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THÈSE

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Existence et Contrôlabilité pour des Equations Différentielles Aléatoires

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Publications

1. M. Benchohra, F. Bouazzaoui, Controllability of Functional Differential Equations with Delay and Random Effect, *Communications on Applied Nonlinear Analysis* Vol.**25** No.1(2018), 52-65.

2. M. Benchohra, F. Bouazzaoui, Controllability of Functional Differential Equations with State-Dependent Delay and Random Effects, *Romanian Journal of Mathematics and Computer Science* Vol.**8** No.1(2018), 38-51.

3. S. Abbas, M. Benchohra, F. Bouazzaoui, Controllability of Second Order Functional Differential Equations with Delay and Random Effect, (submitted).

Abstract

The objective of this thesis is to present the results of controllability of mild solutions of an semilinear First and Second Order functional differential equations with delay and Random Effect in different situations (constant and state-dependent delay). All problems have been studied with infinite delay and in a Banach space.

The technique used is to reduce the study of ours problem to research of fixed point of an integral operator properly constructed. By applying the famous fixed point theorems combined with the theory of semigroup and cosinus family.

Key words : Semilinear functional differential equations, Semilinear functional differential equations of second order, random mild solution, random fixed-point, infinite delay, state-dependent delay, semigroup, cosinus family, measure of noncompactness, controllability, Banach space.

AMS Subject Classification : 34G20, 34G25, 34K20, 34K30.

Résumé

L'objectif de cette thèse est de présenter des résultats de la contrôlabilité des solutions faibles d'équations différentielles fonctionnelles semi-linéaires de premier et de second ordre avec retard et effet aléatoire dans différentes situations (retard constant et dépendant de l'état). Tous les problèmes ont été étudiés avec un retard infini et dans un espace de Banach.

La technique utilisée est de réduire l'étude de notre problème à la recherche de point fixe d'un opérateur intégral correctement construit. En appliquant les fameux théorèmes des points fixes combinés à la théorie de la famille des semigroupes et des cosinus.

Mots clé : Equations différentielles fonctionnelles semi-linéaire , Equations différentielles fonctionnelles de second ordre, solutions faible aléatoire, points fixes aléatoires, retard infini, retard dépendant de l'état, semi-groupes, famille cosinus, mesure de non compacité , contrôllabilité, espace de Banach .

Classification AMS : 34G20, 34G25, 34K20, 34K30.

Contents

Introduction

Functional differential equations arise in a variety of areas of biological, physical, and engineering applications, see, for example, the books of Hale [50], Hale and Verduyn Lunel [52], Kolmanovskii and Myshkis, Wu [61], and the references therein. Delay differential equations is one of the oldest branches of the theory of infinite dimensional dynamical systems - theory which describes qualitative properties of systems changing in time. We refer to the classical monographs on the theory of ordinary delay equations [38,50]. However, complicated situations in which the delay depends on the unknown functions have been proposed in modelling in recent years. These equations are frequently called equations with state-dependent delay. Often, it has been assumed that the delay is either a fixed constant or is given as an integral in which case is called distributed delay; see for instance the books by Hale and Verduyn Lunel [52], Kol-Manovskii and Myshkis [61], Smith [81], Abbas and Benchohra [1] , and Wu [90], and the references therein. Existence results and among other things were derived recently for functional differential equations when the solution is depending on the delay on a bounded interval for impulsive problems. We refer the reader to the papers by Hernandez *et al* [56] and Li *et al*. [68]. Very recently, Baghli et al. considered when the solution is depending in the delay for evolution equations in [9]. It should be pointed out that, to study the abstract functional differential equations with infinite delay, people usually employ an axiomatic definition of the phase space introduced by Hale and Kato [51], but defined as in the book [57].

Functional evolution equations with state-dependent delay appear frequently in mathematical modeling of several real of problems and for this reason the study of this type of equations has received great attention in the last few years, see for instance [39, 54, 55]. An extensive theory is developed for evolution equations [4, 42]. Uniqueness and existence results have been established recently for different evolution problems in the papers by Baghli and Benchohra for finite and infinite delay in [7, 8, 9].

