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Contents

Introduction

Stabilization of evolution problems

Problems of global existence and stability in time of Partial Differential Equations made object, recently, of many work. In this thesis we were interested in study of the global existence and the stabilization of some evolution equations.

The purpose of stabilization is to attenuate the vibrations by feedback, thus it consists in guaranteeing the decrease of energy of the solutions to 0 in a more or less fast way by a mechanism of dissipation

More precisely, the problem of stabilization consists in determining the asymptotic behaviour of the energy by $E(t)$, to study its limits in order to determine if this limit is null or not and if this limit is null, to give an estimate of the decay rate of the energy to zero.

This problem has been studied by many authors for various systems. In our study, we obtain several type of stabilization

1) Strong stabilization: $E(t) \to 0$, as $t \to \infty$.

2) Logarithmic stabilization: $E(t) \leq c(\log(t))^{-\delta}, \forall t > 0, (c, \delta > 0)$.

3) polynomial stabilization: $E(t) \leq ct^{-\delta}, \forall t > 0, (c, \delta > 0)$

4) uniform stabilization: $E(t) \leq ce^{-\delta t}, \forall t > 0, (c, \delta > 0)$.

For wave equation with dissipation of the form $u'' - \Delta_x u + g(u') = 0$, stabilization problems have been investigated by many authors:
When $g : \mathbb{R} \to \mathbb{R}$ is continuous and increasing function such that $g(0) = 0$, global existence of solutions is known for all initial conditions (u_0, u_1) given in $H_4(\Omega) \times L^2(\Omega)$. This result is, for instance, a consequence of the general theory of nonlinear semi-groups of contractions generated by a maximal monotone operator (see Brézis [**16**]).

Moreover, if we impose on the control the condition $\forall \lambda \neq 0, g(\lambda) \neq 0$, then strong asymptotic stability of solutions occurs in $H_0^1(\Omega) \times L^2(\Omega)$, i.e.,

$$(u, u') \to (0, 0) \text{ strongly in } H_0^1(\Omega) \times L^2(\Omega),$$

without speed of convergence. If the solution goes to 0 as time goes to ∞, how to get energy decay rates?

Dafermos has written in 1978 "Another advantage of this approach is that it is so simplistic that it requires only quite weak assumptions on the dissipative mechanism. The corresponding drawback is that the deduced information is also weak, never yielding, for example, decay rates of solutions."

Many authors have worked since then on energy decay rates. First results were obtained for linear stabilization, then for polynomial stabilization (see M. Nakao A. Haraux **[30]**, E. Zuazua and V. Komornik **[36]**) and then extended to arbitrary growing feedbacks (close to 0). In the same time, geometrical aspects were considered.

By combining the multiplier method with the techniques of micro-local analysis, Lasiecka et al have investigated different dissipative systems of partial differential equations (with Dirichlet and Neumann boundary conditions) under general geometrical conditions with nonlinear feedback without any growth restrictions near the origin or at infinity. The computation of decay rates is reduced to solving an appropriate explicitly given ordinary differential equation of monotone type. More precisely, the following explicit decay estimate of the energy is obtained:

$$E(t) \leq h(\frac{t}{t_0} - 1), \quad \forall t \geq t_0, \tag{1}$$

where $t_0 > 0$ and h is the solution of the following differential equation:

$$h'(t) + q(h(t)) = 0, \quad \forall t \geq 0 \quad \text{and} \quad h(0) = E(0) \tag{2}$$

and the function q is determined entirely from the behavior at the origin of the nonlinear feedback by proving that E satisfies

$$(Id - q)^{-1}\Big(E((m+1)t_0)\Big) \leq E(mt_0), \quad \forall m \in \mathbb{N}.$$

In this thesis, the main objective is to give a global existence and stabilization results.
This work consists in three chapter.

Chapter 1: The Euler-Bernoulli beam equation with dynamic boundary control of fractional derivative type

In this chapter, we consider the Timoshenko beam system with dynamic controls of fractional derivative type, that is,

$$(P) \qquad \varphi_{tt}(x,t) + \varphi_{xxxx}(x,t) = 0 \text{ in }]0, L[\times]0, +\infty[$$

where $(x,t) \in (0,L) \times (0,+\infty)$. This system is subject to the boundary conditions

$$\begin{array}{ll} \varphi(0,t) = \varphi_x(0,t) = 0 & \text{in } (0,+\infty) \\ \varphi_{xx}(L,t) = 0 & \text{in } (0,+\infty) \\ -m\varphi_{tt}(L,t) + \varphi_{xxx}(L,t) = \gamma\partial_t^{\alpha,\eta}\varphi(L,t) & \text{in } (0,+\infty) \end{array}$$

We prove a global existence result using the semi-group theory based on maximum monotone method. Furthermore, we show that our system is not uniformly stable in general, since it is the case of the interval, more precisely we show that an infinite number of eigenvalues approach the imaginary axis. Also, we look for a polynomial decay rate for smooth initial data for our system by applying a frequency domain approach combining with a multiplier method.

Chapter 2: Optimal decay rates for the acoustic wave motions with boundary memory damping

In this chapter, we consider the Timoshenko beam system with dynamic controls of fractional derivative type, that is,

$$(P) \qquad \begin{cases} y_{tt}(x,t) - y_{xx}(x,t) = 0 & \text{in } (0,L)\times(0,+\infty), \\ y(0,t) = 0 & \text{in } (0,+\infty), \\ y_x(L,t) = z_t(t) & \text{in } (0,+\infty), \\ y_t(L,t) + mz(t) + \gamma\partial_t^{\alpha,\eta} z(t) = 0 & \text{in } (0,+\infty), \\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x) & \text{in } (0,L), \end{cases}$$

where $(x,t) \in (0,L)\times(0,+\infty), m > 0, \gamma > 0, \eta \geq 0$ and the initial data are taken in suitable spaces. We prove a global existence result using the semi-group theory based on maximum monotone method. Furthermore, we show that our system is not uniformly stable in general, since it is the case of the interval, more precisely we show that an infinite number of eigenvalues approach the imaginary axis. Also, we look for a polynomial decay rate for smooth initial data for our system by applying a frequency domain approach combining with a multiplier method.

Chapter 3:Exponential decay of the Timoshenko beam system with fractional time delays

In this chapter, we consider the same system as above

$$(P) \quad \begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) + a_1\partial_t^{\sigma_1,\kappa_1}\varphi(x,t-\tau_1) + a_2\varphi_t(x,t) = 0, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x+\psi)(x,t) + \tilde{a}_1\partial_t^{\sigma_2,\kappa_2}\psi(x,t-\tau_2) + \tilde{a}_2\psi_t(x,t) = 0, \end{cases}$$

where $(x,t) \in (0,L)\times(0,+\infty)$. This system is subject to the boundary conditions

$$\varphi(0,t) = \varphi(L,t) = 0, \quad \psi(0,t) = \psi(L,t) = 0 \text{ in } (0,+\infty).$$

we prove exponential energy decay estimate using the semigroup theory of linear operators and an estimate on the resolvent of the generator associated with the semigroup.

Chapter 1

PRELIMINARIES

As the analysis done in this P.H.D thesis local on the semigroup and spectral analysis theories, we recall, in this chapter, some basic definitions and theorems which will be used in the following chapters. we refer to

1.1 Bounded and Unbounded linear operators

We start this chapter by give some well known results abounded and unbounded operators. We are not trying to give development, but review the basic definitions and theorem, mostly without proof. Let $E, (\|.\|_E)$ and $(F, \|.\|_F)$ be two Banach spaces over $\mathbb{C}$, and H will always denote a Hilbert space equipped with the inner scalar product $< ., . >_H$ and the corresponding norm $\|.\|_H$.
A linear operator $T : E \mapsto F$ is a transformation which maps lineary E in F, that is

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v), \quad \forall u, v \in E \quad and \quad \alpha, \beta \in \mathbb{C}$$

Definition 1.1.1 *A linear operator $T : E \mapsto F$ is said to be bounded there exists $C \geq 0$ such that*

$$\|Tu\|_F < C\|u\|_E \quad \forall u \in E$$

The set of all bounded linear operators from E into F is denoted by $\mathcal{L}(E, F)$. Moreover, the set of all bounded linear operators from E into F is denoted by $\mathcal{L}(E)$

Definition 1.1.2 *A bounded operator $T \in \mathcal{L}(E, F)$ is said to be co;pact if for each sequence $(x)_{n\in\mathbb{N}} \in E$ with $\|x_n\|_E = 1$ for each $n \in \mathbb{N}$, the sequence $(Tx_n)_{n\in\mathbb{N}}$ has a subsequence which converges in F.*
The set of all compact operators from E into F is denoted by $\mathcal{K}(E, F)$. For simplicity one writes $\mathcal{K}(E) = \mathcal{K}(E, F)$

Definition 1.1.3 *Let $T \in \mathcal{L}(E, F)$, we define*

- *Range of T by*

$$\mathcal{R}(T) = \{Tu : \quad u \in E\} \subset F.$$

- *Kernel of T by*

$$\ker(T) = \{u \in E : \quad Tu = 0\} \subset E.$$

Theorem 1.1.1 *(fredholm alternative) If $T \in \mathcal{K}(E)$, then*

- $\ker(I - T)$ *is finite dimension, (I is the identity operator on E)*
- $\mathcal{R}(I - T)$ *is closed*
- $\ker(I - T) = 0 \Leftrightarrow \mathcal{R}(I - T) = E$

Definition 1.1.4 *An unbounded linear operator T from E into F is a pair (T,D(T)), consisting of a subspace $D(T) \subset E$ (called the domain of T) and a linear transformation*

$$T : D(T) \subset E \to F$$

In the case when $E = F$ then we say (T,D(T)) is an unbounded linear operator on E. If $D(T) = E$ then $T \in \mathcal{L}(E, F)$.

Definition 1.1.5 *Let $T : D(T) \subset E \to F$ be an unbounded linear operator.*

- *The range of T is defined by*

$$\mathcal{R}(T) = \{Tu : u \in D(T)\} \subset F$$

- *The kernel of T is defined by*

$$\ker(T) = \{u \in D(T) : Tu = 0\} \subset E$$

- *the graph of T is defined by*

$$G(T) = \{(u, Tu) : u \in D(T)\} \subset E \times F$$

Definition 1.1.6 *A map T is said to be closed if $G(T)$ is closed in $E \times F$.the closeness of an unbounded linear operator T can be characterize as following*
if $u_n \in D(T)$ such that $u_n \to u$ in E and $Tu_n \to v$ in F, then $u \in D(T)$ and $Tu = v$

Definition 1.1.7 *Let $T : D(T) \subset E \to F$ be a closed unbounded linear operator.*

- *The resolvent set of T is defined by*

$$\rho(T) = \{\lambda \in \mathbb{C} : \lambda I - T \quad is \quad bijective \quad from \quad D(T) \quad onto \quad F\}$$

- *The resolvent of T is defined by*

$$\mathrm{R}(\lambda, T) = \{(\lambda I - T)^{-1} : \lambda \in \rho(T)\}$$

- *The spectrum set of T is the complement of the resolvent set in $\mathbb{C}$, denoted by*

$$\sigma(T) = \mathbb{C}/\rho(T)$$

Definition 1.1.8 *Let $T : D(T) \subset E \to F$ be a closed unbounded linear operator. we can split the spectrum $\sigma(T)$ of T into three disjoint sets, given by*

- *The ponctuel spectrum of T is define by*

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : \ker(\lambda I - T) \neq \{0\}\}$$

in this case λ is called an eignvalue of T

- *The continuous spectrum of T is define by*

$$\sigma_c(T) = \{\lambda \in \mathbb{C} : \ker(\lambda I - T) = 0, \overline{\mathcal{R}(\lambda I - T)} = F \quad \text{and} \quad (\lambda I - T)^{-1} \quad \text{is} \quad \text{not} \quad \text{bounded}\}.$$

- *The residual spectrum of T is define by*

$$\sigma_c(T) = \{\lambda \in \mathbb{C} : \ker(\lambda I - T) = 0 \quad \text{and} \quad \mathcal{R}(\lambda I - T) \quad \text{is} \quad \text{not} \quad \text{dense} \quad \text{in} \quad F\}$$

Definition 1.1.9 *Let $T : D(T) \subset E \to F$ be a closed unbounded linear operator and let λ be an eigenvalue of A. A non-zero element $e \in E$ is called a generalized eigenvalue of T associated with the eigenvalue λ, if there exists $n \in \mathbb{N}^*$ such that*

$$(\lambda I - T)^n e = 0 \quad \text{and} \quad (\lambda I - T)^{n-1} e \neq 0$$

. If $n = 1$, then is called an eigenvector.

Definition 1.1.10 *Let $T : D(T) \subset E \to F$ be a closed unbounded linear operator. We say that T has compact resolvent, if there exist $\lambda_0 \in \rho(T)$ such that $(\lambda_0 I - T)^{-1}$ is compact.*

Theorem 1.1.2 *Let $(T, D(T))$be a closed unbounded linear operator on H then the space $(D(T), \|.\|_{D(T)})$ where $\|u\|_{D(A)} = \|Tu\|_H + \|u\|_H \quad \forall u \in D(T)$ is Banach space.*

Theorem 1.1.3 *Let (T,D(T)) be a closed unbounded linear operator on H then, $\rho(T)$ is an open set of* $\mathbb{C}$

1.2 Semigroups, Existence and uniqueness of solution

In this section, we start by introducing some basic concepts concerning the semigroups. The vast majority of the evolution equations can be reduced to the form

$$\begin{cases} U_t = \mathcal{A}U, & t > 0 \\ U(0) = U_0, \end{cases} \tag{1.1}$$

where A is the infinitesimal generator of C_0-semigroup $S(T)$ over a Hilbert space H. Lets start by basic definitions and theorems.
Let $(X, \|.\|_X)$ be a Banach space, and H be a Hlbert space equipped with the inner product $< ., . >_H$ and the induced norm $\|.\|_H$

Definition 1.2.1 *A family $(S(t))_{t\geq 0}$ of bounded linear operations in X is called a strong continuous semigroup (in short, a C_0-semigroup) if*

- $S(0) = I$ *(I is called identity operator on X).*
- $S(t + s) = S(t)S(s), \quad \forall t, s \geq 0$
- *For each* $u \in H, S(t)u$ *is continuous in* t *on* $[0, +\infty[$

Sometimes we also denote $S(t)$ by e^{tA}

Definition 1.2.2 *For a semigroup $(S(t))_{t\geq 0}$, we define an linear operator A with domain D(A) consisting of points u such that the limit*

$$Au := \lim_{t\to 0^+} \frac{S(t)u - u}{t}, \quad u \in D(A)$$

exists. Then A is called the infinitesimal generator of the semigroup in X. Then there exist a constant $M \geq 1$ and $\omega \geq 0$ such that

$$\|S(t)\|_{\mathcal{L}(X)} \geq Me^{\omega t}, \forall t \geq 0$$

If $\omega = 0$ then the corresponding semigroup is uniformly bounded. moreover, if $M = 1$ then $(S(t))_{t\geq 0}$ is said to be a C_0-semigroup of contractions.

Definition 1.2.3 *An unbounded linear operator (A,D(A)) on H, is said to be dissipative if*

$$\mathrm{R} < Au, u >_H \geq 0, \quad \forall u \in D(A)$$

Definition 1.2.4 *An unbounded linear operator $(A, D(A))$ on X, is said to be m-dissipative if*

- *A is dissipative operator*
- $\exists \lambda_0 >$ *such that* $\mathcal{R}(\lambda_0 I - A) = X$

Theorem 1.2.1 *Let A be a m-dissipative operator, then*

- $\mathcal{R}(\lambda_0 I - A) = X, \quad \forall \lambda > 0$
- $]0, \infty[\subseteq \rho(A)$

Theorem 1.2.2 *(Hill-Yoshida) An unbounded linear operator (A,D(A)) on X, is the infinitesimal generator of a C_0-semigroup of contractions $(S(t))_{t\geq 0}$ if and only if*

- *A is closed and $\overline{D(A)} = X$*
- *The resolvent set $\rho(A)$ of contains $\mathbb{R}^+$, and for all $\lambda > 0$,*

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} \leq \lambda^{-1}$$

Theorem 1.2.3 *(Lumer-philips) Let (A,D(A)) be an unbounded linear operator on X, with dense domain D(A) in X. A is the infinitesimal generator of c_0-semigroup of contractions if and only if is a m-dissipative operator.*

Theorem 1.2.4 *Let (A,D(A)) be an unbounded linear operator on X. If A is dissipative with $\mathcal{R}(I - A) = X$, and X is reflexive then $\overline{D(A)} = X$*

Corollary 1.2.1 *Let $(A, D(A))$ be an unbounded linear operator on H. A is the infinitesimal generator of a C_0-semigroup of contractions if and only if A is m-dissipative operator.*

Theorem 1.2.5 *Let A be a linear operator with dense domain $D(A)$ in a Hilbert space H. If A is dissipative and $0 \in \rho(A)$, then A is the infinitesimal generator of C_0-semigroup of contractions on H.*

Theorem 1.2.6 *(Hill-Yoshida) Let $(A, D(A))$ be an unbounded linear operator on H. Assume that A is the infinitesimal generator of C_0-semigroup of contractions$(S(t))_{t\geq 0}$*

1. *For $U_0 \in D(A)$, the problem (1.1) admits a unique strong solution*

$$U(t) = S(t)U_0 \in C^0(\mathbb{R}^+, D(A)) \cap C^1(\mathbb{R}^+, H)$$

.

2. *For $U_0 \in H$, the problem (1.1) admits a unique weak solution*

$$U(t) \in C^0(\mathbb{R}^+, H)$$

.

1.3 Stability of semigroups

In this section we start by introducing definition about strong, exponential and polynomial stability of a C_0-semigroup. Then we collect some results about the stability of C_0-semigroup.

Let $(X, \|.\|_H)$ be a Banach space, and H be a Hilbert space equipped with the inner product $< ., . >_H$ and the induced norm $\|.\|_H$.

Definition 1.3.1 *Assume that A is the operator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on X. We say that the C_0-semigroup $(S(t))_{t\geq 0}$ is*

- *Strongly stable if*

$$\lim_{t\to+\infty} \|S(t)u\|_X = 0 \quad \forall u \in X$$

- *Uniformly stable if*

$$\lim_{t\to+\infty} \|S(t)u\|_{L(X)} = 0$$

- *Exponentially stable if there exist two positive constants M and ϵ such that*

$$\|S(t)u\|_X \leq Me^{-\epsilon t}\|u\|_X \quad \forall t > 0, \forall u \in X$$

- *Polynomially stable if there exist two positive constants C and α such that*

$$\|S(t)u\|_X \leq Ct^{-\alpha}\|u\|_X \quad \forall t > 0, \forall u \in X$$

Proposition 1.3.1 *Assume that A is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on X. The following statements are equivalent*

- *$(S(t))_{t\geq 0}$ is uniformly stable*
- *$(S(t))_{t\geq 0}$ is exponentially stable*

First, we look for the necessary conditions of strong stability of a C_0-semigroup. The result was obtained by Arendt and Batty

Theorem 1.3.1 *(Arendt and Batty) Assume that A is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on a reflexive Banach space X. If*

- *A has no pure imaginary eigenvalues.*
- *$\sigma(A) \cap i\mathbb{R}$ is countable.*

Then S(t)is strongly stable.

Remark 1.3.1 *If the resolvent $(I-T)^{-1}$ of T is compact, then $\sigma(T)=\sigma_p(T)$. Thus, the state of Theorem (1.3) lessens to $\sigma_p(A)\cap i\mathbb{R}=\emptyset$.*

Next, when the C_0-semigroup is strongly stable, we look for the necessary and suffient conditions of exponential stability of C_0-semigroup. In fact, exponential stability results are obtained using different methods like: multipliers method, frequency domain approach, Riez basis approach, Fourier analysis or a combination of them. In this thesis we will review only two methods. The first method is a frequency domain approach method was obtained by Huang-Pruss.

Theorem 1.3.2 *(Huang-Pruss) Assume that A is the generator of strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on H. S(t) is uniformly stable if and only if*

- $i\mathbb{R}\subset\rho(A)$.
- $\sup_{\beta\in\mathbb{R}}\|(i\beta I-A)^{-1}\|_{\mathcal{L}(H)}<+\infty$

The second one, is a classical method based on the spectrum analysis of the operator A.

Definition 1.3.2 *Let (A,D(A)) be an unbounded linear operator on H. Assume that A is the infinitesimal generator of C_0-semigroup of contractions $(S(t))_{t\geq 0}$*

- *The growth bound of A is define by*

$$\omega_0(A)=\omega\in\mathbb{R}:\exists N_\omega\in\mathbb{R}\text{ such that }\quad\forall t\geq 0\quad\text{we have }\|S(t)\|\leq N_\omega e^{\omega t}$$

- *The spectral bound of A is define by*

$$s(A)=\sup\mathrm{R}(\lambda):\lambda\in\sigma(A)$$

Proposition 1.3.2 *Let (A,D(A)) be an unbounded linear operator on H. Assume that A is the infinitesimal generator of C_0-semigroup of contractions $(S(t))_{t\geq 0}$. Then, $(S(t))_{t\geq 0}$ is uniformly exponentially stable if and only if its growth bound $\omega_0(A)<0$*

Proposition 1.3.3 *Let (A,D(A)) be an unbounded linear operator on H. Assume that A is the infinitesimal generator of C_0-semigroup of contractions $(S(t))_{t\geq 0}$. Then, we have*

$$s(A)\leq\omega_0(A)$$

Corollary 1.3.1 *Let (A,D(A)) be an unbounded linear operator on H. Assume that $s(A)=0$, then $(S(t))_{t\geq 0}$ is not uniformly exponentially stable.*
In this case when the C_0-semigroup is not exponentially stable we look for a polynomial one. In general, polynomial stability results also are obtained using different methods like: multipliers method, frequency domain approach, Riez basis approach, Fourier analysis or a combination of them. In this thesis we will review only one method. The first method is a frequency domain approach method was method was obtained by Batty, A.Borichev and Tomilov, Z. Liu and B.Rao.

Theorem 1.3.3 *(Batty, A.Borichev and Y.Tomilov, Z.Liu and B.Rao) Assume that A is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on H. If $i\mathbb{R} \subset \rho(A)$, then for a fixed $l > 0$ the following conditions are equivalent*

1. $\lim\limits_{|\lambda|\to+\infty} \sup \dfrac{1}{\lambda^l} \|(\lambda I - A)^{-1}\|_{\mathcal{L}(H)} < +\infty$

2. $\|S(t)U_0\|_H \leq \dfrac{C}{t^{l-1}} \|U_0\|_{D(A)} \quad \forall t > 0, \quad U_0 \in D(A), \quad for \quad some \quad C > 0$

1.4 Fractional Derivative control

In this part, we introduce the necessary elements for the good understanding of this manuscript. It includes a brief reminder of the basic elements of the theory of fractional computation as well as some examples of applications of this theory in this scientific field. The concept of fractional co;putation is a generalization of ordinary derivation and integration to an arbitrary order. Derivatives of non-integer order are now widely applied in many domains, for example in economics, electronics, mechanics, biology, probability and viscoelasticity.
A particular interest for fractional derivation is related to the mechanical modeling of gums and rubbers. In short, all kinds of materials that preserve the memory of previous deformations in particular viscoelastic. Indeed, the fractional derivation is introduced naturally. There exists a many mathematical definitions of fractional order integration and derivation. These definitions do not always lead to identical results but are equivalent for a wide large of functions. We introduce the fractional integration operator as well as the two most definitions of fractional derivatives, used, namely that Riemann-Liouville and Caputo, by giving the most important properties of the notations. Fractional systems appear in different fields of research. however, the progressive interest in their applications in the basic and applied sciences. It can be noted that for most of the domains presented (automatic, physics, mechanics of continuous media). The fractional operators are used to take into account memory effects. We can mention the works that reroute various applications of fractional computation.
In physics, on of the most remarkable applications of fractional computation in physics was in the context of classical mechanics. Riewe, has shown that the Lagragien of the motion of temporal derivatives of fractional orders leads to an equation of motion with friction forces and nonconservative are essential in macroscopic variational processing such as friction. This result are remarkable because friction forces and non conservative forces are essential in the usual macroscopic variational processing and therefore in the most advances methods classical mechanics. Riewe, has generalized the usual Lagrangian variation which depends on the fractional derivatives in order to deal with the usual non-conservative forces. On the another hand, serval approaches have been developed to generalize the principle of least action and the Euler-Lagrange equation to the case of fractional derivative. The definition of the fractional order derivation is based on that of a fractional order integration, a fractional order derivation takes on a global character in contrast to an integral derivation. It turns out that the derivative of a fractional order integration, a fractional order derivation takes on a global character in contrast to an integral derivation. It turns out that the derivative of a fractional order of a

function requires the knowledge of $f(t)$ over the entire interval $]a,b[$, where in the whole case only the local knowledge of f around around t is necessary. This property allows to interpret fractional order systems as long memory systems, the whole systems being then interpretable as systems with short memory. Now, we give the definition of the fractional derivatives in the sense of Riemann-Liouville as well as some essential properties.

