundary value problem of the form:

$$\begin{cases} {}^cD^{r_k}_{t_k^+}x(t) = f(t,x(t),x(\theta(t))), \quad 1 < r_k \leq 2,\ k = 0,1,...,p,\ t \in J', \\ \\ \Delta x(t_k) = I_k(x(t_k)),\ \ \Delta x'(t_k) = I_k^*(x(t_k)),\ \ k = 1,2,...,p, \\ x(0) = \sum_{k=0}^{p} \lambda_k \mathfrak{I}^{\beta_k}_{t_k^+}x(\eta_k),\ \ x'(0) = 0,\ t_k < \eta_k < t_{k+1}, \end{cases} \tag{2.1}$$

where ${}^cD^{r_k}_{t_k^+}$ is the Caputo fractional derivative of order r_k and $\mathfrak{I}^{\beta_k}_{t_k^+}$ is fractional Riemann-Liouville integral of order $\beta_k > 0$, $f : J \times \mathscr{F} \times \mathscr{F} \longrightarrow \mathscr{F}$ is a given function satisfying some assumptions that will be specified later, $I_k, I_k^* : \mathscr{F} \longrightarrow \mathscr{F}$, deviating argument $\theta \in C(J,J)$, $J = [0,T]$ $(T > 0)$, $0 = t_0 < t_1 < ... < t_k < ... < t_p < t_{p+1} = T$, $J' = J/\{t_0,t_1,...,t_p\}$, and $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, where $x(t_k^+)$ and $x(t_k^-)$ denote the right and the left limits of $x(t)$ at $t = t_k$ $(k = 1,2,...,p)$, respectively. $\Delta x'(t_k)$ have a similar meaning for $x'(t)$.

2.1.2 Main Results

Definition 2.1.1. *A function $x \in PC(J,\mathscr{F})$ is said to be a solution of (2.1) if x satisfies the equation ${}^cD^{r_k}_{t_k^+}x(t) = \psi(t,x(t),x(\theta(t)))$ on J' and conditions :*

$$\Delta x(t_k) = \varphi_k(x(t_k)),\ \ \Delta x'(t_k) = \varphi_k^*(x(t_k)),\ \ k = 1,2,...,p,$$

and

$$x(0) = \sum_{k=0}^{p} \lambda_k \mathfrak{I}^{\beta_k}_{t_k^+}x(\eta_k),\ \ x'(0) = 0,\ t_k < \eta_k < t_{k+1}$$

Lemma 2.1.2. *For a given $\sigma \in C(J,\mathscr{F}_w)$, a function x is a solution of the following impulsive boundary value problem :*

$$\begin{cases} {}^cD^{r_k}_{t_k^+}x(t) = \sigma(t), \quad 1 < r_k \leq 2,\ k = 0,1,...,p,\ t \in J', \\ \\ \Delta x(t_k) = \varphi_k(x(t_k)),\ \ \Delta x'(t_k) = \varphi_k^*(x(t_k)),\ \ k = 1,2,...,p, \\ x(0) = \sum\limits_{k=0}^{p} \lambda_k \mathfrak{I}^{\beta_k}_{t_k^+}x(\eta_k),\ \ x'(0) = 0,\ t_k < \eta_k < t_{k+1}, \end{cases} \tag{2.2}$$

if and only if x is a solution of the impulsive fractional integral equation

$$x(t) = \begin{cases} \frac{1}{\Gamma(r_0)}\int_0^t (t-s)^{r_0-1}\sigma(s)ds + \mathscr{A},\ \ t \in J_0; \\ \int_{t_k}^t \frac{(t-s)^{r_k-1}}{\Gamma(r_k)}\sigma(s)ds + \sum\limits_{i=1}^{k}\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{r_{i-1}-1}}{\Gamma(r_{i-1})}\sigma(s)ds + \varphi_i(x(t_i))\right] \\ +\sum\limits_{i=1}^{k-1}(t_k-t_i)\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)}\sigma(s)ds + \varphi_i^*(x(t_i))\right] \\ +\sum\limits_{i=1}^{k}(t-t_k)\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)}\sigma(s)ds + \varphi_i^*(x(t_i))\right] + \mathscr{A},\ \ t \in J_k,\ k = 1,...,p, \end{cases} \tag{2.3}$$

where

$$\begin{aligned} \mathscr{A} \;=\; & \left(1-\sum_{k=0}^{p}\frac{\lambda_k(\eta_k-t_k)^{\beta_k}}{\Gamma(\beta_k+1)}\right)^{-1} \times \left\{\sum_{k=0}^{p}\lambda_k\int_{t_k}^{\eta_k}\frac{(\eta_k-s)^{r_k+\beta_k-1}}{\Gamma(r_k+\beta_k)}\sigma(s)ds \right. \\ & +\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\eta_k-t_k)^{\beta_k}}{\Gamma(\beta_k+1)} \times \left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{r_{i-1}-1}}{\Gamma(r_{i-1})}\sigma(s)ds + \varphi_i(x(t_i))\right] \\ & +\sum_{k=1}^{p}\sum_{i=1}^{k-1}\frac{\lambda_k(\eta_k-t_k)^{\beta_k}(t_k-t_i)}{\Gamma(\beta_k+1)} \times \left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)}\sigma(s)ds + \varphi_i^*(x(t_i))\right] \\ & \left. +\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\eta_k-t_k)^{\beta_k+1}}{\Gamma(\beta_k+2)} \times \left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)}\sigma(s)ds + \varphi_i^*(x(t_i))\right]\right\}. \end{aligned}$$

For the sake of computational convenience, we set

$$T^* = \max_{0\le i\le p}\{T^{r_i}\}, \Gamma^* = \min_{0\le i\le p}\{\Gamma(r_i)\}, \Lambda_1 = \left|1-\sum_{k=0}^{p}\frac{\lambda_k(\eta_k-t_k)^{\beta_k}}{\Gamma(\beta_k+1)}\right|^{-1},$$
$$\Lambda_2 = \sum_{k=0}^{p}\frac{\lambda_k T^{r_k+\beta_k}}{\Gamma(r_k+\beta_k+1)}, \Lambda_3 = \sum_{k=1}^{p}\frac{\lambda_k T^{\beta_k}}{\Gamma(\beta_k+1)}, \Lambda_4 = \sum_{k=1}^{p}\frac{\lambda_k T^{\beta_k}}{\Gamma(\beta_k+2)},$$
$$\delta_1 = \Lambda_2 + \frac{(2p-1)T^*}{\Gamma^*}\Lambda_3 + \frac{pT^*}{\Gamma^*}\Lambda_4,$$
$$\delta_2 = (1+\Lambda_1\Lambda_3)(p+T(p-1)) + (1+\Lambda_1\Lambda_4)Tp.$$

Proof. Assume x satisfies (2.3). If $t \in J_0$ then

$${}^cD^{r_k}_{t_k^+}x(t) = \sigma(t).$$

Lemma 1.5.18 implies

$$\begin{aligned} x(t) &= \Im_0^{r_0}x(t) - c_1 - c_2t \\ &= \frac{1}{\Gamma(r_0)}\int_0^t (t-s)^{r_0-1}\sigma(s)ds - c_1 - c_2t, \end{aligned} \tag{2.4}$$

for some $c_1, c_2 \in \mathscr{F}$. Differentiating 2.4, we get

$$x'(t) = \frac{1}{\Gamma(r_0-1)}\int_0^t (t-s)^{r_0-2}\sigma(s)ds - c_2 \tag{2.5}$$

If $t \in J_1$, then Lemma 1.5.18 implies

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(r_1)}\int_{t_1}^t (t-s)^{r_1-1}\sigma(s)ds - d_1 - d_2(t-t_1), \\ x'(t) &= \frac{1}{\Gamma(r_1-1)}\int_{t_1}^t (t-s)^{r_1-2}\sigma(s)ds - d_2 \end{aligned}$$

for some $d_1, d_2 \in \mathscr{F}$. Thus,

$$\begin{aligned} x(t_1^-) &= \frac{1}{\Gamma(r_0)}\int_0^{t_1}(t-s)^{r_0-1}\sigma(s)ds - c_1 - c_2 t_1, \\ x(t_1^+) &= -d_1, \\ x'(t_1^-) &= \frac{1}{\Gamma(r_0-1)}\int_0^{t_1}(t-s)^{r_0-2}\sigma(s)ds - c_2, \\ x'(t_1^+) &= -d_2. \end{aligned}$$

Using the impulse conditions,

$$\begin{aligned} \Delta x(t_1) &= x(t_1^+) - x(t_1^-) = \varphi_1(x(t_1)), \\ \Delta x'(t_1) &= x'(t_1^+) - x'(t_1^-) = \varphi_1^*(x(t_1)), \end{aligned}$$

we get

$$\begin{aligned} -d_1 &= \frac{1}{\Gamma(r_0)}\int_0^{t_1}(t-s)^{r_0-1}\sigma(s)ds - c_1 - c_2 t_1 + \varphi_1(x(t_1)), \\ -d_2 &= \frac{1}{\Gamma(r_0-1)}\int_0^{t_1}(t-s)^{r_0-2}\sigma(s)ds - c_2 + \varphi_1^*(x(t_1)), \end{aligned}$$

which means that,

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(r_1)}\int_{t_1}^{t}(t-s)^{r_1-1}\sigma(s)ds + \frac{1}{\Gamma(r_0)}\int_0^{t_1}(t-s)^{r_0-1}\sigma(s)ds \\ &+ \frac{t-t_1}{\Gamma(r_0-1)}\int_0^{t_1}(t-s)^{r_0-2}\sigma(s)ds + \varphi_1(x(t_1)) + (t-t_1)\varphi_1^*(x(t_1)) - c_1 - c_2 t. \end{aligned}$$

By a similar process, we can get

$$\begin{aligned}
x(t) &= \int_{t_k}^{t} \frac{(t-s)^{r_k-1}}{\Gamma(r_k)}\sigma(s)ds + \sum_{i=1}^{k}\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-1}}{\Gamma(r_{i-1})}\sigma(s)ds + \varphi_i(x(t_i))\right] \\
&+ \sum_{i=1}^{k-1}(t_k-t_i)\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)}\sigma(s)ds + \varphi_i^*(x(t_i))\right] \\
&+ \sum_{i=1}^{k}(t-t_k)\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)}\sigma(s)ds + \varphi_i^*(x(t_i))\right] - c_1 - c_2 t, \qquad (2.6)
\end{aligned}$$

for $t \in J_k, = 1, \cdots, p$.

The condition $x'(0) = 0$ implies $c_2 = 0$. For $t \in J_k$, we have

$$\begin{aligned}
\Im_{t_k^+}^{\beta_k} x(t) &= \int_{t_k}^{t} \frac{(t-s)^{r_k+\beta_k-1}}{\Gamma(r_k+\beta_k)}\sigma(s)ds + \sum_{i=1}^{k}\frac{(t-t_k)^{\beta_k}}{\Gamma(\beta_k+1)}\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-1}}{\Gamma(r_{i-1})}\sigma(s)ds + \varphi_i(x(t_i))\right] \\
&+ \sum_{i=1}^{k-1}\frac{(t-t_k)^{\beta_k}(t_k-t_i)}{\Gamma(\beta_k+1)}\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)}\sigma(s)ds + \varphi_i^*(x(t_i))\right] \\
&+ \sum_{i=1}^{k}\frac{(t-t_k)^{\beta_k+1}}{\Gamma(\beta_k+2)}\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)}\sigma(s)ds + \varphi_i^*(x(t_i))\right] - \frac{c_1(t-t_k)^{\beta_k}}{\Gamma(\beta_k+1)},
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{p}\lambda_k \Im_{t_k^+}^{\beta_k} x(\eta_k) &= \sum_{k=0}^{p}\lambda_k \int_{t_k}^{\eta_k} \frac{(t-s)^{r_k+\beta_k-1}}{\Gamma(r_k+\beta_k)}\sigma(s)ds \\
&+ \sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\eta_k-t_k)^{\beta_k}}{\Gamma(\beta_k+1)}\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-1}}{\Gamma(r_{i-1})}\sigma(s)ds + \varphi_i(x(t_i))\right] \\
&+ \sum_{k=1}^{p}\sum_{i=1}^{k-1}\frac{\lambda_k(\eta_k-t_k)^{\beta_k}(t_k-t_i)}{\Gamma(\beta_k+1)}\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)}\sigma(s)ds + \varphi_i^*(x(t_i))\right] \\
&+ \sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\eta_k-t_k)^{\beta_k+1}}{\Gamma(\beta_k+2)}\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)}\sigma(s)ds + \varphi_i^*(x(t_i))\right] \\
&- \frac{c_1\lambda_k(\eta_k-t_k)^{\beta_k}}{\Gamma(\beta_k+1)}.
\end{aligned}$$

Applying the condition $x(0) = \sum\limits_{k=0}^{p} \lambda_k \mathfrak{I}_{t_k^+}^{\beta_k} x(\eta_k)$, gives

$$\begin{aligned}
-c_1 = &\left(1 - \sum_{k=0}^{p} \frac{\lambda_k(\eta_k - t_k)^{\beta_k}}{\Gamma(\beta_k+1)}\right)^{-1} \left\{ \sum_{k=0}^{p} \lambda_k \int_{t_k}^{\eta_k} \frac{(t-s)^{r_k+\beta_k-1}}{\Gamma(r_k+\beta_k)} \sigma(s)ds \right.\\
&+ \sum_{k=1}^{p}\sum_{i=1}^{k} \frac{\lambda_k(\eta_k - t_k)^{\beta_k}}{\Gamma(\beta_k+1)} \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-1}}{\Gamma(r_{i-1})}\sigma(s)ds + \varphi_i(x(t_i))\right]\\
&+ \sum_{k=1}^{p}\sum_{i=1}^{k-1} \frac{\lambda_k(\eta_k - t_k)^{\beta_k}(t_k-t_i)}{\Gamma(\beta_k+1)} \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)}\sigma(s)ds + \varphi_i^*(x(t_i))\right]\\
&\left.+ \sum_{k=1}^{p}\sum_{i=1}^{k} \frac{\lambda_k(\eta_k - t_k)^{\beta_k+1}}{\Gamma(\beta_k+2)} \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)}\sigma(s)ds + \varphi_i^*(x(t_i))\right]\right\}.
\end{aligned}$$

Substituting the value of c_i $(i=1,2)$ in (2.4) and (2.6), we obtain (2.3). Conversely, assume that x is a solution of the impulsive fractional integral equation (2.3); then by a direct computation, it follows that the solution given by (2.3) satisfies)2.2). □

Let $\mathfrak{B}_\alpha = \{x \in PC(J,\mathscr{F}),\ \|x\|_{PC} \leq \alpha\}$, $\alpha > 0$

Let us now list some conditions on the functions involved in the problem (2.1). Assume that

(H1) For each uniformly ACG_* function $x : J \to \mathscr{F}$, the function $\psi(.,x(.),x(\theta(.)))$ is HKP integrable and ψ is weakly-weakly sequentially continuous function.

(H2) For any $\alpha > 0$, there exists a HK-integrable function $M_\alpha : J \to \mathbb{R}^+$ and a nondecreasing continuous function $\Omega : [0,\infty) \to [0,\infty)$ such that, $\|\psi(t,x,y)\| \leq M_\alpha(t)\Omega(\alpha)$ for all $t \in J$, $(x,y) \in \mathfrak{B}_\alpha \times \mathfrak{B}_\alpha$, and there exist positive constants L_1 and L_2 such that

$$|\varphi_k(x)| \leq L_1,\ \ |\varphi_k^*(x)| \leq L_2,\ \ \text{for } t \in J,\ x \in \mathscr{F},\ k = 1,\ldots,p.$$

(H3) For each bounded set $U \subset \mathfrak{B}_\alpha$ and for each closed interval $I \subset J$, for $t \in I$:

$$\beta(\psi(t,U,U)) \leq \rho M_\alpha(t)\beta(U), \quad \rho > 0.$$

(H4) The following family is uniformly HK-integrable over J for every $x \in \mathfrak{B}_\alpha$.

$$\{x^*\psi(.,x(.),x(\theta(.))) : \ x^* \in \mathscr{F}^*, \ \|x^*\| \le 1\}$$

(H5) There exists a constant $\alpha_0 > 0$ such that

$$\frac{\alpha_0}{\Omega(\alpha_0)\|M_{\alpha_0}\|_\infty\left(\frac{3pT^*}{\Gamma^*}+\Lambda_1\Delta_1\right)+\Delta_2} > 1$$

(H6) The family

$$\left\{x^*\int_{J_k} G(t,s)\psi(s,x_n(s),x_n(\theta(s)))ds\right\}_{n=0}^{\infty}$$

is equicontinuous and uniformly ACG_* on J for every $t \in J$.

Theorem 2.1.3. *Assume that assumptions (H_1) to (H_6) hold. If*

$$\rho\|M_{\alpha_0}\|_\infty\left(\frac{3pT^*}{\Gamma^*}+\Lambda_1\Delta_1\right) < 1, \tag{2.7}$$

Then the boundary value problem (2.1) has at least one solution.

Proof. In view of lemma 2.1.2, we define an operator $N : PC(J,\mathscr{F}) \longrightarrow PC(J,\mathscr{F})$ by

$$\begin{aligned}
Nx(t) &= \int_{t_k}^{t} \frac{(t-s)^{r_k-1}}{\Gamma(r_k)} \psi(s,x(s),x(\theta(s)))ds \\
&+ \sum_{i=1}^{k} \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-1}}{\Gamma(r_{i-1})} \psi(s,x(s),x(\theta(s)))ds + \varphi_i(x(t_i)) \right] \\
&+ \sum_{i=1}^{k-1} (t_k-t_i) \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)} \psi(s,x(s),x(\theta(s)))ds + \varphi_i^*(x(t_i)) \right] \\
&+ \sum_{i=1}^{k} (t-t_k) \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)} \psi(s,x(s),x(\theta(s)))ds + \varphi_i^*(x(t_i)) \right] \\
&+ \left(1 - \sum_{k=0}^{p} \frac{\lambda_k(\eta_k-t_k)^{\beta_k}}{\Gamma(\beta_k+1)} \right)^{-1} \left\{ \sum_{k=0}^{p} \lambda_k \int_{t_k}^{\eta_k} \frac{(\eta_k-s)^{r_k+\beta_k-1}}{\Gamma(r_k+\beta_k)} \psi(s,x(s),x(\theta(s)))ds \right. \\
&+ \sum_{k=1}^{p}\sum_{i=1}^{k} \frac{\lambda_k(\eta_k-t_k)^{\beta_k}}{\Gamma(\beta_k+1)} \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-1}}{\Gamma(r_{i-1})} \psi(s,x(s),x(\theta(s)))ds + \varphi_i(x(t_i)) \right] \\
&+ \sum_{k=1}^{p}\sum_{i=1}^{k-1} \frac{\lambda_k(\eta_k-t_k)^{\beta_k}(t_k-t_i)}{\Gamma(\beta_k+1)} \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)} \psi(s,x(s),x(\theta(s)))ds + \varphi_i^*(x(t_i)) \right] \\
&+ \left. \sum_{k=1}^{p}\sum_{i=1}^{k} \frac{\lambda_k(\eta_k-t_k)^{\beta_k+1}}{\Gamma(\beta_k+2)} \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)} \psi(s,x(s),x(\theta(s)))ds + \varphi_i^*(x(t_i)) \right] \right\}.
\end{aligned}$$

Observe that the fixed points of the operator N are solutions of the problem (2.1).