The stochastic differential equation with delay is a special type of stochastic functional differential equations. The stochastic functional differential equations with state-dependent delay have many important applications in mathematical models of real phenomena, and the study of this type of equations has received much attention in recent years. Guendouzi and Benzatout [48] studied the existence of mild solutions for a class of impulsive stochastic differential inclusions with state-dependent delay. Random differential equations, as natural extensions of deterministic ones, arise in many applications and have been investigated by many mathematicians see [66, 67, 69, 83, 93] and references therein. Between them differential equations with random coefficients (see [83, 32]) offer a natural and rational approach (see [82], Chapter 1), since sometimes we can get the random distributions of some main disturbances by historical experiences and data rather than take all random disturbances into account and assume the noise to be white noises. There are real phenomena with anomalous dynamics such as signal transmissions through strong magnetic fields, atmospheric diffusion of pollution, network traffic, the effect of speculations on the protability of stocks in the financial markets and so on where the classical models are not sufficiently good to describe these features. We refer the reader to the monographs [30, 29, 63, 70, 71], and the references therein.

The notion of controllability is of great importance in mathematical control theory. Many fundamental problems of control theory such as pole-assignment, stabilizability and optimal control may be solved under the assumption that the system is controllable. The problem of controllability is to show the existence of control function, which steers the solution of the system from its initial state to the final state, where the initial and final states may vary over the entire space. Controllability of nonlinear systems with and without impulse have studied by several authors, see, for instance [10], [33], [68] and [80]. Controllability problems described as abstract differential systems in infinite-dimensional spaces appear in many branches of physics and technical sciences, such as heat flow in materials with memory, viscoelasticity and other physical phenomena [14]. The controllability of infinite-dimensional systems in Banach spaces has been studied extensively by virtue of the fixed point arguments. The essential part of this method is to transform the controllability problem into a fixed point problem for an appropriate operator in a function space. The problem of controllability for functional differential inclusions in Banach spaces has been extensively studied. For example, Benchohra et al. [21, 25, 27, 26] discussed the controllability for one-order, second-order functional differential and integrodifferential inclusions in Banach spaces with the help of some fixed-point theorem. This thesis is devoted to the controllability of mild random solution for some first and second order functional delay differential equations with random effect in Banach space are studied by Benaissa and Benchohra [5, 16, 17]. Sufficient conditions are considered to transform the controllability problem into a fixed point problem. Our results are obtained using the search for the existence of random fixed point of a continuous random operator with stochastic domain. We have arranged this thesis as follows:

In **Chapter 1**, we introduce notations, definitions, lemmas and fixed point theorems which are used throughout this thesis.

In **Chapter 2**, we prove the controllability of random mild solutions of the following functional differential equation with constant delay and random effects of the form:

$$\begin{aligned} y'(t,w) &= Ay(t,w) + f(t, y_t(\cdot,w), w) + Bu(t,w), \quad \text{a.e. } t \in J := [0,b] \\ y(t,w) &= \phi(t,w), \quad t \in (-\infty, 0], \end{aligned}$$

where $w \in \Omega$, $f : J \times \mathcal{B} \times \Omega \to E$ and $\phi : (-\infty, 0] \times \Omega \to E$ are given random functions. This problem has been considered in the paper [18].

In **Chapter 3**, we prove the controllability of random mild solutions of the following functional differential equations with state-dependent delay and random effects of the form:

$$\begin{aligned} y'(t,w) &= Ay(t,w) + f(t, y_{\rho(t,y_t)}(\cdot,w), w) + Bu\,(t,w)\,, \quad \text{a.e. } t \in J \\ y(t,w) &= \phi(t,w), \quad t \in (-\infty, 0], \end{aligned}$$

where $w \in \Omega$, $f : J \times \mathcal{B} \times \Omega \to E$ and $\phi : (-\infty, 0] \times \Omega \to E$ are given random functions. This problem has been considered in the paper [19].

In the first section of **Chapter 4**, we give controllability result of the following second order functional differential equation with constant delay and random effects of the form:

$$\begin{aligned} y^{''}(t,w) &= Ay(t,w) + f(t, y_t(\cdot,w), w) + Bu(t,w), \quad \text{a.e. } t \in J \\ y(t,w) &= \phi(t,w), \quad t \in (-\infty, 0],\ y^{'}(0,w) = \varphi(w) \in E \end{aligned}$$

where $w \in \Omega$, $f : J \times \mathcal{B} \times \Omega \to E$ and $\phi : (-\infty, 0] \times \Omega \to E$ are given random functions.

Then in section 4.3, we prove the controllability result of the following second order functional differential equation with state-dependent delay and random effects of the form:

$$\begin{aligned} y^{''}(t,w) &= Ay(t,w) + f(t, y_{\rho(t,y_t)}(\cdot,w), w) + Bu(t,w), \quad \text{a.e. } t \in J \\ y(t,w) &= \phi(t,w), \quad t \in (-\infty, 0],\ y^{'}(0,w) = \varphi(w) \in E, \end{aligned}$$

where $w \in \Omega$, $f : J \times \mathcal{B} \times \Omega \to E$ and $\phi : (-\infty, 0] \times \Omega \to E$ are given random functions. This two problems has been considered in the paper [2].