Definition 1.4.1 *The fractional integral of order $\alpha > 0$, in sense Rieamann-Liouville is given by*

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \quad t > a$$

Definition 1.4.2 *The fractional derivative of order $\alpha > 0$, in sens of Rieamann-Liouville of a function f defined on the interval $[a,b]$ is given by*

$$D_{RL,a}^\alpha(t) = D^n I_\alpha^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f(s)ds, \quad n = [\alpha]+1, t > a$$

In particular, if $\alpha = 0$, then

$$D_{RL,a}^0 f(t) = I_a^0 f(t) = f(t)$$

If $\alpha = n \in \mathbb{N}$, then

$$D_{RL,a}^0 f(t) = f^{(n)}(t).$$

moreover, if $0 < \alpha < 1$, then $n = 1$, then

$$D_{RL,a}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-s)^{-\alpha} f(s)ds, \quad t > a$$

Example 1.4.0.1 *Let $\alpha > 0, \quad \gamma > -1$ and $f(t) = (t-a)^\gamma$, then*

$$I_a^\alpha f(t) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} (t-a)^{\gamma+\alpha},$$

$$D_{RL,a}^\alpha f(t) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha+1)} (t-a)^{\gamma-\alpha}$$

In particular, if $\gamma = 0$ and $\alpha > 0$, then $D_{RL,a}^\alpha(C) = C\dfrac{(t-\alpha)^{-\alpha}}{\Gamma(1-\alpha)}$

The derivatives of Riemann-Liouville have certain disadvantages when attempting to model real world phenomena. The problems studied require a definition of the fractional derivatives allowing the use of the physically interpretable initial conditions introducing $y(0), y\prime(0)$, ect. There shortcomings led to an alternatives that satisfies these demands in the last sixties. It was introduced by Caputo.In fact, Caputo and Minardi used this definition in their work on viscoelasticity.
Now, we give the definition of the fractional derivatives in the sense of Caputo as well as some essential properties.

Definition 1.4.3 *The fractional derivative of order $\alpha > 0$, in sense of Caputo, defined on the interval$[a, b]$, is given by*

$$D^{\alpha}_{C,a} f(t) = D^{\alpha}_{RL,a} \left(f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k \right).$$

where

$$n = \begin{cases} [\alpha] + 1 & if \quad \alpha \notin \mathbb{N}, \\ \alpha & if \quad \alpha \in \mathbb{N}^*, \end{cases}$$

In particular, where $0 < \alpha < 1$, the relation (1.4.3) take the form

$$\begin{aligned} D^{\alpha}_{C,a} f(t) &= D^{\alpha}_{C,a}([f(t) - f(a)]) \\ &= I^{1-\alpha}_{a} f\prime(t) \\ &= \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-a)^{-\alpha} f\prime(s) ds \end{aligned}$$

If $\alpha \in \mathbb{N}$, then $f^{(n)}(t) \quad D^{\alpha}_{C,a} f(t) = f^n(t)$ coincides i.e

$$D^{\alpha}_{Ca} f(t) = f^n(t).$$

Example 1.4.0.2 *Let $\alpha > 0$ and $f(t)(t-a)^{\gamma}$ where $\gamma > -1$. then*

$$D^{\alpha}_{Ca} f(t) = \frac{\Gamma(\gamma)}{\Gamma(\gamma - \alpha + 1)} (t-a)^{\gamma - \alpha}$$

In particular, if $\gamma = 0$ and $\alpha > 0$, then $D^{\alpha}_{Ca} C = 0$

Chapter 2

THE EULER-BERNOULLI BEAM EQUATION WITH DYNAMIC BOUNDARY CONTROL OF FRACTIONAL DERIVATIVE TYPE

2.1 Introduction

In this chapter we investigate the existence and decay properties of solutions for the initial boundary value problem of the Euler Bernoulli beam equation of the type

$$(P) \qquad \varphi_{tt}(x,t) + \varphi_{xxxx}(x,t) = 0 \text{ in }]0, L[\times]0, +\infty[$$

where $(x,t) \in (0,L) \times (0,+\infty)$. This system is subject to the boundary conditions

$$\begin{array}{ll} \varphi(0,t) = \varphi_x(0,t) = 0 & \text{in } (0,+\infty) \\ \varphi_{xx}(L,t) = 0 & \text{in } (0,+\infty) \\ -m\varphi_{tt}(L,t) + \varphi_{xxx}(L,t) = \gamma\partial_t^{\alpha,\eta}\varphi(L,t) & \text{in } (0,+\infty) \end{array}$$

where $\gamma > 0$. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α with respect to the time variable. It is defined as follows

$$\partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s)\, ds, \ \eta \geq 0.$$

The system is finally completed with initial conditions

$$\varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x)$$

where the initial data (φ_0, φ_1) belong to a suitable function space.

The problem (P) describes the motion of clamped flexible beam where one end is clamped, and the free end holds a rigid tip mass whose mass m is positive.

Models of this form play a fundamental role in many mechanical systems and thus occur in many applications such as exible robot arms, helicopter rotor blades, spacecraft antennae, airplane wings, high-rise buildings, etc. An important issue is the suppression of vibrations, since undesired oscillations can reduce the performance of the system, or worse, result in damage to the structure. For this reason, the Euler-Bernoulli beam is often coupled with a boundary control, which acts on the tip and is used to dissipate the vibration.

A simple model describing the transverse vibration of a system of non-homogeneous connected Euler-Bernoulli beams, which was developed in [**20**], is given by a system of the form

$$(EB) \qquad \begin{cases} \rho\varphi_{tt}(x,t) + EI\varphi_{xxxx}(x,t) = 0 & \text{in }]0,L[\times]0,+\infty[\\ \varphi(0,t) = 0, & \\ \varphi_x(0,t) = 0, & \\ m\varphi_{tt}(1,t) - EI\varphi_{xxx}(1,t) = -f_e, & k_1 \in \mathbb{R}, \\ -EI\varphi_{xx}(1,t) - J\varphi_{xtt}(1,t) = \tau_e, & k_2 \in \mathbb{R}, \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), & x \in (0,L) \end{cases}$$

where ρ denotes the mass density per unit length, EI is the flexural rigidity coefficient, m is the mass of a of the tip mass, J is the mass moment of inertia, and the following variables have engineering meanings:

$$\begin{array}{rcl} \varphi & = & \text{vertical displacement,} \\ \varphi_x & = & \text{rotation,} \\ -EI\varphi_{xx} & = & \text{bending moment} \\ -EI\varphi_{xxx} & = & \text{shear} \end{array}$$

at a point x, at time t.
f_e and τ_e describe the external torque and force acting on the tip mass.

Control of elastic systems is one of the main themes in control engineering. The case of the wave equation with linear and nonlinear boundary feedback has attracted a lot of attention in recent years. The bibliography of works in the direction is truly long (see [**3**], [**17**], [**18**], [**19**], [**30**], [**36**], [**42**]) and many energy estimates have been derived for arbitrary growing feedbacks (polynomial, exponential or logarithmic decay).

For plates, also with linear and nonlinear boundary feedback acting through shear forces and moments, we refer to [**39**]), [**40**] for stabilization results and [**37**] for estimates of the decay. The more difficult case of control by moment only has been studied in [**41**]. All these papers are based on multiplied techniques.

The case of homogeneous beam [**20**] has also been considered, in the more difficult case of control by moment only and an exponential stability is proved.

In [**1**] Z. Achouri et all study the decay rate of the energy of the Euler-Bernoulli equation with a boundary fractional derivative control as in this study (with $m = 0$). Using energy methods, they prove strong asymptotic stability under the condition $\eta = 0$ and a polynomial type decay rate $E(t) \leq C/t^{1/(1-\alpha)}$ if $\eta \neq 0$.

The boundary feedback under the consideration are of fractional type and are described by the fractional derivatives

$$\partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s)\, ds$$

The order of our derivatives is between 0 and 1. Very little attention has been paid to this type of feedback. In addition to being nonlocal, fractional derivatives involve singular and nonintegrable kernels ($t^{-\alpha}, 0 < \alpha < 1$). This leads to substantial mathematical difficulties since all the previous methods developed for convolution terms with regular and/or integrable kernels are no longer valid.

It has been shown (see [**49**]) that, as ∂_t, the fractional derivative ∂_t^α forces the system to become dissipative and the solution to approach the equilibrium state. Therefore, when applied on the boundary, we can consider them as controllers which help to reduce the vibrations.

In recent years, the application of fractional calculus has become a new interest in research areas such as viscoelasticity, chaos, biology, wave propagation, fluid flow, electromagnetics, automatic control, and signal processing (see [**59**]). For example, in viscoelasticity, due to the nature of the material microstructure, both elastic solid and viscous fluid like response qualities are involved. Using Boltzmann assumptions, we end up with a stress-strain relationship defined by a time convolution. Viscoelastic response occurs in a variety of materials, such as soils, concrete, rubber, cartilage, biological tissue, glasses, and polymers (see [**8**], [**9**], [**10**] and [**46**]). In our case, the fractional dissipations may come from a viscoelastic surface of the beam or simply describe an active boundary viscoelastic damper designed for the purpose of reducing the vibrations (see [**48**], [**49**]).

Our purpose in this chapter is to give a global solvability in Sobolev spaces and energy decay estimates of the solutions to the problem (P) with a boundary control of fractional derivative type. We think that the interaction of the tip mass and boundary control have an effect on the result of [**1**].

To obtain global solutions to the problem (P), we use the argument combining the semigroup theory (see [**16**]) with the energy estimate method. For decay estimates, Under the condition $\eta = 0$, using a spectral analysis, we prove non-uniform stability. On the other hand if $\eta \neq 0$, we also show a polynomial type decay rate using a frequency domain approach and a recent theorem of A. Borichev and Y. Tomilov.

2.2 Augmented model

This section is concerned with the reformulation of the model (P) into an augmented system. For that, we need the following claims.

Theorem 2.2.1 (see [48]) *Let μ be the function:*

$$\mu(\xi) = |\xi|^{(2\alpha-1)/2}, \quad -\infty < \xi < +\infty,\ 0 < \alpha < 1. \tag{2.1}$$

Then the relationship between the 'input' U and the 'output' O of the system

$$\partial_t \phi(\xi,t) + (\xi^2 + \eta)\phi(\xi,t) - U(t)\mu(\xi) = 0, \quad -\infty < \xi < +\infty, \eta \geq 0, t > 0, \tag{2.2}$$

$$\phi(\xi,0) = 0, \tag{2.3}$$

$$O(t) = (\pi)^{-1}\sin(\alpha\pi)\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi,t)\,d\xi \tag{2.4}$$

is given by

$$O = I^{1-\alpha,\eta}U = D^{\alpha,\eta}U \tag{2.5}$$

where

$$[I^{\alpha,\eta}f](t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}e^{-\eta(t-\tau)}f(\tau)\,d\tau.$$

Lemma 2.2.1 *If* $\lambda \in D = \{\lambda \in \mathbb{C} : Re\lambda + \eta > 0\} \cup \{\lambda \in \mathbb{C} : Im\lambda \neq 0\}$ *then*

$$\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}\,d\xi = \frac{\pi}{\sin\alpha\pi}(\lambda+\eta)^{\alpha-1}.$$

Proof Let us set

$$f_\lambda(\xi) = \frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}.$$

We have

$$\left|\frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}\right| \le \begin{cases} \dfrac{\mu^2(\xi)}{Re\lambda+\eta+\xi^2} & \text{or} \\ \dfrac{\mu^2(\xi)}{|Im\lambda|+\eta+\xi^2} \end{cases}$$

Then the function f_λ is integrable. Moreover

$$\left|\frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}\right| \le \begin{cases} \dfrac{\mu^2(\xi)}{\eta_0+\eta+\xi^2} & \text{for all } Re\lambda \ge \eta_0 > -\eta \\ \dfrac{\mu^2(\xi)}{\tilde{\eta}_0+\xi^2} & \text{for all } |Im\lambda| \ge \tilde{\eta}_0 > 0 \end{cases}$$

From Theorem 1.16.1 in **[69]**, the function

$$f_\lambda : D \to \mathbb{C} \text{ is holomorphe.}$$

For a real number $\lambda > -\eta$, we have

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}\,d\xi = \int_{-\infty}^{+\infty} \frac{|\xi|^{2\alpha-1}}{\lambda+\eta+\xi^2}\,d\xi = \int_0^{+\infty} \frac{x^{\alpha-1}}{\lambda+\eta+x}\,dx \text{ (with } \xi^2 = x)\\
&= (\lambda+\eta)^{\alpha-1}\int_1^{+\infty} y^{-1}(y-1)^{\alpha-1}\,dy \text{ (with } y = x/(\lambda+\eta)+1)\\
&= (\lambda+\eta)^{\alpha-1}\int_0^1 z^{-\alpha}(1-z)^{\alpha-1}\,dz \text{ (with } z = 1/y)\\
&= (\lambda+\eta)^{\alpha-1}B(1-\alpha,\alpha) = (\lambda+\eta)^{\alpha-1}\Gamma(1-\alpha)\Gamma(\alpha) = (\lambda+\eta)^{\alpha-1}\frac{\pi}{\sin\pi\alpha}.
\end{aligned}$$

Both holomorphic functions f_λ and $\lambda \mapsto (\lambda+\eta)^{\alpha-1}\dfrac{\pi}{\sin\pi\alpha}$ coincide on the half line $]-\infty,-\eta[$, hence on D following the principe of isolated zeroes.

We are now in a position to reformulate system (P). Indeed, by using Theorem 2.2.1, system (P) may be recast into the augmented model:

$$
(P') \qquad \begin{cases}
\varphi_{tt} + \varphi_{xxxx} = 0, \\
\partial_t \phi(\xi,t) + (\xi^2+\eta)\phi(\xi,t) - \varphi_t(L,t)\mu(\xi) = 0, \\
\varphi(0,t) = \varphi_x(0,t) = 0, \\
\varphi_{xx}(L,t) = 0, \\
\varphi_{xxx}(L,t) - m\varphi_{tt}(L,t) = \zeta \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi,t)\, d\xi, \\
\varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x).
\end{cases}
$$

We define the energy associated to the solution of the problem (P') by the following formula:

$$
E(t) = \frac{1}{2}\|\varphi_t\|_2^2 + \frac{1}{2}\|\varphi_{xx}\|_2^2 + \frac{\gamma}{2}(\pi)^{-1}\sin(\alpha\pi)\int_{-\infty}^{+\infty}(\phi(\xi,t))^2\, d\xi + \frac{m}{2}|\varphi_t(L,t)|^2. \tag{2.6}
$$

Lemma 2.2.2 *Let (φ,ϕ) be a solution of the problem (P'). Then, the energy functional defined by (2.6) satisfies*

$$
E'(t) = -(\pi)^{-1}\sin(\alpha\pi)\gamma\int_{-\infty}^{+\infty}(\xi^2+\eta)(\phi(\xi,t))^2\, d\xi \le 0. \tag{2.7}
$$

Proof

Multiplying the first equation in (P') by φ_t, integrating over $(0,L)$ and using integration by parts, we get

$$
\frac{1}{2}\frac{d}{dt}\|\varphi_t\|_2^2 + \int_0^L \varphi_{xxxx}\varphi_t dx = 0.
$$

Then

$$
\frac{d}{dt}\left(\frac{1}{2}\|\varphi_t\|_2^2 + \frac{1}{2}\|\varphi_{xx}\|_2^2\right) + m\varphi_t(L,t)\varphi_{tt}(L,t) + \zeta\varphi_t(L,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi,t)\, d\xi = 0. \tag{2.8}
$$

Multiplying the second equation in (P') by $\gamma(\pi)^{-1}\sin(\alpha\pi)\phi_t$ and integrating over $(-\infty,+\infty)$, to obtain:

$$
\frac{\zeta}{2}\frac{d}{dt}\|\phi\|_2^2 + \zeta\int_{-\infty}^{+\infty}(\xi^2+\eta)(\phi(\xi,t))^2\, d\xi - \zeta\varphi_t(L,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi,t)\, d\xi = 0. \tag{2.9}
$$

From (2.6), (2.8) and (2.9) we get

$$
E'(t) = -\zeta\int_{-\infty}^{+\infty}(\xi^2+\eta)(\phi(\xi,t))^2\, d\xi
$$

where $\zeta = (\pi)^{-1}\sin(\alpha\pi)\gamma$. This completes the proof of the lemma.

2.3 Global existence

In this section we will give well-posedness results for problem (P') using semigroup theory. Let us introduce the semigroup representation of the (P'). Let $U = (\varphi, \varphi_t, \phi, v)^T$ and rewrite (P') as

$$\begin{cases} U' = \mathcal{A}U, \\ U(0) = (\varphi_0, \varphi_1, \phi_0, v_0), \end{cases} \tag{2.10}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} \varphi \\ u \\ \phi \\ v \end{pmatrix} = \begin{pmatrix} u \\ -\varphi_{xxxx} \\ -(\xi^2+\eta)\phi + u(L)\mu(\xi) \\ \dfrac{1}{m}\varphi_{xxx}(L) - \dfrac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)d\xi \end{pmatrix} \tag{2.11}$$

with domain

$$D(\mathcal{A}) = \left\{ \begin{array}{l} (\varphi, u, \phi, v)^T \text{ in } \mathcal{H} : \varphi \in H^4(0,L) \cap H_L^2(0,L), u \in H_L^2(0,L), \\ \varphi_{xx}(L) = 0, -(\xi^2+\eta)\phi + u(L)\mu(\xi) \in L^2(-\infty,+\infty), \\ |\xi|\phi \in L^2(-\infty,+\infty), \ v = u(L) \end{array} \right\}. \tag{2.12}$$

where the energy space $\mathcal{H}$ is defined as

$$\mathcal{H} = H_L^2(0,L) \times L^2(0,L) \times L^2(-\infty,+\infty) \times \mathbb{C}$$

where

$$H_L^2(0,L) = \{\varphi \in H^2(0,L) : \varphi(0) = \varphi_x(0) = 0\}$$

For $U = (\varphi, u, \phi, v)^T, \overline{U} = (\overline{\varphi}, \overline{u}, \overline{\phi}, \overline{v})^T$, we define the following inner product in $\mathcal{H}$

$$\langle U, \overline{U}\rangle_{\mathcal{H}} = \int_0^L (u\overline{u} + \varphi_{xx}\overline{\varphi}_{xx})\, dx + \zeta\int_{-\infty}^{+\infty}\phi\overline{\phi}\, d\xi + mv\overline{v}.$$

We show that the operator $\mathcal{A}$ generates a C_0- semigroup in $\mathcal{H}$. In this step, we prove that the operator $\mathcal{A}$ is dissipative. Let $U = (\varphi, u, \phi, v)^T$. Using (2.10), (2.7) and the fact that

$$E(t) = \frac{1}{2}\|U\|_{\mathcal{H}}^2, \tag{2.13}$$

we get

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\zeta\int_{-\infty}^{+\infty}(\xi^2+\eta)(\phi(\xi))^2\, d\xi. \tag{2.14}$$

Consequently, the operator $\mathcal{A}$ is dissipative. Now, we will prove that the operator $\lambda I - \mathcal{A}$ is surjective for $\lambda > 0$. For this purpose, let $(f_1, f_2, f_3, f_4)^T \in \mathcal{H}$, we seek $U = (\varphi, u, \phi, v)^T \in D(\mathcal{A})$ solution of the following system of equations

$$\begin{cases} \lambda\varphi - u = f_1, \\ \lambda u + \varphi_{xxxx} = f_2, \\ \lambda\phi + (\xi^2+\eta)\phi - u(L)\mu(\xi) = f_3, \\ \lambda\varphi - \dfrac{1}{m}\varphi_{xxx}(L) + \dfrac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty}\phi(\xi)u(\xi)\ d\xi = f_4. \end{cases} \tag{2.15}$$

Suppose that we have found φ. Therefore, the first equation in (2.15) gives

$$u = \lambda\varphi - f_1. \tag{2.16}$$

It is clear that $u \in H_L^2(0, L)$. Furthermore, by (2.15) we can find ϕ as

$$\phi = \frac{f_3(\xi) + \mu(\xi)u(L)}{\xi^2 + \eta + \lambda}. \tag{2.17}$$

By using (2.15) and (2.16) the function φ satisfying the following system

$$\lambda^2\varphi + \varphi_{xxxx} = f_2 + \lambda f_1. \tag{2.18}$$

Solving system (2.18) is equivalent to finding $\varphi \in H^4 \cap H_L^2(0, L)$ such that

$$\int_0^L (\lambda^2\varphi w + \varphi_{xxxx}w)\,dx = \int_0^L (f_2 + \lambda f_1)w\,dx, \tag{2.19}$$

for all $w \in H_L^2(0, L)$. By using (2.19) and (2.17) the function φ satisfying the following system

$$\left\{\begin{array}{l} \displaystyle\int_0^L (\lambda^2\varphi w + \varphi_{xx}w_{xx})\,dx + (\lambda m + \tilde{\zeta})u(L)w(L) + \\ \displaystyle = \int_0^L (f_2 + \lambda f_1)w\,dx + mf_4w(L) - \zeta\int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + \lambda} f_3(\xi)\,d\xi\; w(L) \end{array}\right. \tag{2.20}$$

where $\tilde{\zeta} = \zeta\displaystyle\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2 + \eta + \lambda}\,d\xi$. Using again (2.16), we deduce that

$$u(L) = \lambda\varphi(L) - f_1(L). \tag{2.21}$$

Inserting (2.21) into (2.20), we get

$$\left\{\begin{array}{l} \displaystyle\int_0^L (\lambda^2\varphi w + \varphi_{xx}w_{xx})\,dx + \lambda(\lambda m + \tilde{\zeta})\varphi(L)w(L) \\ \displaystyle = \int_0^L (f_2 + \lambda f_1)w\,dx + mf_4w(L) + (\lambda m + \tilde{\zeta})f_1(L)w(L) - \zeta\int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + \lambda} f_3(\xi)\,d\xi\; w(L) \end{array}\right. \tag{2.22}$$

Consequently, problem (2.22) is equivalent to the problem

$$a(\varphi, w) = L(w) \tag{2.23}$$

where the bilinear form $a : [H_L^2(0, L) \times H_L^2(0, L)] \to \mathbb{R}$ and the linear form
$L : H_L^2(0, L) \to \mathbb{R}$ are defined by

$$a(\varphi, w) = \int_0^L (\lambda^2\varphi w + \varphi_{xx}w_{xx})\,dx + \lambda(\lambda m + \tilde{\zeta})\varphi(L)w(L)$$

and

$$L(w) = \int_0^L (f_2 + \lambda f_1)w\,dx + mf_4w(L) + (\lambda m + \tilde{\zeta})f_1(L)w(L) - \zeta\int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + \lambda} f_3(\xi)\,d\xi\; w(L).$$

It is easy to verify that a is continuous and coercive, and L is continuous. So applying the Lax-Milgram theorem, we deduce that for all $w \in H_L^2(0,L)$ problem (2.23) admits a unique solution $\varphi \in H_L^2(0,L)$. Applying the classical elliptic regularity, it follows from (2.22) that $\varphi \in H^4(0,L)$. Therefore, the operator $\lambda I - A$ is surjective for any $\lambda > 0$. Consequently, using Hille-Yosida theorem, we have the following results.

Theorem 2.3.1 (Existence and uniqueness)

(1) If $U_0 \in D(\mathcal{A})$, then system (2.10) has a unique strong solution

$$U \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

(2) If $U_0 \in \mathcal{H}$, then system (2.10) has a unique weak solution

$$U \in C^0(\mathbb{R}_+, \mathcal{H}).$$

2.4 Lack of exponential stability

In order to state and prove our stability results, we need some lemmas.