If $\aleph$ is a family of uniformly HK-integrable functions over J. Then $\Big($analogously to the proof of Kurtz and Swartz[35], theorem 4.28$\Big)$, $\aleph$ satisfies uniformly the Cauchy criterion over any closed sub interval $I \subset J$. Similarly to Kurtz and Swartz[35], theorem 4.28, the condition

$$\forall \varepsilon > 0 \ \exists \text{gauge } \rho \text{ on } J, \ \forall P_1, P_2 \ \rho \text{ fine-partitions}, \ \forall \psi \in \aleph \ |\underline{S}(\psi,P_1) - \underline{S}(\psi,P_2)| < \varepsilon,$$

implies

$$\forall \varepsilon > 0 \ \exists \text{gauge } \gamma \text{ on } J, \ \forall P_1, P_2 \ \gamma \text{ fine-partitions}, \ \forall \psi \in \aleph, \ \left| \underline{S}(\psi,P) - (HK)\int_J \psi(t)dt \right| < \varepsilon.$$

Thus the family $\aleph$ is uniformly HK-integrable over $[0,\tau]$ for every $\tau < T$. Whence, under the assumption (H_4), the family

$$\{x^*(\psi(.,x(.),x(\theta(.)))) : x^* \in \mathscr{F}^*, \ \|x^*\| \leq 1\}$$

is uniformly HK-integrable over any sub interval $[0,\tau] \subset J$, for every $x \in \mathfrak{B}_{\alpha_0}$. This means that

$$x^* \in \mathscr{F}^* \to (HK) \int_0^\tau L(t,s) x^* \psi(s,x(s),x(\theta(s)))ds$$

is weak*-continuous linear functional for all $\tau \in J$ and any function $L(t,s) \in L^\infty$, which means that there is $x_{t,\tau} \in \mathscr{F}$ satisfying

$$x^* x_{t,\tau} = (HK) \int_0^\tau x^* L(t,s) \psi(s,x(s),x(\theta(s)))ds, \ \ \forall x^* \in \mathscr{F}^*,$$

In clear, the function $L(t,.)\psi(.,x(.),x(\theta(.)))$ is HKP-integrable on J and since

$$s \mapsto \frac{(t-s)^{r_k-1}}{\Gamma(r_k)} \in L^\infty(J), \qquad s \mapsto \frac{(t_i-s)^{r_{i-1}-1}}{\Gamma(r_{i-1})} \in L^\infty(J),$$
$$s \mapsto \frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)} \in L^\infty(J), \qquad s \mapsto \frac{(\eta_k-s)^{r_k+\beta_k-1}}{\Gamma(r_k+\beta_k)} \in L^\infty(J).$$

then the operator N makes sense.
Let $\alpha_0 > 0$, and consider the set

$$\mathfrak{B} = \left\{ \begin{array}{l} x \in \mathfrak{B}_{\alpha_0} : \|x\|_\infty \leq \alpha_0, \ \forall \tau_1, \tau_2 \in J, \tau_1 < \tau_2 : \\ \|x(\tau_2) - x(\tau_1)\| \leq \Omega(\alpha_0) \|M_{\alpha_0}\|_\infty \left(\frac{(\tau_2-t_k)^{r_k} - (\tau_1-t_k)^{r_k}}{\Gamma(r_k+1)} + p(\tau_2-\tau_1)\frac{T^*}{\Gamma^*} \right) + pL_2(\tau_2-\tau_1) \end{array} \right\}$$

Clearly, the subset $\mathfrak{B} \subset \mathfrak{B}_\alpha \subset PC(J,\mathscr{F})$ is closed, convex and equicontinuous. We shall show that N satisfies the assumptions of Theorem (1.7.1), the proof will be given in several steps.

Step 1.
N maps $\mathfrak{B}$ into itself. Take $x \in \mathfrak{B}$ and $t \in J$ and assume that $Nx(t) \neq 0$. By the Hahn-

Banach theorem, there exists $x^* \in \mathscr{F}^*$ such that $\|x^*\| = 1$ and $\|Nx(t)\| = |x^*(Nx(t))|$.

$$
\begin{aligned}
\|Nx(t)\| &= |x^*(Nx(t))| \\
&\le \int_{t_k}^{t} \frac{(t-s)^{r_k-1}}{\Gamma(r_k)} |x^*(\psi(s,x(s),x(\theta(s))))| ds \\
&\quad + \sum_{i=1}^{k} \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-1}}{\Gamma(r_{i-1})} |x^*(\psi(s,x(s),x(\theta(s))))| ds + |(\varphi_i(x(t_i))| \right] \\
&\quad + \sum_{i=1}^{k-1} (t_k-t_i) \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)} |x^*(\psi(s,x(s),x(\theta(s))))| ds + |\varphi_i^*(x(t_i))| \right] \\
&\quad + \sum_{i=1}^{k} (t-t_k) \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)} |x^*(\psi(s,x(s),x(\theta(s))))| ds + |\varphi_i^*(x(t_i))| \right] \\
&\quad + \Lambda_1 \Bigg\{ \sum_{k=0}^{p} \lambda_k \int_{t_k}^{\eta_k} \frac{(\eta_k-s)^{r_k+\beta_k-1}}{\Gamma(r_k+\beta_k)} |x^*(\psi(s,x(s),x(\theta(s))))| ds \\
&\quad + \sum_{k=1}^{p} \sum_{i=1}^{k} \frac{\lambda_k(\eta_k-t_k)^{\beta_k}}{\Gamma(\beta_k+1)} \times \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-1}}{\Gamma(r_{i-1})} |x^*(\psi(s,x(s),x(\theta(s))))| ds + |\varphi_i(x(t_i))| \right] \\
&\quad + \sum_{k=1}^{p} \sum_{i=1}^{k-1} \frac{\lambda_k(\eta_k-t_k)^{\beta_k}(t_k-t_i)}{\Gamma(\beta_k+1)} \times \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)} |x^*(\psi(s,x(s),x(\theta(s))))| ds + |\varphi_i^*(x(t_i))| \right] \\
&\quad + \sum_{k=1}^{p} \sum_{i=1}^{k} \frac{\lambda_k(\eta_k-t_k)^{\beta_k+1}}{\Gamma(\beta_k+2)} \times \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)} |x^*(\psi(s,x(s),x(\theta(s))))| ds + |\varphi_i^*(x(t_i))| \right] \Bigg\} \\
&\le \Omega(\alpha_0)\|M_{\alpha_0}\|_\infty \int_{t_k}^{t} \frac{(t-s)^{r_k-1}}{\Gamma(r_k)} ds \\
&\quad + \sum_{i=1}^{k} \left[\Omega(\alpha_0)\|M_{\alpha_0}\|_\infty \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-1}}{\Gamma(r_{i-1})} ds + L_1 \right] \\
&\quad + \sum_{i=1}^{k-1} (t_k-t_i) \left[\Omega(\alpha_0)\|M_{\alpha_0}\|_\infty \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)} ds + L_2 \right] \\
&\quad + \sum_{i=1}^{k} (t-t_k) \left[\Omega(\alpha_0)\|M_{\alpha_0}\|_\infty \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)} ds + L_2 \right] \\
&\quad + \Lambda_1 \Bigg\{ \Omega(\alpha_0)\|M_{\alpha_0}\|_\infty \sum_{k=0}^{p} \lambda_k \int_{t_k}^{\eta_k} \frac{(\eta_k-s)^{r_k+\beta_k-1}}{\Gamma(r_k+\beta_k)} ds \\
&\quad + \sum_{k=1}^{p} \sum_{i=1}^{k} \frac{\lambda_k(\eta_k-t_k)^{\beta_k}}{\Gamma(\beta_k+1)} \times \left[\Omega(\alpha_0)\|M_{\alpha_0}\|_\infty \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-1}}{\Gamma(r_{i-1})} ds + L_1 \right]
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=1}^{p}\sum_{i=1}^{k-1}\frac{\lambda_k(\eta_k-t_k)^{\beta_k}(t_k-t_i)}{\Gamma(\beta_k+1)}\times\left[\Omega(\alpha_0)\|M_{\alpha_0}\|_\infty\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)}ds+L_2\right]\\
&+\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\eta_k-t_k)^{\beta_k+1}}{\Gamma(\beta_k+2)}\times\left[\Omega(\alpha_0)\|M_{\alpha_0}\|_\infty\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)}ds+L_2\right]\Bigg\}\\
\le{}&\Omega(\alpha_0)\|M_{\alpha_0}\|_\infty\sum_{i=1}^{p+1}\frac{(t_i-t_{i-1})^{r_{i-1}}}{\Gamma(r_{i-1}+1)}+pL_1+T\Omega(\alpha_0)\|M_{\alpha_0}\|_\infty\sum_{i=1}^{p-1}\frac{(t_i-t_{i-1})^{r_{i-1}-1}}{\Gamma(r_{i-1})}\\
&+(p-1)TL_2+T\Omega(\alpha_0)\|M_{\alpha_0}\|_\infty\sum_{i=1}^{p}\frac{(t_i-t_{i-1})^{r_{i-1}-1}}{\Gamma(r_{i-1})}+pTL_2\\
&+\Lambda_1\Bigg\{\Omega(\alpha_0)\|M_{\alpha_0}\|_\infty\sum_{k=0}^{p}\lambda_k\frac{(\eta_k-t_k)^{r_k+\beta_k}}{\Gamma(r_k+\beta_k+1)}\\
&+\Omega(\alpha_0)\|M_{\alpha_0}\|_\infty\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\eta_k-t_k)^{\beta_k}(t_i-t_{i-1})^{r_{i-1}}}{\Gamma(\beta_k+1)\Gamma(r_{i-1}+1)}+L_1\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\eta_k-t_k)^{\beta_k}}{\Gamma(\beta_k+1)}\\
&+\Omega(\alpha_0)\|M_{\alpha_0}\|_\infty\sum_{k=1}^{p}\sum_{i=1}^{k-1}\frac{\lambda_k(\eta_k-t_k)^{\beta_k}(t_k-t_i)(t_i-t_{i-1})^{r_{i-1}-1}}{\Gamma(\beta_k+1)\Gamma(r_{i-1})}\\
&+L_2\sum_{k=1}^{p}\sum_{i=1}^{k-1}\frac{\lambda_k(\eta_k-t_k)^{\beta_k}(t_k-t_i)}{\Gamma(\beta_k+1)}+\Omega(\alpha_0)\|M_{\alpha_0}\|_\infty\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\eta_k-t_k)^{\beta_k+1}(t_i-t_{i-1})^{r_{i-1}-1}}{\Gamma(\beta_k+2)\Gamma(r_{i-1})}\\
&+L_2\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\eta_k-t_k)^{\beta_k+1}}{\Gamma(\beta_k+2)}\Bigg\}\\
\le{}&\Omega(\alpha_0)\|M_{\alpha_0}\|_\infty\sum_{i=1}^{p+1}\frac{T^*}{\Gamma^*}+pL_1+T\Omega(\alpha_0)\|M_{\alpha_0}\|_\infty\sum_{i=1}^{p-1}\frac{T^*}{\Gamma^*}+(p-1)TL_2\\
&+T\Omega(\alpha_0)\|M_{\alpha_0}\|_\infty\sum_{i=1}^{p}\frac{T^*}{\Gamma^*}+pTL_2+\Lambda_1\Bigg\{\Omega(\alpha_0)\|M_{\alpha_0}\|_\infty\sum_{k=0}^{p}\frac{\lambda_kT^{r_k+\beta_k}}{\Gamma(r_k+\beta_k+1)}\\
&+p\Omega(\alpha_0)\|M_{\alpha_0}\|_\infty\times\sum_{k=1}^{p}\frac{\lambda_kT^{\beta_k}T^*}{\Gamma(\beta_k+1)\Gamma^*}+pL_1\sum_{k=1}^{p}\frac{\lambda_kT^{\beta_k}}{\Gamma(\beta_k+1)}\\
&+(p-1)\Omega(\alpha_0)\|M_{\alpha_0}\|_\infty\sum_{k=1}^{p}\frac{\lambda_kT^{\beta_k}T^*}{\Gamma(\beta_k+1)\Gamma^*}+(p-1)L_2\sum_{k=1}^{p}\frac{\lambda_kT^{\beta_k+1}}{\Gamma(\beta_k+1)}\\
&+p\Omega(\alpha_0)\|M_{\alpha_0}\|_\infty\sum_{k=1}^{p}\frac{\lambda_kT^{\beta_k}T^*}{\Gamma(\beta_k+2)\Gamma^*}+pL_2\sum_{k=1}^{p}\frac{\lambda_kT^{\beta_k+1}}{\Gamma(\beta_k+2)}\Bigg\}\\
\le{}&\Omega(\alpha_0)\|M_{\alpha_0}\|_\infty\left(\frac{3pT^*}{\Gamma^*}+\Lambda_1\left[\Lambda_2+\frac{(2p-1)T^*}{\Gamma^*}\Lambda_3+\frac{pT^*}{\Gamma^*}\Lambda_4\right]\right)+(1+\Lambda_3)pL_1\\
&+\Big[(p-1)T\Lambda_3+pT\Lambda_4\Big]L_2\\
\le{}&\alpha_0\ \Big(\text{Using (H2) and (H5)}\Big)
\end{aligned}
$$

Let $\tau_1, \tau_2 \in J_k,\ k = 1,...,p,\ \tau_1 < \tau_2,\ x \in D$, so $Nx(\tau_2) - Nx(\tau_1) \neq 0$. Then there exist $x^* \in E^*$ such that

$$\|Nx(\tau_2) - Nx(\tau_1)\| = |x^*(Nx(\tau_2) - Nx(\tau_1))|.$$

Thus

$$\begin{aligned}
\|Nx(\tau_2) - Nx(\tau_1)\| &= \Bigg| x^* \Bigg(\int_{t_k}^{\tau_2} \frac{(\tau_2 - s)^{r_k - 1}}{\Gamma(r_k)} \psi(s, x(s), x(\theta(s)))ds - \int_{t_k}^{\tau_1} \frac{(\tau_2 - s)^{r_k - 1}}{\Gamma(r_k)} \psi(s, x(s), x(\theta(s)))ds \\
&\quad + \int_{t_k}^{\tau_1} \frac{(\tau_2 - s)^{r_k - 1}}{\Gamma(r_k)} \psi(s, x(s), x(\theta(s)))ds - \int_{t_k}^{\tau_1} \frac{(\tau_1 - s)^{r_k - 1}}{\Gamma(r_k)} \psi(s, x(s), x(\theta(s)))ds \\
&\quad + (\tau_2 - \tau_1) \sum_{i=1}^{k} \left[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{r_{i-1} - 2}}{\Gamma(r_{i-1} - 1)} \psi(s, x(s), x(\theta(s))) + \varphi_i^*(x(t_i)) \right] \Bigg) \Bigg| \\
&= \Bigg| x^* \Bigg(\int_{\tau_1}^{\tau_2} \frac{(\tau_2 - s)^{r_k - 1}}{\Gamma(r_k)} \psi(s, x(s), x(\theta(s)))ds \\
&\quad + \int_{t_k}^{\tau_1} \frac{(\tau_2 - s)^{r_k - 1} - (\tau_1 - s)^{r_k - 1}}{\Gamma(r_k)} \psi(s, x(s), x(\theta(s)))ds \\
&\quad + (\tau_2 - \tau_1) \sum_{i=1}^{k} \left[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{r_{i-1} - 2}}{\Gamma(r_{i-1} - 1)} \psi(s, x(s), x(\theta(s))) + \varphi_i^*(x(t_i)) \right] \Bigg) \Bigg| \\
&\leq \int_{\tau_1}^{\tau_2} \frac{(\tau_2 - s)^{r_k - 1}}{\Gamma(r_k)} \Big| x^*(\psi(s, x(s), x(\theta(s)))) \Big| ds \\
&\quad + \int_{t_k}^{\tau_1} \frac{(\tau_2 - s)^{r_k - 1} - (\tau_1 - s)^{r_k - 1}}{\Gamma(r_k)} \Big| x^*(\psi(s, x(s), x(\theta(s)))) \Big| ds \\
&\quad + (\tau_2 - \tau_1) \sum_{i=1}^{k} \left[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{r_{i-1} - 2}}{\Gamma(r_{i-1} - 1)} \Big| x^*(\psi(s, x(s), x(\theta(s)))) \Big| + \Big| \varphi_i^*(x(t_i)) \Big| \right] \\
&\leq \Omega(\alpha_0)\|M_{\alpha_0}\|_\infty \Bigg(\int_{\tau_1}^{\tau_2} \frac{(\tau_2 - s)^{r_k - 1}}{\Gamma(r_k)} ds + \int_{t_k}^{\tau_1} \frac{(\tau_2 - s)^{r_k - 1} - (\tau_1 - s)^{r_k - 1}}{\Gamma(r_k)} ds \\
&\quad + (\tau_2 - \tau_1) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{r_{i-1} - 2}}{\Gamma(r_{i-1} - 1)} ds \Bigg) + (\tau_2 - \tau_1) p L_2 \\
&\leq \Omega(\alpha_0)\|M_{\alpha_0}\|_\infty \left(\frac{(\tau_2 - t_k)^{r_k} - (\tau_1 - t_k)^{r_k}}{\Gamma(r_k + 1)} + (\tau_2 - \tau_1) \sum_{i=1}^{k} \frac{(t_i - t_{i-1})^{r_{i-1} - 1}}{\Gamma(r_{i-1})} \right) \\
&\quad + pL_2(\tau_2 - \tau_1) \\
&\leq \Omega(\alpha_0)\|M_{\alpha_0}\|_\infty \left(\frac{(\tau_2 - t_k)^{r_k} - (\tau_1 - t_k)^{r_k}}{\Gamma(r_k + 1)} + (\tau_2 - \tau_1) \sum_{i=1}^{p} \frac{T^*}{\Gamma^*} \right) + pL_2(\tau_2 - \tau_1) \\
&\leq \Omega(\alpha_0)\|M_{\alpha_0}\|_\infty \left(\frac{(\tau_2 - t_k)^{r_k} - (\tau_1 - t_k)^{r_k}}{\Gamma(r_k + 1)} + p(\tau_2 - \tau_1) \frac{T^*}{\Gamma^*} \right) + pL_2(\tau_2 - \tau_1).
\end{aligned}$$

Hence $N(\mathfrak{B}) \subset \mathfrak{B}$.

Step 2.

N is weakly sequentially continuous.

Let (x_n) be a sequence in $\mathfrak{B}$ and let $x_n(t) \to x(t)$ in $\mathscr{F}_\omega)$. Since ψ satisfies assumptions (H1), we have $\psi(t, x_n(t), x_n(\theta(t)))$ converging weakly uniformly to $\psi(t, x(t), x(\theta(t)))$. By assumption (H6) and Theorem (1.3.3), we have

$$\lim_{n\to\infty} Nx_n(t) = Nx(t)$$

i.e. $Nx_n(t) \to Nx(t)$. Then $N : \mathfrak{B} \to \mathfrak{B}$ is weakly sequentially continuous.