An exemple is given in the end of each section to illustrate the theoretical result.

Chapter 1

Preliminaries

In this section, we review some fundamental concepts, notations, definitions, fixed point theorems and properties required to establish our main results.

1.1 Notations and definitions

Let E be a Banach space with the norm $|\cdot|$, $J = [0, b]$ be a real interval and $C(J, E)$ be the Banach space of continuous functions from J into E with the usual supremum norm

$$\|y\| = \sup_{t \in J} |y(t)|.$$

Let $B(E)$ denotes the Banach space of bounded linear operators from E into E.

A measurable function $y : J \to E$ is said to be Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For the Bochner integral properties, see the classical monograph of Yosida [94]).
Let $L^1(J, E)$ denotes the Banach space of measurable functions $y : J \to E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^b |y(t)| dt$$

.

Definition 1.1.1. *A map $f : J \times \mathcal{B} \times \Omega \to E$ is said to be Carathéodory if*

(i) $t \to f(t, y, w)$ *is measurable for all $y \in \mathcal{B}$ and for all $w \in \Omega$;*

(ii) $y \to f(t, y, w)$ *is continuous for almost each $t \in J$ and for all $w \in \Omega$;*

(iii) $w \to f(t, y, w)$ *is measurable for all $y \in \mathcal{B}$ and almost each $t \in J$.*

1.2 Semigroups

Let X be a Banach space and $B(X)$ be the Banach space of linear bounded operators.

Definition 1.2.1. *One parameter family $\{T(t)|t \geq 0\} \subset B(X)$ satisfying the conditions:*

1. $T(0) = I$, *(I denotes the identity operator in X) ;*
2. $T(t + s) = T(t)oT(s)$, *for* $t, s \geq 0$ *;*
3. *The map $t \longrightarrow T(t)(y)$ is strongly continuous, for each $y \in X$, i.e;*

$$\lim_{t \to 0} T(t)y = y \quad \forall\, y \in X.$$

A semigroup of bounded linear operators $T(t)$, is uniformly continuous if

$$\lim_{t \to 0} \|T(t) - I\| = 0.$$

Definition 1.2.2. *Let $T(t)$ be a semigroup of class (C_0) defined on X. The infinitesimal generator A of $T(t)$ is the linear operator defined by*

$$A(y) = \lim_{h \to 0} \frac{T(h)y - y}{h} \quad for\ y \in D(A),$$

where

$$D(A) = \left\{ y \in X / \lim_{h \to 0} \frac{T(h)(y) - y}{h} \quad exists\ in\ X \right\}.$$

Example 1.2.1. *Let X be the space of continuous functions $\phi : [0,1] \to \mathbb{R}$ endowed with the sup norm. Then the family $(T(t))_{t\geq 0}$ defined by*

$$(T(t)\phi)(x) = \phi(xe^{-t}),\ t \geq 0,\ \phi \in X,\ x \in [0,1],$$

is a $C_0 - semigroup$ on X and its infinitesimal generator A is defined on

$$\begin{aligned} D(A) &= \{\phi \in C([0,1],\mathbb{R}) : \phi'(x)\ exists\ and\ is\ continuous\ for\ x \in [0,1]\} \\ &= C^1([0,1],\mathbb{R}) \end{aligned}$$

by

$$A\phi = \phi'.$$

1.3 Cosine family

Definition 1.3.1. *A one parameter family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators in the Banach space X is called a strongly continuous cosine family if and only if*

- *$C(0) = I$ (I is the identity operator);*
- *$C(t)x$ is strongly continuous in t on $\mathbb{R}$ for each fixed $x \in X$;*
- *$C(t+s) + C(t-s) = 2C(t)C(s)$ for all t, $s \in \mathbb{R}$.*

If $\{C(t) : t \in \mathbb{R}\}$ is a strongly continuous cosine family in X, then we define the associated sine family $\{S(t) : t \in \mathbb{R}\}$ by

$$S(t)x = \int_0^t C(s)x ds,\ \ x \in X, t \in \mathbb{R}$$

The infinitesimal generator $A : X \to X$ of a cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by

$$Ax = \frac{d^2}{dt^2} C(t)x|_{t=0};\ x \in D(A),$$

where $D(A) = \{x \in X : C(\cdot)x \in C^2(\mathbb{R}, X)\}$. For more details on strongly continuous cosine and sine families we refer the reader to the papers of Fattorini [43] and Travis and Webb [85, 86].