Theorem 2.4.1 ([60]) *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on Hilbert space. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{i\beta : \beta \in \mathbb{R}\} \equiv i\mathbb{R}$$

and

$$\overline{\lim_{|\beta|\to\infty}} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

Theorem 2.4.2 ([14]) *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup on a Hilbert space. If*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \text{ and } \sup_{|\beta|\geq 1} \frac{1}{\beta^l}\|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < M$$

for some l, then there exist c such that

$$\|e^{\mathcal{A}t}U_0\|^2 \leq \frac{c}{t^{\frac{2}{l}}}\|U_0\|^2_{D(\mathcal{A})}.$$

Theorem 2.4.3 ([44]) *Let $\mathcal{A}$ be the generator of a uniformly bounded C_0. semigroup $\{S(t)\}_{t\geq 0}$ on a Hilbert space $\mathcal{H}$. If:*

(i) $\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.

(ii) The intersection of the spectrum $\sigma(\mathcal{A})$ with $i\mathbb{R}$ is at most a countable set,

then the semigroup $\{S(t)\}_{t\geq 0}$ is asymptotically stable, i.e, $\|S(t)z\|_{\mathcal{H}} \to 0$ as $t \to \infty$ for any $z \in \mathcal{H}$.

Theorem 2.4.4 *The semigroup generated by the operator $\mathcal{A}$ is not exponentially stable.*

Proof: We will examine two cases.
Case 1 $\eta = 0$: We shall show that $i\lambda = 0$ is not in the resolvent set of the operator $\mathcal{A}$. Indeed, noting that $(x\sin x, 0, 0, 0)^T \in \mathcal{H}$, and denoting by $(\varphi, u, \phi, v)^T$ the image of $(x\sin x, 0, 0, 0)^T$ by $\mathcal{A}^{-1}$, we see that $\phi(\xi) = |\xi|^{\frac{2\alpha-5}{2}} L\sin L$. But, then $\phi \notin L^2(-\infty, +\infty)$, since $\alpha \in]0,1[$. And so $(\varphi, u, \phi, v)^T \notin D(\mathcal{A})$.
• **Case 2** $\eta \neq 0$: We aim to show that an infinite number of eigenvalues of $\mathcal{A}$ approach the imaginary axis which prevents the Euler-Bernoulli system (P) from being exponentially stable. Indeed we first compute the characteristic equation that gives the eigenvalues of $\mathcal{A}$. Let λ be an eigenvalue of $\mathcal{A}$ with associated eigenvector $U = (\varphi, u, \phi, v)^T$. Then $\mathcal{A}U = \lambda U$ is equivalent to

$$\begin{cases} \lambda\varphi - u = 0, \\ \lambda u + \varphi_{xxxx} = 0, \\ \lambda\phi + (\xi^2 + \eta)\phi - u(L)\mu(\xi) = 0, \\ \lambda v - \dfrac{1}{m}\varphi_{xxx}(L) + \dfrac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi = 0 \end{cases} \tag{2.24}$$

From $(2.24)_1 - (2.24)_2$ for such λ, we find

$$\lambda^2\varphi + \varphi_{xxxx} = 0. \tag{2.25}$$

Using $(2.24)_3$ and $(2.24)_4$, we get

$$\begin{cases} \varphi(0) = 0, \varphi_x(0) = 0, \varphi_{xx}(L) = 0 \\ \lambda v - \dfrac{1}{m}\varphi_{xxx}(L) + \dfrac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty} \dfrac{\mu^2(\xi)}{\xi^2 + \lambda + \eta}\, d\xi u(L) \\ \qquad\qquad = -\dfrac{1}{m}\varphi_{xxx}(L) + \left(\lambda + \dfrac{\gamma}{m}(\lambda + \eta)^{\alpha-1}\right)\lambda\varphi(L) = 0. \end{cases} \tag{2.26}$$

The caracteristics polynomiale of (2.25) is

$$s^4 + \lambda^2 = 0.$$

We find the roots

$$t_1(\lambda) = \frac{1}{\sqrt{2}}(1+i)\sqrt{\lambda}, \quad t_2(\lambda) = -t_1, t_3(\lambda) = it_1, t_4(\lambda) = -t_3.$$

Here and below, for simplicity we denote $t_i(\lambda)$ by t_i. The solution φ is given by

$$\varphi(x) = \sum_{i=1}^{4} c_i e^{t_i x}. \tag{2.27}$$

Thus the boundary conditions may be written as the following system:

$$M(\lambda)C(\lambda) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ t_1 & t_2 & t_3 & t_4 \\ t_1^2 e^{t_1 L} & t_2^2 e^{t_2 L} & t_3^2 e^{t_3 L} & t_4^2 e^{t_4 L} \\ h(t_1)e^{t_1 L} & h(t_2)e^{t_2 L} & h(t_3)e^{t_3 L} & h(t_4)e^{t_4 L} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.28}$$

where we have set

$$h(r) = r^3 - \lambda \left(m\lambda + \gamma(\lambda + \eta)^{\alpha - 1} \right).$$

Hence a non-trivial solution φ exists if and only if the determinant of $M(\lambda)$ vanishes. Set $f(\lambda) = detM(\lambda)$, thus the characteristic equation is $f(\lambda) = 0$.

Our purpose in the sequel is to prove, thanks to Rouché's Theorem, that there is a subsequence of eigenvalues for which their real part tends to 0.

In the sequel, since $\mathcal{A}$ is dissipative, we study the asymptotic behavior of the large eigenvalues λ of $\mathcal{A}$ in the strip $-\alpha_0 \leq \mathcal{R}(\lambda) \leq 0$, for some $\alpha_0 > 0$ large enough and for such λ, we remark that $e^{t_i L}, i = 1, \ldots, 4$ remains bounded.

Lemma 2.4.1 *There exists $N \in \mathbb{N}$ such that*

$$\{\lambda_k\}_{k \in \mathbf{Z}^*, |k| \geq N} \subset \sigma(\mathcal{A}) \tag{2.29}$$

where

$$\lambda_k = i \left(\left(\frac{\left(k + \frac{1}{4}\right)\pi}{L} \right)^2 + \frac{1}{Lm} - \frac{1}{\pi(2k + \frac{1}{2})m^2} - \frac{1}{\pi^2(2k + \frac{1}{2})^2 m^2} \left(1 - \frac{2}{3}\frac{L}{m}\right) \right)$$
$$+ \frac{\tilde{\alpha}}{k^{4-2\alpha}} + \frac{\beta}{k^{4-2\alpha}} + o\left(\frac{1}{k^{4-2\alpha}}\right), |k| \geq N, \tilde{\alpha} \in i\mathbb{R}, \beta \in \mathbb{R}, \beta < 0.$$

Moreover for all $|k| \geq N$, the eigenvalues λ_k are simple.

Proof

$$\begin{aligned} f(\lambda) &= 8it_1^6 + 2it_1^6 e^{(1-i)Lt_1} + 2it_1^6 e^{(1+i)Lt_1} + 2it_1^6 e^{-(1+i)Lt_1} + 2it_1^6 e^{(-1+i)Lt_1} \\ &\quad - (2+2i)\, m\lambda^2 t_1^3 e^{(1-i)Lt_1} + 2(1-i)m\lambda^2 t_1^3 e^{(1+i)Lt_1} - 2(1-i)m\lambda^2 t_1^3 e^{-(1+i)Lt_1} \\ &\quad + 2(1+i)m\lambda^2 t_1^3 e^{(-1+i)Lt_1} - 2(1+i)qr\lambda t_1^3 e^{(1-i)Lt_1} + 2(1-i)qr\lambda t_1^3 e^{(1+i)Lt_1} \\ &\quad - 2(1-i)qr\lambda t_1^3 e^{-(1+i)Lt_1} + 2(1+i)qr\lambda t_1^3 e^{(-1+i)Lt_1} \\ &= 2im\sqrt{2\lambda}t_1^6 e^{-iL\sqrt{2\lambda}} \Bigg(e^{(1+i)L\sqrt{2\lambda}} - i + \frac{1}{m\sqrt{2\lambda}} \left(e^{(1+i)L\sqrt{2\lambda}} + 1 \right) + \frac{\gamma r}{m\lambda} \left(e^{(1+i)L\sqrt{2\lambda}} - i \right) \\ &\quad + ie^{2iL\sqrt{2\lambda}} - e^{(-1+i)L\sqrt{2\lambda}} + \frac{\gamma r}{m\lambda} \left(ie^{2iL\sqrt{2\lambda}} - e^{(-1+i)L\sqrt{2\lambda}} \right) \\ &\quad + \frac{1}{m\sqrt{2\lambda}} \left(e^{2iL\sqrt{2\lambda}} + e^{(-1+i)L\sqrt{2\lambda}} + 4e^{iL\sqrt{2\lambda}} \right) \Bigg) \end{aligned} \tag{2.30}$$

Since all the eigenvalues locate on the open left-half complex plane, and since λ is symmetric with respect to the real axis, we need only to consider the case where $\pi/2 \leq \theta \leq \pi$. Since $\sqrt{\lambda} = \sqrt{|\lambda|}(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2})$, we see that

$$e^{-\sqrt{2\lambda}} = O(e^{-\mu}\sqrt{|\lambda|}), \quad e^{i\sqrt{2\lambda}} = O(e^{-\mu}\sqrt{|\lambda|}), \quad \mu > 0.$$

We set

$$\begin{aligned}\tilde{f}(\lambda) &= \Bigg(e^{(1+i)L\sqrt{2\lambda}} - i + \frac{1}{m\sqrt{2\lambda}}(e^{(1+i)L\sqrt{2\lambda}}+1) + \frac{\gamma r}{m\lambda}(e^{(1+i)L\sqrt{2\lambda}} - i) \\ &\quad + ie^{2iL\sqrt{2\lambda}} - e^{(-1+i)L\sqrt{2\lambda}} + \frac{\gamma r}{m\lambda}(ie^{2iL\sqrt{2\lambda}} - e^{(-1+i)L\sqrt{2\lambda}}) \\ &\quad + \frac{1}{m\sqrt{2\lambda}}(e^{2iL\sqrt{2\lambda}} + e^{(-1+i)L\sqrt{2\lambda}} + 4e^{iL\sqrt{2\lambda}})\Bigg) \\ &= f_0(\lambda) + \tfrac{f_1(\lambda)}{\lambda^{1/2}} + \tfrac{f_2(\lambda)}{\lambda^{2-\alpha}} + \tfrac{f_3(\lambda)}{\lambda^{3-\alpha}} + o\left(\tfrac{1}{\lambda^{3-\alpha}}\right)\end{aligned} \tag{2.31}$$

where

$$f_0(\lambda) = e^{(1+i)L\sqrt{2\lambda}} - i, \tag{2.32}$$

$$f_1(\lambda) = \frac{1}{\sqrt{2}m}(e^{(1+i)L\sqrt{2\lambda}} + 1). \tag{2.33}$$

$$f_2(\lambda) = \frac{\gamma}{m}(e^{(1+i)L\sqrt{2\lambda}} - i). \tag{2.34}$$

$$f_3(\lambda) = \frac{\gamma\eta(\alpha-1)}{m}(e^{(1+i)L\sqrt{2\lambda}} - i). \tag{2.35}$$

Note that f_0 and f_1 remain bounded in the strip $-\alpha_0 \leq \mathcal{R}(\lambda) \leq 0$.

Step 2. We look at the roots of f_0. From (2.32), f_0 has one familie of roots that we denote λ_k^0.

$$f_0(\lambda) = 0 \Leftrightarrow e^{(1+i)L\sqrt{2\lambda}} = i.$$

Hence

$$(1+i)L\sqrt{2\lambda} = i\left(2k + \frac{1}{2}\right)\pi, \quad k \in \mathbf{Z},$$

i.e.,

$$\lambda_k^0 = \frac{i}{4L^2}\left(2k + \frac{1}{2}\right)^2 \pi^2, \qquad k \in \mathbf{Z}.$$

Now with the help of Rouché's Theorem, we will show that the roots of $\tilde{f}$ are close to those of f_0. Changing in (2.31) the unknown λ by $u = (1+i)L\sqrt{2\lambda}$ then (2.31) becomes

$$\tilde{f}(u) = (e^u + 1) + O\left(\frac{1}{u}\right) = f_0(u) + O\left(\frac{1}{u}\right).$$

The roots of f_0 are $u_k = \dfrac{i}{4L^2}(2k + \dfrac{1}{2})^2\pi^2, k \in \mathbf{Z}$, and setting $u = u_k + re^{it}, t \in [0, 2\pi]$, we can easily check that there exists a constant $C > 0$ independent of k such that $|e^u + 1| \geq Cr$ for r small enough. This allows to apply Rouché's Theorem. Consequently, there exists a subsequence of roots of $\tilde{f}$ which tends to the roots u_k of f_0. Equivalently, it means that there exists $N \in \mathbb{N}$ and a subsequence $\{\lambda_k\}_{|k|\geq N}$ of roots of $f(\lambda)$, such that $\lambda_k = \lambda_k^0 + o(1)$ which tends to the roots $\dfrac{i}{4L^2}(2k + \dfrac{1}{2})^2\pi^2$ of f_0. Finally for $|k| \geq N, \lambda_k$ is simple since λ_k^0 is.

Step 3. From Step 2, we can write

$$\lambda_k = \frac{i}{4L^2}(2k+\frac{1}{2})^2\pi^2 + \varepsilon_k. \tag{2.36}$$

Using (2.36), we get

$$e^{(1+i)L\sqrt{2\lambda}} = i + \frac{2iL^2}{\pi}\frac{\varepsilon_k}{(2k+\frac{1}{2})} + o(\frac{\varepsilon_k}{k}). \tag{2.37}$$

Substituting (2.37) into (2.31), using that $\tilde{f}(\lambda_k) = 0$, we get:

$$\tilde{f}(\lambda_k) = \frac{2iL^2}{\pi}\frac{\varepsilon_k}{(2k+\frac{1}{2})} + \frac{2L}{m(2k+\frac{1}{2})\pi} + o(\frac{\varepsilon_k}{k}) = 0 \tag{2.38}$$

and hence

$$\varepsilon_k = \frac{i}{Lm}. \tag{2.39}$$

This step shows that ε_k is equivalent to a pure imaginary number. Since we are interested in the asymptotic behavior of the real part of the ε_k. we need to find the next term in the development.

Step 4. From Step 2, we can write

$$\lambda_k = \frac{i}{4L^2}(2k+\frac{1}{2})^2\pi^2 + \frac{i}{Lm} + \varepsilon_k. \tag{2.40}$$

Using (3.21) and (2.40), we obtain

$$\tilde{f}(\lambda_k) = \frac{2iL^2}{\pi}\frac{\varepsilon_k}{(2k+\frac{1}{2})} - \frac{L^2}{m^2(2k+\frac{1}{2})^2\pi^2} + o(\frac{\varepsilon_k}{k}) + o(\frac{1}{k^2}) = 0 \tag{2.41}$$

and hence

$$\varepsilon_k = -\frac{i}{\pi(2k+\frac{1}{2})m^2}. \tag{2.42}$$

Step 5. From Step 2, we can write

$$\lambda_k = \frac{i}{4L^2}(2k+\frac{1}{2})^2\pi^2 + \frac{i}{Lm} - \frac{i}{\pi(2k+\frac{1}{2})m^2} + \varepsilon_k. \tag{2.43}$$

Using (3.21) and (2.40), we obtain

$$\tilde{f}(\lambda_k) = \frac{2iL^2}{\pi}\frac{\varepsilon_k}{(2k+\frac{1}{2})} + \frac{2}{3}L^2\frac{2L-3m}{\pi^3(2k+\frac{1}{2})^3m^3} + o(\frac{\varepsilon_k}{k}) + o(\frac{1}{k^3}) = 0 \tag{2.44}$$

and hence

$$\varepsilon_k = -\frac{i}{\pi^2(2k+\frac{1}{2})^2m^2}\left(1 - \frac{2}{3}\frac{L}{m}\right). \tag{2.45}$$

Step 6. From Step 2, we can write

$$\lambda_k = \frac{i}{4L^2}(2k+\frac{1}{2})^2\pi^2 + \frac{i}{Lm} - \frac{i}{\pi(2k+\frac{1}{2})m^2} - \frac{i}{\pi^2(2k+\frac{1}{2})^2m^2}\left(1-\frac{2}{3}\frac{L}{m}\right) + \varepsilon_k. \tag{2.46}$$

Using (2.41), we obtain

$$\begin{array}{c} \tilde{f}(\lambda_k) = \frac{2iL^2}{\pi}\frac{\varepsilon_k}{(2k+\frac{1}{2})} + \frac{8}{\pi^4}\frac{L^3}{(2k+\frac{1}{2})^4m^3} + \frac{2\gamma L}{\pi m^2}\left(\frac{4L}{\pi}\right)^{4-2\alpha}\frac{i^\alpha}{(2k+\frac{1}{2})^{5-2\alpha}} \\ +o(\frac{\varepsilon_k}{k}) + O(\frac{1}{k^5}) = 0 \end{array} \tag{2.47}$$

hence

$$\begin{array}{c} \varepsilon_k = i\left(\frac{4L}{(\pi(2k+\frac{1}{2})m)^3} + \frac{\gamma}{Lm^2}\left(\frac{4L}{\pi}\right)^{4-2\alpha}\frac{\cos\alpha\frac{\pi}{2}}{(2k+\frac{1}{2})^{4-2\alpha}}\right) \\ -\frac{\gamma}{Lm^2}\left(\frac{4L}{\pi}\right)^{4-2\alpha}\frac{\sin\alpha\frac{\pi}{2}}{(2k+\frac{1}{2})^{4-2\alpha}}. \end{array} \tag{2.48}$$

From (2.39) we have in that case $|k|^{2(2-\alpha)}\mathcal{R}\lambda_k \sim \beta$, with

$$\beta = -\frac{\gamma}{Lm^2}\left(\frac{2L}{\pi}\right)^{4-2\alpha}\sin\alpha\frac{\pi}{2}.$$

The operator $\mathcal{A}$ has a non exponential decaying branch of eigenvalues. Thus the proof is complete.

2.5 Asymptotic stability

Lemma 2.5.1 *$\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.*

Proof
We will argue by contraction. Let us suppose that there $\lambda \in \mathbb{R}, \lambda \neq 0$ and $U \neq 0$, such that $\mathcal{A}U = i\lambda U$. Then, we get

$$\begin{cases} i\lambda\varphi - u = 0, \\ i\lambda u + \varphi_{xxxx} = 0, \\ i\lambda\phi + (\xi^2+\eta)\phi - u(L)\mu(\xi) = 0, \\ i\lambda v - \frac{1}{m}\varphi_{xxx}(L) + \frac{\zeta}{m}\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)\,d\xi = 0 \end{cases} \tag{2.49}$$

Then, from (2.14) we have

$$\phi \equiv 0. \tag{2.50}$$

From $(2.49)_3$, we have

$$u(L) = 0. \tag{2.51}$$

Hence, from $(2.49)_1$ we obtain

$$\varphi(L) = 0 \text{ and } \varphi_{xxx}(L) = 0. \tag{2.52}$$

From $(2.49)_1$ and $(2.49)_2$, we have

$$-\lambda^2\varphi + \varphi_{xxxx} = 0. \tag{2.53}$$

Now, we prove that $\varphi_x(L) = 0$. We have the following Lemma.

Lemma 2.5.2 ([23]) *Let $\varphi \in H^2(0,L)$ a solution of equation (2.53). Assume there exists $\zeta \in [0, L[$ such that $\varphi(\zeta), \varphi_x(\zeta), \varphi_{xx}(\zeta)$ are ≥ 0, and $\varphi(\zeta) + \varphi_x(\zeta) > 0$. Then $\varphi, \varphi_x, \varphi_{xx}$ are > 0 on $]\zeta, L]$.*

Proof We integrate equation (2.53) from $\zeta < x$ to x:

$$\begin{aligned} \varphi_{xxx}(x) &= \varphi_{xxx}(\zeta) + \int_\zeta^x \varphi_{xxxx}(t)\,dt \\ &= \varphi_{xxx}(\zeta) + \lambda^2 \int_\zeta^x \varphi(t)\,dt. \end{aligned}$$

Integrating once more, we get

$$\begin{aligned} \varphi_{xx}(x) - \varphi_{xx}(\zeta) &= (x-\zeta)\varphi_{xxx}(\zeta) + \lambda^2 \int_\zeta^x \int_\zeta^t \varphi(z)\,dz\,dt \\ &= (x-\zeta)\varphi_{xxx}(\zeta) + \lambda^2 \int_\zeta^x (x-t)\varphi(t)\,dt. \end{aligned}$$

$$\varphi_{xx}(x) = \varphi_{xx}(\zeta) + (x-\zeta)\varphi_{xxx}(\zeta) + \lambda^2 \int_\zeta^x (x-t)\varphi(t)\,dt. \tag{2.54}$$

Since $\varphi(\zeta) + \varphi_x(\zeta) > 0$ and $\varphi(\zeta) \geq 0$, there exists $\eta > 0$ such that $\varphi > 0$ on $]\zeta, \eta]$. Let $\eta \leq 1$ as large as possible, and suppose $\eta < 1$, that is, $\varphi(\eta) = 0$. By (2.54) and assumptions in Lemma 2.5.2, $\varphi_{xx} \geq 0$ on $[\zeta, \eta]$. Thus φ_x is nondecreasing, and therefore ≥ 0 on $[\zeta, \eta]$.

Then φ is also nondecreasing on $[\zeta, \eta]$. But this contradicts $\varphi(\eta) = 0$. Thus $\eta = 1$ and φ is > 0 on $]\zeta, \eta]$. The same is true for $\varphi_{xxx}, \varphi_{xx}$ and φ_x.

Corollary 2.5.1 *Soit $\varphi \in H^2(0,L)$ a solution of equation (2.53) such that $\varphi(L) \geq 0, \varphi_x(L) \leq 0, \varphi_{xx}(L) \geq 0$, and $\varphi(L) - \varphi_x(L) > 0$. Then $\varphi > 0$ on $[0, L[$.*

Proof We set $\psi(x) = \varphi(L-x)$. Then ψ satisfies (2.54). Then applying Lemma 2.5.2.

Now, as $\varphi(L) = 0$, assume $\varphi_x(L) \neq 0$, for instance $\varphi_x(L) < 0$, without restriction. By corollary 2.5.1, $\varphi > 0$ on $[0, L[$, thus $\varphi(0) > 0$, which is a contradiction. Therefore, $\varphi(L) = \varphi_x(L) = \varphi_{xx}(L) = \varphi_{xxx}(L) = 0$.

Consider $X = (\varphi, \varphi_x, \varphi_{xx}, \varphi_{xxx})$. Then we can rewrite (2.52) and (2.53) as the initial value problem

$$\begin{aligned} &\frac{d}{dx}X = \mathcal{B}X \\ &X(L) = 0, \end{aligned} \tag{2.55}$$

where

$$\mathcal{B} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\lambda^2 & 0 & 0 & 0 \end{pmatrix}$$

By the Picard Theorem for ordinary differential equations the system (2.55) has a unique solution $X = 0$. Therefore $\varphi = 0$. It follows from (2.49), that $u = 0$ and $\phi = 0$, i.e., $U = 0$. Consequently, $\mathcal{A}$ does not have purely imaginary eigenvalues, so the condition (i) of Theorem 2.4.3 holds. The condition (ii) of Theorem 2.4.3 will be satisfied if we show that $\sigma(\mathcal{A}) \cap \{i\mathbb{R}\}$ is at most a countable set. We have the following lemma.

Lemma 2.5.3 *We have*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \text{ if } \eta \neq 0,$$
$$i\mathbb{R}^* \subset \rho(\mathcal{A}) \text{ if } \eta = 0$$

where $\mathbb{R}^* = \mathbb{R} - \{0\}$.