Step 3.

The operator $N : \mathfrak{B} \to \mathfrak{B}$ is power-convex condensing.

Let us take a bounded, convex and closed subset $\mathfrak{D}$ of $\mathfrak{B}$ as $\mathfrak{D} = \overline{co}N(\mathfrak{B})$. we have $N(\overline{co}N(\mathfrak{B})) \subset N(\mathfrak{D}) \subset \overline{co}N(\mathfrak{B})$, i.e $N : \mathfrak{D} \to \mathfrak{D}$. Lemma (.) confirms that $\mathfrak{D}$ is equicontinuous in $C(J, \mathfrak{F}_w)$. Obviously, N is bounded and continuous. Take $x_0 \in \mathfrak{D}$, we are going to prove that there is n_0, such that, for any bounded $U \subset \mathfrak{D}$,

$$\beta(N^{(n,x_0)}(U)) \leq \beta(U).$$

By $U \subset \mathfrak{D} \subset \mathfrak{B}$, $N(U)$ is equicontinuous. Then $N^{(2,x_0)}(U)$ is equicontinuous from $N^{(2,x_0)}(U) = N(\overline{co}N(\mathfrak{B}), x_0) \subset N(\mathfrak{B})$. In general, $\forall n \in \mathbb{N}$, $N^{(n,x_0)}(U)$ is equicontinuous. Since $N^{(n,x_0)}(U)$ is bounded, By Lemma (1.4.5)

$$\beta(N^{(n,x_0)}(U)) = \max_{t\in J}(N^{(n,x_0)}(U)(t)), \quad n = 2, 3, \cdots \tag{2.8}$$

We have

$$\begin{aligned}
\beta(N^{(1,x_0)}(U)(t)) &= \beta(NU(t)) \\
&\leq \int_{t_k}^{t} \frac{(t-s)^{r_k-1}}{\Gamma(r_k)} \beta(\psi(s, U(s), U(\theta(s))))ds \\
&\quad + \sum_{i=1}^{k} \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-1}}{\Gamma(r_{i-1})} \beta(\psi(s, U(s), U(\theta(s))))ds \right]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{k-1}(t_k-t_i)\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)}\beta(\psi(s,U(s),U(\theta(s))))ds\right]\\
&+\sum_{i=1}^{k}(t-t_k)\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)}\beta(\psi(s,U(s),U(\theta(s))))ds\right]\\
&+\left|1-\sum_{k=0}^{p}\frac{\lambda_k(\eta_k-t_k)^{\beta_k}}{\Gamma(\beta_k+1)}\right|^{-1}\times\left\{\sum_{k=0}^{p}\lambda_k\int_{t_k}^{\eta_k}\frac{(\eta_k-s)^{r_k+\beta_k-1}}{\Gamma(r_k+\beta_k)}\beta(\psi(s,U(s),U(\theta(s))))ds\right.\\
&+\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\eta_k-t_k)^{\beta_k}}{\Gamma(\beta_k+1)}\times\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{r_{i-1}-1}}{\Gamma(r_{i-1})}\beta(\psi(s,U(s),U(\theta(s))))ds\right]\\
&+\sum_{k=1}^{p}\sum_{i=1}^{k-1}\frac{\lambda_k(\eta_k-t_k)^{\beta_k}(t_k-t_i)}{\Gamma(\beta_k+1)}\times\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)}\beta(\psi(s,U(s),U(\theta(s))))ds\right]\\
&\left.+\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\eta_k-t_k)^{\beta_k+1}}{\Gamma(\beta_k+2)}\times\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)}\beta(\psi(s,U(s),U(\theta(s))))ds+\right]\right\}\\
\leq\ &\rho\left\|M_{\alpha_0}\right\|\beta(U)\int_{t_k}^{t}\frac{(t-s)^{r_k-1}}{\Gamma(r_k)}ds+\sum_{i=1}^{k}\left[\rho\left\|M_{\alpha_0}\right\|\beta(U)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{r_{i-1}-1}}{\Gamma(r_{i-1})}ds\right]\\
&+\sum_{i=1}^{k-1}(t_k-t_i)\left[\rho\left\|M_{\alpha_0}\right\|\beta(U)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)}ds\right]\\
&+\sum_{i=1}^{k}(t-t_k)\left[\rho\left\|M_{\alpha_0}\right\|\beta(U)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)}ds\right]\\
&+\Lambda_1\times\left\{\sum_{k=0}^{p}\lambda_k\rho\left\|M_{\alpha_0}\right\|\beta(U)\int_{t_k}^{\eta_k}\frac{(\eta_k-s)^{r_k+\beta_k-1}}{\Gamma(r_k+\beta_k)}ds\right.\\
&+\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\eta_k-t_k)^{\beta_k}}{\Gamma(\beta_k+1)}\times\left[\rho\left\|M_{\alpha_0}\right\|\beta(U)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{r_{i-1}-1}}{\Gamma(r_{i-1})}ds\right]\\
&+\sum_{k=1}^{p}\sum_{i=1}^{k-1}\frac{\lambda_k(\eta_k-t_k)^{\beta_k}(t_k-t_i)}{\Gamma(\beta_k+1)}\times\left[\rho\left\|M_{\alpha_0}\right\|\beta(U)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)}ds\right]\\
&\left.+\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\eta_k-t_k)^{\beta_k+1}}{\Gamma(\beta_k+2)}\times\left[\rho\left\|M_{\alpha_0}\right\|\beta(U)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)}ds\right]\right\}\\
\leq\ &\rho\|M_{\alpha_0}\|_\infty\beta(U)\left(\sum_{i=1}^{p+1}\frac{(t_i-t_{i-1})^{r_{i-1}}}{\Gamma(r_{i-1}+1)}+T\sum_{i=1}^{p-1}\frac{(t_i-t_{i-1})^{r_{i-1}-1}}{\Gamma(r_{i-1})}+T\sum_{i=1}^{p}\frac{(t_i-t_{i-1})^{r_{i-1}-1}}{\Gamma(r_{i-1})}\right.\\
&+\Lambda_1\times\left\{\sum_{k=0}^{p}\frac{\lambda_k(\eta_k-t_k)^{r_k+\beta_k}}{\Gamma(r_k+\beta_k+1)}+\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\eta_k-t_k)^{\beta_k}}{\Gamma(\beta_k+1)}\times\frac{(t_i-t_{i-1})^{r_{i-1}-1}}{\Gamma(r_{i-1})}\right.\\
&+\sum_{k=1}^{p}\sum_{i=1}^{k-1}\frac{\lambda_k(\eta_k-t_k)^{\beta_k}(t_k-t_i)}{\Gamma(\beta_k+1)}\times\frac{(t_i-t_{i-1})^{r_{i-1}-1}}{\Gamma(r_{i-1})}\\
&\left.\left.+\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\eta_k-t_k)^{\beta_k+1}}{\Gamma(\beta_k+2)}\times\frac{(t_i-t_{i-1})^{r_{i-1}-1}}{\Gamma(r_{i-1})}\right\}\right)\\
\leq\ &\rho\|M_{\alpha_0}\|_\infty\beta(U)\left(\sum_{i=1}^{p+1}\frac{T^*}{\Gamma^*}+T\sum_{i=1}^{p-1}\frac{T^*}{\Gamma^*}+T\sum_{i=1}^{p}\frac{T^*}{\Gamma^*}\right.
\end{aligned}
$$

$$
\begin{aligned}
& +\Lambda_1 \times \left\{ \sum_{k=0}^{p} \frac{\lambda_k T^{r_k+\beta_k}}{\Gamma(r_k+\beta_k+1)} + p \sum_{k=1}^{p} \frac{\lambda_k T^{\beta_k} T^*}{\Gamma(\beta_k+1)\Gamma^*} \right. \\
& \left. \left. +(p-1) \sum_{k=1}^{p} \frac{\lambda_k T^{\beta_k} T^*}{\Gamma(\beta_k+1)\Gamma^*} + p \sum_{k=1}^{p} \frac{\lambda_k T^{\beta_k} T^*}{\Gamma(\beta_k+2)\Gamma^*} \right\} \right) \\
\leq \;& \rho \|M_{\alpha_0}\|_\infty \beta(U) \left(\frac{3pT^*}{\Gamma^*} + \Lambda_1 \left[\Lambda_2 + \frac{(2p-1)T^*}{\Gamma^*} \Lambda_3 + \frac{pT^*}{\Gamma^*} \Lambda_4 \right] \right)
\end{aligned}
$$

which means that

$$
\beta(N^{(1,x_0)}(U)(t)) \leq \rho \|M_{\alpha_0}\|_\infty \left(\frac{3pT^*}{\Gamma^*} + \Lambda_1 \Delta_1 \right) \beta(U).
$$

$L(t,s)\psi(s, \overline{\text{co}}\{N(U)(s), x_0\}(s), \overline{\text{co}}\{N(U)(t), x_0\}(\theta(s)))$ is equicontinuous as $N^{(1,x_0)}(U)(t) = N(U)$ is equicontinuous. Thus,

$$
\begin{aligned}
\beta(N^{(2,x_0)}(U)(t)) \;=\;& \beta(N\overline{\text{co}}\{N^{(1,x_0)}(U)(t)\}) = \beta(N\overline{\text{co}}\{N(U)(t)\}) \\
\leq \;& \rho \left\|M_{\alpha_0}\right\| \beta(N(U)) \int_{t_k}^{t} \frac{(t-s)^{r_k-1}}{\Gamma(r_k)} ds + \sum_{i=1}^{k} \left[\rho \left\|M_{\alpha_0}\right\| \beta(N(U)) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-1}}{\Gamma(r_{i-1})} ds \right] \\
& + \sum_{i=1}^{k-1} (t_k - t_i) \left[\rho \left\|M_{\alpha_0}\right\| \beta(N(U)) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)} ds \right] \\
& + \sum_{i=1}^{k} (t - t_k) \left[\rho \left\|M_{\alpha_0}\right\| \beta(N(U)) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)} ds \right] \\
& + \Lambda_1 \times \left\{ \sum_{k=0}^{p} \lambda_k \rho \left\|M_{\alpha_0}\right\| \beta(N(U)) \int_{t_k}^{\eta_k} \frac{(\eta_k-s)^{r_k+\beta_k-1}}{\Gamma(r_k+\beta_k)} ds \right. \\
& + \sum_{k=1}^{p} \sum_{i=1}^{k} \frac{\lambda_k (\eta_k - t_k)^{\beta_k}}{\Gamma(\beta_k+1)} \times \left[\rho \left\|M_{\alpha_0}\right\| \beta(N(U)) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-1}}{\Gamma(r_{i-1})} ds \right] \\
& + \sum_{k=1}^{p} \sum_{i=1}^{k-1} \frac{\lambda_k (\eta_k - t_k)^{\beta_k} (t_k - t_i)}{\Gamma(\beta_k+1)} \times \left[\rho \left\|M_{\alpha_0}\right\| \beta(N(U)) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)} ds \right] \\
& \left. + \sum_{k=1}^{p} \sum_{i=1}^{k} \frac{\lambda_k (\eta_k - t_k)^{\beta_k+1}}{\Gamma(\beta_k+2)} \times \left[\rho \left\|M_{\alpha_0}\right\| \beta(N(U)) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{r_{i-1}-2}}{\Gamma(r_{i-1}-1)} ds \right] \right\}
\end{aligned}
$$

$$\begin{aligned}
&\leq \rho\|M_{\alpha_0}\|_\infty \beta(N(U))\left(\sum_{i=1}^{p+1}\frac{(t_i-t_{i-1})^{r_{i-1}}}{\Gamma(r_{i-1}+1)}+T\sum_{i=1}^{p-1}\frac{(t_i-t_{i-1})^{r_{i-1}-1}}{\Gamma(r_{i-1})}+T\sum_{i=1}^{p}\frac{(t_i-t_{i-1})^{r_{i-1}-1}}{\Gamma(r_{i-1})}\right.\\
&\quad+\Lambda_1\times\left\{\sum_{k=0}^{p}\frac{\lambda_k(\eta_k-t_k)^{r_k+\beta_k}}{\Gamma(r_k+\beta_k+1)}+\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\eta_k-t_k)^{\beta_k}}{\Gamma(\beta_k+1)}\times\frac{(t_i-t_{i-1})^{r_{i-1}-1}}{\Gamma(r_{i-1})}\right.\\
&\quad+\sum_{k=1}^{p}\sum_{i=1}^{k-1}\frac{\lambda_k(\eta_k-t_k)^{\beta_k}(t_k-t_i)}{\Gamma(\beta_k+1)}\times\frac{(t_i-t_{i-1})^{r_{i-1}-1}}{\Gamma(r_{i-1})}\\
&\quad\left.\left.+\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\eta_k-t_k)^{\beta_k+1}}{\Gamma(\beta_k+2)}\times\frac{(t_i-t_{i-1})^{r_{i-1}-1}}{\Gamma(r_{i-1})}\right\}\right)\\
&\leq \rho\|M_{\alpha_0}\|_\infty \beta(N(U))\left(\sum_{i=1}^{p+1}\frac{T^*}{\Gamma^*}+T\sum_{i=1}^{p-1}\frac{T^*}{\Gamma^*}+T\sum_{i=1}^{p}\frac{T^*}{\Gamma^*}\right.\\
&\quad+\Lambda_1\times\left\{\sum_{k=0}^{p}\frac{\lambda_k T^{r_k+\beta_k}}{\Gamma(r_k+\beta_k+1)}+p\sum_{k=1}^{p}\frac{\lambda_k T^{\beta_k}T^*}{\Gamma(\beta_k+1)\Gamma^*}\right.\\
&\quad\left.\left.+(p-1)\sum_{k=1}^{p}\frac{\lambda_k T^{\beta_k}T^*}{\Gamma(\beta_k+1)\Gamma^*}+p\sum_{k=1}^{p}\frac{\lambda_k T^{\beta_k}T^*}{\Gamma(\beta_k+2)\Gamma^*}\right\}\right)\\
&\leq \rho\|M_{\alpha_0}\|_\infty \beta(N(U))\left(\frac{3pT^*}{\Gamma^*}+\Lambda_1\left[\Lambda_2+\frac{(2p-1)T^*}{\Gamma^*}\Lambda_3+\frac{pT^*}{\Gamma^*}\Lambda_4\right]\right),
\end{aligned}$$

which gives

$$\beta(N^{(2,x_0)}(U)(t))\leq\left[\rho\|M_{\alpha_0}\|_\infty\left(\frac{3pT^*}{\Gamma^*}+\Lambda_1\Delta_1\right)\right]^2\beta(U).$$

Generally,

$$\beta(N^{(n,x_0)}(U)(t))\leq\left[\rho\|M_{\alpha_0}\|_\infty\left(\frac{3pT^*}{\Gamma^*}+\Lambda_1\Delta_1\right)\right]^n\beta(U).$$

Hence, there is $n_0 \in \mathbb{N}^*$ such that $\forall t \in J$,

$$\beta(N^{(n_0,x_0)}(U)(t)) \leq \left[\rho \|M_{\alpha_0}\|_\infty \left(\frac{3pT^*}{\Gamma^*} + \Lambda_1\Delta_1\right)\right]_0^n \beta(U) \leq \beta(U), \text{ from (2.7).}$$

Using (2.8), $\beta(N^{(n,x_0)}(U)) \leq \beta(U)$, thus $N : U \to u$ is convex-power condensing. Applying Theorem (1.7.1) we conclude that N has a fixed point which is a solution of the problem (2.1). □

2.1.3 Example

For $r_0 = \frac{3}{2}$, $r_1 = 2$, $\beta_0 = 2$, $\beta_1 = 1$, $\lambda_0 = \frac{1}{2}$, $\lambda_1 = \frac{7}{3}$; $\eta_0 = \frac{2}{5}$, $\eta_1 = \frac{7}{2}$ and $t_1 = \frac{1}{2}$, we consider the following impulsive multiorders fractional differential equation:

$$\begin{cases} {}^cD^{r_k}_{t_k^+} x(t) = \frac{1}{2e^t}\left(\frac{(x(t)+x(cos(t)))}{1+||x||}\right), \ t \in [0,1], \ t \neq \frac{1}{2}, \ k = 0,1 \\ \\ \Delta x(\frac{1}{2}) = \frac{2x(\frac{1}{2})}{1+||x||}, \ \ \Delta x'(\frac{1}{2}) = \frac{x(\frac{1}{2})}{1+||x||} \\ x(0) = \sum\limits_{k=0}^{1} \lambda_k \mathfrak{I}^{\beta_k}_{t_k^+} x(\eta_k), \ \ x'(0) = 0, \ t_k < \eta_k < t_{k+1}, \end{cases}$$

we have, for each $x \in \mathfrak{F}$,

$$|\psi(t, x(t), x(\theta(t)))| \leq \frac{1}{e^t}\left(\frac{||x||}{5+||x||}\right), t \in [0,1].$$

$$\frac{2x(\frac{1}{2})}{1+||x||} \leq 2, \ \frac{x(\frac{1}{2})}{1+||x||} \leq 1.$$

Hence conditions (H1), (H2), (H4) and (H6) are satisfied with

$$\begin{aligned} M_r(t) &= \frac{1}{e^t} \\ \Omega(u) &= \frac{u}{5+u}, \quad u \in [0,\infty) \\ L_1 = 2 \quad &, \quad L_2 = 1 \end{aligned}$$

By (.), for any bounded set $U \subset \mathfrak{F}$, we have

$$\beta(\psi(t,U,U) \leq \frac{1}{e^t} \times \frac{1}{5+||x||}\beta(U) \leq \frac{1}{5e^t} \times \beta(U).$$

Hence (H3) is satisfied.

Bu a simple computation, we get $\Lambda_1 \simeq 0.40,\ \Lambda_2 \simeq 0.10,\ \Lambda_3 \simeq 1.25,\ \Lambda_4 \simeq 0.41,\ \|M_{\alpha_0}\|_\infty = 1,\ \Delta_1 \simeq 1.97,\ \Delta \simeq 4.91$, then (2.7) is satisfied.

We have

$$\frac{\alpha_0}{\Omega(r_0)\|M_{\alpha_0}\|_\infty \left(\frac{3pT^*}{\Gamma^*} + \Lambda_1\Delta_1\right) + \Delta_2} > 1,$$

implies

$$\frac{\alpha_0 + 5}{\left(\frac{6}{\sqrt{\pi}} + 0.4 \times 1.97\right) + 4.91} > 1,$$

that is $\alpha_0 > 4.1$ and $(H5)$ is satisfied.

$\rho\|M_{\alpha_0}\|_\infty \left(\frac{3pT^*}{\Gamma^*} + \Lambda_1\Delta_1\right) = \frac{1}{5}\left(\frac{6}{\sqrt{\pi}} + 0.4 \times 1.97\right) \simeq 0.83 < 1$. As a consequence of Theorem (1.7.1) the problem (2.1) has a solution defined on $[0,1]$.