1.4 Some properties of phase spaces

In this thesis, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [51] and follow the terminology used in [57]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ into E, and satisfying the following axioms :

(A_1) If $y : (-\infty, b) \to E, b > 0$, is continuous on J and $y_0 \in \mathcal{B}$, then for every $t \in J$, the following conditions hold :
(i) $y_t \in \mathcal{B}$;
(ii) There exists a positive constant H such that $|y(t)| \leq H\|y_t\|_{\mathcal{B}}$;
(iii) There exist two functions $L(\cdot) and M(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of y with L continuous and bounded and M locally bounded such that :

$$\|y_t\|_{\mathcal{B}} \leq L(t) \sup\{\ |y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_{\mathcal{B}}.$$

(A_2) For the function y in (A_1), y_t is a $\mathcal{B}-$ valued continuous function on J.

(A_3) The space $\mathcal{B}$ is complete.

Denote

$$K_b = \sup\{L(t) : t \in J\},$$

and

$$M_b = \sup\{M(t) : t \in J\}.$$

Remark 1.4.1. 1. *(ii) is equivalent to $|\phi(0)| \leq H\|\phi\|_{\mathcal{B}}$ for every $\phi \in \mathcal{B}$.*

2. *Since $\|\cdot\|_{\mathcal{B}}$ is a seminorm, two elements $\phi, \psi \in \mathcal{B}$ can verify $\|\phi - \psi\|_{\mathcal{B}} = 0$ without necessarily $\phi(\theta) = \psi(\theta)$ for all $\theta \leq 0$.*

3. *From the equivalence in the first remark, we can see that for all $\phi, \psi \in \mathcal{B}$ such that $\|\phi - \psi\|_{\mathcal{B}} = 0$: We necessarily have that $\phi(0) = \psi(0)$.*

We now indicate some examples of phase spaces. For other details, we refer, for instance to the book by Hino *et al* [57].

Example 1.4.1. *Let:*

BC *the space of bounded and continuous functions defined from* $(-\infty, 0]$ *to* E;

BCU *the space of bounded uniformly continuous functions defined from* $(-\infty, 0]$ *to* E;

C^∞ $:= \{\phi \in BC : \lim_{\theta\to-\infty} \phi(\theta)$ *exists in* $E\}$;

C^0 $:= \{\phi \in BC : \lim_{\theta\to-\infty} \phi(\theta) = 0\}$, *endowed with the uniform norm*

$$\|\phi\| = \sup\{|\phi(\theta)| : \theta \le 0\}.$$

We have that the spaces BUC, C^∞ *and* C^0 *satisfy conditions* $(A_1)-(A_3)$. *However,* BC *satisfies* $(A_1), (A_3)$ *but* (A_2) *is not satisfied.*

Example 1.4.2. *The spaces* C_g, UC_g, C_g^∞ *and* C_g^0.

Let g *be a positive continuous function on* $(-\infty, 0]$. *We define:*

$$C_g := \left\{\phi \in C((-\infty,0],E) : \frac{\phi(\theta)}{g(\theta)} \text{ is bounded on } (-\infty,0]\right\};$$

$$C_g^0 := \left\{\phi \in C_g : \lim_{\theta\to-\infty} \frac{\phi(\theta)}{g(\theta)} = 0\right\}, \text{ endowed with the uniform norm}$$

$$\|\phi\| = \sup\left\{\frac{|\phi(\theta)|}{g(\theta)} : \theta \le 0\right\}.$$

Then we have that the spaces C_g *and* C_g^0 *satisfy conditions* (A_3). *We consider the following condition on the function* g.

(g_1) *For all* $a > 0, \sup_{0\le t\le a} \sup\left\{\frac{g(t+\theta)}{g(\theta)} : -\infty < \theta \le -t\right\} < \infty.$

They satisfy conditions (A_1) *and* (A_2) *if* (g_1) *holds.*

Example 1.4.3. *The space* C_γ.