Proof

Let $\lambda \in \mathbb{R}$. Let $F = (f_1, f_2, f_3)^T \in \mathcal{H}$ be given, and let $X = (\varphi, u, \phi)^T \in D(\mathcal{A})$ be such that

$$(i\lambda I - \mathcal{A})X = F. \tag{2.56}$$

Equivalently, we have

$$\begin{cases} i\lambda\varphi - u = f_1, \\ i\lambda u + \varphi_{xxxx} = f_2, \\ i\lambda\phi + (\xi^2 + \eta)\phi - u(L)\mu(\xi) = f_3, \\ i\lambda v - \dfrac{1}{m}\varphi_{xxx}(L) + \dfrac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi = f_4 \end{cases} \tag{2.57}$$

From $(2.57)_1$ and $(2.57)_2$, we have

$$-\lambda^2\varphi + \varphi_{xxxx} = (f_2 + i\lambda f_1) \tag{2.58}$$

Suppose that $\lambda \neq 0$. It is enough to consider $\lambda > 0$. Let $\lambda = \tau^2$. Taking into account the domain boundary conditions $\varphi(0) = \varphi_x(0) = 0$, implies that the general solution for (2.58) is of the form

$$\begin{aligned} \varphi(x) = {} & A(\cosh \tau x - \cos \tau x) + B(\sinh \tau x - \sin \tau x) \\ & + \frac{1}{2\tau^3}\int_0^x (f_2(\sigma) + i\tau^2 f_1(\sigma))(\sinh \tau(x-\sigma) - \sin \tau(x-\sigma))\, d\sigma. \end{aligned} \tag{2.59}$$

Hence

$$\begin{aligned} \varphi_x(x) = {} & \tau[A(\sinh \tau x + \sin \tau x) + B(\cosh \tau x - \cos \tau x)] \\ & + \frac{1}{2\tau^2}\int_0^x (f_2(\sigma) + i\tau^2 f_1(\sigma))(\cosh \tau(x-\sigma) - \cos \tau(x-\sigma))\, d\sigma, \end{aligned} \tag{2.60}$$

$$\begin{aligned} \varphi_{xx}(x) = {} & \tau^2[A(\cosh \tau x + \cos \tau x) + B(\sinh \tau x + \sin \tau x)] \\ & + \frac{1}{2\tau}\int_0^x (f_2(\sigma) + i\tau^2 f_1(\sigma))(\sinh \tau(x-\sigma) + \sin \tau(x-\sigma))\, d\sigma, \end{aligned} \tag{2.61}$$

$$\begin{aligned}\varphi_{xxx}(x) &= \tau^3[A(\sinh\tau x - \sin\tau x) + B(\cosh\tau x + \cos\tau x)] \\ &+\frac{1}{2}\int_0^x (f_2(\sigma) + i\tau^2 f_1(\sigma))(\cosh\tau(x-\sigma) + \cos\tau(x-\sigma))\,d\sigma.\end{aligned} \tag{2.62}$$

Taking the remaining boundary condition $\varphi_{xx}(L) = 0$, we obtain

$$\begin{aligned}&A(\cosh\tau L + \cos\tau L) + B(\sinh\tau L + \sin\tau L) \\ &\qquad = -\frac{1}{2\tau^3}\int_0^L (f_2(\sigma) + i\tau^2 f_1(\sigma))(\sinh\tau(L-\sigma) + \sin\tau(L-\sigma))\,d\sigma.\end{aligned} \tag{2.63}$$

From $(2.57)_3$, we have

$$\phi(\xi) = \frac{u(L)\mu(\xi) + f_3(\xi)}{i\lambda + \xi^2 + \eta}.$$

Then

$$\left(i\lambda + \frac{\zeta}{m}\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{i\lambda + \xi^2 + \eta}\,d\xi\right)u(L) - \frac{1}{m}\varphi_{xxx}(L) + \frac{\zeta}{m}\int_{-\infty}^{+\infty}\frac{\mu(\xi)f_3(\xi)}{i\lambda + \xi^2 + \eta}\,d\xi = f_4. \tag{2.64}$$

Since

$$\zeta\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{i\lambda + \xi^2 + \eta}\,d\xi = \gamma(i\lambda + \eta)^{\alpha-1}$$

and

$$u(L) = i\lambda\varphi(L) - f_1(L),$$

using (2.64), we get

$$\begin{aligned}&i\lambda\left(i\lambda + \frac{\gamma}{m}(i\lambda+\eta)^{\alpha-1}\right)\varphi(L) - \frac{1}{m}\varphi_{xxx}(L) \\ &= \left(i\lambda + \frac{\gamma}{m}(i\lambda+\eta)^{\alpha-1}\right)f_1(L) - \frac{\zeta}{m}\int_{-\infty}^{+\infty}\frac{\mu(\xi)f_3(\xi)}{i\lambda + \xi^2 + \eta}\,d\xi + f_4.\end{aligned}$$

Then

$$\begin{aligned}&A[i\tau^2(i\tau^2 + \frac{\gamma}{m}(i\lambda+\eta)^{\alpha-1})(\cosh\tau L - \cos\tau L) - \frac{\tau^3}{m}(\sinh\tau L - \sin\tau L)] \\ &+B[i\tau^2(i\tau^2 + \frac{\gamma}{m}(i\lambda+\eta)^{\alpha-1})(\sinh\tau L - \sin\tau L) - \frac{\tau^3}{m}(\cosh\tau L + \cos\tau L)] \\ &= \frac{1}{2m}\int_0^L (f_2(\sigma) + i\tau^2 f_1(\sigma))(\cosh\tau(L-\sigma) + \cos\tau(L-\sigma))\,d\sigma \\ &-i(i\tau^2 + \frac{\gamma}{m}(i\tau^2+\eta)^{\alpha-1})\frac{1}{2\tau}\int_0^L (f_2(\sigma) + i\tau^2 f_1(\sigma))(\sinh\tau(L-\sigma) - \sin\tau(L-\sigma))\,d\sigma \\ &+(i\tau^2 + \frac{\gamma}{m}(i\tau^2+\eta)^{\alpha-1})f_1(L) - \frac{\zeta}{m}\int_{-\infty}^{+\infty}\frac{\mu(\xi)f_3(\xi)}{i\tau^2 + \xi^2 + \eta}\,d\xi + f_4.\end{aligned} \tag{2.65}$$

Using (2.65) and (2.63), a linear system in A and B is obtained

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22}\end{pmatrix}\begin{pmatrix} A \\ B\end{pmatrix} = \begin{pmatrix}\tilde{C}_1 \\ \tilde{C}_2\end{pmatrix} \tag{2.66}$$

where

$$\begin{array}{rcl} m_{11} & = & (\cosh \tau L + \cos \tau L), \\ m_{12} & = & (\sinh \tau L + \sin \tau L), \\ m_{21} & = & [i\tau^2(i\tau^2 + \frac{\gamma}{m}(i\lambda+\eta)^{\alpha-1})(\cosh \tau L - \cos \tau L) - \frac{\tau^3}{m}(\sinh \tau L - \sin \tau L)], \\ m_{22} & = & [i\tau^2(i\tau^2 + \frac{\gamma}{m}(i\lambda+\eta)^{\alpha-1})(\sinh \tau L - \sin \tau L) - \frac{\tau^3}{m}(\cosh \tau L + \cos \tau L)]. \end{array}$$

$$\begin{array}{rcl} \tilde{C}_1 & = & -\frac{1}{2\tau^3}\int_0^L (f_2(\sigma) + i\tau^2 f_1(\sigma))(\sinh \tau(L-\sigma) + \sin \tau(L-\sigma))\, d\sigma \\ \tilde{C}_2 & = & \frac{1}{2m}\int_0^L (f_2(\sigma) + i\tau^2 f_1(\sigma))(\cosh \tau(L-\sigma) + \cos \tau(L-\sigma))\, d\sigma \\ & & -i(i\tau^2 + \frac{\gamma}{m}(i\tau^2+\eta)^{\alpha-1})\frac{1}{2\tau}\int_0^L (f_2(\sigma) + i\tau^2 f_1(\sigma))(\sinh \tau(L-\sigma) - \sin \tau(L-\sigma))\, d\sigma \\ & & +(i\tau^2 + \frac{\gamma}{m}(i\tau^2+\eta)^{\alpha-1})f_1(L) - \frac{\zeta}{m}\int_{-\infty}^{+\infty} \frac{\mu(\xi) f_3(\xi)}{i\tau^2+\xi^2+\eta}\, d\xi + f_4. \end{array}$$

Let the determinant of the linear system given in (2.66) be denoted by D. Then the following is obtained:

$$\begin{array}{rcl} D & = & m_{11}m_{22} - m_{12}m_{21} \\ & = & -\frac{\tau^3}{m}(\cosh \tau L + \cos \tau L)^2 + \frac{\tau^3}{m}(\sinh \tau L + \sin \tau L)(\sinh \tau L - \sin \tau L) \\ & & -2i\tau^2(i\tau^2 + \frac{\gamma}{m}(i\lambda+\eta)^{\alpha-1})[\cosh \tau L \sin \tau L - \sinh \tau L \cos \tau L] \\ & = & -2\frac{\tau^3}{m}(1 + \cosh \tau L \cos \tau L) - 2i\tau^2(i\tau^2 + \frac{\gamma}{m}(i\lambda+\eta)^{\alpha-1})[\cosh \tau L \sin \tau L - \sinh \tau L \cos \tau L] \\ & = & -2\frac{\tau^3}{m}(1 + \cosh \tau L \cos \tau L) + 2\tau^4[\cosh \tau L \sin \tau L - \sinh \tau L \cos \tau L] \\ & & -2\frac{\gamma}{m}\tau^2(\tau^4+\eta^2)^{\frac{\alpha-1}{2}} \sin(1-\alpha)\theta\ [\cosh \tau L \sin \tau L - \sinh \tau L \cos \tau L] \\ & & -2i\frac{\gamma}{m}\tau^2(\tau^4+\eta^2)^{\frac{\alpha-1}{2}} \cos(1-\alpha)\theta\ [\cosh \tau L \sin \tau L - \sinh \tau L \cos \tau L] \end{array}$$

where $\theta \in]-\pi/2, \pi/2[$ such that

$$\begin{array}{rcl} \cos\theta & = & \frac{\eta}{\sqrt{\lambda^2+\eta^2}} \\ \sin\theta & = & \frac{\lambda}{\sqrt{\lambda^2+\eta^2}}. \end{array}$$

The roots of

$$[\cosh \varpi \sin \varpi - \sinh \varpi \cos \varpi] = 0$$

are of the form $\varpi_k = \delta_k + k\pi,\ \delta_k < \pi/4,\ k \in \mathbb{N}^*$. Hence

$$1 + \cosh \varpi_k \cos \varpi_k \neq 0 \quad \forall k \in \mathbb{N}^*.$$

Then

$$D \neq 0 \quad \forall \lambda \in \mathbb{R}^*.$$

Hence $i\lambda-\mathcal{A}$ is surjective for all $\lambda \in \mathbb{R}^*$. Now, if $\lambda = 0$ and $\eta \neq 0$, the system (2.57) is reduced to the following system

$$\begin{cases} u = -f_1, \\ \varphi_{xxxx} = f_2, \\ (\xi^2+\eta)\phi - u(L)\mu(\xi) = f_3, \\ -\dfrac{1}{m}\varphi_{xxx}(L) + \dfrac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\,d\xi = f_4 \end{cases} \tag{2.67}$$

We deduce from $(2.67)_2$

$$\varphi_{xxx}(x) = \int_0^x f_2(s)\,ds + C.$$

$$\varphi_{xx}(x) = \int_0^x \int_0^s f_2(r)\,dr\,ds + Cx + C'.$$

$$\varphi_x(x) = \int_0^x \int_0^s \int_0^r f_2(z)\,dz\,dr\,ds + \frac{C}{2}x^2 + C'x + C''.$$

$$\varphi(x) = \int_0^x \int_0^s \int_0^r \int_0^z f_2(w)\,dw\,dz\,dr\,ds + \frac{C}{6}x^3 + \frac{C'}{2}x^2 + C''x + C'''.$$

As $\varphi(0) = \varphi_x(0) = 0$, we find $C'' = C''' = 0$.

From $(2.67)_1$ and $(2.67)_3$, we have

$$\begin{aligned} \varphi_{xxx}(L) &= \zeta \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2+\eta}\,d\xi u(L) + \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi) f_3(\xi)}{\xi^2+\eta}\,d\xi - m f_4 \\ &= -\gamma\eta^{\alpha-1} f_1(L) + \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi) f_3(\xi)}{\xi^2+\eta}\,d\xi - m f_4. \end{aligned}$$

We find

$$C = -\int_0^L f_2(r)\,dr - \gamma\eta^{\alpha-1} f_1(L) + \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi) f_3(\xi)}{\xi^2+\eta}\,d\xi - m f_4.$$

Because $\varphi_{xx}(L) = 0$, we find

$$C' = -CL - \int_0^L \int_0^s f_2(r)\,dr\,ds.$$

Hence $\mathcal{A}$ is surjective.

Lemma 2.5.4 *Let $\mathcal{A}$ be defined by (2.11). Then*

$$\mathcal{A}^* \begin{pmatrix} \varphi \\ u \\ \phi \\ v \end{pmatrix} = \begin{pmatrix} -u \\ \varphi_{xxxx} \\ -(\xi^2+\eta)\phi - u(L)\mu(\xi) \\ -\dfrac{1}{m}\varphi_{xxx}(L) + \dfrac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\,d\xi \end{pmatrix} \tag{2.68}$$

with domain

$$D(\mathcal{A}^*) = \left\{ \begin{array}{l} (\varphi, u, \phi, v)^T \ \text{in } \mathcal{H} : \varphi \in H^4(0,L) \cap H_L^2(0,L), u \in H_L^2(0,L), v \in \mathbb{C} \\ \varphi_{xx}(L) = 0, -(\xi^2+\eta)\phi - u(L)\mu(\xi) \in L^2(-\infty,+\infty), u(L) = v \\ |\xi|\phi \in L^2(-\infty,+\infty). \end{array} \right\}. \tag{2.69}$$

Proof
Let $U=(\varphi,u,\phi,v)^T$ and $V=(\tilde\varphi,\tilde u,\tilde\phi,\tilde v)^T$. We have $<\mathcal{A}U,V>_{\mathcal{H}}=<U,\mathcal{A}^*V>_{\mathcal{H}}$.

$$\begin{aligned}<\mathcal{A}U,V>_{\mathcal{H}} &= \int_0^L u_{xx}\tilde\varphi_{xx}\,dx-\int_0^L \tilde u\varphi_{xxxx}\,dx+\zeta\int_{-\infty}^{+\infty}[-(\xi^2+\eta)\phi+u(L)\mu(\xi)]\tilde\phi\,d\xi\\ &\quad +m\left(\frac{1}{m}\varphi_{xxx}(L)-\frac{\zeta}{m}\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)\,d\xi\right)\tilde v\\ &= \int_0^L u\tilde\varphi_{xxxx}\,dx-\int_0^L\tilde u_{xx}\varphi_{xx}\,dx-\tilde\varphi_{xxx}(L)u(L)+\tilde\varphi_{xx}(L)u_x(L)-\varphi_{xxx}(L)\tilde u(L)\\ &\quad -\zeta\int_{-\infty}^{+\infty}\phi[(\xi^2+\eta)\tilde\phi]\,d\xi+\zeta u(L)\int_{-\infty}^{+\infty}\mu(\xi)\tilde\phi\,d\xi+\varphi_{xxx}(L)\tilde v-\zeta\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)\,d\xi\tilde v.\end{aligned}$$

As $v=u(L)$ and if we set $\tilde v=\tilde u(L)$ and $\tilde\varphi_{xx}(L)=0$, we find

$$\begin{aligned}<\mathcal{A}U,V>_{\mathcal{H}} &= \int_0^L u\tilde\varphi_{xxxx}\,dx-\int_0^L\tilde u_{xx}\varphi_{xx}\,dx-\zeta\int_{-\infty}^{+\infty}\phi[(\xi^2+\eta)\tilde\phi+\mu(\xi)\tilde u(L)]\,d\xi\\ &\quad +mv\left(-\frac{1}{m}\varphi_{xxx}(L)+\frac{\zeta}{m}\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)\,d\xi\right).\end{aligned}$$

Theorem 2.5.1 *$\sigma_r(\mathcal{A})=\emptyset$, where $\sigma_r(\mathcal{A})$ denotes the set of residual spectrum of $\mathcal{A}$.*

Proof
Since $\lambda\in\sigma_r(\mathcal{A}),\overline{\lambda}\in\sigma_p(\mathcal{A}^*)$ the proof will be accomplished if we can show that $\sigma_p(\mathcal{A})=\sigma_p(\mathcal{A}^*)$. This is because obviously the eigenvalues of A are symmetric on the real axis. From (2.68), the eigenvalue problem $\mathcal{A}^*Z=\lambda Z$ for $\lambda\in\mathbb{C}$ and $0\neq Z=(\varphi,u,\phi,v)\in D(\mathcal{A}^*)$ we have

$$\begin{cases}\lambda\varphi+u=0,\\ \lambda u-\varphi_{xxxx}=0,\\ \lambda\phi+(\xi^2+\eta)\phi+u(L)\mu(\xi)=0,\\ \lambda v+\dfrac{1}{m}\varphi_{xxx}(L)-\dfrac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)\,d\xi=0\end{cases}\tag{2.70}$$

From $(2.70)_1$ and $(2.70)_2$, we find
$$\lambda^2\varphi+\varphi_{xxxx}=0,\tag{2.71}$$
As $v=u(L)=-\lambda\varphi(L)$, we deduce from $(2.70)_3$ and $(2.70)_3$ that

$$\varphi_{xxx}(L)\;=\;-m\lambda u(L)+\zeta\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)\,d\xi\;=\;\lambda(m\lambda+\gamma(\lambda+\eta)^{\alpha-1})\varphi(L).\tag{2.72}$$

with the following conditions

$$\varphi(0)=0,\ \varphi_x(0)=0,\ \varphi_{xx}(L)=0.\tag{2.73}$$

System (2.71)-(2.73) is the same as (2.25) and (2.26). Hence $\mathcal{A}^*$ has the same eigenvalues with $\mathcal{A}$. The proof is complete.
•**Case2** $\eta\neq 0$:

Theorem 2.5.2 *The semigroup $S_{\mathcal{A}}(t)_{t\geq 0}$ is polynomially stable and*

$$\|S_{\mathcal{A}}(t)U_0\|_{\mathcal{H}} \leq \frac{1}{t^{1/(4-2\alpha)}}\|U_0\|_{D(\mathcal{A})}.$$

Proof

We will need to study the resolvent equation $(i\lambda - \mathcal{A})U = F$, for $\lambda \in \mathbb{R}$, namely

$$\begin{cases} i\lambda\varphi - u = f_1, \\ i\lambda u + \varphi_{xxxx} = f_2, \\ i\lambda\phi + (\xi^2+\eta)\phi - u(L)\mu(\xi) = f_3, \\ i\lambda v - \dfrac{1}{m}\varphi_{xxx}(L) + \dfrac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\,d\xi = f_4 \end{cases} \tag{2.74}$$

where $F = (f_1, f_2, f_3, f_4)^T$. Taking inner product in $\mathcal{H}$ with U and using (2.14) we get

$$|Re\langle \mathcal{A}U, U\rangle| \leq \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{2.75}$$

This implies that

$$\zeta\int_{-\infty}^{+\infty}(\xi^2+\eta)(\phi(\xi,t))^2\,d\xi \leq \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{2.76}$$

and, applying $(2.74)_1$, we obtain

$$||\lambda||\varphi(L)| - |f_1(L)||^2 \leq |u(L)|^2.$$

We deduce that

$$|\lambda|^2|\varphi(L)|^2 \leq c|f_1(L)|^2 + c|u(L)|^2.$$

Moreover, from $(2.74)_4$, we have

$$\varphi_{xxx}(L) = im\lambda u(L) + \zeta\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)\,d\xi - mf_4.$$

Then

$$\begin{aligned} &|\varphi_{xxx}(L)|^2 \leq 2m^2|\lambda|^2|u(L)|^2 + 2m^2 f_4^2 + 2\zeta^2\left|\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)\,d\xi\right|^2 \\ &\leq 2m^2|\lambda|^2|u(L)|^2 + 2m^2 f_4^2 + 2\zeta^2\left(\int_{-\infty}^{+\infty}(\xi^2+\eta)^{-1}|\mu(\xi)|^2\,d\xi\right)\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi_i(\xi)|^2\,d\xi \\ &\leq 2m^2|\lambda|^2|u(L)|^2 + c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c'\|F\|_{\mathcal{H}}^2. \end{aligned} \tag{2.77}$$

From $(2.74)_3$, we obtain

$$u(L)\mu(\xi) = (i\lambda + \xi^2 + \eta)\phi - f_3(\xi). \tag{2.78}$$

By multiplying $(2.78)_1$ by $(i\lambda+\xi^2+\eta)^{-1}\mu(\xi)$, we get

$$(i\lambda+\xi^2+\eta)^{-1}u(L)\mu^2(\xi) = \mu(\xi)\phi - (i\lambda+\xi^2+\eta)^{-1}\mu(\xi)f_3(\xi). \tag{2.79}$$

Hence, by taking absolute values of both sides of (2.79), integrating over the interval $]-\infty,+\infty[$ with respect to the variable ξ and applying Cauchy-Schwartz inequality, we obtain

$$S|u(L)| \leq U\left(\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi|^2\,d\xi\right)^{\frac{1}{2}} + V\left(\int_{-\infty}^{+\infty}|f_3(\xi)|^2\,d\xi\right)^{\frac{1}{2}} \tag{2.80}$$

where

$$S = \int_{-\infty}^{+\infty}(|\lambda|+\xi^2+\eta)^{-1}|\mu(\xi)|^2\,d\xi$$

$$U = \left(\int_{-\infty}^{+\infty}(\xi^2+\eta)^{-1}|\mu(\xi)|^2\,d\xi\right)^{\frac{1}{2}}$$

$$V = \left(\int_{-\infty}^{+\infty}(|\lambda|+\xi^2+\eta)^{-2}|\mu(\xi)|^2\,d\xi\right)^{\frac{1}{2}}.$$

Thus, by using again the inequality $2PQ \leq P^2+Q^2, P\geq 0, Q\geq 0$, we get

$$S^2|u(L)|^2 \leq 2U^2\left(\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi|^2\,d\xi\right) + 2V^2\left(\int_{-\infty}^{+\infty}|f_3(\xi)|^2\,d\xi\right). \tag{2.81}$$

We deduce that

$$|u(L)|^2 \leq c|\lambda|^{2-2\alpha}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c\|F\|^2_{\mathcal{H}}. \tag{2.82}$$

Let us introduce the following notation

$$\mathcal{I}_\varphi(\alpha) = |u(\alpha)|^2 + |\varphi_{xx}(\alpha)|^2$$

$$\mathcal{E}_\varphi(L) = \int_0^L \mathcal{I}_\varphi(s)\,ds.$$

Lemma 2.5.5 *Let $q \in H^1(0,L)$. We have that*

$$\int_0^L q_x[|u(x)|^2+3|\varphi_{xx}(x)|^2]\,dx + 2\int_0^L q_{xx}\varphi_{xx}\overline{\varphi}_x\,dx \leq [q\mathcal{I}_\varphi]_0^L - 2L\varphi_{xxx}(L)\varphi_x(L) + R \tag{2.83}$$

where R satisfies

$$|R| \leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

for a positive constant C.