2.2 BVP of order $r \in (1;2]$

2.2.1 Introduction

we consider the following fractional integral boundary value problem of the form:

$$\begin{cases} {}^cD^r x(t) = \psi(t, x(t), Hx(t), Sx(t)), \quad 1 < r \leq 2,\ k = 0,1,...,m,\ t \in J', \\ \\ \Delta x(t_k) = \varphi_k(x(t_k), x'(t_k)),\ \ \Delta x'(t_k) = \varphi_k^*(x(t_k), x'(t_k)),\ \ k = 1,2,...,m, \\ \delta x(0) + \mu x'(0) = \int_0^T \sigma_1(s)ds, \delta x(T) + \mu x'(T) = \int_0^T \sigma_2(s)ds, \end{cases} \tag{2.9}$$

where ${}^cD^r$ is the Caputo fractional derivative of order r, $J = [0,T]$, $\psi \in C(J \times \mathscr{F} \times \mathscr{F} \times \mathscr{F}, \mathscr{F})$ is a given function satisfying some assumptions that will be specified later, $\varphi_k, \varphi_k^* \in C(\mathscr{F} \times \mathscr{F}, \mathscr{F})$, $0 = t_0 < t_1 < ... < t_k < ... < t_{m-1} < t_m = T$, $J' = J/\{t_0, t_1, ..., t_m\}$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, where $x(t_k^+)$ and $x(t_k^-)$ denote the right and the left limits of $x(t)$ at $t = t_k$ $(k = 1,2,...,m)$, respectively. $\Delta x'(t_k)$ have a similar meaning for $x'(t)$, $\sigma_1, \sigma_2 : \mathscr{F} \longrightarrow \mathscr{F}$, $\delta > 0$, $\mu \geq 0$ and $\mathscr{F}$ is a Banach space with norm $\|.\|$. Moreover $Hx(t) = \int_0^t g(t,s)x(s)ds$, $Sx(t) = \int_0^t h(t,s)x(s)ds$ where g and h will be specified later.

2.2.2 Main Results

Definition 2.2.1. *A map $x \in PC^1(J, \mathscr{F})$ is called a solution of problem (2.9) if it satisfies the equation ${}^cD^r x(t) = \psi(t, x(t), Hx(t), Sx(t))$ on J' and the conditions $\delta x(0) + \mu x'(0) = \int_0^T \sigma_1(x(s))ds$, $\delta x(T) + \mu x'(T) = \int_0^T \sigma_2(x(s))ds$, $\Delta x(t_k) = \varphi_k(x(t_k), x'(t_k))$, and $\Delta x'(t_k) = \varphi_k^*(x(t_k), x'(t_k))$, for $k = 0,1,2,...,m$.*

Lemma 2.2.2. *For any $\sigma, \sigma_1, \sigma_2 \in C(J, \mathscr{F})$, a function x is a solution of the impulsive boundary value problem (2.10)-(2.13) :*

$${}^cD^r x(t) = \sigma(t), \quad 1 < r \leq 2,\ k = 0,1,...,m,\ t \in J', \tag{2.10}$$

$$\Delta x(t_k) = \varphi_k(x(t_k), x'(t_k)), \;\; \Delta x'(t_k) = \varphi_k^*(x(t_k), x'(t_k)), \;\; k = 1, 2, ..., m, \tag{2.11}$$

$$\delta x(0) + \mu x'(0) = \int_0^T \sigma_1(s) ds, \tag{2.12}$$

$$\delta x(T) + \mu x'(T) = \int_0^T \sigma_2(s) ds, \tag{2.13}$$

if and only if x is a solution of the following impulsive fractional integral equation

$$x(t) = \begin{cases} \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} \sigma(s) ds + \mathscr{A}, & t \in J_0 \\ \\ \frac{1}{\Gamma(r)} \int_{t_k}^t (t-s)^{r-1} \sigma(s) ds + \mathscr{A} \\ + \sum\limits_{0<t_k<t} \left(\frac{1}{\Gamma(r)} \left[\int_{t_{k-1}}^{t_k} (t_k - s)^{r-1} \sigma(s) ds + \varphi_k(x(t_k), x'(t_k)) \right] \right) \\ + \sum\limits_{0<t_k<t} \left(\frac{(t-t_k)}{\Gamma(r-1)} \left[\int_{t_{k-1}}^{t_k} (t_k - s)^{r-2} \sigma(s) ds + \varphi_k^*(x(t_k), x'(t_k)) \right] \right), & t \in J_k \end{cases} \tag{2.14}$$

where

$$\begin{aligned} \mathscr{A} = &+ \frac{1}{T} \left(\frac{\mu}{\delta} - t \right) \left(\frac{1}{\Gamma(r)} \int_{t_k}^T (T-s)^{r-1} \sigma(s) ds + \frac{\mu}{\delta} \frac{1}{\Gamma(r-1)} \int_{t_k}^T (T-s)^{r-2} \sigma(s) ds \right. \\ &+ \sum_{0<t_k<T} \left[\frac{1}{\Gamma(r)} \int_{t_{k-1}}^{t_k} (t_k - s)^{r-1} \sigma(s) ds + \varphi_k(x(t_k), x'(t_k)) \right] \\ &+ \left. \sum_{0<t_k<T} \left[\frac{(\frac{\mu}{\delta} + T - t_k)}{\Gamma(r-1)} \int_{t_{k-1}}^{t_k} (t_k - s)^{r-2} \sigma(s) ds + \varphi_k^*(x(t_k), x'(t_k)) \right] \right) \\ &+ \frac{1}{T\delta^2} \left[(\delta(T-t) + \mu) \int_0^T \sigma_1(s) ds + (\delta t - \mu) \int_0^T \sigma_2(s) ds \right] \end{aligned}$$

Proof. By Lemma 1.5.18 and for a_0, $a_1 \in \mathfrak{P}$, the solution of (3.1) can be written as

$$x(t) = \mathfrak{I}^r \sigma(t) - a_0 - a_1 t = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} \sigma(s) ds - a_0 - a_1 t, \quad t \in [0, t_1]$$

in view of the relations ${}^cD^{r_1} \mathfrak{I}^{r_1} x(t) = x(t)$ and $\mathfrak{I}^{r_1} \mathfrak{I}^{r_2} x(t) = \mathfrak{I}^{r_1 + r_2} x(t)$ for $r_1, r_2 > 0$, we obtain

$$x'(t) = \frac{1}{\Gamma(r-1)} \int_0^t (t-s)^{r-2} \sigma(s) ds - a_1$$

Applying the boundary condition (2.12), we find that

$$x(t)=\frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}\sigma(s)ds+a_1(\frac{\mu}{\delta}-t)+\frac{1}{\delta}\int_0^T \sigma_1(s)ds, \quad t\in[0,t_1]. \tag{2.15}$$

If $t\in(t_1,t_2]$, then we have

$$x(t)=\frac{1}{\Gamma(r)}\int_{t_1}^t (t-s)^{r-1}\sigma(s)ds-b_0-b_1(t-t_1),$$

where $b_0,b_1\in\mathfrak{P}$.

Using the impulse conditions (2.11), we have

$$\begin{aligned}-b_0=&\frac{1}{\Gamma(r)}\int_0^{t_1}(t_1-s)^{r-1}\sigma(s)ds+a_1(\frac{\mu}{\delta}-t_1)\\&+\frac{1}{\delta}\int_0^T\sigma_1(s)ds+\varphi_1(x(t_1),x'(t_1))\\-b_1=&\frac{1}{\Gamma(r)}\int_0^{t_1}(t_1-s)^{r-2}\sigma(s)ds-a_1+\varphi_1^*(x(t_1),x'(t_1)).\end{aligned}$$

Thus,

$$\begin{aligned}x(t)=&\frac{1}{\Gamma(r)}\int_{t_1}^t (t-s)^{r-1}\sigma(s)ds+\frac{1}{\Gamma(r)}\int_0^{t_1}(t_1-s)^{r-1}\sigma(s)ds\\&+\frac{1}{\delta}\int_0^T\sigma_1(s)ds+\varphi_1(x(t_1),x'(t_1))\\&+a_1(\frac{\mu}{\delta}-t)+(t-t_1)\Bigg[\frac{1}{\Gamma(r-1)}\int_0^{t_1}(t_1-s)^{r-1}\sigma(s)ds\\&+\varphi_1^*(x(t_1),x'(t_1))\Bigg], \quad t\in(t_1,t_2].\end{aligned}$$

Repeating the process in this way, the solution $x(t)$ for $t \in (t_k, t_{k+1}]$ can be written as

$$
\begin{aligned}
x(t) = & \frac{1}{\Gamma(r)} \int_{t_k}^{t} (t-s)^{r-1} \sigma(s) ds + a_1 (\frac{\mu}{\delta} - t) + \frac{1}{\delta} \int_0^1 \sigma_1(s) ds \\
& + \sum_{0<t_k<t} \left[\frac{1}{\Gamma(r)} \int_{t_{k-1}}^{t_k} (t_k - s)^{r-1} \sigma(s) ds + \varphi_k(x(t_k), x'(t_k)) \right] \\
& + \sum_{0<t_k<t} (t - t_k) \left[\frac{1}{\Gamma(r-1)} \int_{t_{k-1}}^{t_k} (t_k - s)^{r-2} \sigma(s) ds + \varphi_k^*(x(t_k), x'(t_k)) \right] \qquad (2.16)
\end{aligned}
$$

Applying the boundary condition (2.13), we find that

$$
\begin{aligned}
a_1 = & \frac{1}{T} \left[\frac{1}{\Gamma(r)} \int_{t_k}^{T} (T-s)^{r-1} \sigma(s) ds + \frac{\mu}{\delta} \frac{1}{\Gamma(r-1)} \int_{t_k}^{T} (T-s)^{r-2} \sigma(s) ds \right. \\
& + \sum_{0<t_k<T} \left(\frac{1}{\Gamma(r)} \int_{t_{k-1}}^{t_k} (t_k - s)^{r-1} \sigma(s) ds + \varphi_k(x(t_k), x'(t_k)) \right) \\
& + \frac{1}{\delta} \left(\int_0^T \sigma_1(s) ds - \int_0^T \sigma_2(s) ds \right) \\
& \left. + \sum_{0<t_k<T} \frac{(\frac{\mu}{\delta} + T - t_k)}{\Gamma(r-1)} \int_{t_{k-1}}^{t_k} (t_k - s)^{r-2} \sigma(s) ds + \varphi_k^*(x(t_k), x'(t_k)) \right]
\end{aligned}
$$

Substituting the value of a_1 in (2.15) and (2.16), we obtain (2.14). Conversely, assume that x satisfies the impulsive fractional integral equation (2.14).

If $t \in [0,t_1]$, then

$$\begin{aligned}
\delta x(0)+\mu x'(0) &= \frac{\mu}{T}\left(\frac{1}{\Gamma(r)}\int_{t_k}^{T}(T-s)^{r-1}\sigma(s)ds+\frac{\mu}{\delta}\frac{1}{\Gamma(r-1)}\int_{t_k}^{T}(T-s)^{r-2}\sigma(s)ds\right.\\
&+\sum_{0<t_k<T}\left[\frac{1}{\Gamma(r)}\int_{t_{k-1}}^{t_k}(t_k-s)^{r-1}\sigma(s)ds+\varphi_k(x(t_k),x'(t_k))\right]\\
&\left.+\sum_{0<t_k<T}\left[\frac{(\frac{\mu}{\delta}+T-t_k)}{\Gamma(r-1)}\int_{t_{k-1}}^{t_k}(t_k-s)^{r-2}\sigma(s)ds+\varphi_k^*(x(t_k),x'(t_k))\right]\right)\\
&+\frac{1}{T\delta}\left[(\delta T+\mu)\int_0^T\sigma_1(s)ds-\mu\int_0^T\sigma_2(s)ds\right]\\
&+\frac{-\mu}{T}\left(\frac{1}{\Gamma(r)}\int_{t_k}^{T}(T-s)^{r-1}\sigma(s)ds+\frac{\mu}{\delta}\frac{1}{\Gamma(r-1)}\int_{t_k}^{T}(T-s)^{r-2}\sigma(s)ds\right.\\
&+\sum_{0<t_k<T}\left[\frac{1}{\Gamma(r)}\int_{t_{k-1}}^{t_k}(t_k-s)^{r-1}\sigma(s)ds+\varphi_k(x(t_k),x'(t_k))\right]\\
&\left.+\sum_{0<t_k<T}\left[\frac{(\frac{\mu}{\delta}+T-t_k)}{\Gamma(r-1)}\int_{t_{k-1}}^{t_k}(t_k-s)^{r-2}\sigma(s)ds+\varphi_k^*(x(t_k),x'(t_k))\right]\right)\\
&+\frac{\mu}{T\delta^2}\left[-\delta\int_0^T\sigma_1(s)ds+\delta\int_0^T\sigma_2(s)ds\right]\\
&=\int_0^T\sigma_1(s)ds,
\end{aligned}$$

and, using the fact that ${}^cD^r$ is the left inverse of I^r, we get

$$ {}^cD^r x(t)=\sigma(t), \text{ for each } t\in[0,t_1].$$

If $t\in(t_k,t_{k+1}]$, $k=1,...m$, and using the fact that ${}^cD^rC=0$ and ${}^cD^rt=0$. where C is a constant, we get

$$ {}^cD^r x(t)=\sigma(t), \text{ for each } t\in(t_k,t_{k+1}].$$

$$\begin{aligned}
\delta x(T)+\mu x'(T) &= \frac{\delta}{\Gamma(r)}\int_{t_k}^{T}(T-s)^{r-1}\sigma(s)ds+\delta\mathscr{A}_{(t=T)} \\
&+\delta\sum_{0<t_k<T}\left(\frac{1}{\Gamma(r)}\left[\int_{t_{k-1}}^{t_k}(t_k-s)^{r-1}\sigma(s)ds+\varphi_k(x(t_k),x'(t_k))\right]\right) \\
&+\delta\sum_{0<t_k<T}\left(\frac{(T-t_k)}{\Gamma(r-1)}\left[\int_{t_{k-1}}^{t_k}(t_k-s)^{r-2}\sigma(s)ds+\varphi_k^*(x(t_k),x'(t_k))\right]\right) \\
&+\frac{\mu}{\Gamma(r-1)}\int_{t_k}^{T}(T-s)^{r-2}\sigma(s)ds+\mu\mathscr{A}'_{(t=T)} \\
&+\frac{\mu}{\Gamma(r-1)}\int_{t_k}^{T}(T-s)^{r-2}\sigma(s)ds \\
&+\mu\sum_{0<t_k<T}\left(\frac{1}{\Gamma(r-1)}\left[\int_{t_{k-1}}^{t_k}(t_k-s)^{r-2}\sigma(s)ds+\varphi_k^*(x(t_k),x'(t_k))\right]\right) \\
&= \frac{\delta}{\Gamma(r)}\int_{t_k}^{T}(T-s)^{r-1}\sigma(s)ds \\
&+\frac{\delta}{T}\left(\frac{\mu}{\delta}-T\right)\left(\frac{1}{\Gamma(r)}\int_{t_k}^{T}(T-s)^{r-1}\sigma(s)ds+\frac{\mu}{\delta}\frac{1}{\Gamma(r-1)}\int_{t_k}^{T}(T-s)^{r-2}\sigma(s)ds\right. \\
&+\sum_{0<t_k<T}\left[\frac{1}{\Gamma(r)}\int_{t_{k-1}}^{t_k}(t_k-s)^{r-1}\sigma(s)ds+\varphi_k(x(t_k),x'(t_k))\right] \\
&\left.+\sum_{0<t_k<T}\left[\frac{(\frac{\mu}{\delta}+T-t_k)}{\Gamma(r-1)}\int_{t_{k-1}}^{t_k}(t_k-s)^{r-2}\sigma(s)ds+\varphi_k^*(x(t_k),x'(t_k))\right]\right) \\
&+\frac{1}{T\delta}\left[\mu\int_0^T\sigma_1(s)ds+(\delta T-\mu)\int_0^T\sigma_2(s)ds\right] \\
&+\delta\sum_{0<t_k<T}\left(\frac{1}{\Gamma(r)}\left[\int_{t_{k-1}}^{t_k}(t_k-s)^{r-1}\sigma(s)ds+\varphi_k(x(t_k),x'(t_k))\right]\right) \\
&+\delta\sum_{0<t_k<T}\left(\frac{(T-t_k)}{\Gamma(r-1)}\left[\int_{t_{k-1}}^{t_k}(t_k-s)^{r-2}\sigma(s)ds+\varphi_k^*(x(t_k),x'(t_k))\right]\right) \\
&-\frac{\mu}{T}(\left(\frac{1}{\Gamma(r)}\int_{t_k}^{T}(T-s)^{r-1}\sigma(s)ds+\frac{\mu}{\delta}\frac{1}{\Gamma(r-1)}\int_{t_k}^{T}(T-s)^{r-2}\sigma(s)ds\right. \\
&+\sum_{0<t_k<T}\left[\frac{1}{\Gamma(r)}\int_{t_{k-1}}^{t_k}(t_k-s)^{r-1}\sigma(s)ds+\varphi_k(x(t_k),x'(t_k))\right]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{0<t_k<T}\left[\frac{(\frac{\mu}{\delta}+T-t_k)}{\Gamma(r-1)}\int_{t_{k-1}}^{t_k}(t_k-s)^{r-2}\sigma(s)ds+\varphi_k^*(x(t_k),x'(t_k))\right]\Bigg)\\
&+\frac{\mu}{T\delta^2}\left[-\delta\int_0^T\sigma_1(s)ds+\delta\int_0^T\sigma_2(s)ds\right]\\
&+\frac{\mu}{\Gamma(r-1)}\int_{t_k}^T(T-s)^{r-2}\sigma(s)ds\\
&+\mu\sum_{0<t_k<T}\left(\frac{1}{\Gamma(r-1)}\left[\int_{t_{k-1}}^{t_k}(t_k-s)^{r-2}\sigma(s)ds+\varphi_k^*(x(t_k),x'(t_k))\right]\right)\\
=&\int_0^T\sigma_2(s)ds
\end{aligned}
$$

This completes the proof. □

Let $D_\alpha=\{x\in PC(J,\mathscr{F}),\ \|x\|\leq\alpha\}$, $\alpha>0$ and let G be the function defined in $[0,T]$ by $G(t,s)=\dfrac{(t-s)^\eta}{\Gamma(\eta+1)}$, $\eta\in\{r-1,r\}$, $t\in J_k$, $s\in[t_{k-1},t]$.

To prove our result, we need the following assumptions:

(H1) For each uniformly ACG_* function $x:J\to\mathscr{F}$, the functions $g(t,.)x(.)$, $h(t,.)x(.)$, $\psi(.,x(.),Hx(.),Sx(.))$ is HKP integrable and ψ is a weakly weakly sequentially continuous function.

(H2) For any $\alpha>0$, there exists a HK-integrable function $M_\alpha:J\to\mathbb{R}^+$ and a non-decreasing continuous function $\Omega:[0,\infty)\to[0,\infty)$ such that $\|\psi(t,x,y,z)\|\leq M_\alpha(t)\Omega(\alpha)$ for all $t\in J$, $(x,y,z)\in D_\alpha\times D_\alpha\times D_\alpha$, and there exist positive constants L_1 and L_2 such that $|\varphi_k(x(.),x'(.))|\leq L_1$, $|\varphi_k^*(x(.),x'(.))|\leq L_2$ for $t\in J,\ x\in D_\alpha,\ k=1,...,m$.

(H3) There exists constants $c_1,c_2>0$ such that

$$\|\sigma_1\|\leq c_1\ ,\ \|\sigma_2\|\leq c_2$$

(H4) For each bounded sets $V,U,W\subset D_\alpha$ and for each closed interval $I\subset J,\ t\in I$, there exist positive constants $\zeta_1,\zeta_2\in J$:

$$\beta(g(I,I)V(I)) \le \zeta_1 \beta(V(I)), \quad \beta(h(I,I)V(I)) \le \zeta_2 \beta(V(I))$$

and

$$\beta(\psi(t,V,U,W)) \le M_\alpha(t) \max(\beta(V),\beta(U),\beta(W)).$$

(H5) The family $\{x^*\psi(.,x(.),Hx(.),Sx(.)) : x^* \in \mathscr{F}^*, \ \|x^*\| \le 1\}$ is uniformly HK-integrable over J for every $x \in D_r$.

(H6) The families $\left\{x^* \int_{J_k} g(t,s)x_n(s)ds\right\}_{n=0}^{\infty}$, $\left\{x^* \int_{J_k} h(t,s)x_n(s)ds\right\}_{n=0}^{\infty}$ and $\left\{x^* \int_{J_k} G(t,s)\psi(.,x_n(s),Hx_n(s),Sx_n(s))ds\right\}_{n=0}^{\infty}$ are uniformly ACG_* and equicontinuous on J for every $t \in J$.

Theorem 2.2.3. *Let $\mathscr{F}$ be a Banach space. If*

$$\|M_\alpha\| \max(1,T\zeta_1,T\zeta_2)\frac{T^*}{\Gamma(r)}\left((m+1)(\frac{\mu}{\delta})^2+(3m+2)\frac{\mu}{\delta}+4m+2\right) < 1, \qquad (2.17)$$

Then the boundary value problem (2.9) has at least one solution.

Proof. In view of Lemma 2.2.2, we define an operator $N : PC(J,\mathscr{F}) \longrightarrow PC(J,\mathscr{F})$ by

$$\begin{aligned}
Nx(t) &= \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}\psi(s,x(s),Hx(s),Sx(s))ds \\
&+ \frac{1}{T}\left(\frac{\mu}{\delta}-t\right)\left(\frac{1}{\Gamma(r)}\int_{t_k}^T (T-s)^{r-1}\psi(s,x(s),Hx(s),Sx(s))ds \right.\\
&+ \frac{\mu}{\delta}\frac{1}{\Gamma(r-1)}\int_{t_k}^T (T-s)^{r-2}\psi(s,x(s),Hx(s),Sx(s))ds \\
&+ \sum_{0<t_k<T}\left[\frac{1}{\Gamma(r)}\int_{t_{k-1}}^{t_k}(t_k-s)^{r-1}\psi(s,x(s),Hx(s),Sx(s))ds+\varphi_k(x(t_k),x'(t_k))\right] \\
&+ \left.\sum_{0<t_k<T}\left[\frac{(\frac{\mu}{\delta}+T-t_k)}{\Gamma(r-1)}\int_{t_{k-1}}^{t_k}(t_k-s)^{r-2}\psi(s,x(s),Hx(s),Sx(s))ds+\varphi_k^*(x(t_k),x'(t_k))\right]\right) \\
&+ \sum_{0<t_k<t}\left(\frac{1}{\Gamma(r)}\left[\int_{t_{k-1}}^{t_k}(t_k-s)^{r-1}\psi(s,x(s),Hx(s),Sx(s))ds+\varphi_k(x(t_k),x'(t_k))\right]\right) \\
&+ \sum_{0<t_k<t}\left(\frac{(t-t_k)}{\Gamma(r-1)}\left[\int_{t_{k-1}}^{t_k}(t_k-s)^{r-2}\psi(s,x(s),Hx(s),Sx(s))ds+\varphi_k^*(x(t_k),x'(t_k))\right]\right) \\
&+ \frac{1}{T\delta^2}\left[(\delta(T-t)+\mu)\int_0^T \sigma_1(s)ds+(\delta t-\mu)\int_0^T \sigma_2(s)ds\right]
\end{aligned}$$

Observe that the fixed points of the operator N are solutions of the problem (2.9). For convenience, we will give some notations: Let $\aleph$ be a family of functions which are uniformly HK-integrable over an interval J. Then following the proof given in [43], Theorem 4.28, $\aleph$ satisfies uniformly the Cauchy criterion over any closed sub-interval $I \subset J$. Analogously to [43], Theorem 4.27, the condition

$\forall \varepsilon > 0$; $\exists$gauge γ; on J; $\forall P_1, P_2$ γ fine-partitions; $\forall \psi \in \aleph$; $|\underline{S}(\psi, P_1) - \underline{S}(\psi, P_2)| < \varepsilon$, implies

$\forall \varepsilon > 0$; $\exists$gauge γ; on J; $\forall P_1, P_2$ γ fine-partitions; $\forall \psi \in \aleph$;

$|\underline{S}(\psi, P) - (HK)\int_J f(t)dt| < \varepsilon$.

Therefore, if $\aleph$ is a family of uniformly HK-integrable functions over the interval $[0, b]$, then the family $\aleph$ is uniformly HK-integrable over $[0, \tau]$ for every $\tau < b$. Consequently, in view of assumption (H4) the family

$$\{x^* \psi(., x(.), Hx(.), Sx(.)) : \ x^* \in \mathscr{F}^*, \ \|x^*\| \leq 1\}$$

is uniformly HK-integrable over any sub-interval $[0, \tau] \subset J$, for every $x \in D_\alpha$. This entails the weak*-continuity of the linear functional

$$x^* \in \mathscr{F}^* \to (HK)\int_0^\tau G(t,s) x^* \psi(s, x(s), Hx(s), Sx(s))ds, \ \forall \tau \in J.$$

This latter means that there is $x_{t,\tau} \in \mathscr{F}$ such that

$$x^* x_{t,\tau} = (HK)\int_0^\tau x^* G(t,s) \psi(s, x(s), Hx(s), Sx(s))ds, \ \forall x^* \in \mathscr{F}^*.$$

That is the function $G(t,.)\psi(,x(.), Hx(.), Sx(.))$ is HKP-integrable on J and thus the operator N makes sense. Consider the set

$$D = D_\alpha \bigcap \{x \in PC(J, \mathscr{F}) : \forall \tau_1, \tau_2 \in J, \tau_1 < \tau_2 : \|x(\tau_2) - x(\tau_1)\| \leq P + \frac{1}{\delta}(\tau_2 - \tau_1)(c_1 + c_2)\}$$

where

$$P = \frac{\|M_\alpha\|\Omega(\alpha)}{\Gamma(r)}\left[(\tau_2^r - \tau_1^r) + (\tau_2 - \tau_1)T^*\left(1 + 5m + (m+1)\frac{\mu}{\delta}\right)\right]$$

. Clearly, the subset D is closed, convex and equicontinuous.

We shall show that N satisfies all the conditions of Theorem **??**. The proof will be given in several steps.

- First we prove that N maps D into itself:

Take $x \in D$ and $t \in J$ and assume that $Nx(t) \neq 0$. By the Hahn-Banach theorem, there exists $x^* \in \mathscr{F}^*$ with $\|x^*\| = 1$ and $\|Nx(t)\| = |x^*(Nx(t))|$. Thus

$$\begin{aligned}
\|Nx(t)\| \leq \Bigg(& \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}|x^*(\psi(s,x(s),Hx(s),Sx(s)))|ds \\
& + \frac{1}{T}\left|\frac{\mu}{\delta} - t\right|\left[\frac{1}{\Gamma(r)}\int_{t_k}^T (T-s)^{r-1}|x^*(\psi(s,x(s),Hx(s),Sx(s)))|ds\right. \\
& + \frac{\mu}{\delta}\frac{1}{\Gamma(r-1)}\int_{t_k}^T (T-s)^{r-2}|x^*(\psi(s,x(s),Hx(s),Sx(s)))|ds \\
& + \sum_{0<t_k<T}\left(\frac{1}{\Gamma(r)}\int_{t_{k-1}}^{t_k}(t_k-s)^{r-1}|x^*(\psi(s,x(s),Hx(s),Sx(s)))|ds + |\varphi_k(x(t_k),x'(t_k))|\right) \\
& \left.+ \sum_{0<t_k<T}(\frac{\mu}{\delta} + T)\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{r-2}}{\Gamma(r-1)}|x^*(\psi(s,x(s),Hx(s),Sx(s)))|ds + |\varphi_k^*(x(t_k),x'(t_k))|\right)\right] \\
& + \sum_{0<t_k<t}\left(\frac{1}{\Gamma(r)}\int_{t_{k-1}}^{t_k}(t_k-s)^{r-1}|x^*(\psi(s,x(s),Hx(s),Sx(s)))|ds + |\varphi_k(x(t_k),x'(t_k))|\right) \\
& + \sum_{0<t_k<t}(t-t_k)\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{r-2}}{\Gamma(r-1)}|x^*(\psi(s,x(s),Hx(s),Sx(s)))|ds + |\varphi_k^*(x(t_k),x'(t_k))|\right) \\
& + \frac{1}{T\delta^2}\left[(\delta(T-t)+\mu)\int_0^T|\sigma_1(s)|ds + |\delta t - \mu|\int_0^T|\sigma_2(s)|ds\right]\Bigg) \\
\leq & \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}M_\alpha(s)\Omega(\alpha)ds + \frac{1}{T}(\frac{\mu}{\delta}+T)\left[\int_{t_k}^T\frac{(T-s)^{r-1}}{\Gamma(r)}M_\alpha(s)\Omega(\alpha)ds\right. \\
& + \frac{\mu}{\delta}\int_{t_k}^T\frac{(T-s)^{r-2}}{\Gamma(r-1)}M_\alpha(s)\Omega(\alpha)ds \\
& + \sum_{0<t_k<T}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{r-1}}{\Gamma(r)}M_\alpha(s)\Omega(\alpha)ds + |\varphi_k(x(t_k),x'(t_k))|\right) \\
& \left.+ \sum_{0<t_k<T}(\frac{\mu}{\delta}+T)\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{r-2}}{\Gamma(r-1)}M_\alpha(s)\Omega(\alpha)ds + |\varphi_k^*(x(t_k),x'(t_k))|\right)\right] \\
& + \sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{r-1}}{\Gamma(r)}M_\alpha(s)\Omega(\alpha)ds + |\varphi_k(x(t_k),x'(t_k))|\right)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{0<t_k<t}(t-t_k)\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{r-2}}{\Gamma(r-1)}M_\alpha(s)\Omega(\alpha)ds+|\varphi_k^*(x(t_k),x'(t_k))|\right)\\
&+\frac{\delta T+\mu}{T\delta^2}\left[\int_0^T|\sigma_1(s)|ds+\int_0^T|\sigma_2(s)|ds\right]\\
\leq\ & ||M_\alpha||\Omega(\alpha)\frac{1}{\Gamma(r)}\int_0^t(t-s)^{r-1}ds+\frac{1}{T}(\frac{\mu}{\delta}+T)\left[||M_\alpha||\Omega(\alpha)\int_{t_k}^T\frac{(T-s)^{r-1}}{\Gamma(r)}ds\right.\\
&+\frac{\mu}{\delta}||M_\alpha||\Omega(\alpha)\int_{t_k}^T\frac{(T-s)^{r-2}}{\Gamma(r-1)}ds+\sum_{0<t_k<T}\left(||M_\alpha||\Omega(\alpha)\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{r-1}}{\Gamma(r)}ds+L_1\right)\\
&\left.+\sum_{0<t_k<T}(\frac{\mu}{\delta}+T)\left(||M_\alpha||\Omega(\alpha))\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{r-2}}{\Gamma(r-1)}ds+L_2\right)\right]\\
&+\sum_{0<t_k<t}\left(||M_\alpha||\Omega(\alpha)\int_{t_{k-1}}^{t_k}(\frac{t_k-s)^{r-1}}{\Gamma(r)}ds+L_1\right)\\
&+\sum_{0<t_k<t}(t-t_k)\left(||M_\alpha||\Omega(\alpha)\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{r-2}}{\Gamma(r-1)}ds+L_2\right)+\frac{\delta T+\mu}{T\delta^2}[Tc_1+Tc_2]\\
\leq\ & ||M_\alpha||\Omega(\alpha)\frac{T^r}{\Gamma(r+1)}+\frac{1}{T}(\frac{\mu}{\delta}+T)[||M_\alpha||\Omega(\alpha)\frac{T^r}{\Gamma(r+1)}+\frac{\mu}{\delta}||M_\alpha||\Omega(\alpha)\frac{T^{r-1}}{\Gamma(r)}\\
&+mT\left(||M_\alpha||\Omega(\alpha)\frac{T^r}{\Gamma(r+1)}+L_1\right)+mT(\frac{\mu}{\delta}+T)\left(||M_\alpha||\Omega(\alpha)\frac{T^{r-1}}{\Gamma(r)}+L_2\right)]\\
&+mT\left(||M_\alpha||\Omega(\alpha)\frac{T^r}{\Gamma(r+1)}+L_1\right)+mT\left(||M_\alpha||\Omega(\alpha)\frac{T^r}{\Gamma(r)}+L_2\right)+\frac{\delta T+\mu}{\delta^2}[c_1+c_2]\\
\leq\ & ||M_\alpha||\Omega(\alpha)(\left(\frac{T^r}{\Gamma(r)})+\frac{\mu}{\delta}\frac{T^{r-1}}{\Gamma(r)}+\frac{T^r}{\Gamma(r)}+(\frac{\mu}{\delta})^2\frac{T^{r-2}}{\Gamma(r)}+\frac{\mu}{\delta}\frac{T^{r-1}}{\Gamma(r)}+\frac{m\mu}{\delta}\frac{T^r}{\Gamma(r)}\right.\\
&+\frac{mT^{r+1}}{\Gamma(r)}+m(\frac{\mu}{\delta})^2\frac{T^{r-1}}{\Gamma(r)}+2m\frac{\mu}{\delta}\frac{T^r}{\Gamma(r)}+m\frac{T^r}{\Gamma(r)}+\frac{mT^{r+1}}{\Gamma(r)}+\frac{mT^{r+1}}{\Gamma(r)}\\
&\left.+m(\frac{\mu}{\delta}+2T)L_1+m((\frac{\mu}{\delta}+T)^2+T)L_2\right)+\frac{\delta T+\mu}{\delta^2}[c_1+c_2]\\
\leq\ & ||M_\alpha||\Omega(\alpha)\frac{T^*}{\Gamma(r)}\left(((m+1)(\frac{\mu}{\delta})^2+(3m+2)\frac{\mu}{\delta}+4m+2)+m(\frac{\mu}{\delta}+2T^*)L_1\right.\\
&\left.+m((\frac{\mu}{\delta}+T^*)^2+T^*)L_2\right)+\frac{\delta T^*+\mu}{\delta^2}[c_1+c_2]\\
\leq\ & r_0.
\end{aligned}
$$

Let $\tau_1,\tau_2\in J_k,\ k=1,...,m,\ \tau_1<\tau_2,\ x\in D$, so $Nx(\tau_2)-Nx(\tau_1)\neq 0$. Then there exists

$x^* \in \mathscr{F}^*$ such that $\|Nx(\tau_2) - Nx(\tau_1)\| = |x^*(Nx(\tau_2) - Nx(\tau_1))|$. Thus

$$
\begin{aligned}
\|Nx(\tau_2)-Nx(\tau_1)\| \leq & \int_0^{\tau_2} \frac{(\tau_2-s)^{r-1}-(\tau_1-s)^{r-1}}{\Gamma(r)}|x^*(\psi(s,x(s),Hx(s),Sx(s)))|ds \\
& + \int_{\tau_1}^{\tau_2} \frac{(\tau_2-s)^{r-1}}{\Gamma(r)}|x^*(\psi(s,x(s),Hx(s),Sx(s)))|ds \\
& + \frac{(\tau_2-\tau_1)}{T}\Bigg[\int_{t_k}^{T} \frac{(T-s)^{r-1}}{\Gamma(r)}|x^*(\psi(s,x(s),Hx(s),Sx(s)))|ds \\
& + \frac{\mu}{\delta}\int_{t_k}^{T} \frac{(T-s)^{r-2}}{\Gamma(r-1)}|x^*(\psi(s,x(s),Hx(s),Sx(s)))|ds \\
& + \sum_{0<t_k<T}\left(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{r-1}}{\Gamma(r)}|x^*(\psi(s,x(s),Hx(s),Sx(s)))|ds + |x^*(\Delta x(t_k))|\right) \\
& + \sum_{0<t_k<T}(\frac{\mu}{\delta}+T-t_k)\Bigg(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{r-2}}{\Gamma(r-1)}|x^*(\psi(s,x(s),Hx(s),Sx(s)))|ds \\
& + |x^*(\Delta x'(t_k))|\Bigg)\Bigg] + \sum_{0<t_k<\tau_2-\tau_1}\left(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{r-1}}{\Gamma(r)}|x^*(\psi(s,x(s),Hx(s),Sx(s)))|ds + |x^*(\Delta x(t_k))|\right) \\
& + (\tau_2-\tau_1)\sum_{0<t_k<\tau_1}\left(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{r-2}}{\Gamma(r-1)}|x^*(\psi(s,x(s),Hx(s),Sx(s)))| + |x^*(\Delta x'(t_k))|\right) \\
& + \sum_{0<t_k<\tau_2-\tau_1}(\tau_2-t_k)\Bigg(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{r-2}}{\Gamma(r-1)}|x^*(\psi(s,x(s),Hx(s),Sx(s)))|ds \\
& + |x^*(\Delta x'(t_k))|\Bigg) + \frac{1}{T\delta}(\tau_2-\tau_1)\left[\int_0^T |(x^*(p_1(s))| + |x^*(p_2(s))|)ds\right] \\
\leq & \int_0^{\tau_1} \frac{(\tau_2-s)^{r-1}-(\tau_1-s)^{r-1}}{\Gamma(r)}||M_\alpha||\Omega(\alpha)ds + \int_{\tau_1}^{\tau_2} \frac{(\tau_2-s)^{r-1}}{\Gamma(r)}||M_\alpha||\Omega(\alpha)ds \\
& + \frac{(\tau_2-\tau_1)}{T}\Bigg[\int_{t_k}^{T} \frac{(T-s)^{r-1}}{\Gamma(r)})||M_\alpha||\Omega(\alpha)ds + \frac{\mu}{\delta}\int_{t_k}^{T} \frac{(T-s)^{r-2}}{\Gamma(r-1)}||M_\alpha||\Omega(\alpha)ds \\
& + \sum_{0<t_k<T}(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{r-1}}{\Gamma(r)}||M_\alpha||\Omega(\alpha)ds + L_1) \\
& + \sum_{0<t_k<T}(\frac{\mu}{\delta}+T-t_k)(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{r-2}}{\Gamma(r-1)}||M_\alpha||\Omega(\alpha)ds + L_2)\Bigg] \\
& + \sum_{0<t_k<\tau_2-\tau_1}(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{r-1}}{\Gamma(r)}||M_\alpha||\Omega(\alpha)ds + L_1) \\
& + (\tau_2-\tau_1)\sum_{0<t_k<\tau_1}(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{r-2}}{\Gamma(r-1)}||M_\alpha||\Omega(\alpha) + L_2) \\
& + \sum_{0<t_k<\tau_2-\tau_1}(\tau_2-t_k)(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{r-2}}{\Gamma(r-1)}||M_\alpha||\Omega(\alpha)ds + L_2) \\
& + \frac{1}{T\delta}(\tau_2-\tau_1)\left[\int_0^T |(x^*(p_1(s))| + |x^*(p_2(s))|)ds\right]
\end{aligned}
$$