Proof

To get (2.83), let us multiply the equation $(2.74)_2$ by $q\overline{\varphi}_x$ Integrating on $(0,L)$ we obtain

$$i\lambda\int_0^L uq\overline{\varphi}_x\,dx + \int_0^L \varphi_{xxxx}q\overline{\varphi}_x\,dx = \int_0^L f_2 q\overline{\varphi}_x\,dx$$

or

$$-\int_0^L uq(\overline{i\lambda\varphi_x})\,dx + \int_0^L q\varphi_{xxxx}\overline{\varphi}_x\,dx = \int_0^L f_2 q\overline{\varphi}_x\,dx.$$

Since $i\lambda\varphi_x = u_x + f_{1x}$ taking the real part in the above equality results in

$$-\frac{1}{2}\int_0^L q\frac{d}{dx}|u|^2\,dx - \frac{1}{2}\int_0^L q\frac{d}{dx}|\varphi_{xx}|^2\,dx + [\varphi_{xxx}\varphi_x q]_0^L + \int_0^L q_{xx}\varphi_{xx}\overline{\varphi}_x\,dx + \int_0^L q_x|\varphi_{xx}|^2\,dx$$
$$= Re\int_0^L f_2 q\overline{\varphi}_x\,dx + Re\int_0^L uq\overline{f}_{1x}\,dx.$$

Performing an integration by parts we get

$$\int_0^L q_x[|u(x)|^2 + 3|\varphi_{xx}(x)|^2]\,dx + 2\int_0^L q_{xx}\varphi_{xx}\overline{\varphi}_x\,dx = [q\mathcal{I}_\varphi]_0^L - 2q(L)\varphi_{xxx}(L)\varphi_x(L) + R$$

where

$$R = 2Re\int_0^L f_2 q\overline{\varphi}_x\,dx + 2Re\int_0^L uq\overline{f}_{1x}\,dx.$$

It is clear that

$$|R| \le C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{2.84}$$

If we take $q(x) = x$ in Lemma 2.5.5 we arrive at

$$\mathcal{E}_\varphi(L) \le L\mathcal{I}_\varphi(L) - 2L\varphi_{xxx}(L)\varphi_x(L) + R. \tag{2.85}$$

Using the continuous embeddings from $H^2(0,L)$ into $C^1([0,L])$ we deduce

$$|\varphi_x(L)| \le C\|\varphi\|_{H^2(0,L)} \le C'\|\varphi_{xx}\|_{L^2(0,L)} \le C'\|U\|_{\mathcal{H}}.$$

Using inequalities (2.85) and (2.84) we conclude that there exists a positive constant C such that

$$\int_0^L \mathcal{I}_\varphi(s)\,ds \le L\mathcal{I}_\varphi(L) + C(|\lambda|^{4-2\alpha}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + |\lambda|^2\|F\|_{\mathcal{H}}^2)^{\frac{1}{2}}\|U\|_{\mathcal{H}} + C'\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{2.86}$$

Since that

$$\int_{-\infty}^{+\infty}(\phi(\xi))^2\,d\xi \le C\int_{-\infty}^{+\infty}(\xi^2+\eta)(\phi(\xi))^2\,d\xi \le C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

Substitution of inequalities (2.82) into (2.86) we get that

$$\|U\|_{\mathcal{H}}^2 \le C(|\lambda|^{2-2\alpha}+1)\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C'(|\lambda|^{4-2\alpha}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + |\lambda|^2\|F\|_{\mathcal{H}}^2)^{\frac{1}{2}}\|U\|_{\mathcal{H}} + C''\|F\|_{\mathcal{H}}^2.$$

So we have

$$\|U\|_{\mathcal{H}} \le C|\lambda|^{4-2\alpha}\|F\|_{\mathcal{H}}.$$

The conclusion then follows by applying the Theorem 2.4.2.

Chapter 3

OPTIMAL DECAY RATES FOR THE ACOUSTIC WAVE MOTIONS WITH BOUNDARY MEMORY DAMPING

3.1 Introduction

In this chapter we investigate the existence and decay properties of solutions for the initial boundary value problem of the wave equation of the type

$$(P) \qquad \begin{cases} y_{tt}(x,t) - y_{xx}(x,t) = 0 & \text{in } (0,L) \times (0,+\infty), \\ y(0,t) = 0 & \text{in } (0,+\infty), \\ y_x(L,t) = z_t(t) & \text{in } (0,+\infty), \\ y_t(L,t) + mz(t) + \gamma \partial_t^{\alpha,\eta} z(t) = 0 & \text{in } (0,+\infty), \\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x) & \text{in } (0,L), \end{cases}$$

where $(x,t) \in (0,L) \times (0,+\infty), m > 0, \gamma > 0, \eta \geq 0$ and the initial data are taken in suitable spaces. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α, $0 < \alpha < 1$, with respect to the time variable (see Choi and MacCamy [**22**]). It is defined as follows

$$\partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s)\, ds, \quad \eta \geq 0.$$

The problem (P) describes sound wave propagation in a domain which is full of some kind of medium and with a portion of boundary made of light-weight viscoelastic material.

Acoustic model was proposed by Morse and Ingard [**51**], and improved in a rigorous mathematical way by Beale and Rosencrans [**11**]. Under the assumption that each local-reacting boundary point acts as a spring, the author analyzed the model in both bounded and exterior domains in [**12**], [**13**]. Uniform energy decay rates were studied in [**15**], [**57**] for acoustic wave systems with both internal and boundary memory damping terms. To our knowledge, there has been few work about the decay rates of acoustic wave energies when only one memory damping acting on the acoustic boundary.

Recently, In **[33]**, the authors considered the following initial boundary value problem with memory type acoustic boundary conditions,

$$(P)\qquad \begin{cases} y_{tt}(x,t) - \Delta y(x,t) = 0 & \text{in } \Omega \times (0,+\infty), \\ y(x,t) = 0 & \text{in } \Gamma_0 \times (0,+\infty), \\ \frac{\partial y}{\partial \nu}(x,t) = z_t(x,t) & \text{in } \Gamma_1 \times (0,+\infty), \\ y_t(x,t) + mz(x,t) + \gamma \partial_t^{\alpha,0} z(x,t) = 0 & \text{in } \Gamma_1 \times (0,+\infty), \\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x) & \text{in } (0,L), \end{cases}$$

They proved well-posedness and strong stability of the system (P) without giving an energy decay rate. Very Recently, in **[32]** the authors proved that the energy is polynomially stable but without obtaining the precise exponent.

The aim of the present chapter is to obtain more precise rates of decay. This can be achieved via some theorems about operator semigroups. We provide a standard method of going from resolvent estimates for a suitable PDE to rates of decay of classical (strong) solutions.

We should mention here that the approach in **[33]** and **[32]**, which is based on Laplace transform is different from ours. By redescribing the fractional derivative term by means of a suitable diffusion equation as in **[48]**, the original model is transformed into an augmented system which can be more easily tackled by the energy method.

3.2 Augmented model

This section is concerned with the reformulation of the model (P) into an augmented system. For that, we need the following claims.

Theorem 3.2.1 (see [48]) *Let μ be the function:*

$$\mu(\xi) = |\xi|^{(2\alpha-1)/2}, \quad -\infty < \xi < +\infty,\ 0 < \alpha < 1. \tag{3.1}$$

Then the relationship between the 'input' U and the 'output' O of the system

$$\partial_t \phi(\xi,t) + (\xi^2 + \eta)\phi(\xi,t) - U(t)\mu(\xi) = 0, \quad -\infty < \xi < +\infty, \eta \geq 0, t > 0, \tag{3.2}$$

$$\phi(\xi,0) = 0, \tag{3.3}$$

$$O(t) = (\pi)^{-1} \sin(\alpha\pi) \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi,t)\, d\xi \tag{3.4}$$

is given by

$$O = I^{1-\alpha,\eta} U = D^{\alpha,\eta} U \tag{3.5}$$

where

$$[I^{\alpha,\eta} f](t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{-\eta(t-\tau)} f(\tau)\, d\tau.$$

Lemma 3.2.1 (see [1]) *If $\lambda \in D_\eta = \mathbb{C}\backslash]-\infty, -\eta]$ then*

$$\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}\, d\xi = \frac{\pi}{\sin \alpha\pi}(\lambda+\eta)^{\alpha-1}.$$

We are now in a position to reformulate system (P). Indeed, by using Theorem 3.2.1, system (P) may be recast into the augmented model:

$$(P') \qquad \begin{cases} y_{tt}(x,t) - y_{xx}(x,t) = 0 & \text{in } (0,L)\times(0,+\infty) \\ \partial_t\phi(\xi,t) + (\xi^2+\eta)\phi(\xi,t) - z_t(t)\mu(\xi) = 0 & \text{in } (-\infty,+\infty)\times(0,+\infty) \\ y(0,t) = 0 & \text{in } (0,+\infty) \\ y_x(L,t) = z_t(t) & \text{in } (0,+\infty) \\ y_t(L,t) + mz(t) + \zeta \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi,t)\, d\xi = 0 & \text{in } (0,+\infty) \\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x) & \text{in } (0,L) \end{cases}$$

We define the energy associated to the solution of the problem (P') by the following formula:

$$E(t) = \frac{1}{2}\|y_t\|_2^2 + \frac{1}{2}\|y_x\|_2^2 + \frac{m}{2}|z(t)|^2 + \frac{\zeta}{2}\int_{-\infty}^{+\infty} |\phi(\xi,t)|^2\, d\xi. \tag{3.6}$$

Lemma 3.2.2 *Let (y,ϕ) be a solution of the problem (P'). Then, the energy functional defined by (3.6) satisfies*

$$E'(t) = -\zeta\int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi,t)|^2\, d\xi \le 0. \tag{3.7}$$

Proof
Multiplying the first equation in (P') by $\overline{y}_t$, integrating over $(0,L)$ and using integration by parts, we get

$$\frac{1}{2}\frac{d}{dt}\|y_t\|_2^2 - \Re\int_0^L y_{xx}\overline{y}_t dx = 0.$$

Then

$$\frac{d}{dt}\left(\frac{1}{2}\|y_t\|_2^2 + \frac{1}{2}\|y_x\|_2^2\right) + \Re z_t(t)\left(m\overline{z}(t) + \zeta\int_{-\infty}^{+\infty} \mu(\xi)\overline{\phi}(\xi,t)\, d\xi\right) = 0. \tag{3.8}$$

Multiplying the second equation in (P') by $\zeta\overline{\phi}_t$ and integrating over $(-\infty,+\infty)$, to obtain:

$$\frac{\zeta}{2}\frac{d}{dt}\|\phi\|_2^2 + \zeta\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,t)|^2\, d\xi - \zeta\Re z_t(t)\int_{-\infty}^{+\infty}\mu(\xi)\overline{\phi}(\xi,t)\, d\xi = 0. \tag{3.9}$$

From (3.6), (3.8) and (3.9) we get

$$E'(t) = -\zeta\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,t)|^2\, d\xi.$$

This completes the proof of the lemma.

3.3 Well-posedness

The energy space associated to system (P) is

$$\mathcal{H} = H_L^1(0,L) \times L^2(0,L) \times L^2(-\infty,+\infty) \times \mathbb{C}, \quad H_L^1(0,L) = \{y \in H^1(0,L),\ y(0) = 0\}$$

equipped with the inner product

$$< U, \tilde{U} >_{\mathcal{H}} = \int_\Omega \left(v\bar{\tilde{v}} + y_x \bar{\tilde{y}}_x \right) dx + mz\bar{\tilde{z}} + \zeta \int_{-\infty}^{+\infty} \phi \bar{\tilde{\phi}} d\xi,$$

where $U = (y, v, \phi, z)^T, \tilde{U} = (\tilde{y}, \tilde{v}, \tilde{\phi}, \tilde{z})^T \in \mathcal{H}$.

Let $U = (y, y_t, \phi, z)^T$ and rewrite (P') as

$$\begin{cases} U' = \mathcal{A}U, \\ U(0) = (y_0, y_1, \phi_0, z_0), \end{cases} \tag{3.10}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A} \begin{pmatrix} y \\ v \\ \phi \\ z \end{pmatrix} = \begin{pmatrix} v \\ y_{xx} \\ -(\xi^2 + \eta)\phi + y_x(L)\mu(\xi) \\ y_x(L) \end{pmatrix} \tag{3.11}$$

with domain

$$D(\mathcal{A}) = \left\{ \begin{array}{l} (y, v, \phi, z)^T \text{ in } \mathcal{H} : y \in H^2(0,L) \cap H_L^1(0,L), v \in H_L^1(0,L), z \in \mathbb{C}, \\ -(\xi^2 + \eta)\phi + y_x(L)\mu(\xi) \in L^2(-\infty,+\infty), \\ v(L) + mz + \zeta \displaystyle\int_{-\infty}^{\infty} \mu(\xi)\phi(\xi)\, d\xi = 0, \\ |\xi|\phi \in L^2(-\infty,+\infty) \end{array} \right\}. \tag{3.12}$$

Now, we will give well-posedness results for problem (P) using semigroup theory. We show that the operator $\mathcal{A}$ generates a C_0- semigroup in $\mathcal{H}$. We prove that $\mathcal{A}$ is a maximal dissipative operator. For this purpose we need the following two lemmas.

Lemma 3.3.1 *The operator $\mathcal{A}$ is dissipative and satisfies, for any $U \in D(\mathcal{A})$,*

$$\Re\langle \mathcal{A}U, U \rangle_{\mathcal{H}} = -\zeta \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi(\xi)|^2\, d\xi. \tag{3.13}$$

Proof For any $U = (y, u, \phi, v)^T \in D(A)$, Using (3.10) and the fact that

$$\|(y, y_t, \phi, v)\|_{\mathcal{H}}^2 = \|U\|_{\mathcal{H}}^2, \tag{3.14}$$

estimate (3.13) easily follows.

Lemma 3.3.2 *The operator $\lambda I - \mathcal{A}$ is surjective for all $\lambda > 0$.*

Proof We need to show that for all $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$, there exists $U = (y, u, \phi, v)^T \in D(\mathcal{A})$ such that

$$\lambda U - \mathcal{A}U = F, \tag{3.15}$$

that is

$$\begin{cases} \lambda y - v = f_1, \\ \lambda v - y_{xx} = f_2, \\ \lambda \phi + (\xi^2 + \eta)\phi - y_x(L)\mu(\xi) = f_3, \\ \lambda z - y_x(L) = f_4. \end{cases} \tag{3.16}$$

Suppose that we have found y. Therefore, the first equation in (3.16) gives

$$v = \lambda y - f_1. \tag{3.17}$$

It is clear that $u \in H_L^1(0, L)$. Furthermore, by (3.16) we can find ϕ as

$$\phi = \frac{f_3(\xi) + \mu(\xi) y_x(L)}{\xi^2 + \eta + \lambda}. \tag{3.18}$$

By using (3.16) and (3.17) the function y satisfying the following system

$$\lambda^2 y - y_{xx} = f_2 + \lambda f_1. \tag{3.19}$$

Solving system (3.19) is equivalent to finding $y \in H^2 \cap H_L^1(0, L)$ such that

$$\int_0^L (\lambda^2 y \overline{w} - y_{xx} \overline{w})\, dx = \int_0^L (f_2 + \lambda f_1) \overline{w}\, dx, \tag{3.20}$$

for all $w \in H_L^1(0, L)$. By using (3.20) and (3.18) the function y satisfying the following system

$$\begin{cases} \displaystyle\int_0^L (\lambda^2 y \overline{w} + y_x \overline{w}_x)\, dx + \frac{\lambda^2}{m + \gamma\lambda(\lambda+\eta)^{\alpha-1}} y(L)\overline{w}(L) \\ \displaystyle = \int_0^L (f_2 + \lambda f_1)\overline{w}\, dx + \frac{1}{m + \gamma\lambda(\lambda+\eta)^{\alpha-1}} (\lambda f_1(L) - \zeta\lambda \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + \lambda} f_3(\xi)\, d\xi - m f_4)\overline{w}(L) \end{cases} \tag{3.21}$$

Consequently, problem (3.21) is equivalent to the problem

$$a(y, w) = L(w) \tag{3.22}$$

where the sesquilinear form $a : [H_L^1(0, L) \times H_L^1(0, L)]^2 \to \mathrm{I\!R}$ and the antilinear form $L : H_L^1(0, L) \times H_L^1(0, L) \to \mathrm{I\!R}$ are defined by

$$a(y, w) = \int_0^L (\lambda^2 y \overline{w} + y_x \overline{w}_x)\, dx + \frac{\lambda^2}{m + \gamma\lambda(\lambda+\eta)^{\alpha-1}} y(L)\overline{w}(L)$$

and

$$L(w) = \int_0^L (f_2 + \lambda f_1)\overline{w}\, dx + \frac{1}{m + \gamma\lambda(\lambda+\eta)^{\alpha-1}} (\lambda f_1(L) - \zeta\lambda \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + \lambda} f_3(\xi)\, d\xi - m f_4)\overline{w}(L).$$

It is easy to verify that a is continuous and coercive, and L is continuous. So applying the Lax-Milgram theorem, we deduce that for all $w \in H_L^1(0,L)$ problem (3.22) admits a unique solution $y \in H_L^1(0,L)$. Applying the classical elliptic regularity, it follows from (3.21) that $y \in H^2(0,L)$. Therefore, the operator $\lambda I - \mathcal{A}$ is surjective for any $\lambda > 0$. Consequently, using Hille-Yosida Theorem, we have the following well-posedness result:

Theorem 3.3.1 (Existence and uniqueness)

(1) If $U_0 \in D(\mathcal{A})$, then system (3.10) has a unique strong solution

$$U \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

(1) If $U_0 \in \mathcal{H}$, then system (3.10) has a unique weak solution

$$U \in C^0(\mathbb{R}_+, \mathcal{H}).$$

3.4 Lack of exponential stability

In order to state and prove our stability results, we need the following well known theorems.

Theorem 3.4.1 ([60]-[31]) *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on Hilbert space. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{i\beta : \beta \in \mathbb{R}\} \equiv i\mathbb{R}$$

and

$$\overline{\lim_{|\beta|\to\infty}} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

Theorem 3.4.2 ([14]) *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup on a Hilbert space. If*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \text{ and } \sup_{|\beta|\geq 1} \frac{1}{\beta^\delta} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < M$$

for some $\delta > 0$, then there exist c such that

$$\|e^{\mathcal{A}t}U_0\|^2 \leq \frac{c}{t^{\frac{2}{\delta}}} \|U_0\|^2_{D(\mathcal{A})}.$$

Theorem 3.4.3 ([7]-[44]) *Let $\mathcal{A}$ be the generator of a uniformly bounded C_0. semigroup $\{S(t)\}_{t\geq 0}$ on a Hilbert space $\mathcal{H}$. If:*

(i) $\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.

(ii) The intersection of the spectrum $\sigma(\mathcal{A})$ with $i\mathbb{R}$ is at most a countable set,

then the semigroup $\{S(t)\}_{t\geq 0}$ is asymptotically stable, i.e, $\|S(t)z\|_H \to 0$ as $t \to \infty$ for any $z \in \mathcal{H}$.

Our main first result is

Theorem 3.4.4 *The semigroup generated by the operator $\mathcal{A}$ is not exponentially stable.*

Proof: We will examine two cases.
Case 1 $\eta = 0$: We shall show that $i\lambda = 0$ is not in the resolvent set of the operator $\mathcal{A}$. Indeed, noting that $(0,0,0,\cos L)^T \in \mathcal{H}$, and denoting by $(y,v,\phi,z)^T$ the image of $(0,0,0,\cos L)^T$ by $\mathcal{A}^{-1}$, we see that $\phi(\xi) = |\xi|^{\frac{2\alpha-5}{2}}\cos L$. But, then $\phi \notin L^2(-\infty,+\infty)$, since $\alpha \in]0,1[$. Hence $(y,v,\phi,z)^T \notin D(\mathcal{A})$.
• **Case 2** $\eta \neq 0$: We aim to show that an infinite number of eigenvalues of $\mathcal{A}$ approach the imaginary axis which prevents the wave system (P) from being exponentially stable. Indeed We first compute the characteristic equation that gives the eigenvalues of $\mathcal{A}$. Let λ be an eigenvalue of $\mathcal{A}$ with associated eigenvector $U = (y,v,\phi,z)^T$. Then $\mathcal{A}U = \lambda U$ is equivalent to

$$\begin{cases} \lambda y - v = 0, \\ \lambda v - y_{xx} = 0, \\ \lambda\phi + (\xi^2+\eta)\phi - y_x(L)\mu(\xi) = 0, \\ \lambda z - y_x(L) = 0, \\ v(L) + mz + \zeta \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\,d\xi = 0, \end{cases} \tag{3.23}$$

From $(3.23)_1 - (3.23)_2$ for such λ, we find

$$\lambda^2 y - y_{xx} = 0. \tag{3.24}$$

Since $v = \lambda y(L)$, using $(3.23)_3$ and $(3.23)_4$, we get

$$\begin{cases} y(0) = 0, \\ \lambda^2 y(L) + (m + \gamma\lambda(\lambda+\eta)^{\alpha-1})y_x(L) = 0. \end{cases} \tag{3.25}$$

The solution y is given by

$$y(x) = \sum_{i=1}^{2} c_i e^{t_i x}, \tag{3.26}$$

where

$$t_1(\lambda) = \lambda, \quad t_2(\lambda) = -\lambda.$$

Thus the boundary conditions may be written as the following system:

$$M(\lambda)C(\lambda) = \begin{pmatrix} 1 & 1 \\ h(t_1)e^{t_1 L} & h(t_2)e^{t_2 L} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.27}$$

where we have set

$$h(r) = (m + \gamma\lambda(\lambda+\eta)^{\alpha-1})r + \lambda^2.$$

Hence a non-trivial solution y exists if and only if the determinant of $M(\lambda)$ vanishes. Set $f(\lambda) = det M(\lambda)$, thus the characteristic equation is $f(\lambda) = 0$.

Our purpose in the sequel is to prove, thanks to Rouché's Theorem, that there is a subsequence of eigenvalues for which their real part tends to 0.

In the sequel, since $\mathcal{A}$ is dissipative, we study the asymptotic behavior of the large eigenvalues λ of $\mathcal{A}$ in the strip $-\alpha_0 \leq \mathcal{R}(\lambda) \leq 0$, for some $\alpha_0 > 0$ large enough and for such λ, we remark that $e^{t_i}, i = 1, 2$ remains bounded.

Lemma 3.4.1 *There exists $N \in \mathbb{N}$ such that*

$$\{\lambda_k\}_{k \in \mathbf{Z}^*, |k| \geq N} \subset \sigma(\mathcal{A}) \tag{3.28}$$

where

$$\lambda_k = i\frac{k\pi}{L} + \frac{\tilde{\alpha}}{k^{1-\alpha}} + \frac{\beta}{k^{1-\alpha}} + o\left(\frac{1}{k^{1-\alpha}}\right), |k| \geq N, \tilde{\alpha} \in i\mathbb{R}, \beta \in \mathbb{R}, \beta < 0.$$

Moreover for all $|k| \geq N$, the eigenvalues λ_k are simple.

Proof

$$\begin{aligned} f(\lambda) &= e^{t_2}h(t_2) - e^{t_1}h(t_1) \\ &= -e^{-\lambda L}((m + \gamma\lambda(\lambda+\eta)^{\alpha-1}) + \lambda)\left(e^{2\lambda L} - \frac{\lambda - (m + \gamma\lambda(\lambda+\eta)^{\alpha-1})}{\lambda + (m + \gamma\lambda(\lambda+\eta)^{\alpha-1})}\right) \\ &= -e^{-\lambda L}((m + \gamma\lambda(\lambda+\eta)^{\alpha-1}) + \lambda)\left(e^{2\lambda L} - 1 + 2\frac{m + \gamma\lambda(\lambda+\eta)^{\alpha-1}}{m + \lambda + \gamma\lambda(\lambda+\eta)^{\alpha-1}}\right). \end{aligned} \tag{3.29}$$

We set

$$\begin{aligned} \tilde{f}(\lambda) &= e^{2\lambda L} - 1 + 2\frac{m + \gamma\lambda(\lambda+\eta)^{\alpha-1}}{m + \lambda + \gamma\lambda(\lambda+\eta)^{\alpha-1}} \\ &= f_0(\lambda) + \tfrac{f_1(\lambda)}{\lambda^{1-\alpha}} + o\left(\tfrac{1}{\lambda^{1-\alpha}}\right) \end{aligned} \tag{3.30}$$

where

$$f_0(\lambda) = e^{2\lambda L} - 1, \tag{3.31}$$

$$f_1(\lambda) = 2\gamma. \tag{3.32}$$

Note that f_0 and f_1 remain bounded in the strip $-\alpha_0 \leq \mathcal{R}(\lambda) \leq 0$.
Step 2. We look at the roots of f_0. From (3.31), f_0 has one familie of roots that we denote λ_k^0.

$$f_0(\lambda) = 0 \Leftrightarrow e^{2\lambda L} = 1.$$

Hence

$$2\lambda L = i2k\pi, \quad k \in \mathbf{Z},$$

i.e.,

$$\lambda_k^0 = \frac{ik\pi}{L}, \quad k \in \mathbf{Z}.$$

Now with the help of Rouché's Theorem, we will show that the roots of $\tilde{f}$ are close to those of f_0. Changing in (3.30) the unknown λ by $u = 2\lambda L$ then (3.30) becomes

$$\tilde{f}(u) = (e^u - 1) + O\left(\frac{1}{u}\right) = f_0(u) + O\left(\frac{1}{u}\right).$$

The roots of f_0 are $u_k = \frac{ik}{L}\pi, k \in \mathbf{Z}$, and setting $u = u_k + re^{it}, t \in [0, 2\pi]$, we can easily check that there exists a constant $C > 0$ independent of k such that $|e^u - 1| \geq Cr$ for r small enough. This allows to apply Rouché's Theorem. Consequently, there exists a subsequence of roots of $\tilde{f}$ which tends to the roots u_k of f_0. Equivalently, it means that there exists $N \in \mathbb{N}$ and a subsequence $\{\lambda_k\}_{|k|\geq N}$ of roots of $f(\lambda)$, such that $\lambda_k = \lambda_k^0 + o(1)$ which tends to the roots $\frac{ik}{L}\pi$ of f_0. Finally for $|k| \geq N, \lambda_k$ is simple since λ_k^0 is.