$$
\begin{aligned}
&\leq ||M_\alpha||\Omega(\alpha)\frac{1}{\Gamma(r+1)}(\tau_2^r-\tau_1^r)+\frac{(\tau_2-\tau_1)}{T}\Bigg[||M_\alpha||\Omega(\alpha)\frac{(T-t_k)^r}{\Gamma(r+1)}\\
&+\frac{\mu}{\delta}\frac{T^{r-1}}{\Gamma(r)}||M_\alpha||\Omega(\alpha)+Tm\left(\frac{(t_k-t_{k-1})^r}{\Gamma(r+1)}||M_\alpha||\Omega(\alpha)+L_1\right)\\
&+(\frac{\mu}{\delta}+T)mT\left(\frac{(t_k-t_{k-1})^{r-1}}{\Gamma(r)}||M_\alpha||\Omega(\alpha)+L_2\right)\Bigg]\\
&+m(\tau_2-\tau_1)\left(\frac{(t_k-t_{k-1})^r}{\Gamma(r+1)}||M_\alpha||\Omega(\alpha)+L_1\right)\\
&+(\tau_2-\tau_1)m\tau_1\left(\frac{(t_k-t_{k-1})^{r-1}}{\Gamma(r)}||M_\alpha||\Omega(\alpha)+L_2\right)\\
&+m(\tau_2-\tau_1)(\tau_2-t_k)\left(\frac{(t_k-t_{k-1})^{r-1}}{\Gamma(r)}||M_\alpha||\Omega(\alpha)+L_2\right)+\frac{1}{\delta}(\tau_2-\tau_1)T(c_1+c_2)\\
&\leq ||M_\alpha||\Omega(\alpha)\frac{1}{\Gamma(r)}(\tau_2^r-\tau_1^r)+\frac{(\tau_2-\tau_1)}{T}\Bigg[||M_\alpha||\Omega(\alpha)\frac{T^r}{\Gamma(r)}+\frac{\mu}{\delta}\frac{T^{r-1}}{\Gamma(r)}||M_\alpha||\Omega(\alpha)\\
&+Tm\left(\frac{T^r}{\Gamma(r)}||M_\alpha||\Omega(\alpha)+L_1\right)+(\frac{\mu}{\delta}+T)mT(\frac{T^{r-1}}{\Gamma(r)}||M_\alpha||\Omega(\alpha)+L_2)\Bigg]\\
&+m(\tau_2-\tau_1)\left(\frac{T^r}{\Gamma(r)}||M_\alpha||\Omega(\alpha)+L_1\right)+(\tau_2-\tau_1)mT\left(\frac{T^{r-1}}{\Gamma(r)}||M_\alpha||\Omega(\alpha)+L_2\right)\\
&+m(\tau_2-\tau_1)T\left(\frac{T^{r-1}}{\Gamma(r)}||M_\alpha||\Omega(\alpha)+L_2\right)+\frac{1}{\delta}(\tau_2-\tau_1)(c_1+c_2)\\
&\leq ||M_\alpha||\Omega(\alpha)\frac{1}{\Gamma(r)}\left[(\tau_2^r-\tau_1^r)+(\tau_2-\tau_1)T^*\left(1+5m+(m+1)\frac{\mu}{\delta}\right)\right]\\
&+\frac{1}{\delta}(\tau_2-\tau_1)(c_1+c_2)\\
&\leq P+\frac{1}{\delta}(\tau_2-\tau_1)(c_1+c_2)
\end{aligned}
$$

Hence $N(D)\subset D$.

- Then we show that N is weakly sequentially continuous.

Let (x_n) be a sequence in D and let $x_n(t)\to x(t)$ in $C(J,\mathscr{F}_w)$. By (H6), we have

$$\int_0^T g(t,s)x_n(s)ds \to \int_0^T g(t,s)x(s)ds$$
$$\int_0^T h(t,s)x_n(s)ds \to \int_0^T h(t,s)x(s)ds$$

Weakly in $\mathscr{F}$, for each $t\in J$ and $Hx_n(t)\to Hx(t)$, $Sx_n(t)\to Sx(t)$. Therefore, the operators

H, S are weakly sequentially continuous in D. Moreover, because ψ is weakly weakly sequentially continuous,

$$\psi(t, x_n(t), Hx_n(t), Sx_n(t)) \to \psi(t, x(t), Hx(t), Sx(t))$$

weakly in $\mathscr{F}$, for each $t \in J$. Now, applying assumption (H4), Theorem 5 in [30] and Lemma 25 in [56], the function $G(t,.)\psi(., x_n(.), Hx_n(.), Sx_n(.))$ is HKP-integrable on J for every $n \geq 1$. By Theorem 1.3.3 and assumption (H6), we have

$$\lim_{n\to\infty} Nx_n(t) = Nx(t)$$

i.e. $Nx_n(t) \to Nx(t)$ in $PC(J, \mathscr{F})$.

- After that we shall show that the operator $N : D \to D$ is power-convex condensing.

Let $B = \overline{\text{co}}N(D) \subset D$. Obviously, B is bounded, convex, closed, and $N(\overline{\text{co}}N(D)) \subset N(B) \subset \overline{\text{co}}N(D)$, i.e $N : B \to B$. By Lemma 1.4.5, B is equicontinuous in $C(J, \mathscr{F}_w)$. Obviously, N is bounded and continuous. Set $x_0 \in B$, we will prove that for all $n \in \mathbb{N}$ and for any bounded $V \subset B$,

$$\beta(N^{(n_0,x_0)}(V)) \leq \beta(V).$$

By $V \subset B \subset D$, $N(V)$ is equicontinuous. Then $N^{(2,x_0)}(V)$ is equicontinuous from $N^{(2,x_0)}(V) = N(\overline{\text{co}}N(D), x_0) \subset N(D)$. Generally, $\forall n \in \mathbb{N}$, $N^{(n,x_0)}(V)$ is equicontinuous. Since $N^{(n,x_0)}(V)$ is bounded, By Lemma 1.4.5

$$\beta(N^{(n,x_0)}(V)) = \max_{t\in J}(N^{(n,x_0)}(V)(t)), \;\; n = 2, 3, \ldots \tag{2.18}$$

We have

$$\begin{aligned}
\beta(N^{(1,x_0)}(V)(t)) &= \beta(NV(t))\\
&\leq \int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)}\beta(\psi(s,V(s),HV(s),SV(s)))ds\\
&+\frac{1}{T}(\frac{\mu}{\delta}-t)\Bigg[\int_{t_k}^{T}\frac{(T-s)^{r-1}}{\Gamma(r)}\beta(\psi(s,V(s),HV(s),SV(s)))ds\\
&+\frac{\mu}{\delta}\int_{t_k}^{T}\frac{(T-s)^{r-2}}{\Gamma(r-1)}\beta(\psi(s,V(s),HV(s),SV(s)))ds\\
&+\sum_{0<t_k<T}\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{r-1}}{\Gamma(r)}\beta(\psi(s,V(s),HV(s),SV(s)))ds\\
&+\sum_{0<t_k<T}(\frac{\mu}{\delta}+T-t_k)\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{r-2}}{\Gamma(r-1)}\beta(\psi(s,V(s),HV(s),SV(s)))ds\Bigg]\\
&+\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{r-1}}{\Gamma(r)}\beta(\psi(s,V(s),HV(s),SV(s)))ds\\
&+\sum_{0<t_k<t}(t-t_k)\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{r-2}}{\Gamma(r-1)}\beta(\psi(s,V(s),HV(s),SV(s)))ds\\
&\leq \|M_\alpha\|\max(\beta(V),\beta(HV),\beta(SV))\int_0^t\frac{(t-s)^{r-1}}{\Gamma(r)}ds\\
&+\frac{1}{T}|\frac{\mu}{\delta}-t|\Bigg[\|M_\alpha\|\max(\beta(V),\beta(HV),\beta(SV))\int_{t_k}^{T}\frac{(T-s)^{r-1}}{\Gamma(r)}ds\\
&+\frac{\mu}{\delta}\|M_\alpha\|\max(\beta(V),\beta(HV),\beta(SV))\int_{t_k}^{T}\frac{(T-s)^{r-2}}{\Gamma(r-1)}ds\\
&+\sum_{0<t_k<T}\|M_\alpha\|\max(\beta(V),\beta(HV),\beta(SV))\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{r-1}}{\Gamma(r)}ds\\
&+\sum_{0<t_k<T}(\frac{\mu}{\delta}+T-t_k)\|M_\alpha\|\max(\beta(V),\beta(HV),\beta(SV))\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{r-2}}{\Gamma(r-1)}ds\Bigg]\\
&+\sum_{0<t_k<t}\|M_\alpha\|\max(\beta(V),\beta(HV),\beta(SV))\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{r-1}}{\Gamma(r)}ds\\
&+\sum_{0<t_k<t}(t-t_k)\|M_\alpha\|\max(\beta(V),\beta(HV),\beta(SV))\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{r-2}}{\Gamma(r-1)}ds\\
&\leq \|M_\alpha\|\max(\beta(V),T\zeta_1\beta(V),T\zeta_2\beta(V))\Bigg(\frac{t^r}{\Gamma(r+1)}+\frac{1}{T}(\frac{\mu}{\delta}+T)\Bigg[\frac{(T-t_k)^r}{\Gamma(r+1)}+\frac{\mu}{\delta}\frac{(T-t_k)^{r-1}}{\Gamma(r)}\\
&+Tm\frac{(t_k-t_{k-1})^r}{\Gamma(r+1)}+Tm(\frac{\mu}{\delta}+T)\frac{(t_k-t_{k-1})^{r-1}}{\Gamma(r)}\Bigg]+mT\frac{(t_k-t_{k-1})^r}{\Gamma(r+1)}+mT^2\frac{(t_k-t_{k-1})^{r-1}}{\Gamma(r)}\Bigg)\\
&\leq \|M_\alpha\|\beta(V)\max(1,T\zeta_1,T\zeta_2)\Bigg(\frac{T^r}{\Gamma(r)}+(\frac{\mu}{\delta}+T)\Bigg[\frac{T^{r-1}}{\Gamma(r)}+\frac{\mu}{\delta}\frac{T^{r-2}}{\Gamma(r)}+m\frac{T^r}{\Gamma(r)}+m(\frac{\mu}{\delta}+T)\frac{T^{r-1}}{\Gamma(r)}\Bigg]\\
&+m\frac{T^{r+1}}{\Gamma(r)}+m\frac{r+1}{\Gamma(r)}\Bigg)
\end{aligned}$$

$$\leq \|M_\alpha\|\beta(V)\max(1,T\zeta_1,T\zeta_2)\frac{T^*}{\Gamma(r)}\Big(1+(\frac{\mu}{\delta}+1)+((\frac{\mu}{\delta})^2+\frac{\mu}{\delta})+(m\frac{\mu}{\delta}+m)$$
$$+m((\frac{\mu}{\delta})^2+2\frac{\mu}{\delta}+1)+2m\Big)$$
$$\leq \|M_\alpha\|\beta(V)\max(1,T\zeta_1,T\zeta_2)\frac{T^*}{\Gamma(r)}\left((m+1)(\frac{\mu}{\delta})^2+(3m+2)\frac{\mu}{\delta}+4m+2\right).$$

Which means that

$$\beta(N^{(1,x_0)}(V)(t))$$
$$\leq \|M_\alpha\|\max(1,T\zeta_1,T\zeta_2)\frac{T^*}{\Gamma(r)}\left((m+1)(\frac{\mu}{\delta})^2+(3m+2)\frac{\mu}{\delta}+4m+2\right)\beta(V).$$

By the equicontinuity of $N^{(1,x_0)}(V)(t)=N(V)$, $G(t,s)\psi(s,\overline{\text{co}}\{N(V)(s),x_0\}$, $H\overline{\text{co}}\{N(V)(s),x_0\}$ and $S\overline{\text{co}}\{N(V)(s),x_0\})$ are equicontinuous. Therefore,

$$\beta(N^{(2,x_0)}(V)(t))=\beta(N\overline{\text{co}}\{N^{(1,x_0)}(V)(t),x_0\})$$
$$=\beta\Bigg(\int_0^t\frac{(t-s)^{r-1}}{\Gamma(r)}\psi(s,\overline{\text{co}}\{N(V)(s),x_0\},H\overline{\text{co}}\{N(V)(s),x_0\},S\overline{\text{co}}\{N(V)(s),x_0\})ds$$
$$+\frac{1}{T}(\frac{\mu}{\delta}-t)\Bigg[\int_{t_k}^T\frac{(T-s)^{r-1}}{\Gamma(r)}\psi(s,\overline{\text{co}}\{N(V)(s),x_0\},H\overline{\text{co}}\{N(V)(s),x_0\},S\overline{\text{co}}\{N(V)(s),x_0\})ds$$
$$+\frac{\mu}{\delta}\int_{t_k}^T\frac{(T-s)^{r-2}}{\Gamma(r-1)}\psi(s,\overline{\text{co}}\{N(V)(s),x_0\},H\overline{\text{co}}\{N(V)(s),x_0\},S\overline{\text{co}}\{N(V)(s),x_0\})ds$$
$$+\sum_{0<t_k<T}\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{r-1}}{\Gamma(r)}\psi(s,\overline{\text{co}}\{N(V)(s),x_0\},H\overline{\text{co}}\{N(V)(s),x_0\},S\overline{\text{co}}\{N(V)(s),x_0\})ds$$
$$+\sum_{0<t_k<T}(\frac{\mu}{\delta}+T-t_k)\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{r-2}}{\Gamma(r-1)}\psi(s,\overline{\text{co}}\{N(V)(s),x_0\},H\overline{\text{co}}\{N(V)(s),x_0\},S\overline{\text{co}}\{N(V)(s),x_0\})ds\Bigg]$$
$$+\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{r-1}}{\Gamma(r)}\psi(s,\overline{\text{co}}\{N(V)(s),x_0\},H\overline{\text{co}}\{N(V)(s),x_0\},S\overline{\text{co}}\{N(V)(s),x_0\})ds$$
$$+\sum_{0<t_k<t}(t-t_k)\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{r-2}}{\Gamma(r-1)}\psi(s,\overline{\text{co}}\{N(V)(s),x_0\},H\overline{\text{co}}\{N(V)(s),x_0\},S\overline{\text{co}}\{N(V)(s),x_0\})ds\Bigg)$$
$$\leq\Bigg(\int_0^t\frac{(t-s)^{r-1}}{\Gamma(r)}\|M_\alpha\|\max(\beta(N(V)),\beta(HN(V)),\beta(SN(V)))ds$$
$$+\frac{1}{T}(\frac{\mu}{\delta}-t)\Bigg[\int_{t_k}^T\frac{(T-s)^{r-1}}{\Gamma(r)}\|M_\alpha\|\max(\beta(N(V)),\beta(HN(V)),\beta(SN(V)))ds$$

$$+\frac{\mu}{\delta}\int_{t_k}^{T}\frac{(T-s)^{r-2}}{\Gamma(r-1)}\|M_\alpha\|\max(\beta(N(V)),\beta(HN(V)),\beta(SN(V)))ds$$

$$+\sum_{0<t_k<T}\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{r-1}}{\Gamma(r)}\|M_\alpha\|\max(\beta(N(V)),\beta(HN(V)),\beta(SN(V)))ds$$

$$+\sum_{0<t_k<T}(\frac{\mu}{\delta}+T-t_k)\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{r-2}}{\Gamma(r-1)}\|M_\alpha\|\max(\beta(N(V)),\beta(HN(V)),\beta(SN(V)))ds\Bigg]$$

$$+\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{r-1}}{\Gamma(r)}\|M_\alpha\|\max(\beta(N(V)),\beta(HN(V)),\beta(SN(V)))ds$$

$$+\sum_{0<t_k<t}(t-t_k)\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{r-2}}{\Gamma(r-1)}\|M_\alpha\|\max(\beta(N(V)),\beta(HN(V)),\beta(SN(V)))ds\Bigg)$$

$$\leq \|M_\alpha\|\max(1,T\zeta_1,T\zeta_2)B(N(V))\int_0^t\frac{(t-s)^{r-1}}{\Gamma(r)}ds$$

$$+\frac{1}{T}|\frac{\mu}{\delta}-t|\Bigg[\|M_\alpha\|\max(1,T\zeta_1,T\zeta_2)B(N(V))\int_{t_k}^{T}\frac{(T-s)^{r-1}}{\Gamma(r)}ds$$

$$+\frac{\mu}{\delta}\|M_\alpha\|\max(1,T\zeta_1,T\zeta_2)B(N(V))\int_{t_k}^{T}\frac{(T-s)^{r-2}}{\Gamma(r-1)}ds$$

$$+\sum_{0<t_k<T}\|M_\alpha\|\max(1,T\zeta_1,T\zeta_2)B(N(V))\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{r-1}}{\Gamma(r)}ds$$

$$+\sum_{0<t_k<T}(\frac{\mu}{\delta}+T-t_k)\|M_\alpha\|\max(1,T\zeta_1,T\zeta_2)B(N(V))\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{r-2}}{\Gamma(r-1)}ds\Bigg]$$

$$+\sum_{0<t_k<t}\|M_\alpha\|\max(1,T\zeta_1,T\zeta_2)B(N(V))\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{r-1}}{\Gamma(r)}ds$$

$$+\sum_{0<t_k<t}(t-t_k)\|M_\alpha\|\max(1,T\zeta_1,T\zeta_2)B(N(V))\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{r-2}}{\Gamma(r-1)}ds$$

$$\leq \|M_\alpha\|\max(1,T\zeta_1,T\zeta_2)B(N(V))\Bigg(\frac{T^r}{\Gamma(r+1)}+\frac{1}{T}(\frac{\mu}{\delta}+T)\Bigg[\frac{T^r}{\Gamma(r+1)}+\frac{\mu}{\delta}\frac{T^{r-1}}{\Gamma(r)}$$

$$+mT\frac{T^r}{\Gamma(r+1)}+mT(\frac{\mu}{\delta}+T)\frac{T^{r-1}}{\Gamma(r)}\Bigg]+mT\frac{T^r}{\Gamma(r+1)}+mT^2\frac{T^{r-1}}{\Gamma(r)}\Bigg)$$

$$\leq \|M_\alpha\|\max(1,T\zeta_1,T\zeta_2)\frac{T^*}{\Gamma(r)}\Bigg((1+(\frac{\mu}{\delta}+1)+(\frac{\mu}{\delta})^2+\frac{\mu}{\delta})+(m\frac{\mu}{\delta}+m)+m((\frac{\mu}{\delta})^2$$

$$+2\frac{\mu}{\delta}+1)+2m\Bigg)\beta(N(V))$$

$$\leq \|M_\alpha\|\max(1,T\zeta_1,T\zeta_2)\frac{T^*}{\Gamma(r)}\Bigg((m+1)(\frac{\mu}{\delta})^2+(3m+2)\frac{\mu}{\delta}+4m+2\Bigg)\beta(N(V))$$

$$\leq \Bigg[\|M_\alpha\|\max(1,T\zeta_1,T\zeta_2)\frac{T^*}{\Gamma(r)}\Bigg((m+1)(\frac{\mu}{\delta})^2+(3m+2)\frac{\mu}{\delta}+4m+2\Bigg)\Bigg]^2\beta(V).$$

Generally, $\beta(N^{(n,x_0)}(V)(t))$ is equal to

$$\left[\|M_\alpha\|\max(1,T\zeta_1,T\zeta_2)\frac{T^*}{\Gamma(r)}\left((m+1)(\frac{\mu}{\delta})^2+(3m+2)\frac{\mu}{\delta}+4m+2\right)\right]^n\beta(V)\leq\beta(V)$$

By (2.18), $\beta(N^{(n,x_0)}(V))\leq\beta(V)$, therefore $N:V\to V$ is convex-power condensing.