Step 3. From Step 2, we can write

$$\lambda_k = i\frac{1}{L}k\pi + \varepsilon_k. \tag{3.33}$$

Using (3.33), we get

$$e^{2\lambda_k L} = 1 + 2L\varepsilon_k + 2L^2\varepsilon_k^2 + o(\varepsilon_k^2). \tag{3.34}$$

Substituting (3.34) into (3.30), using that $\tilde{f}(\lambda_k) = 0$, we get:

$$\begin{aligned} \tilde{f}(\lambda_k) &= 2L\varepsilon_k + \frac{2\gamma}{(\frac{k\pi i}{L} + \varepsilon_k)^{(1-\alpha)}} + o(\varepsilon_k) + o(1/k) \\ &= 2L\varepsilon_k + \frac{2\gamma}{(\frac{k\pi}{L}i)^{(1-\alpha)}} + o\left(\frac{1}{k}\right) = 0 \end{aligned} \tag{3.35}$$

and hence

$$\varepsilon_k = -\frac{\gamma}{L^\alpha}\frac{1}{(k\pi)^{(1-\alpha)}}\left(\cos(1-\alpha)\frac{\pi}{2} - i\sin(1-\alpha)\frac{\pi}{2}\right) + o\left(\frac{1}{k^{1-\alpha}}\right) \text{ for } k \succeq 0.$$

From (3.35) we have in that case $|k|^{1-\alpha}\mathcal{R}\lambda_k \sim \beta$, with

$$\beta = -\frac{\gamma}{L^\alpha \pi^{1-\alpha}}\cos(1-\alpha)\frac{\pi}{2}.$$

The operator $\mathcal{A}$ has a non exponential decaying branch of eigenvalues. Thus the proof is complete.

3.5 Polynomial Stability and Optimality(for $\eta \neq 0$)

In the previous section,we have shown that the transmission wave system is not exponentially stable. In this section, we prove that it is polynomially stable with an optimal rate of decay when $\eta > 0$. To achieve this, we use a recent result by Borichev and Tomilov **[14]**. Accordingly, if we consider a bounded C_0-semigroup $S(t) = e^{\mathcal{A}t}$ on a Hilbert space. If

$$i\mathbb{R} \subset \rho(\mathcal{A}) \text{ and } \overline{\lim}_{|\beta|\to\infty}\frac{1}{\beta^\delta}\|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$$

for some $\delta > 0$, then there exist c such that

$$\|e^{\mathcal{A}t}U_0\|^2 \leq \frac{c}{t^{\frac{2}{\delta}}}\|U_0\|^2_{D(\mathcal{A})}.$$

Our main result is as follows.

Theorem 3.5.1 *The semigroup $S_{\mathcal{A}}(t)_{t\geq 0}$ is polynomially stable and*

$$E(t) = \|S_{\mathcal{A}}(t)U_0\|^2_{\mathcal{H}} \leq \frac{1}{t^{2/(1-\alpha)}}\|U_0\|^2_{D(\mathcal{A})}.$$

Moreover, the rate of energy decay $t^{-2/(1-\alpha)}$ is optimal for any initial data in $D(\mathcal{A})$.

Proof
We will need to study the resolvent equation $(i\lambda - \mathcal{A})U = F$, for $\lambda \in \mathbb{R}$, namely

$$\begin{cases} i\lambda y - v = f_1, \\ i\lambda v - y_{xx} = f_2, \\ i\lambda\phi + (\xi^2+\eta)\phi - y_x(L)\mu(\xi) = f_3, \\ i\lambda z - y_x(L) = f_4 \end{cases} \tag{3.36}$$

with the boundary condition

$$v(L) + mz + \zeta\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)\,d\xi = 0. \tag{3.37}$$

We divide the proof into three steps, as follows:
Step 1. Inserting $(3.36)_1$ into $(3.36)_2$, we get

$$\lambda^2 y + y_{xx} = -(f_2 + i\lambda f_1).$$

As $y(0) = 0$, then

$$y(x) = c_1\sin\lambda x - \frac{1}{\lambda}\int_0^x (f_2(\sigma) + i\lambda f_1(\sigma))\sin\lambda(x-\sigma)\,d\sigma, \tag{3.38}$$

and hence

$$y_x(x) = c_1\lambda\cos\lambda x - \int_0^x (f_2(\sigma) + i\lambda f_1(\sigma))\cos\lambda(x-\sigma)\,d\sigma, \tag{3.39}$$

Step 2. With the third equation of (3.36), we get

$$\phi(\xi) = \frac{y_x(L)\mu(\xi) + f_3(\xi)}{i\lambda + \xi^2 + \eta}. \tag{3.40}$$

Inserting (3.40) in the boundary condition (3.37), we easy to check that

$$-\lambda^2 y(L) + (m + \gamma i\lambda(i\lambda+\eta)^{\alpha-1})y_x(L) = i\lambda f_1(L) - mf_4 - \zeta i\lambda\int_{-\infty}^{+\infty}\frac{\mu(\xi)f_3(\xi)}{i\lambda+\xi^2+\eta}\,d\xi. \tag{3.41}$$

Using (3.38) and (3.39), we can rewrite (3.41) as an equation in the unknown c_1

$$
\begin{aligned}
&c_1(-\lambda^2 \sin \lambda L + \lambda(m + \gamma i\lambda(i\lambda + \eta)^{\alpha-1}) \cos \lambda L) \\
&= i\lambda f_1(L) - m f_4 - \zeta i\lambda \int_{-\infty}^{+\infty} \frac{\mu(\xi) f_3(\xi)}{i\lambda + \xi^2 + \eta}\, d\xi - \lambda \int_0^L (f_2(\sigma) + i\lambda f_1(\sigma)) \sin \lambda(L-\sigma)\, d\sigma \\
&+(m + \gamma i\lambda(i\lambda + \eta)^{\alpha-1}) \int_0^L (f_2(\sigma) + i\lambda f_1(\sigma)) \cos \lambda(L-\sigma)\, d\sigma.
\end{aligned}
\tag{3.42}
$$

Step 3. We set

$$
g(\lambda) = -\lambda \sin \lambda L + (m + \gamma i\lambda(i\lambda + \eta)^{\alpha-1}) \cos \lambda L. \tag{3.43}
$$

As $f_1 \in H^1_*(0, L)$ and $f_2 \in L^1(0, L)$, we have

$$
\left| \int_0^L (f_2(\sigma) + i\lambda f_1(\sigma)) \sin \lambda(L-\sigma)\, d\sigma \right| \leq c(\|f_2\|_{L^2(0,L)} + \|f_1\|_{H^1(0,L)}).
$$

$$
\left| \int_0^L (f_2(\sigma) + i\lambda f_1(\sigma)) \cos \lambda(L-\sigma)\, d\sigma \right| \leq c(\|f_2\|_{L^2(0,L)} + \|f_1\|_{H^1(0,L)}).
$$

As $g(\lambda) \neq 0$ for all λ (if $\eta = 0$ then for all $\lambda \neq 0$) , then c_1 is uniqueley determined by (3.42). Hence the operator $i\lambda - \mathcal{A}$ is surjective for all λ (if $\eta = 0$ then for all $\lambda \neq 0$). Moreover, taking account of Lemma 3.4.1, the operator $i\lambda - \mathcal{A}$ is injective for all λ. Then $i\mathbb{R} \subset \rho(\mathcal{A})$ (if $\eta = 0$ then $i\mathbb{R}^* \subset \rho(\mathcal{A})$).

Moreover, we can easily prove that

$$
|g(\lambda)| \geq c|\lambda|^\alpha \text{ for } \lambda \text{ large.}
$$

Hence

$$
|c_1| \leq c|\lambda|^{-\alpha} \text{ for } \lambda \text{ large.}
$$

Then, we deduce that

$$
\begin{aligned}
\|y_x\|_{L^2(0,L)} &\leq c|\lambda|^{1-\alpha} \text{ for } \lambda \text{ large.} \\
\|v\|_{L^2(0,L)} &\leq c|\lambda|^{1-\alpha} \text{ for } \lambda \text{ large.} \\
|z| &\leq c|\lambda|^{-\alpha} \text{ for } \lambda \text{ large.}
\end{aligned}
$$

Moreover from (2.14), we have

$$
\|\phi\|^2_{L^2(-\infty,\infty)} \leq \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi(\xi)|^2\, d\xi \leq c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.
$$

Thus, we conclude that

$$
\|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{H}} \leq c|\lambda|^{1-\alpha} \quad \text{as } |\lambda| \to \infty. \tag{3.44}
$$

The conclusion then follows by applying the Theorem 2.4.2. Besides, we prove that the decay rate is optimal. Indeed, the decay rate is consistent with the asymptotic expansion of eigenvalues which show a behavior of the real part like $k^{-(1-\alpha)}$.

Remark 3.5.1 *The method developed in this chapter is direct and very flexible; it can be applied to various dissipative problems. In particual, we will consider in the futur more general acoustic wave motions and also multidimensional cases under some geometric control conditions.*

Chapter 4

EXPONENTIAL DECAY OF THE TIMOSHENKO BEAM SYSTEM WITH FRACTIONAL TIME DELAYS

4.1 Introduction

In this chapter we investigate the well-posedness and the internal stabilization of the Timoshenko beam system in bounded interval $(0, L)$ in the presence of time fractional delay. The system is given by the two coupled hyperbolic equations

$$(P) \quad \begin{cases} \rho_1 \varphi_{tt}(x,t) - K(\varphi_x + \psi)_x(x,t) + a_1 \partial_t^{\sigma_1,\kappa_1} \varphi(x, t-\tau_1) + a_2 \varphi_t(x,t) = 0, \\ \rho_2 \psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x + \psi)(x,t) + \tilde{a}_1 \partial_t^{\sigma_2,\kappa_2} \psi(x, t-\tau_2) + \tilde{a}_2 \psi_t(x,t) = 0, \end{cases}$$

where t denotes the time variable and x is the space variable in $]0, L[$. The unknowns $\varphi = \varphi(x,t)$ and $\psi = \psi(x,t)$ represent, respectively, the transverse displacement of the beam and the rotation angle of the filament of the beam. In (P), $\rho_1 = \rho$,$\rho_2 = I_p$, $b = EI$, where ρ, I_p, E, I and K are respectively, the density (the mass per unit length), the polar moment of inertia of a cross-section, Young's modulus of elasticity, the moment of inertia of a cross-section and the Shear modulus. The notation $\partial_t^{\alpha,\eta}$ ($\alpha \in (0,1)$ and $\eta > 0$) stands for the generalized Caputo's fractional derivative of order α with respect to the time variable. It is defined as follows

$$\partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s)\, ds, \ \eta > 0,$$

where the parameters $a_1, a_2, \tilde{a}_1$ and $\tilde{a}_2$ are positive constants. Also, we consider the following initial conditions:

$$(P_c) \quad \begin{cases} \varphi(x,0) = \varphi_0, \quad \varphi_t(x,0) = \varphi_1, \quad \psi(x,0) = \psi_0, \quad \psi_t(x,0) = \psi_1, \quad x \in (0,L), \\ \varphi_t(x, t-\tau_1) = f_0(x, t-\tau_1), \quad \psi_t(x, t-\tau_2) = g_0(x, t-\tau_2) \quad x \in (0,L), \quad t \in (0, \tau_i). \end{cases}$$

where $\tau_1, \tau_2 > 0$ are the time delays. Systeme (P) is subjected to the following boundary conditions:

$$(CO) \qquad \varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = 0$$

The initial data $(\varphi_0, \varphi_1, f_0, \psi_0, \psi_1, g_0)$ belongs to a suitable functional space. There are many works concerning the stabilization of hyperbolic systems and many authors have shown that an arbitrary small delay may destabilize a systeme which is asymptotically stable in the absence of delay see **[25]**. Namely, **[34]** proved the exponential stability of the Timoshenko with $a_i = 0, i = 1, 2$. The result of **[34]** holds under some conditions betwen b and K. **[67]** showed the exponential stability of the uniform Thimoshenko beam by using one distributed feedback. **[66]** considred the case of the uniform Timoshenko beam under two locally distributed feedbacks and proved an exponential stability result. **[68]** proved the uniform stabilization of the Timoshenko beam under one locally distributed feed-back of $b(x)\psi_t$ acting in the left-hand side of the second equation (P), where b is a positive and continuous function, which satisfies

$$b(x) \geq b_0 > 0, \quad \forall x \in [a_0, a_1] \subset [0, L]$$

and proved that the uniform stability holds if and only if the wave speeds are equal $(\frac{K}{\rho} = \frac{E\ I}{I_p})$. Other-wise, only the asymptotic stability has been proved. **[70]** proved an exponential stability of the uniform Timoshenko beam by two pointwise control. Recenly, **[26]** have treated the Timoshenko-type systems with internal frictional dampings and discrete time delays

$$(TM) \quad \begin{cases} \rho_1 \varphi_{tt}(x,t) - K(\varphi_x + \psi)_x(x,t) + a_1 \varphi_t(x, t-\tau_1) + a_2 \varphi_t(x,t) = 0, \\ \rho_2 \psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x + \psi)(x,t) + \tilde{a}_1 \psi_t(x, t-\tau_2) + \tilde{a}_2 \psi_t(x,t) = 0, \\ \varphi(x,0) = \varphi_0, \quad \varphi_t(x,0) = \varphi_1 \\ \psi(x,0) = \psi_0, \quad \psi_t(x,0) = \psi_1 \\ \varphi_t(x, t-\tau_1) = f_0(x, t-\tau_1), \quad \psi_t(x, t-\tau_2) = g_0(x, t-\tau_2). \end{cases}$$

It is showed an exponential stability regardless to the speeds of wave propagation of the system if the weights of the time delays are smaller than the ones of the corresponding dampings, respectively.

The chapter is organized as follows. Section 2 deals with the well-posedness of the problem while, in section 3, we prove the uniform decay of the delayed system $(P)-(P_c)$ by constructing an appropriate Lyapunov functional and by assuming that the weights of the delay are small enough as in **[8]**.

4.2 Preliminary

This section is concerned with the reformulation of the model (P) into an augmented system. For that, we need the following claims.

Theorem 4.2.1 (see [48]) *Let ω be the function:*

$$\omega(\xi) = |\xi|^{(2\sigma-1)/2}, \quad -\infty < \xi < +\infty, \ 0 < \sigma < 1. \tag{4.1}$$

Then the relationship between the 'input' U and the 'output' O of the system

$$\partial_t \psi(\xi, t) + (\xi^2 + \kappa)\psi(\xi, t) - U(t)\omega(\xi) = 0, \quad -\infty < \xi < +\infty, \kappa > 0, t > 0, \tag{4.2}$$

$$\psi(\xi, 0) = 0, \tag{4.3}$$

$$O(t) = (\pi)^{-1} \sin(\sigma\pi) \int_{-\infty}^{+\infty} \omega(\xi)\psi(\xi, t)\, d\xi \tag{4.4}$$

is given by

$$O = I^{1-\sigma,\kappa} U = D^{\sigma,\kappa} U, \tag{4.5}$$

where

$$[I^{\sigma,\kappa} f](t) = \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} e^{-\kappa(t-s)} f(s)\, ds.$$

Lemma 4.2.1 (see [1]) *If* $\lambda \in D_\kappa = \mathbb{C} \backslash]-\infty, -\kappa]$ *then*

$$\int_{-\infty}^{+\infty} \frac{\omega^2(\xi)}{\lambda + \kappa + \xi^2}\, d\xi = \frac{\pi}{\sin \sigma\pi} (\lambda + \kappa)^{\sigma - 1}.$$

We make the following hypotheses on the damping and the delay functions:

$$\begin{cases} a_1 \kappa_1^{\sigma_1 - 1} & < & a_2, \\ \tilde{a}_1 \kappa_2^{\sigma_2 - 1} & < & \tilde{a}_2. \end{cases} \tag{4.6}$$

We define the energy of the solution by:

$$\begin{aligned} E(t) = \frac{1}{2} \Big(& \rho_1 \|\varphi_t\|^2_{L^2(0,L)} + \rho_2 \|\psi_t\|^2_{L^2(0,L)} + +k\|(\varphi_x + \psi)\|^2_{L^2(0,L)} + b\|\psi_x\|^2_{L^2(0,L)} \\ & + \nu_1 \|z_1(x,p)\|^2_{L^2(0,L)\times(0,1)} + \nu_2 \|z_2(x,p)\|^2_{L^2(0,L)\times(0,1)} + \zeta_1 \|\phi_1\|^2_{L^2(0,L)\times(-\infty,+\infty)} \\ & + \zeta_2 \|\phi_2\|^2_{L^2(0,L)\times(-\infty,+\infty)} \Big) \end{aligned} \tag{4.7}$$

where ν_i are positive constants verifying

$$\begin{cases} \tau_1 \zeta_1 \left(\displaystyle\int_{-\infty}^{+\infty} \frac{\omega_1^2(\xi)}{\xi^2 + \kappa_1}\, d\xi \right) < \nu_1 < \tau_1 \left(2a_2 - \zeta_1 \left(\displaystyle\int_{-\infty}^{+\infty} \frac{\omega_1^2(\xi)}{\xi^2 + \kappa_1}\, d\xi \right) \right), \\ \tau_2 \zeta_2 \left(\displaystyle\int_{-\infty}^{+\infty} \frac{\omega_2^2(\xi)}{\xi^2 + \kappa_2}\, d\xi \right) < \nu_2 < \tau_2 \left(2\tilde{a}_2 - \zeta_2 \left(\displaystyle\int_{-\infty}^{+\infty} \frac{\omega_2^2(\xi)}{\xi^2 + \kappa_2}\, d\xi \right) \right) \end{cases} \tag{4.8}$$

Remark 4.2.1 *Using Lemma 4.2.1, the condition (4.8) means that*

$$\begin{cases} \tau_1 a_1 \kappa_1^{\sigma_1 - 1} < \nu_1 < \tau_1 (2a_2 - a_1 \kappa_1^{\sigma - 1}), \\ \tau_2 \tilde{a}_1 \kappa_2^{\sigma_2 - 1} < \nu_2 < \tau_2 (2\tilde{a}_2 - \tilde{a}_1 \kappa_2^{\sigma_2 - 1}). \end{cases}$$

Lemma 4.2.2 *Let* $(\varphi, \phi_1, z_1, \psi, \phi_2, z_2)$ *be a regular solution of the problem* (P'). *Then there exists a positive constant* C *such that the energy functional defined by (4.7) satisfies*

$$E'(t) \le -C \sum_{j=1}^{n} \int_\Omega \left(u_t^2 + z(x,1,t)^2 \right) dx. \tag{4.9}$$

Proof. Multiplying the first equation in (P) by φ_t, integrating over $(0,L)$ and using integration by parts, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\varphi_t\|_2^2 - k\int_0^L (\varphi_x+\psi)_x\varphi_t\,dx\\&+\zeta\int_0^L\varphi_t\int_{-\infty}^{+\infty}\omega_1(\xi)\phi_1(x,\xi,t)\,d\xi\,dx + a_2\int_0^L|\varphi_t(t)|^2\,dx = 0.\end{aligned}$$

Then

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\varphi_t\|_2^2 + k\int_0^L (\varphi_x+\psi)\varphi_{xt}\,dx\\&+\zeta\int_0^L\varphi_t\int_{-\infty}^{+\infty}\omega_1(\xi)\phi_1(x,\xi,t)\,d\xi\,dx + a_2\int_0^L|\varphi_t(t)|^2\,dx = 0.\end{aligned}\tag{4.10}$$

Multiplying the second equation in (P) by ψ_t, integrating over $(0,L)$ and using integration by parts, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\psi_t\|_2^2 - b\int_0^L \psi_{xx}\psi_t\,dx + k\int_0^L(\varphi_x+\psi)\varphi_t\,dx\\&+\tilde{\zeta}\int_0^L\psi_t\int_{-\infty}^{+\infty}\omega_2(\xi)\phi_2(x,\xi,t)\,d\xi\,dx + \tilde{a}_2\int_0^L|\psi_t(t)|^2\,dx = 0.\end{aligned}$$

Then

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\psi_t\|_2^2 + k\int_0^L \psi_x\psi_{xt}\,dx + k\int_0^L(\varphi_x+\psi)\psi_t\,dx\\&+\tilde{\zeta}\int_0^L\psi_t\int_{-\infty}^{+\infty}\omega_2(\xi)\phi_2(x,\xi,t)\,d\xi\,dx + \tilde{a}_2\int_0^L|\psi_t(t)|^2\,dx = 0.\end{aligned}\tag{4.11}$$

Multiplying the five equation in (P') by $\zeta\phi_1$ and integrating over $(0,L)\times(-\infty,+\infty)$, we obtain:

$$\begin{aligned}&\frac{\zeta}{2}\frac{d}{dt}\|\phi_1\|^2_{L^2((0,L)\times(-\infty,+\infty))} + \zeta\int_0^L\int_{-\infty}^{+\infty}(\xi^2+\kappa_1)|\phi_1(x,\xi,t)|^2\,d\xi\,dx\\&\qquad\qquad -\zeta\int_0^L z_1(x,1,t)\int_{-\infty}^{+\infty}\omega_1(\xi)\phi_1(x,\xi,t)\,d\xi\,dx = 0.\end{aligned}\tag{4.12}$$

Multiplying the six equation in (P') by $\tilde{\zeta}\phi_2$ and integrating over $(0,L)\times(-\infty,+\infty)$, we obtain:

$$\begin{aligned}&\frac{\tilde{\zeta}}{2}\frac{d}{dt}\|\phi_2\|^2_{L^2((0,L)\times(-\infty,+\infty))} + \tilde{\zeta}\int_0^L\int_{-\infty}^{+\infty}(\xi^2+\kappa_2)|\phi_2(x,\xi,t)|^2\,d\xi\,dx\\&\qquad\qquad -\tilde{\zeta}\int_0^L z_2(x,1,t)\int_{-\infty}^{+\infty}\omega_2(\xi)\phi_2(x,\xi,t)\,d\xi\,dx = 0.\end{aligned}\tag{4.13}$$

Multiplying the third equation in (P') by $\nu_1 z_1$ and integrating over $(0,L)\times(0,1)$, we get:

$$\frac{1}{2}\frac{d}{dt}\|z_1\|^2_{L^2((0,L)\times(0,1))} + \frac{\tau_1^{-1}}{2}\int_0^L\left(z_1^2(x,1,t)-\varphi_t^2(x,t)\right) = 0.\tag{4.14}$$

Multiplying the forth equation in (P') by $\nu_2 z_2$ and integrating over $(0,L)\times(0,1)$, we get:

$$\frac{1}{2}\frac{d}{dt}\|z_2\|^2_{L^2((0,L)\times(0,1))} + \frac{\tau_2^{-1}}{2}\int_0^L\left(z_2^2(x,1,t)-\psi_t^2(x,t)\right) = 0.\tag{4.15}$$

From (4.7), (4.10) and (4.14) we get

$$
\begin{aligned}
E'(t) &= -a_2\|\varphi_t\|^2_{L^2} - \tilde{a}_2\|\psi_t\|^2_{L^2} - \zeta\int_0^L\int_{-\infty}^{+\infty}(\xi^2+\kappa_1)|\phi_1(x,\xi,t)|^2\,d\xi\,dx \\
&-\tilde{\zeta}\int_0^L\int_{-\infty}^{+\infty}(\xi^2+\kappa_2)|\phi_2(x,\xi,t)|^2\,d\xi\,dx - \zeta\int_0^L\varphi_t\int_{-\infty}^{+\infty}\omega_1(\xi)\phi_1(x,\xi,t)\,d\xi\,dx \\
&-\tilde{\zeta}\int_0^L\psi_t\int_{-\infty}^{+\infty}\omega_2(\xi)\phi_2(x,\xi,t)\,d\xi\,dx + \zeta\int_0^L z_1(x,1,t)\int_{-\infty}^{+\infty}\omega_1(\xi)\phi_1(x,\xi,t)\,d\xi\,dx \\
&+\tilde{\zeta}\int_0^L z_2(x,1,t)\int_{-\infty}^{+\infty}\omega_2(\xi)\phi_1(x,\xi,t)\,d\xi\,dx + \frac{\nu_1\tau_1^{-1}}{2}\int_0^L\varphi_t^2(x,t)\,dx \\
&-\frac{\nu_1\tau_1^{-1}}{2}\int_0^L z_1^2(x,1,t)\,dx + \frac{\nu_2\tau_2^{-1}}{2}\int_0^L\psi_t^2(x,t)\,dx - \frac{\nu_2\tau_2^{-1}}{2}\int_0^L z_2^2(x,1,t)\,dx
\end{aligned}
\tag{4.16}
$$

Moreover, we have

$$
\left|\int_{-\infty}^{+\infty}\omega_j(\xi)\phi_j(x,\xi,t)\,d\xi\right| \le \left(\int_{-\infty}^{+\infty}\frac{\omega_j^2(\xi)}{\xi^2+\kappa_j}\,d\xi\right)^{\frac{1}{2}}\left(\int_{-\infty}^{+\infty}(\xi^2+\kappa_j)|\phi_j(x,\xi,t)|^2\,d\xi\right)^{\frac{1}{2}}.
$$

Then

$$
\begin{aligned}
&\left|\int_0^L z_j(x,1,t)\int_{-\infty}^{+\infty}\omega_j(\xi)\overline{\phi}_j(x,\xi,t)\,d\xi\,dx\right| \\
&\le \left(\int_{-\infty}^{+\infty}\frac{\omega_j^2(\xi)}{\xi^2+\kappa_j}\,d\xi\right)^{\frac{1}{2}}\|z_j(x,1,t)\|_{L^2(0,L)}\left(\int_0^L\int_{-\infty}^{+\infty}(\xi^2+\kappa_j)|\phi_j(x,\xi,t)|^2\,dx\,d\xi\right)^{\frac{1}{2}}
\end{aligned}
$$

and

$$
\begin{cases}
\left|\int_0^L\overline{\varphi}_t(x,t)\int_{-\infty}^{+\infty}\omega_1(\xi)\phi_1(x,\xi,t)\,d\xi\,dx\right| \\
\le \left(\int_{-\infty}^{+\infty}\frac{\omega_1^2(\xi)}{\xi^2+\kappa_1}\,d\xi\right)^{\frac{1}{2}}\|\varphi_t(x,t)\|_{L^2(0,L)}\left(\int_0^L\int_{-\infty}^{+\infty}(\xi^2+\kappa_1)|\phi_1(x,\xi,t)|^2\,dx\,d\xi\right)^{\frac{1}{2}}. \\
\left|\int_0^L\overline{\psi}_t(x,t)\int_{-\infty}^{+\infty}\omega_2(\xi)\phi_1(x,\xi,t)\,d\xi\,dx\right| \\
\le \left(\int_{-\infty}^{+\infty}\frac{\omega_2^2(\xi)}{\xi^2+\kappa_2}\,d\xi\right)^{\frac{1}{2}}\|\psi_t(x,t)\|_{L^2(0,L)}\left(\int_0^L\int_{-\infty}^{+\infty}(\xi^2+\kappa_2)|\phi_2(x,\xi,t)|^2\,dx\,d\xi\right)^{\frac{1}{2}}.
\end{cases}
$$

Applying the Cauchy-Schwarz inequality we obtain

$$
\begin{aligned}
E'(t) \le &\left(-a_2+\frac{\zeta I_1}{2}+\frac{\nu_1\tau_1^{-1}}{2}\right)\int_0^L\varphi_t^2(x,t)\,dx + \left(\frac{\zeta I_1}{2}-\frac{\nu_1\tau_1^{-1}}{2}\right)\int_0^L z_1^2(x,1,t)\,dx, \\
&\left(-\tilde{a}_2+\frac{\tilde{\zeta}I_2}{2}+\frac{\nu_2\tau_2^{-1}}{2}\right)\int_0^L\psi_t^2(x,t)\,dx + \left(\frac{\tilde{\zeta}I_2}{2}-\frac{\nu_2\tau_2^{-1}}{2}\right)\int_0^L z_2^2(x,1,t)\,dx,
\end{aligned}
$$

where $I_j = \int_{-\infty}^{+\infty}\frac{\omega_j^2(\xi)}{\xi^2+\kappa_j}\,d\xi$, which implies

$$
\begin{aligned}
E'(t) \le &-C_1\int_\Omega\left(\varphi_t^2(x,t)+z_1^2(x,1,t)\right)\,dx \\
&-C_2\int_\Omega\left(\psi_t^2(x,t)+z_2^2(x,1,t)\right)\,dx
\end{aligned}
$$

with

$$\begin{cases} C_1 &= \min\left\{\left(a_2 - \frac{\zeta I_1}{2} - \frac{\nu_1\tau_1^{-1}}{2}\right), \left(-\frac{\zeta I_1}{2} + \frac{\nu_1\tau_1^{-1}}{2}\right)\right\} \\ C_2 &= \min\left\{\left(\tilde{a}_2 - \frac{\tilde{\zeta} I_2}{2} - \frac{\nu_2\tau_2^{-1}}{2}\right), \left(-\frac{\tilde{\zeta} I_2}{2} + \frac{\nu_2\tau_2^{-1}}{2}\right)\right\} \end{cases}$$

Since ν_j is chosen satisfying assumption (4.8), the constant C_j is positive. This completes the proof of the lemma.

4.3 Well-posedness of the problem

In this section we prove the global existence and the uniqueness of the solution of system $(P)-(P_c)$. for this purpose, we adopt the technique of Nicaise and Pignotti to prove that the operator $\mathcal{A}$ generates a contraction semi-group on the Hilbert space $\mathcal{H}$.
So, let us introduce the following new variables:

$$z_1(x,p,t) = \varphi_t(x, t-\tau_1 p), x \in (0,L), p \in (0,1), t > 0. \tag{4.17}$$

and

$$z_2(x,p,t) = \psi_t(x, t-\tau_2 p), x \in (0,L), p \in (0,1), t > 0. \tag{4.18}$$

Then,it is easy to check that

$$\tau_1 z_{1t}(x,p,t) + z_{1p}(x,p,t) = 0 \text{ in } (0,L)\times(0,1)\times(0,+\infty). \tag{4.19}$$

$$\tau_2(z_{2t}(x,p,t) + (z_{2p}(x,p,t) = 0 \text{ in } (0,L)\times(0,1)\times(0,+\infty). \tag{4.20}$$

Therefore, our problem (P) is equivalent to
(P')

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) + \zeta\int_{-\infty}^{+\infty}\omega_1(\xi)\phi_1(x,\xi,t)\,d\xi + a_2\varphi_t(t) = 0 \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x+\psi)(x,t) + \tilde{\zeta}\int_{-\infty}^{+\infty}\omega_2(\xi)\phi_2(x,\xi,t)\,d\xi + \tilde{a}_2\psi_t(t) = 0 \\ \tau_1 z_{1t}(x,p,t) + z_{1p}(x,p,t) = 0 \\ \tau_2 z_{2t}(x,p,t) + z_{2p}(x,p,t) = 0 \\ \partial_t\phi_1(x,\xi,t) + (\xi^2+\kappa_1)\phi_1(\xi,t) - z_1(x,1,t)\omega_1(\xi) = 0 \\ \partial_t\phi_2(x,\xi,t) + (\xi^2+\kappa_2)\phi_2(\xi,t) - z_2(x,1,t)\omega_2(\xi) = 0 \end{cases}$$

Now, we will give well-posedness results for problem (P') using semigroup theory. Let us introduce the semigroup representation of the (P'). Let $U = (\varphi, \widetilde{\varphi}, z_1, \phi_1, \psi, \widetilde{\psi}, z_2, \phi_2)^T$; then U satisfies the problem

$$\begin{cases} U' = \mathcal{A}U, \\ U(0) = (\varphi_0, \varphi_1, f_0(.,-.\tau), (\phi_1)_0, \psi_0, \psi_1, g_0(.,-.\tau), (\phi_2)_0)^T, \end{cases} \tag{4.21}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix}\varphi\\ \tilde{\varphi}\\ z_1\\ \phi_1\\ \psi\\ \tilde{\psi}\\ z_2\\ \phi_2\end{pmatrix}=\begin{pmatrix}\tilde{\varphi}\\ \frac{k}{\rho_1}(\varphi_x+\psi)_x-\frac{\zeta}{\rho_1}\int_{-\infty}^{+\infty}\omega_1(\xi)\phi_1(x,\xi)\,d\xi-\frac{a_2}{\rho_1}\tilde{\varphi}\\ \frac{-z_{1p}}{\tau_1}\\ -(\xi^2+\kappa_1)\phi_1+z_1(x,1)\omega_1(\xi)\\ \tilde{\psi}\\ \frac{b}{\rho_2}\psi_{xx}-\frac{k}{\rho_2}(\varphi_x+\psi)-\frac{\tilde{\zeta}}{\rho_2}\int_{-\infty}^{+\infty}\omega_2(\xi)\phi_2(x,\xi)\,d\xi-\frac{\tilde{a}_2}{\rho_2}\tilde{\psi}\\ \frac{-z_{2p}}{\tau_2}\\ -(\xi^2+\kappa_2)\phi_2+z_2(x,1)\omega_2(\xi)\end{pmatrix}\tag{4.22}$$

with domain

$$D(\mathcal{A})=\left\{\begin{array}{l}(\varphi,\tilde{\varphi},z_1,\phi_1,\psi,\tilde{\psi},z_2,\phi_2)^T\ \text{ in } H:(\tilde{\varphi},\tilde{\psi})=(z_1,z_2)(.,0)\text{ on }(0,L),\\ -(\xi^2+\kappa_1)\varphi+z_1(x,1)\omega_1(\xi)\in L^2((0,L)\times(-\infty,+\infty),\\ -(\xi^2+\kappa_2)\psi+z_2(x,1)\omega_2(\xi)\in L^2((0,L)\times(-\infty,+\infty),\\ |\xi|\phi_1\in L^2((0,L)\times(-\infty,+\infty))\\ |\xi|\phi_2\in L^2((0,L)\times(-\infty,+\infty))\end{array}\right\}.\tag{4.23}$$

where

$$\begin{aligned}H=H^2(0,L)\cap H_0^1(0,L)\times H_0^1(0,L)\times(H^2(0,L)\cap H_0^1(0,L))\times H_0^1(0,L)\\ \times L^2(0,1,H^1(0,L))\times L^2(0,1,H^1(0,L)).\end{aligned}$$

and

$$H_0^1(0,L)=\{f\in H^1(0,L):f(0)=f(L)=0\}$$

Now,the energy space $\mathcal{H}$ is defined as follows:

$$\mathcal{H}:=(H_0^1(0,L)\times L^2(0,L)\times L^2((0,L)\times(0,1))\times L^2(-\infty,+\infty))^2.\tag{4.24}$$

For $U=(\varphi,\tilde{\varphi},z_1,\phi_1,\psi,\tilde{\psi},z_2,\phi_2)^T$, $\overline{U}=(\overline{\varphi},\overline{\tilde{\varphi}},\overline{z_1},\overline{\phi_1},\overline{\psi},\overline{\tilde{\psi}},\tilde{\psi},\overline{z_2},\overline{\phi_2})^T$ for ζ_1 and ζ_2 positive constans,we define the following inner product in $\mathcal{H}$ as follows:

$$\begin{aligned}\langle U,\overline{U}\rangle_{\mathcal{H}}=\int_0^L\{\rho_1\tilde{\varphi}\overline{\tilde{\varphi}}+\rho_2\tilde{\psi}\overline{\tilde{\psi}}+k(\varphi_x+\psi)(\overline{\varphi_x}+\overline{\psi})+b\ \psi_x\overline{\psi_x}\}\,dx+\nu_1\int_0^L\int_0^1 z_1(x,p)\overline{z_1}(x,p)dp\ dx\\ +\nu_2\int_0^L\int_0^1 z_2(x,p)\overline{z_2}(x,p)dp\ dx+\zeta_1\int_0^L\int_{-\infty}^{+\infty}\phi_1\overline{\phi_1}d\xi\,dx+\zeta_2\int_0^L\int_{-\infty}^{+\infty}\phi_2\overline{\phi_2}d\xi\,dx\end{aligned}$$

The existence and uniqueness result reads as follows.

Theorem 4.3.1 *For any $U_0\in\mathcal{H}$, there exists a unique solutions $U(x,t)\in C([0,+\infty),\mathcal{H})$ of problem* (4.21)*, Moreover , if $U_0\in D(\mathcal{A})$, then*

$$U\in C([0,+\infty),D(\mathcal{A}))\cap C^1([0,+\infty),\mathcal{H}).$$

Proof: in order to prove the result stated in (4.3.1) , we will use the semi-groupe approach. That is, we will show that the operator $\mathcal{A}$ generates a C_0-semigroup in $\mathcal{H}$. In this step, we concern ourselves to prove that the operator $\mathcal{A}$ is dissipative. Indeed, for $U=(\varphi,\widetilde{\varphi},z_1,\phi_1,\psi,\widetilde{\psi},z_2,\phi_2)^T\in\mathcal{D}(\mathcal{A})$ we have

$$\begin{array}{l}\langle\mathcal{A}U,U\rangle_{\mathcal{H}}=-a_2\int_0^L|\widetilde{\varphi}|^2\,dx-\tilde{a}_2\int_0^L|\widetilde{\psi}|^2\,dx\\ -\zeta\int_0^L(\int_{-\infty}^{+\infty}\omega_1(\xi)\phi_1(x,\xi)\,d\xi)\widetilde{\varphi}\,dx-\tilde{\zeta}\int_0^L(\int_{-\infty}^{+\infty}\omega_2(\xi)\phi_2(x,\xi)\,d\xi)\widetilde{\psi}\,dx\\ -\frac{\nu_1}{\tau_1}\int_0^L\int_0^1 z_1(x,p)z_{1p}(x,p)dp\ dx-\frac{\nu_2}{\tau_2}\int_0^L\int_0^1 z_2(x,p)z_{2p}(x,p)dp\ dx\\ +\zeta_1\int_0^L\int_{-\infty}^{+\infty}[-(\xi^2+\kappa_1)\phi_1(x,\xi)+z_1(x,1)\omega_1(\xi)]\phi_1(x,\xi)\ d\xi\,dx\\ +\zeta_2\int_0^L\int_{-\infty}^{+\infty}[-(\xi^2+\kappa_2)\phi_2(x,\xi)+z_2(x,1)\omega_2(\xi)]\phi_2(x,\xi)\ d\xi\,dx.\end{array}\tag{4.25}$$

Where

$$\begin{array}{l}\frac{\nu_1}{\tau_1}\int_0^L\int_0^1 z_1(x,p)z_{1p}(x,p)dp\ dx+\frac{\nu_2}{\tau_2}\int_0^L\int_0^1 z_2(x,p)z_{2p}(x,p)dp\ dx\\ \quad=\frac{\nu_1}{\tau_1}\int_0^L\int_0^1\frac{1}{2}\frac{\partial}{\partial\ p}z_1^2(x,p)dp\ dx+\frac{\nu_2}{\tau_2}\int_0^L\int_0^1\frac{1}{2}\frac{\partial}{\partial\ p}z_2^2(x,p)dp\ dx\\ \quad=\frac{\nu_1}{2\tau_1}\int_0^L\{z_1^2(x,1)-z_1^2(x,0)\}dx+\frac{\nu_2}{2\tau_2}\int_0^L\{z_2^2(x,1)-z_2^2(x,0)\}dx\end{array}\quad.\tag{4.26}$$

Applying the Cauchy-Schwarz inequality we obtain

$$\begin{array}{rcl}\langle\mathcal{A}U,U\rangle_{\mathcal{H}}&\leq&\left(-a_2+\frac{\zeta I_1}{2}+\frac{\nu_1\tau_1^{-1}}{2}\right)\int_0^L|\widetilde{\varphi}(x)|^2\,dx+\left(\frac{\zeta I_1}{2}-\frac{\nu_1\tau_1^{-1}}{2}\right)\int_0^L z_1^2(x,1)\,dx\\ &&+\left(-\tilde{a}_2+\frac{\tilde{\zeta}I_2}{2}+\frac{\nu_2\tau_2^{-1}}{2}\right)\int_0^L|\widetilde{\psi}(x)|^2\,dx+\left(\frac{\tilde{\zeta}I_2}{2}-\frac{\nu_2\tau_2^{-1}}{2}\right)\int_0^L z_2^2(x,1)\,dx\\ &\leq&0\end{array}$$

where $I_j=\int_{-\infty}^{+\infty}\frac{\omega_j^2(\xi)}{\xi^2+\kappa_j}\,d\xi$.

Consequently,(4.25) becomes

$$\begin{array}{l}\langle\mathcal{A}U,U\rangle_{\mathcal{H}}=-\gamma_1(\varphi_t(L))^2-\tilde{\gamma}_1(\psi_t(L))^2-a_1\int_0^L z_1(x,1)\varphi_t(x,t)\ dx-a_2\int_0^L z_2(x,1)\psi_t(x,t)\ dx\\ \qquad-\frac{\zeta_1}{2\tau}\int_0^L\{z_1^2(x,1)-z_1^2(x,0)\}dx-\frac{\zeta_2}{2\tau}\int_0^L\{z_2^2(x,1)-z_2^2(x,0)\}\ dx\\ \qquad-\zeta\int_{-\infty}^{+\infty}(\xi^2+\eta)\ \phi_1^2(\xi,t)\ d\xi_1-\tilde{\zeta}\int_{-\infty}^{+\infty}(\xi^2+\eta)\ \phi_2^2(\xi,t)\ d\xi_2\end{array}.\tag{4.27}$$

By using Young inequality, we obtain from (4.27), that

$$\begin{aligned}
\langle \mathcal{A}U, U\rangle_{\mathcal{H}} &\leq -\gamma_1(\varphi_t(L))^2 - \tilde{\gamma_1}(\psi_t(L))^2 - \zeta\int_{-\infty}^{+\infty}(\xi^2+\eta)\ \phi_1^2\ d\xi_1 - \tilde{\zeta}\int_{-\infty}^{+\infty}(\xi^2+\eta)\ \phi_2^2\ d\xi_2\\
&+(\tfrac{a_1}{2}+\tfrac{\zeta_1}{2\tau})\int_0^L \varphi_t^2(x,t)\ dx + (\frac{-\zeta_1}{2\tau})\int_0^L z_1^2(x,1)\ dx\\
&+(\tfrac{a_2}{2}+\tfrac{\zeta_2}{2\tau})\int_0^L \psi_t^2(x,t)\ dx + (\frac{-\zeta_2}{2\tau})\int_0^L z_2^2(x,1)\ dx\\
&\leq max\{\tfrac{1}{\rho_1}(\tfrac{a_1}{2}+\tfrac{\zeta_1}{2\tau});\tfrac{1}{\rho_2}(\tfrac{a_2}{2}+\tfrac{\zeta_2}{2\tau})\}\langle U,U\rangle_{\mathcal{H}}\\
&\leq C\langle U,U\rangle_{\mathcal{H}}
\end{aligned}$$

Consequently, the operator $\mathcal{A}-\mathcal{CI}$ is dissipative. Now, we will prove that the operator $\lambda I-\mathcal{A}$ is surjective for $\lambda>0$. For this purpose, let $(f_1, f_2, g_1, g_2, h_1, h_2, M_1, M_2)^T \in \mathcal{H}$, we seek $U=(\varphi,\tilde{\varphi},z_1,\phi_1,\psi,\tilde{\psi},z_2,\phi_2)^T\in D(\mathcal{A})$ solution of the following system of equations

$$\begin{cases}
\lambda\varphi-\tilde{\varphi}=f_1,\\
\lambda\tilde{\varphi}-\frac{k}{\rho_1}(\varphi_x+\psi)_x+\frac{\zeta}{\rho_1}\int_{-\infty}^{+\infty}\omega_1(\xi)\phi_1(x,\xi)\,d\xi+\frac{a_2}{\rho_1}\tilde{\varphi}=f_2,\\
\lambda z_1+\frac{z_{1p}}{\tau_1}=f_3,\\
\lambda\phi_1+(\xi^2+\kappa_1)\phi_1-z_1(x,1)\omega_1(\xi)=f_4,\\
\lambda\psi-\tilde{\psi}=f_5,\\
\lambda\tilde{\psi}-\frac{b}{\rho_2}\psi_{xx}+\frac{k}{\rho_2}(\varphi_x+\psi)+\frac{\tilde{\zeta}}{\rho_2}\int_{-\infty}^{+\infty}\omega_2(\xi)\phi_2(x,\xi)\,d\xi+\frac{\tilde{a}_2}{\rho_2}\tilde{\psi}=f_6,\\
\lambda z_2+\frac{z_{2p}}{\tau_2}=f_7,\\
\lambda\phi_2+(\xi^2+\kappa_2)\phi_2-z_2(x,1)\omega_2(\xi)=f_8,
\end{cases}\tag{4.28}$$

Suppose that we have found (φ,ψ) with the appropriate regularity,then

$$\begin{aligned}\tilde{\varphi}&=\lambda\varphi-f_1,\\ \tilde{\psi}&=\lambda\psi-f_5\end{aligned}\quad.\tag{4.29}$$

It is clear that $\tilde{\varphi}\in H_0^1(0,L)$ and $\tilde{\psi}\in H_0^1(0,L)$. Furthermore, by (4.28) we can find (z_1,z_2) as

$$(z_1,z_2)(x,0)=(\tilde{\varphi},\tilde{\psi})(x), x\in(0,L)$$

Following the same approach as in Nicaise and Pignotti (2008), we obtain, by using (4.28)

$$\begin{aligned}z_1(x,p)&=e^{-\lambda p\tau_1}\tilde{\varphi}(x)+\tau_1e^{-\lambda p\tau_1}\int_0^p e^{\lambda s\tau_1}f_3(x,s)\ ds,\\ z_2(x,p)&=e^{-\lambda p\tau_2}\tilde{\psi}(x)+\tau_2e^{-\lambda p\tau_2}\int_0^p e^{\lambda s\tau_2}f_7(x,s)\ ds\end{aligned}\quad.$$

From (4.29) we obtain

$$\begin{aligned}z_1(x,p)&=\lambda\varphi(x)e^{-\lambda p\tau_1}-f_1(x)e^{-\lambda p\tau_1}+\tau_1e^{-\lambda p\tau_1}\int_0^p e^{\lambda s\tau_1}f_3(x,s)\ ds,\\ z_2(x,p)&=\lambda\psi(x)e^{-\lambda p\tau_2}-f_5(x)e^{-\lambda p\tau_2}+\tau_2e^{-\lambda p\tau_2}\int_0^p e^{\lambda s\tau_2}f_7(x,s)\ ds\end{aligned}\quad.$$

By using (4.28) and (4.29) the functions φ and ψ satisfying the following equations:

$$\begin{aligned}&\lambda^2\varphi-\frac{k}{\rho_1}(\varphi_x+\psi)_x+\frac{\zeta}{\rho_1}\int_{-\infty}^{+\infty}\omega_1(\xi)\phi_1(x,\xi)\,d\xi+\frac{a_2}{\rho_1}\tilde{\varphi}=f_2+\lambda f_1,\\ &\lambda^2\psi-\frac{b}{\rho_2}\psi_{xx}+\frac{k}{\rho_2}(\varphi_x+\psi)+\frac{\tilde{\zeta}}{\rho_2}\int_{-\infty}^{+\infty}\omega_2(\xi)\phi_2(x,\xi)\,d\xi+\frac{\tilde{a}_2}{\rho_2}\tilde{\psi}=f_6+\lambda f_5\end{aligned}\quad.\tag{4.30}$$

Using the following

$$
\begin{aligned}
z_1(x,1) &= \tilde{\varphi}(x)e^{-\lambda\tau_1} + \tau_1 e^{-\lambda\tau_1}\int_0^1 e^{\lambda s\tau_1} f_3(x,s)\; ds \\
&= \lambda\varphi(x)e^{-\lambda\tau_1} + (z_1)_0(x)
\end{aligned}
$$

$$
\begin{aligned}
z_2(x,1) &= \tilde{\psi}(x)e^{-\lambda\tau_2} + \tau_2 e^{-\lambda\tau_2}\int_0^1 e^{\lambda s\tau_2} f_7(x,s)\; ds \\
&= \lambda\psi(x)e^{-\lambda\tau_2} + (z_2)_0(x)
\end{aligned}
$$

where for $x \in (0,L)$,

$$
(z_1)_0(x) = -f_1(x)e^{-\lambda\tau_1} + \tau_1 e^{-\lambda\tau_1}\int_0^1 e^{\lambda\tau_1 s} f_3(x,s)\; ds \tag{4.31}
$$

$$
(z_2)_0(x) = -f_5(x)e^{-\lambda\tau_2} + \tau_2 e^{-\lambda\tau_2}\int_0^1 e^{\lambda\tau_2 s} f_7(x,s)\; ds \tag{4.32}
$$