Applying Theorem 1.7.1 we conclude that N has a fixed point which is a solution of the problem (2.9). □

2.2.3 Example

For $r=\frac{3}{2}$, $\mu=2,\delta=3,t_1=\frac{1}{2}$, $g(t,s)=2\frac{(t-s)^{\frac{3}{2}}}{\Gamma(\frac{3}{2})}$, $h(t,s)=\frac{(t-s)^{\frac{1}{2}}}{\Gamma(\frac{1}{2})}$, $\sigma_1(t)=\frac{x(t)}{12+||x(t)||}$ and $\sigma_2(t)=\frac{x(t)}{15+||x(t)||}$, we consider the following impulsive multi-orders fractional differential equation:

$$\begin{cases} {}^cD^{\frac{3}{2}}x(t)=\frac{1}{20+e^t}(\frac{x(t)}{1+||x(t)||})+\int_0^t g(t,s)x(t)ds+\int_0^t h(t,s)x(t)ds,\ t\in[0,1],\ t\neq\frac{1}{2},\ k=0,1\\ \Delta x(\frac{1}{2})=\frac{2\left(x(\frac{1}{2})+x'(\frac{1}{2})\right)}{1+||x||+||x'||},\quad \Delta x'(\frac{1}{2})=\frac{\left(x(\frac{1}{2})+x'(\frac{1}{2})\right)}{1+||x||+||x'||}\\ 3x(0)+2x'(0)=\int_0^1\sigma_1(x(s))ds,\quad 3x(1)+2x'(1)=\int_0^1\sigma_2(x(s))ds, \end{cases} \tag{2.19}$$

For each $x\in\mathscr{F}$ and $t\in[0,1]$, we have

$$|\psi(t,x(t),y(t),z(t))|\leq\frac{1}{20+e^t}max(||x||,||y||,||z||),t\in[0,1], \tag{2.20}$$

$$||\Delta x(\frac{1}{2})||\leq 2,\ ||\Delta x'(\frac{1}{2})||\leq 1,\quad ||\sigma_1||\leq\frac{1}{12},\ ||\sigma_2||\leq\frac{1}{15}$$

Hence conditions (H1), (H2), (H3), (H5), (H6) are satisfied with

$$\begin{aligned} M_\alpha(t) &= \frac{1}{20+e^t} \\ \Omega(u) &= 3u, \;\; u \in [0,\infty) \\ L_1 = 2 \quad &, \quad L_2 = 1 \\ c_1 = \frac{1}{12} \quad &, \quad c_2 = \frac{1}{15} \end{aligned}$$

By (3.10), for each bounded sets $V, U, W \subset D_\alpha$ and for each closed interval $I \subset J$, $t \in I$, we have

$$\beta(g(I,I)V(I)) \leq \frac{2}{\Gamma(\frac{3}{2})}\beta(V(I)), \quad \beta(h(I,I)V(I)) \leq \frac{1}{\Gamma(\frac{1}{2})}\beta(V(I))$$

and

$$\beta(\psi(t,V,U,W)) \leq M_\alpha(t)\max(\beta(V),\beta(U),\beta(W)),$$

where $\zeta_1 = \frac{2}{\Gamma(\frac{3}{2})} = \frac{4}{\sqrt{\pi}}$, $\quad \zeta_2 = \frac{1}{\Gamma(\frac{1}{2})} = \frac{1}{\sqrt{\pi}}$ Hence (H4) is satisfied.
For a simple calculus, we get

$$\|M_\alpha\| \max(1, T\zeta_1, T\zeta_2)\frac{T^*}{\Gamma(r)}\left((m+1)(\tfrac{\mu}{\delta})^2 + (3m+2)\tfrac{\mu}{\delta} + 4m+2\right) = \frac{1}{20\Gamma(\frac{3}{2})}(\tfrac{92}{9}) = \frac{46}{45\sqrt{\pi}} < 1,$$

then (2.17) is satisfied.

Consequently, Theorem ([38]) implies that problem (2.19) has a solution defined on $[0,1]$.

3

Weak solutions for fractional differential equations in Banach spaces

This chapter is devoted to study the existence of weak solutions for a class of initial value problems for the impulsive fractional differential equations involving the weak fractional Caputo derivative in a Banach space, based on Article [4] and techniques of weak noncompactness measures.

3.0.1 Weak fractional derivatives

Proposition 3.0.1. *[4] For an R-integrable function $x(.) : J \to \mathscr{F}$ on J, we have*

$$I^{\rho}x(t) = (P)\int_0^t \frac{(t-s)^{\rho-1}}{\Gamma(\rho)}x(s)ds, t \in J, \rho \in (0,1),$$

that is, $I^{\rho}x(t)$ exists on J as a fractional Pettis integral.

Proposition 3.0.2. *[4] If the strong derivative $x'(.)$ of a strongly differentiable function $x(.) : J \to \mathscr{F}$ on J, is R-integrable on J, then*

$$D^{\rho}x(t) = (B)\int_0^t \frac{(t-s)^{-\rho}}{\Gamma(1-\rho)}x'(s)ds, t \in J, \rho \in (0,1)$$

that is, $D^{\rho}x(t)$ exists on J as a fractional Bochner integral.

If $x(.) : J \to \mathscr{F}$ is weakly continuous on J, then the function $y(.) : J \to \mathscr{F}$, given by

$$y(t) = (\omega)\int_0^t x(s)ds, t \in J,$$

is weakly differentiable on J and $y'_{\omega}(t) = x(t)$ for every $t \in J$.

If $x(.) : J \to \mathscr{F}$ is weakly differentiable on J and $x'_{\omega}(.)$ is weakly continuous on J, then

$$x(t) = x(0) + (\omega)\int_0^t x'_{\omega}(s)ds, t \in J.$$

If $x(.) : J \to \mathscr{F}$ is RP-integrable on J, then

$$I_\omega^\rho x(t) = \int_0^t \frac{(t-s)^{\rho-1}}{\Gamma(\rho)} x(s) ds, t \in J,$$

exists on J as a fractional Pettis integral. Also, I_ω^ρ is a linear operator from $RP(J,\mathscr{F})$ into $P^\infty(J,\mathscr{F})$ and for $\rho > 0$, $\beta > 0$ we have

$$I_\omega^\rho I_\omega^\beta x(t) = I_\omega^{\rho+\beta} x(t), t \in J.$$

Remark 3.0.3. *If $x(.) : J \to \mathscr{F}$ is R-integrable on J, then $I_\omega^\rho x(t)$ exists on J as a fractional Bochner integral and $I_\omega^\rho x(t) = I^\rho x(t)$ for $t \in J$. Also if $x(.) \in C_\omega(J,\mathscr{F})$, then $x(.)$ is bounded on J. Then if we put $\mathscr{K} := sup_{x\in J} \|x(t)\|$ and fix $x^* \in \mathscr{F}^*$, we have for $t,s \in J, s \le t$*

$$\left|\left\langle x^*, I_\omega^\rho x(t)\right\rangle - \left\langle x^*, I_\omega^\rho x(s)\right\rangle\right| \le \frac{2\mathscr{K}\,\|x^*\|}{\Gamma(1+\rho)}(t-s)^\rho$$

Consequently the real-valued function $t \longmapsto \left\langle x^*, I_\omega^\rho x(t)\right\rangle$ is continuous on J for every $x^* \in \mathscr{F}^*$, and I_ω^ρ is a linear operator from $C_\omega(J,\mathscr{F})$ into $C_\omega(J,\mathscr{F})$.

Lemma 3.0.4. *[?] If $x(.) : J \to \mathscr{F}$ is weakly differentiable on J for every $x^* \in \mathscr{F}^*$ and $x'_\omega(.)$ is RP-integrable on J and $\rho \in (0,1)$, then*

1. The function

$$x_{1-\rho}(t) = \int_0^t \frac{(t-s)^{-\rho}}{\Gamma(1-\rho)} x(s) ds, \;\; t \in J$$

is ωAC and weakly differentiable a.e. on J. Also $(x_{1-\rho})'_\omega(.)$ is RP-integrable and

$$(x_{1-\rho})'_\omega(t) = \frac{t^{-\rho}}{\Gamma(1-\rho)} y(o) + I_\omega^{1-\rho}(t) \;\; a.e. \;\; on \;\; J$$

which can be written as

$$D^{\rho}_{\omega}x(t) = (x_{1-\rho})'_{\omega}(t) - \frac{t^{-\rho}}{\Gamma(1-\rho)}y(o) \ \ a.e. \ \ on \ \ J$$

$D^{\rho}_{\omega}x(t)$ is called the weak Caputo derivative of $x(.)$, $(x_{1-\rho})'_{\omega}(t)$ denoted also by ${}^{RL}D^{\rho}_{\omega}x(t)$ is called the weak Riemann-Liouville derivative of $x(.)$.

2. *$I^{\rho}_{\omega}D^{\rho}_{\omega}x(t) = x(t) - x(0)$ on J*

3. *$D^{\rho}_{\omega}I^{\rho}_{\omega}x(t) = x(t)$ on J.*

Theorem 3.0.5. *[4] If $x(.) \in RP(J, \mathscr{F})$, then the Abel integral equation*

$$\int_0^t \frac{(t-s)^{\rho-1}}{\Gamma(\rho)}x(s)ds = y(t) \ \ t \in J$$

has a solution in $x(.) \in RP(J, \mathscr{F})$ if and only if the function $y_{1-\rho}(.)$ has the following properties :

1. $y_{1-\rho}(.)$ is ωAC on J

2. $y_{1-\rho}(.)$ is weakly differentiable a.e on J and

$$x(t) = (y_{1-\rho})'_{\omega}(t), \ \ for y_{1-\rho} a.e \ \ t \in J.$$

3. $y_{1-\rho}(0) = 0$.

3.0.2 Introduction

We consider the boundary value problem for fractional differential equations,

$${}^{c}D^{\alpha}_{w}x(t) = f(t,x), \text{ for each } \ t \in J = [0,T], \ t \neq t_k, \ k = 1,\dots,m, \ 0 < \alpha \leq 1, \tag{3.1}$$

$$\Delta x|_{t=t_k} = I_k(x(t_k^-)), \ k = 1,\dots,m, \tag{3.2}$$

$$x(0) = x_0, \tag{3.3}$$

where ${}^cD^\alpha_w$ is the fractional Caputo weak derivative of the function $x(.) : J \to E$, $f(.,.) : J \times E \to E$ is a given function, $I_k : E \to E$, $k = 1,\dots,m$ and $x_0 \in E$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim_{h\to 0^+} y(t_k+h)$ and $x(t_k^-) = \lim_{h\to 0^-} x(t_k+h)$ represent the right and left limits of $x(t)$ at $t = t_k$, $k = 1,\dots,m$.

3.0.3 Main Results

Let us start by defining what we mean by a weak solution of the problem 3.1 - 3.3.

Definition 3.0.6. *A function $x(.) \in PC(J,\mathscr{F})$ is said to be a weak solution of 3.1 - 3.3 if $x(.)$ is weakly differentiable, $x'_\omega(.)$ is RP-integrable, ${}^cD^\rho_\omega x(t) = \varphi(t,x(t))$for a.e.$t \in J' = [0,T] - t_1,...,t_m$, and conditions $\Delta x|_{t=t_k} = I_k(x(t_k^-)), k = 1,...,m$, and $x(0) = x_0$ are satisfied.*

Lemma 3.0.7. *let $\varphi(.,.) : J \times \mathscr{F} \to \mathscr{F}$ be a function such that $\varphi(.,x(.))$ is weakly continuous for every function $x(.) \in PC(J,\mathscr{F})$. Then a function $x(.) \in PC(J,\mathscr{F})$ is a weak solution of 3.1 - 3.3 if and only if it satisfies the integral equation*

$$x(t) = \begin{cases} x_0 + \frac{1}{\Gamma(\alpha)}(p)\int_0^t (t-s)^{\alpha-1}\varphi(s,x(s))ds, & if\, t \in [0,t_1] \\ x_0 + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{i=k}(p)\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\varphi(s,x(s))ds & \\ +\frac{1}{\Gamma(\alpha)}(p)\int_{t_k}^t (t-s)^{\alpha-1}\varphi(s,x(s))ds & \\ +\sum_{i=1}^{i=k} I_i(x(t_i^-)), & if\, t \in (t_k,t_{k+1}], k = 1,...,m \end{cases} \tag{3.4}$$

Proof. Assume that the function $x(.) \in PC(J,\mathscr{F})$ is a weak solution of 3.1 - 3.3.
If $t \in [0,t_1]$ then Lemma 3.0.4 implies that $x(t) - x(0) = I^\rho_\omega \varphi(t,x(t))$ on $[0,t_1]$, that is $x(.)$ satisfies the integral equation

$$x_0 + \frac{1}{\Gamma(\alpha)}(p)\int_0^t (t-s)^{\alpha-1}\varphi(s,x(s))ds.$$

If $t \in (t_1, t_2]$ then from Lemma 3.0.4 it follows that $x(t) - x(t_1^+) = I_\omega^\alpha \varphi(t, x(t))$ on $(t_1, t_2]$, that is $x(.)$ satisfies the integral equation

$$\begin{aligned} x(t) &= x(t_1^+) + \frac{1}{\Gamma(\alpha)}(p)\int_{t_1}^{t}(t-s)^{\alpha-1}\varphi(s,x(s))ds \\ &= \Delta x|_{t=t_1} + x(t_1^-) + \frac{1}{\Gamma(\alpha)}(p)\int_{t_1}^{t}(t-s)^{\alpha-1}\varphi(s,x(s))ds \\ &= I_1(x(t_1^-)) + x_0 + \frac{1}{\Gamma(\alpha)}(p)\int_{0}^{t_1}(t-s)^{\alpha-1}\varphi(s,x(s))ds \\ &+ \frac{1}{\Gamma(\alpha)}(p)\int_{t_1}^{t}(t-s)^{\alpha-1}\varphi(s,x(s))ds. \end{aligned}$$

If $t \in (t_2, t_3]$ then from Lemma 3.0.4 it follows that $x(t) - x(t_2^+) = I_\omega^\alpha \varphi(t, x(t))$ on $(t_2, t_3]$, that is $x(.)$ satisfies the integral equation

$$\begin{aligned} x(t) &= x(t_2^+) + \frac{1}{\Gamma(\alpha)}(p)\int_{t_2}^{t}(t-s)^{\rho-1}\varphi(s,x(s))ds \\ &= \Delta x|_{t=t_2} + x(t_2^-) + \frac{1}{\Gamma(\alpha)}(p)\int_{t_2}^{t}(t-s)^{\alpha-1}\varphi(s,x(s))ds \\ &= I_2(x(t_2^-)) + I_1(x(t_1^-)) + x_0 + \frac{1}{\Gamma(\alpha)}(p)\int_{0}^{t_1}(t-s)^{\alpha-1}\varphi(s,x(s))ds \\ &+ \frac{1}{\Gamma(\alpha)}(p)\int_{t_1}^{t_2}(t-s)^{\alpha-1}\varphi(s,x(s))ds + x(t_1^+) + \frac{1}{\Gamma(\alpha)}(p)\int_{t_2}^{t}(t-s)^{\alpha-1}\varphi(s,x(s))ds. \end{aligned}$$

If $t \in (t_k, t_{k+1}]$ then again from Lemma 3.0.4 we get 3.4. Conversely, suppose that the function $x(.) \in PC(J, \mathscr{F})$ satisfies the integral equation 3.4.

If $t \in [0, t_1]$ then the weakly continuous function $u(.) = \varphi(., x(.))$ satisfies the Abel equation

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)ds = v(t), t \in [0, t_1],$$

where $v(t) = x(t) - x_0, t \in [0, t_1]$, From Theorem 3.0.5 it follows that $v_{1-\alpha}(.)$ is weakly differentiable a.e. $t \in [0, t_1]$ and

$$u(t) = (v_{1-\alpha})'_\omega(t) = (x_{1-\alpha})'_\omega(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)}x_0 \text{ for a.e. } t \in [0, t_1].$$

Then by Lemma 3.0.4 we have that $u(t) = {}^cD_\omega^\alpha x(t)$ for a.e $t \in [0, t_1]$; that is

$$^{c}D^{\alpha}_{\omega}x(t) = \varphi(t,x(t)).$$

If $t \in (t_k,t_{k+1}], k = 1,...,m$, and using the fact that $^{c}D^{\alpha}_{\omega}K = 0$, where K is a constant, we get

$$\begin{aligned} ^{c}D^{\alpha}_{\omega}x(t) &= \underbrace{^{c}D^{\alpha}_{\omega}(x_0 + \sum_{i=1}^{i=k}(p)\int_{t_{i-1}}^{t_i}\frac{(t_i - s)^{\rho-1}}{\Gamma(\alpha)}\varphi(s,x(s))ds + \sum_{i=1}^{i=k} I_i(x(t_i^-)))}_{0} \\ &+ \; ^{c}D^{\alpha}_{\omega}((p)\int_{t_k}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi(s,x(s))ds) \end{aligned}$$

Thus

$$^{c}D^{\alpha}_{\omega}x(t) = ^{c}D^{\alpha}_{\omega}((p)\int_{t_k}^{t}\tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi(s,x(s))ds).$$

Using a simple change of variable, we have

$$^{c}D^{\alpha}_{\omega}x(t) = ^{c}D^{\alpha}_{\omega}((p)\int_{0}^{t-t_k}\tfrac{(t-t_k-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi(s+t_k,x(s+t_k))ds)$$

consequently and using the Lemma (.), we have

$$^{c}D^{\alpha}_{\omega}x(t+t_k) = D^{\alpha}_{\omega}I^{\alpha}_{\omega}f(t+t_k,x(t+t_k)) = \varphi(t+t_k,x(t+t_k)),$$

That is to say

$$^{c}D^{\alpha}_{\omega}x(t) = \varphi(t,x(t)), t \in (t_k,t_{k+1}].$$

□

Theorem 3.0.8. *Let $\varphi(.,.) : J \times \mathscr{F} \to \mathscr{F}$ satisfies the following conditions:*

(A_1) $\varphi(.,.)$ *is weakly-weakly continuous;*

(A_2) $\varphi(.,.)$ *is bounded, that is, there exists $\mathscr{K} > 0$ such that, $\|\varphi(t,x)\| \leq \mathscr{K}$ for all $(t,x) \in J \times \mathscr{F}$;*

(A_3) *There exists $c > 0$ such that $\|I_k\| \leq c$ for each $x \in \mathscr{F}$ and $k \in \{1,...,m\}$;*

(A_4) $\phi : \mathbb{R}^*_+ \to \mathbb{R}^*_+$ *is a non decreasing continuous function such that $\phi(0) = 0$ and $\phi(t) < t$ for all $t > 0$;*

(A_5) *For each bounded set $\Omega \subset \mathscr{F}$ we have $\beta(I_k(\Omega)) \leq c\beta(\Omega), k \in \{1,...,m\}$ and $\beta(\varphi(J \times$*

$\Omega) \leq \phi(\beta(\Omega))$.