The problem (4.30) can be reformulated as

$$
\begin{aligned}
&\int_0^L \left(\lambda^2\varphi - \frac{k}{\rho_1}(\varphi_x+\psi)_x + \lambda\varphi\left(\frac{a_1}{\rho_1}(\lambda+\kappa_1)^{\sigma_1-1}e^{-\lambda\tau_1} + \frac{a_2}{\rho_1}\right)\right).w_1\; dx \\
&= \int_0^L \left(f_2 + \lambda f_1 - \frac{a_1}{\rho_1}(\lambda+\kappa_1)^{\sigma_1-1}(z_1)_0(x) + \frac{a_2}{\rho_1}f_1 - \frac{\zeta}{\rho_1}\int_{-\infty}^{+\infty}\frac{\omega_1(\xi)}{\lambda+\xi^2+\kappa_1}f_4(x,\xi)\,d\xi\right) w_1\; dx. \\
&\int_0^L \left(\lambda^2\psi - \frac{b}{\rho_2}\psi_{xx} + \frac{k}{\rho_2}(\varphi_x+\psi) + \lambda\psi\left(\frac{\tilde{a}_1}{\rho_2}(\lambda+\kappa_1)^{\sigma_2-1}\; e^{-\lambda\tau_2} + \frac{\tilde{a}_2}{\rho_2}\right)\right).w_2\; dx \\
&= \int_0^L \left(f_6 + \lambda f_5 - \frac{\tilde{a}_1}{\rho_2}(\lambda+\kappa_2)^{\sigma_2-1}(z_2)_0(x) + \frac{\tilde{a}_2}{\rho_2}f_5 - \frac{\tilde{\zeta}}{\rho_2}\int_{-\infty}^{+\infty}\frac{\omega_2(\xi)}{\lambda+\xi^2+\kappa_2}f_8(x,\xi)\,d\xi\right) w_2\; dx.
\end{aligned} \tag{4.33}
$$

for all $w_1, w_2 \in H_0^1(0,L)$. Integrating the first equation in (4.33) by parts, we obtain

$$
\begin{aligned}
&\int_0^L \left(\rho_1\lambda^2\varphi w_1 + k(\varphi_x+\psi)w_{1x} + \lambda\left(a_1(\lambda+\kappa_1)^{\sigma_1-1}e^{-\lambda\tau_1} + a_2\right)\varphi w_1\right)\; dx \\
&= \int_0^L \left(\rho_1(f_2+\lambda f_1) - a_1(\lambda+\kappa_1)^{\sigma_1-1}(z_1)_0(x) + a_2 f_1 - \zeta\int_{-\infty}^{+\infty}\frac{\omega_1(\xi)}{\lambda+\xi^2+\kappa_1}f_4(x,\xi)\,d\xi\right) w_1\; dx.
\end{aligned} \tag{4.34}
$$

Integrating the second equation in (4.33) by parts, we get

$$
\begin{aligned}
&\int_0^L \left(\rho_2\lambda^2\psi w_2 + b\psi_x w_{2x} + k(\varphi_x+\psi)w_2 + \lambda\left(\tilde{a}_1(\lambda+\kappa_1)^{\sigma_2-1}\; e^{-\lambda\tau_2} + \tilde{a}_2\right)\psi w_2\right)\; dx \\
&= \int_0^L \left(\rho_2(f_6+\lambda f_5) - \tilde{a}_1(\lambda+\kappa_2)^{\sigma_2-1}(z_2)_0(x) + \tilde{a}_2 f_5 - \tilde{\zeta}\int_{-\infty}^{+\infty}\frac{\omega_2(\xi)}{\lambda+\xi^2+\kappa_2}f_8(x,\xi)\,d\xi\right) w_2\; dx.
\end{aligned} \tag{4.35}
$$

Using (4.34) and (4.35), the probleme (4.33) is equivalent to the probleme

$$
a((\varphi,\psi),(w_1,w_2)) = L(w_1,w_2), \tag{4.36}
$$

where the bilinear form $a : (H_0^1(0,L) \times H_0^1(0,L))^2 \to \mathbb{R}$ and the linear form $L : H_0^1(0,L) \times H_0^1(0,L) \to \mathbb{R}$ are defined by

$$a((\varphi,\psi),(w_1,w_2)) = \int_0^L \left(\rho_1\lambda^2\varphi w_1 + \rho_2\lambda^2\psi w_2 + k(\varphi_x+\psi)(w_{1x}+w_2) + b\psi_x w_{2x}\right)\, dx$$
$$+ \int_0^L \left(\lambda\left(a_1(\lambda+\kappa_1)^{\sigma_1-1}e^{-\lambda\tau_1} + a_2\right)\varphi w_1 + \lambda\left(\tilde{a}_1(\lambda+\kappa_1)^{\sigma_2-1}\, e^{-\lambda\tau_2} + \tilde{a}_2\right)\psi w_2\right)\, dx$$

and

$$L(w_1,w_2) = \int_0^L \left(\rho_1(f_2+\lambda f_1) - a_1(\lambda+\kappa_1)^{\sigma_1-1}(z_1)_0(x) + a_2 f_1 - \zeta\int_{-\infty}^{+\infty}\frac{\omega_1(\xi)}{\lambda+\xi^2+\kappa_1}f_4(x,\xi)\,d\xi\right) w_1\, dx$$
$$+ \int_0^L \left(\rho_2(f_6+\lambda f_5) - \tilde{a}_1(\lambda+\kappa_2)^{\sigma_2-1}(z_2)_0(x) + \tilde{a}_2 f_5 - \tilde{\zeta}\int_{-\infty}^{+\infty}\frac{\omega_2(\xi)}{\lambda+\xi^2+\kappa_2}f_8(x,\xi)\,d\xi\right) w_2\, dx.$$

It is easy to verify that a is continuous and coercive, and L is continuous. So applying the Lax-Milgram theorem, we deduce that for all $(w_1,w_2) \in H_0^1(0,L)$ problem (4.36) admits a unique solution $(\varphi,\psi) \in H_0^1 \times H_0^1(0,L)$.it follows from (4.34) and (4.35) that $(\varphi,\psi) \in (H^2(0,L) \cap H_0^1(0,L)) \times (H^2(0,L) \cap H_0^1(0,L))$. Therefore, the operator $\lambda I - A$ is surjective for any $\lambda > 0$. Hence, the well-posedness result follows from Hille-Yosida theorem.

4.4 Exponential Stability

The necessary and suficient conditions for the exponential stability of the C_0- semigroup of contractions on a Hilbert space were obtained by Gearhart **[60]** and Huang **[31]** independently, see also Pruss **[60]**. We will use the following result due to Gearhart.

Theorem 4.4.1 ([60]- [31]) *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on Hilbert space $\mathcal{H}$. Then $S(t)$ is exponentially stable if and only if*

$$\varrho(\mathcal{A}) \supseteq \{i\beta : \beta \in \mathbb{R}\} \equiv i\mathbb{R} \tag{4.37}$$

and

$$\overline{\lim_{|\beta|\to\infty}} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{4.38}$$

Our main result is as follows.

Theorem 4.4.2 *The semigroup $S_{\mathcal{A}}(t)_{t\geq 0}$ generated by $\mathcal{A}$ is exponentially stable.*

Proof. We will need to study the resolvent equation $(i\tilde{\lambda} - \mathcal{A})U = G$, for $\lambda \in \mathbb{R}$, namely

$$\begin{cases} i\lambda\varphi - \tilde{\varphi} = f_1, \\ i\lambda\tilde{\varphi} - \frac{k}{\rho_1}(\varphi_x+\psi)_x + \frac{\zeta}{\rho_1}\int_{-\infty}^{+\infty}\omega_1(\xi)\phi_1(x,\xi)\,d\xi + \frac{a_2}{\rho_1}\tilde{\varphi} = f_2, \\ i\lambda z_1 + \frac{z_{1p}}{\tau_1} = f_3, \\ i\lambda\phi_1 + (\xi^2+\kappa_1)\phi_1 - z_1(x,1)\omega_1(\xi) = f_4, \\ i\lambda\psi - \tilde{\psi} = f_5, \\ i\lambda\tilde{\psi} - \frac{b}{\rho_2}\psi_{xx} + \frac{k}{\rho_2}(\varphi_x+\psi) + \frac{\tilde{\zeta}}{\rho_2}\int_{-\infty}^{+\infty}\omega_2(\xi)\phi_2(x,\xi)\,d\xi + \frac{\tilde{a}_2}{\rho_2}\tilde{\psi} = f_6, \\ i\lambda z_2 + \frac{z_{2p}}{\tau_2} = f_7, \\ i\lambda\phi_2 + (\xi^2+\kappa_2)\phi_2 - z_2(x,1)\omega_2(\xi) = f_8, \end{cases} \tag{4.39}$$

where $F=(f_1,f_2,f_3,f_4,f_5,f_6,f_7,f_8)^T$. Taking inner product in $\mathcal{H}$ with U and using (4.33) we get

$$|Re\langle \mathcal{A}U,U\rangle| \leq \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{4.40}$$

This implies that

$$\int_0^L \tilde{\varphi}^2(x)\,dx, \int_0^L \tilde{\psi}^2(x)\,dx, \quad \int_0^L z_j^2(x,1)\,dx \leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{4.41}$$

From $(4.39)_3$, we obtain

$$\begin{cases} \phi_1 &= \dfrac{z_1(x,1)\omega_1(\xi)+f_4}{i\lambda+\xi^2+\kappa_1} \\ \phi_2 &= \dfrac{z_2(x,1)\omega_2(\xi)+f_8}{i\lambda+\xi^2+\kappa_2} \end{cases} \tag{4.42}$$

Then

$$\begin{aligned} &\|\phi_1\|_{L^2((0,L)\times(-\infty,+\infty))} \leq \\ &\left\|\frac{\omega_1(\xi)}{i\tilde{\lambda}+\xi^2+\kappa_1}\right\|_{L^2(-\infty,+\infty)} \|z_1(x,1)\|_{L^2(0,L)} + \left\|\frac{f_4}{i\tilde{\lambda}+\xi^2+\kappa_1}\right\|_{L^2((0,L)\times(-\infty,+\infty))} \\ &\leq \left(2(1-\sigma_1)\frac{\pi}{\sin\sigma_1\pi}(|\lambda|+\kappa_1)^{\sigma_1-2}\right)^{\frac{1}{2}} \|z_1(x,1)\|_{L^2(0,L)} + \frac{\sqrt{2}}{|\lambda|+\kappa_1}\|f_4\|_{L^2((0,L)\times(-\infty,+\infty))}. \end{aligned} \tag{4.43}$$

Similarly, we have

$$\begin{aligned} &\|\xi\phi_1\|_{L^2((0,L)\times(-\infty,+\infty))} \leq \\ &\left\|\frac{\xi\omega_1(\xi)}{i\lambda+\xi^2+\kappa_1}\right\|_{L^2(-\infty,+\infty)} \|z_1(x,1)\|_{L^2(0,L)} + \left\|\frac{\xi f_4}{i\lambda+\xi^2+\kappa_1}\right\|_{L^2((0,L)\times(-\infty,+\infty))} \\ &\leq \left(2\sigma_1\frac{\pi}{\sin\sigma_1\pi}(|\lambda|+\kappa_1)^{\sigma_1-1}\right)^{\frac{1}{2}} \|z_1(x,1)\|_{L^2(0,L)} + \frac{\sqrt{2}}{\sqrt{|\lambda|+\kappa_1}}\|f_4\|_{L^2((0,L)\times(-\infty,+\infty))}. \end{aligned} \tag{4.44}$$

Let us introduce the following notation

$$\mathcal{I}_u(x) = |\tilde{\varphi}(x)|^2 + |\tilde{\psi}(x)|^2 + k|\varphi_x(x)+\psi(x)|^2 + b|\psi_x(x)|^2$$

and

$$\mathcal{E}_u = \int_\Omega \mathcal{I}_u(x)\,dx.$$

Lemma 4.4.1 *We have that*

$$\mathcal{E}_u \leq c\|F\|_{\mathcal{H}}^2 + c'\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{4.45}$$

for positive constants c and c'.

Proof. Multiplying the equation $(4.39)_2$ by $\bar{\varphi}$, integrating on $(0,1)$ we obtain

$$\begin{aligned} &-\rho_1\int_0^1 \tilde{\varphi}(\overline{i\lambda\varphi})\,dx + k\int_0^1(\varphi_x+\psi)\overline{\varphi_x}\,dx \\ &+\zeta\int_0^1 \bar{\varphi}\Big(\int_{-\infty}^{+\infty}\omega_1(\xi)\phi_1(x,\xi)\,d\xi\Big)\,dx + a_2\int_0^1 \tilde{\varphi}\bar{\varphi}\,dx = \rho_1\int_0^1 \bar{\varphi}f_2\,dx. \end{aligned} \tag{4.46}$$

From $(4.39)_1$, we have $i\lambda\varphi = \tilde{\varphi} + f_1$. Then

$$\begin{aligned} &-\rho_1 \int_0^1 |\tilde{\varphi}|^2\, dx + k\int_0^1 (\varphi_x + \psi)\overline{\varphi_x}\, dx \\ &+\zeta \int_0^1 \bar{\varphi}(\int_{-\infty}^{+\infty} \omega_1(\xi)\phi_1(x,\xi)\, d\xi)\, dx + a_2\int_0^1 \tilde{\varphi}\bar{\varphi}\, dx = \rho_1 \int_0^1 (\bar{\varphi} f_2 + \tilde{\varphi}\bar{f}_1)\, dx. \end{aligned} \tag{4.47}$$

Multiplying the equation $(4.39)_6$ by $\bar{\psi}$, integrating on $(0,1)$ we obtain

$$\begin{aligned} &-\rho_2 \int_0^1 \tilde{\psi}(\overline{i\lambda\psi})\, dx + b\int_0^1 |\psi_x|^2\, dx + k\int_0^1 (\varphi_x + \psi)\overline{\psi}\, dx \\ &+\tilde{\zeta} \int_0^1 \bar{\psi}(\int_{-\infty}^{+\infty} \omega_2(\xi)\phi_2(x,\xi)\, d\xi)\, dx + \tilde{a}_2\int_0^1 \tilde{\psi}\bar{\psi}\, dx = \rho_2 \int_0^1 \bar{\psi} f_6\, dx. \end{aligned} \tag{4.48}$$

From $(4.39)_1$, we have $i\lambda\psi = \tilde{\psi} + f_1$. Then

$$\begin{aligned} &-\rho_2 \int_0^1 |\tilde{\psi}|^2\, dx + b\int_0^1 |\psi_x|^2\, dx + k\int_0^1 (\varphi_x + \psi)\overline{\psi}\, dx \\ &+\tilde{\zeta} \int_0^1 \bar{\psi}(\int_{-\infty}^{+\infty} \omega_2(\xi)\phi_2(x,\xi)\, d\xi)\, dx + \tilde{a}_2\int_0^1 \tilde{\psi}\bar{\psi}\, dx = \rho_2 \int_0^1 (\bar{\psi} f_6 + \tilde{\psi}\bar{f}_5)\, dx. \end{aligned} \tag{4.49}$$

Hence

$$\begin{aligned} &-\rho_1 \int_0^1 |\tilde{\varphi}|^2\, dx - \rho_2\int_0^1 |\tilde{\psi}|^2\, dx + k\int_0^1 |\varphi_x + \psi|^2\, dx + b\int_0^1 |\psi_x|^2\, dx \\ &+\zeta \int_0^1 \bar{\varphi}(\int_{-\infty}^{+\infty} \omega_1(\xi)\phi_1(x,\xi)\, d\xi)\, dx + \tilde{\zeta}\int_0^1 \bar{\psi}(\int_{-\infty}^{+\infty} \omega_2(\xi)\phi_2(x,\xi)\, d\xi)\, dx \\ &+a_2\int_0^1 \tilde{\varphi}\bar{\varphi}\, dx + \tilde{a}_2\int_0^1 \tilde{\psi}\bar{\psi}\, dx \\ &= \rho_1 \int_0^1 (\bar{\varphi} f_2 + \tilde{\varphi}\bar{f}_1)\, dx + \rho_2\int_0^1 (\bar{\psi} f_6 + \tilde{\psi}\bar{f}_5)\, dx. \end{aligned} \tag{4.50}$$

We can estimate

$$\begin{aligned} &\left|\int_0^1 \bar{\varphi}(\int_{-\infty}^{+\infty} \omega_1(\xi)\phi_1(x,\xi)\, d\xi)\, dx\right| \\ &\leq \|\varphi\|_{L^2(0,1)} \left(\int_{-\infty}^{+\infty} \frac{\omega_1^2(\xi)}{\xi^2+\kappa_1}\, d\xi\right)^{\frac{1}{2}} \left(\int_0^1\int_{-\infty}^{+\infty} (\xi^2+\kappa_1)|\phi_1(x,\xi)|^2\, d\xi\, dx\right)^{\frac{1}{2}} \\ &\leq \frac{\varepsilon}{2}\left(\int_{-\infty}^{+\infty} \frac{\omega_1^2(\xi)}{\xi^2+\kappa_1}\, d\xi\right)\|\varphi\|^2_{L^2(0,1)} + \frac{1}{2\varepsilon}\int_0^1\int_{-\infty}^{+\infty} (\xi^2+\kappa)|\phi_1(x,\xi)|^2\, d\xi\, dx \\ &\leq \frac{\varepsilon}{2}C\left(\int_{-\infty}^{+\infty} \frac{\omega_1^2(\xi)}{\xi^2+\kappa_1}\, d\xi\right)\|\varphi_x\|^2_{L^2(0,1)} + \frac{1}{2\varepsilon}\int_0^1\int_{-\infty}^{+\infty} (\xi^2+\kappa_1)|\phi_1(x,\xi)|^2\, d\xi\, dx, \end{aligned}$$

$$\begin{aligned} \left|\int_0^1 \bar{\varphi}\tilde{\varphi}\, dx\right| &\leq \|\varphi\|_{L^2(0,1)}\|\tilde{\varphi}\|_{L^2(0,1)} \\ &\leq \frac{\varepsilon}{2}C\|\varphi_x\|^2_{L^2(0,1)} + \frac{1}{2\varepsilon}\|\tilde{\varphi}\|^2_{L^2(0,1)}, \end{aligned}$$

$$\left|\int_0^1 \bar{\varphi} f_2\, dx\right| \leq \frac{\varepsilon}{2}C\|\varphi_x\|^2_{L^2(0,1)} + \frac{1}{2\varepsilon}\|f_2\|^2_{L^2(0,1)},$$

$$\left|\int_0^1 \tilde{\varphi}\bar{f}_1\,dx\right| \le \frac{\varepsilon}{2}\|\tilde{\varphi}\|^2_{L^2(0,1)} + \frac{1}{2\varepsilon}\|f_1\|^2_{L^2(0,1)}.$$

Similarly, we have

$$\left|\int_0^1 \bar{\psi}(\int_{-\infty}^{+\infty}\omega_1(\xi)\phi_2(x,\xi)\,d\xi)\,dx\right|$$
$$\le \frac{\varepsilon}{2}C\left(\int_{-\infty}^{+\infty}\frac{\omega_2^2(\xi)}{\xi^2+\kappa_2}\,d\xi\right)\|\psi_x\|^2_{L^2(0,1)} + \frac{1}{2\varepsilon}\int_0^1\int_{-\infty}^{+\infty}(\xi^2+\kappa_2)|\phi_2(x,\xi)|^2\,d\xi\,dx$$

$$\begin{aligned}\left|\int_0^1 \bar{\psi}\tilde{\psi}\,dx\right| &\le \|\psi\|_{L^2(0,1)}\|\tilde{\psi}\|_{L^2(0,1)}\\ &\le \frac{\varepsilon}{2}C\|\psi_x\|^2_{L^2(0,1)} + \frac{1}{2\varepsilon}\|\tilde{\psi}\|^2_{L^2(0,1)},\end{aligned}$$

$$\left|\int_0^1 \bar{\psi}f_6\,dx\right| \le \frac{\varepsilon}{2}C\|\psi_x\|^2_{L^2(0,1)} + \frac{1}{2\varepsilon}\|f_6\|^2_{L^2(0,1)},$$

$$\left|\int_0^1 \tilde{\psi}\bar{f}_5\,dx\right| \le \frac{\varepsilon}{2}\|\tilde{\psi}\|^2_{L^2(0,1)} + \frac{1}{2\varepsilon}\|f_5\|^2_{L^2(0,1)}.$$

Choosing ε small enough, we conclude (4.45).

Moreover, equations $(4.39)_4$ and $(4.39)_4$ have a unique solution

$$\begin{aligned} z_1(x,\varrho) &= e^{-i\tau_1\lambda\varrho}z_1(x,0) + \tau e^{-i\tau_1\lambda\varrho}\int_0^\varrho e^{-i\tau_1\lambda}f_4(x,r)\,dr\\ &= e^{-i\tau_1\lambda\varrho}\tilde{\varphi}(x) + \tau_1 e^{-i\tau_1\lambda\varrho}\int_0^\varrho e^{-i\tau_1\lambda}f_4(x,r)\,dr.\end{aligned}$$

and

$$z_2(x,\varrho) = e^{-i\tau_2\lambda\varrho}\tilde{\psi}(x) + \tau_2 e^{-i\tau_2\lambda\varrho}\int_0^\varrho e^{-i\tau_2\lambda}f_8(x,r)\,dr$$

Then

$$\|z_1(x,\varrho)\|_{L^2((0,1)\times(0,1))} \le \|\tilde{\varphi}(x)\|_{L^2(0,1)} + \tau_1\|f_4(x,\varrho)\|_{L^2((0,1)\times(0,1))}. \tag{4.51}$$

and

$$\|z_2(x,\varrho)\|_{L^2((0,1)\times(0,1))} \le \|\tilde{\psi}(x)\|_{L^2(0,1)} + \tau_2\|f_8(x,\varrho)\|_{L^2((0,1)\times(0,1))}. \tag{4.52}$$

Finally, (4.43), (4.45) and (4.51) imply that

$$\|U\|_{\mathcal{H}} \le C$$

for a positive constant C. The conclusion then follows by applying Theorem 4.4.1.

Remark 4.4.1 *We can extend the results of this paper to more general measure density instead of (4.17), that is ω is an even nonnegative measurable function such that*

$$\int_{-\infty}^{\infty}\frac{\omega(\xi)^2}{1+\xi^2}\,d\xi < \infty. \tag{4.53}$$

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Abstract

In this thesis we considered some evolution problems with the presence of boundary dissipation of fractional derivative type. In particular, we consider dynamic Euler-Bernoulli-beam equation, acoustic wave equation and Timoshenko system. Under assumptions on initial data and boundary conditions, we focused our study on the global existence and asymptotic behavior of solutions where we obtained several results on the decay rate.

Keywords: Euler-Bernoulli beam equation, boundary dissipation of fractional derivative type, acoustic wave equation, Timoshenko system, asymptotic behavior of solution.

Résumé

Dans cette thèse, nous avons considéré quelques problèmes d'évolution hyperbolique avec la présence des termes dissipatifs de type fractionnaires. En particulier on considère l'équation d'Euler-Bernoulli, l'équation des ondes acoustiques et le système de Timoshenko. Sous quelques hypothèses sur les données initiales et aux bords, nous avons concentré notre étude sur l'existence globale et le comportement asymptotique des solutions où nous avons obtenu plusieurs résultats sur la vitesse de décroissance de l'énergie.

Mots clés: l'equation de Euler-Bernoulli, termes dissipatifs de type fractionnaires, l'equation des ondes acoustiques, le système de Timoshenko, le comportement asymptotique des solution.

ملخص

في هذه الأطروحة اقترحنا بعض المسائل الرياضية لمعادلات و جمل معادلات بوجود آليات للتبديد ذات أشكال كسرية من زوايا مختلفة. ندرس خاصة معادلة اولر-رنولي معادلة الموجات وجمل تيموشانكو تحت بعض الفرضيات على الشروط الابتدائية و الشروط الحدية، ركزنا دراستنا على وجود الحلول ودراسة السلوك المقارب للحلول الموجودة عند اللانهاية الزمنية أين توصلنا لإيجاد عدة نتائج حول طريقة تناقص الطاقة

الكلمات المفتاحية:معادلة اولر- برنولي اليات التبديد ذات اشكال كسرية, معادلة الموجة الصوتية, جمل تيموشانكو, السلوك المقارب للحلول.