Then there exists an interval J_ρ=$[0,\rho] \subset J$, *(*$\rho > 0$*) such that the set of weak solutions of* 3.1 - 3.3 *defined on* J_ρ *is non-empty and compact in the space* $C_\omega(J_\rho,\mathscr{F})$.

Proof. Let $\rho \in (0,T]$ be such that

$$\tfrac{(m+1)\rho^\alpha}{\Gamma(\alpha+1)} + mc < 1. \tag{3.5}$$

Consider the nonlinear Operator

$$N : C_\omega(J_\rho,\mathscr{F}) \to C_\omega(J_\rho,\mathscr{F})$$

defined by

$$\begin{aligned}(N(x))(t) \quad &= \quad x_0 + \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(s,x(s))ds \\ &+ \quad \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(s,x(s))ds + \sum_{0<t_k<t} I_k(x(t_k^-))\end{aligned}$$

We remark that the for $x(.) \in C_\omega(J_\rho,\mathscr{F})$, the Operator N is well defined by Remark 3.0.3. the Hahn-Banach theorem, allows as to deduce that there exists $x^* \in \mathscr{F}^*$ satisfying $\|x^*\| = 1$ and $\|(N(x))(t)\| = |\langle x^*,(N(x))(t)\rangle|, t \in J_\rho$. By (A_2), we have

$$\begin{aligned}\|(N(x))(t)\| &\leq \|x_0\| + \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} |\langle x^*,\varphi(s,x(s))\rangle| ds \\ &+ \int_{t_k}^{t} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} |\langle x^*,\varphi(s,x(s))\rangle| ds + \sum_{0<t_k<t} \left|\langle x^*, I_k(x(t_k^-))\rangle\right| \\ &\leq \|x_0\| + \frac{\mathscr{K} m\rho^\alpha}{\Gamma(\alpha+1)} + \frac{\mathscr{K}\rho^\alpha}{\Gamma(\alpha+1)} + \sum_{0<t_k<t} \|I_k(x(t_k^-)\| \\ &\leq \|x_0\| + \frac{\mathscr{K}(m+1)\rho^\alpha}{\Gamma(\alpha+1)} + mc. \\ &< \|x_0\| + \mathscr{K} + mc.\end{aligned}$$

Let $r=\|x_0\|+\mathscr{K}+mc$ and

$$\tilde{B}=\left\{x(.)\in C_{\omega}(J_{\rho},\mathscr{F});\|x(.)\|\leq r\right\}.$$

$\tilde{B}$ is convex and closed topological subspace of $C_{\omega}(J_{\rho},\mathscr{F})$, and $N(\tilde{B})\subset\tilde{B}$.
Now, let $\tau_1,\tau_2\in J_{\rho}$ such that $\tau_1<\tau_2$ and $(N(x))(\tau_2)\neq(N(x))(\tau_1)$. By the Hahn Banach theorem, there exists a $x^*\in\mathscr{F}^*$ with $\|x^*\|=1$ and

$$\|(N(x))(\tau_2)-(N(x))(\tau_1)\|=|\langle x^*,(N(x))(\tau_2)-(N(x))(\tau_1)\rangle|.$$

Then

$$\begin{aligned}\|(N(x))(\tau_2)-(N(x))(\tau_1)\| &\leq \int_0^{\tau_1}\Big(\frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)}-\frac{(\tau_1-s)^{\alpha-1}}{\Gamma(\alpha)}\Big)|\langle x^*,\varphi(s,x(s))\rangle|\,ds\\ &+\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)}|\langle x^*,\varphi(s,x(s))\rangle|\,ds\\ &+\sum_{0<t_k<\tau_2-\tau_1}\left|\left\langle x^*,I_k(x(t_k^-))\right\rangle\right|\\ &\leq \frac{\mathscr{K}}{\Gamma(\alpha+1)}[-(\tau_2-\tau_1)^{\alpha}+\tau_2^{\alpha}-\tau_1^{\alpha}]+\frac{\mathscr{K}}{\Gamma(\alpha+1)}(\tau_2-\tau_1)^{\alpha}\\ &+\sum_{0<t_k<\tau_2-\tau_1}\left\|I_k(x(t_k^-))\right\|\\ &\leq \frac{\mathscr{K}}{\Gamma(\alpha+1)}(\tau_2^{\alpha}-\tau_1^{\alpha})+\sum_{0<t_k<\tau_2-\tau_1}\left\|I_k(x(t_k^-))\right\|\end{aligned}$$

which proves that $N(\tilde{B})$ is an equicontinuous set.
After that we show that N is a continuous operator on $\tilde{B}$. To this end, fix $x(.)\in\tilde{B},\varepsilon>0$ and $x^*\in\mathscr{F}^*$ with $\|x^*\|\leq 1$. By [Lemma 3, [53]] and using the fact that $\varphi(.,.)$ is weakly-weakly continuous, there exists a weak neighborhood $\mathbb{U}$ of 0 in $\mathscr{F}$ such that $|\langle x^*,\varphi(s,x(s))-\varphi(s,y(s))\rangle|\leq\frac{\varepsilon\Gamma(\rho+1)}{2(m+1)\rho^{\rho}}$ for $s\in J_{\rho}$ and $y(.)\in\tilde{B}$ with $x(s)-y(s)\in\mathbb{U}$.This

and the fact that I_k are continuous functions, yields to :

$$
\begin{aligned}
|\langle x^*,(N(x))(t)-(N(y))(t)\rangle| &\leq \sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}|\langle x^*,\varphi(s,x(s))-\varphi(s,y(s))\rangle|ds\\
&+\int_{t_k}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|\langle x^*,\varphi(s,x(s))-\varphi(s,y(s))\rangle|ds\\
&+\sum_{0<t_k<t}\left\|I_k(x(t_k^-))-I_k(y(t_k^-))\right\|\\
&\leq \frac{\varepsilon}{2}\left[\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}ds+\int_{t_k}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right]\\
&+\sum_{0<t_k<t}\frac{\varepsilon}{2m}\\
&\leq \frac{\varepsilon(m+1)\rho^\alpha}{2\Gamma(\alpha+1)}+m\frac{\varepsilon}{2m}\\
&\leq \varepsilon,
\end{aligned}
$$

whence N is a continuous operator on $\tilde{B}$.

Let $L=\overline{co}(\tilde{B})$. L is also bounded and equicontinuous, as $N(\tilde{B})$ is bounded and equicontinuous in $C_\omega([0,\rho],\mathscr{F})$. Let V be a subset of L such that $B_c(V)\neq 0$ and $(NV)(t)=\{(N(x))(t);x\in V\}$. Let $t\in J_\rho$ and $\varepsilon>0$. If we choose $\xi>0$ such that

$$\xi<\left(\frac{\varepsilon\Gamma(\alpha+1)}{2\mathscr{K}(m+1)}\right)^{\frac{1}{\alpha}},\ \sum_{t-\xi<t_k<t}\left\|I_k(x(t_k^-))\right\|\leq\frac{\varepsilon}{2}$$

and

$$\mathscr{I}=\sum_{t-\xi<t_k<t}\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi(s,x(s))ds+\int_{t-\xi}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi(s,x(s))ds+\sum_{t-\xi<t_k<t}I_k(x(t_k^-))\neq 0.$$

By the Hahn Banach theorem, there exists a $x^* \in \mathscr{F}$ with $\|x^*\| = 1$ and

$$\begin{aligned}
\|\mathscr{I}\| &= \left| \left\langle x^*, \sum_{t-\xi<t_k<t} \int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(s,x(s))ds + \int_{t-\xi}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(s,x(s))ds + \sum_{t-\xi<t_k<t} I_k(x(t_k^-)) \right\rangle \right| \\
&\leq \sum_{t-\xi<t_k<t} \int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} |\langle x^*, \varphi(s,x(s))\rangle| ds + \int_{t-\xi}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |\langle x^*, \varphi(s,x(s))\rangle| ds \\
&\quad + \sum_{t-\xi<t_k<t} |\langle x^*, I_k(x(t_k^-))\rangle| \\
&\leq \frac{(m+1)\mathscr{K}\xi^\alpha}{\Gamma(\alpha+1)} + \frac{\varepsilon}{2} \\
&\leq \varepsilon,
\end{aligned}$$

thus using property (i) of the measure of weak noncompactness we infer

$$\beta\left\{ \sum_{t-\xi<t_k<t} \int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(s,V(s))ds + \int_{t-\xi}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(s,V(s))ds + \sum_{t-\xi<t_k<t} I_k(V(t_k^-)) \right\} \leq 2\varepsilon \quad (3.6)$$

Next, As $s \mapsto (t-s)^{\rho-1}$ is continuous on $[0, t-\xi]$ there exists $\lambda > 0$ such that

$$\left|(t-\tau)^{\alpha-1} - (t-s)^{\alpha-1}\right| < \varepsilon, \text{ for all } \tau, s \in [0, t-\xi] \text{ with } |\tau - s| < \lambda.$$

We need to divide the interval $[0, t-\xi]$ into n parts $0 = u_0 < u_1 < ... < u_n = t-\xi$ such that $u_i - u_{i-1} < \mu$ $(i = 1, 2, ..., n)$ and $T_i = [t_{i-1}, t_i]$ and let

$$L_k = \sum_{0<t_k<t-\xi} \int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(s,V(s))ds \text{ and } L = \int_0^{t-\xi} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(s,V(s))ds.$$

It follows that

$$\beta\left(\sum_{0<t_k<t-\xi} L_k + L + \sum_{0<t_k<t-\xi} I_k(V(t_k^-)) \right) \leq \sum_{0<t_k<t-\xi} \beta(L_k) + \beta(L) + \sum_{0<t_k<t-\xi} \beta(I_k(V(t_k^-)))$$

$$\begin{aligned}
\int_0^{t-\xi} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(s,x(s))ds &= \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n} \omega \int_{u_{i-1}}^{u_i} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(s,x(s))ds \\
&\in \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n} (u_i - u_{i-1}) \overline{co}\left\{(t-s)^{\alpha-1}\varphi(s,z); s \in T_i, z \in W\right\},
\end{aligned}$$

where $W=\{x(s);x(.)\in V\}$, so

$$\begin{aligned}\beta\left(\int_0^{t-\xi}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi(s,V(s))ds\right) &\leq \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{n}(u_i-u_{i-1})\beta\left(\{(t-s)^{\rho-1}\varphi(s,z);s\in T_i,z\in W\}\right)\\ &\leq \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{n}(u_i-u_{i-1})(t-u_i)^{\rho-1}\beta\left(\varphi(T_i\times W)\right)\\ &\leq \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{n}(u_i-u_{i-1})(t-u_i)^{\rho-1}\phi(\beta(W)).\end{aligned}$$

Since the real valued functions $s\to(t-s)^{\rho-1}$ is continuous on $[0,t-\xi]$, then

$$\tfrac{1}{\Gamma(\alpha)}(u_i-u_{i-1})(t-u_i)^{\alpha-1}\phi(\beta(W)) \leq \phi(\beta(W))\int_{u_{i-1}}^{u_i}\tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds+\tfrac{\varepsilon(u_i-u_{i-1})}{\Gamma(\alpha)}\beta(W)$$

which means that

$$\tfrac{1}{\Gamma(\alpha)}\sum_{i=1}^{n}(u_i-u_{i-1})(t-u_i)^{\rho-1}\phi(\beta(W))\leq\phi(\beta(W))\int_{u_{i-1}}^{u_i}\tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds+\tfrac{\varepsilon(t-\xi)}{\Gamma(\alpha)}\beta(W).$$

Consequently

$$\beta\left(\int_0^{t-\xi}\tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi(s,V(s))ds\right) \leq \tfrac{\rho^\alpha}{\Gamma(\alpha+1)}\phi(\beta(W))+\tfrac{\varepsilon\rho}{\Gamma(\alpha)}\beta(W)$$

On the other hand,

$$\begin{aligned}\sum_{0<t_k<t-\xi}\beta(L_k) &= \sum_{0<t_k<t-\xi}\beta\left(\int_0^{t_k-t_{k-1}}\frac{(t_k-t_{k-1}-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi(s+t_{k-1},x(s+t_{k-1}))ds\right)\\ &\leq \frac{m\rho^\alpha}{\Gamma(\alpha+1)}g(\beta(W))+\frac{\varepsilon\rho}{\Gamma(\alpha)}\beta(W)\end{aligned}$$

and

$$\sum_{0<t_k<t-\xi} \beta(I_k(V(t_k^-))) \leq \sum_{0<t_k<t-\xi} c\beta((V(t_k^-)))$$
$$\leq mc\beta(W).$$

Whence

$$\begin{aligned}\beta\left(\sum_{0<t_k<t-\xi} L_k + L + \sum_{0<t_k<t-\xi} I_k(W(t_k^-))\right) &\leq \frac{(m+1)\rho^\alpha}{\Gamma(\alpha+1)}\phi(\beta(W)) + \frac{2\varepsilon\rho}{\Gamma(\alpha)}\beta(W) + mc\beta(W)\\ &\leq \frac{(m+1)\rho^\alpha}{\Gamma(\alpha+1)}\beta(W) + mc\beta(W) + \frac{2\varepsilon\rho}{\Gamma(\alpha)}\beta(W)\\ &\leq \left(\frac{(m+1)\rho^\alpha}{\Gamma(\alpha+1)} + mc\right)\beta(W) + \frac{2\varepsilon\rho}{\Gamma(\alpha)}\beta(W)\\ &\leq \beta(W) + \frac{2\varepsilon\rho}{\Gamma(\alpha)}\beta(W) \qquad (3.7)\end{aligned}$$

Since

$$\begin{aligned}(NV)(t) \subset &\left(\sum_{0<t_k<t-\xi} L_k + L + \sum_{0<t_k<t-\xi} I_k(V(t_k^-)\right)\\ &+ \left(\left(\sum_{t-\xi<t_k<t} L_k + \int_{t-\xi}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi(s,V(s))ds + \sum_{t-\xi<t_k<t} I_k(V(t_k^-)\right)\right.\end{aligned}$$

Using (3.6) and (3.7) we have

$$\beta((NV)(t)) \leq \beta(W) + \frac{2\varepsilon\rho}{\Gamma(\alpha)}\beta(W) + 2\varepsilon, t \in J_\rho. \qquad (3.8)$$

Since (3.8) is true for every $\varepsilon > 0$, we conclude that

$$\beta((NV)(t)) \leq \beta(W), t \in J_\rho.$$

Then by Lemma 1.4.5 we get

$$\beta((NV)(t)) \leq \beta(V) \leq \beta_c(V), t \in J_\rho.$$

and thus $\beta_c((NV)) \leq \beta_c(V)$. Following the proof of Theorem 1.7.2 in [11], we can show that L is closed convex subset of $C_\omega(J_\rho, \mathscr{F})$.Therefore, by Brouwer-Schauder-Tichonov type fixed point theorem 1.7.2 it follows, that the set of the fixed points of N in $\tilde{B}$ is non-empty and compact,so the set of solutions of N the problem (3.1)-(3.3) on J_ρ is non-empty and compact in $C_\omega(J_\rho, \mathscr{F})$. □

3.0.4 Example

To illustrate the usefulness of our main results. Let us consider the following impulsive fractional initial value problem

$$^{c}D_{\omega}^{\frac{2}{3}}x(t) = \frac{1}{80+e^t}\frac{x(t)}{1+\|x(t)\|}, \quad \text{for each } t \in [0, \tfrac{1}{8}], t \neq \tfrac{1}{16}, \tag{3.9}$$

$$\Delta x|_{t=\frac{1}{16}} = \frac{1}{40}\frac{x(\frac{1}{16}^-)}{1+\left\|x(\frac{1}{16}^-)\right\|}, \tag{3.10}$$

$$x(0) = x_0, \tag{3.11}$$

Set

$$\varphi(t,u) = \frac{1}{80+e^t}\frac{x}{1+\|x\|}, (t,x) \in [0,1] \times \mathscr{F}$$

and

$$I_k(x) = \frac{1}{40}\frac{x}{1+\|x\|}, x \in \mathscr{F}$$

Let $x, y \in \mathscr{E}$ and $t \in [0, \frac{1}{8}]$. Then we have

$$\|\varphi(t,x)\| \leq \left|\frac{1}{80+e^t}\right| \left\|\frac{x}{1+\|x\|}\right\| \leq \frac{1}{80}$$

and

$$\|I_k(x)\| \leq \frac{1}{40}\frac{\|x\|}{1+\|x\|} \leq \frac{1}{40}$$

Hence the conditions $(A_2)-(A_4)$ holds with $\mathscr{K}=\frac{1}{80}$ and $c=\frac{1}{40}$. We shall check that condition (3.5) with $m=1$ and $\rho=T=\frac{1}{8}$. Indeed

$$\frac{(m+1)\rho^{\alpha}}{\Gamma(\alpha+1)}+mc=\frac{2(\frac{1}{8})^{\frac{2}{3}}}{0.89}+\frac{1}{20}=0.495<1$$

Then by Theorem 3.0.8 the problem (3.9)-(3.11) has a weak solution on $[0,\frac{1}{8}]$.

Conclusion and future study

After presenting the necessary preliminary concepts useful to well understanding the present work, we have shown the results concerning the existence of weak solutions for certain impulsive differential equations of mixed type of fractional order and multi-order relating to the derivative of Caputo in the Banach spaces using the low of non-compactness measurement technique coupled with the Henstock-Kurzweil-Pettis integrals and Krasnosel'skii-type fixed point theorem.

We have also established the existence of a weak solution for a class of initial value problems for the impulsive fractional differential equations involving the weak fractional Caputo derivative in a Banach space based on Brouwer-Schauder-Tichonov type fixed point theorem combined with the technique of measures of weak noncompactness.

A similar technique can be used to the other fractional derivatives or applied them to the differential inclusions, we can also introduce the numerical methods in the resolution of the impulsive equations.

At the end, we hope that the results presented in this thesis will contribute to the development of the study of fractional differential equations, by opening new horizons for scientific research on this emerging theme.

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