For any real positive constant γ, *we define the functional space* C_γ *by*

$$C_\gamma := \left\{\phi \in C((-\infty,0],E) : \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) \text{ exists in } E\right\}$$

endowed with the following norm

$$\|\phi\| = \sup\{e^{\gamma\theta}|\phi(\theta)| : \theta \le 0\}.$$

Then in the space C_γ, *the axioms* $(A_1)-(A_3)$ *are satisfied*

Example 1.4.4. *(The phase space $(C_r \times L^p(g,E))$).*
Let $g : (-\infty, -r) \to \mathbb{R}$ be a positive Lebesgue integrable function and assume that there exists a non-negative and locally bounded function γ on $(-\infty, 0]$ such that $g(\zeta+\theta) \leq \gamma(\zeta)g(\theta)$, for all $\zeta \leq 0$ and $\theta \in (-\infty, -r)\backslash N_\zeta$, where $N_\zeta \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero. The space $(C_r \times L^p(g,E))$ consists of all classes of functions $\phi : (-\infty, 0] \to \mathbb{R}$ such that ϕ is continuous on $[-\infty, 0]$, Lebesgue measurable and $g\left\|\phi\right\|^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm in $C_r \times L^p(g,E)$ is defined by
$\|\phi\|_{\mathcal{B}} := sup\{\|\phi\,(\theta)\| : -r \leq \theta \leq 0\} + \left(\int_{-\infty}^{-r} g(\theta)\left\|\phi\right\|^p d\theta\right)^{\frac{1}{p}}$.
Assume that $g(\cdot)$ verifies the condition $(g-5), (g-6)$ and $(g-7)$ in the nomenclature [57]. In this case, $\mathcal{B} = C_r \times L^p(g,E)$ verifies assumptions $(A1), (A2), (A3)$, Theorem 1.3.8 for details. Moreover, when $r = 0$ and $p = 2$, we have that $H = 1, M(t) = \gamma(-t)^{\frac{1}{2}}$ and $L(t) = 1 + \left(\int_{-t}^{0} g(\theta)d\theta\right)^{\frac{1}{2}}$ for $t \geq 0$.

1.5 Measure of noncompactness

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness.

Definition 1.5.1. *[15] Let E be a Banach space and Ω_E the bounded subsets of E. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_E \to [0, \infty)$ defined by:*

$$\alpha(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^{n} B_i \text{ and } diam(B_i) \leq \epsilon\};\ \text{here } B \in \Omega_E.$$

The Kuratowski measure of noncompactness satisfies the following properties (for more details see [15]).

(a) $\alpha(B) = 0 \Longleftrightarrow \overline{B}$ is compact (B is relatively compact).

(b) $\alpha(B) = \alpha(\overline{B})$.

(c) $A \subset B \Longrightarrow \alpha(A) \leq \alpha(B)$.

(d) $\alpha(A+B) \leq \alpha(A) + \alpha(B)$.

(e) $\alpha(cB) = |c|\alpha(B); c \in \mathbb{R}$.

(f) $\alpha(convB) = \alpha(B)$.

Lemma 1.5.1 ([64, 49]). *If $H \subset C(J, E)$ is bounded and equicontinuous, then $\alpha(H(t))$ is continuous on J and*

$$\alpha\left(\left\{\int_J x(s)ds : x \in H\right\}\right) \leq \int_J \alpha(H(s))ds,$$

where $H(s) = \{x(s) : x \in H\}, t \in J$.

1.6 Random operators

Let (Ω, F, P) a complete probability space where Ω is a set, F is the σ-algebra of Ω and P is the meusure defined on F
Let Y be a separable Banach space with the Borel σ-algebra B_Y. A mapping $y : \Omega \longrightarrow Y$ is said to be a random variable with values in Y if for each $B \in B_Y, y^{-1}(B) \in F$. A mapping $T : \Omega \times Y \longrightarrow Y$ is called a random operator if $T(\cdot, y)$ is measurable for each $y \in Y$ and is generally experessed as $T(w, y) = T(w)y$; we will use these two expressions alternatively.
Next, we will give a very useful random fixed point theorem with stochastic domain.

Definition 1.6.1. *[41] Let C be a mapping from Ω into 2^Y. A mapping $T : \{(w, y) : w \in \Omega \wedge y \in C(w)\} \longrightarrow Y$ is called random operator with stochastic domain C if C is measurable (i.e., for all closed $A \subseteq Y$, $\{w \in \Omega : C(w) \cap A \neq \emptyset\} \in F$ and for all open $D \subseteq Y$ and all $y \in Y$, $\{w \in \Omega : y \in C(w) \wedge T(w, y) \in D\} \in F$. T we be called 'continuous' if every $T(w)$ is continuous.*

For a random operator T, a mapping $y : \Omega \longrightarrow Y$ is called random (stochastic) fixed point of T if for p-almost all $w \in \Omega, y(w) \in C(w)$ and $T(w)y(w) = y(w)$ and for all open $D \subseteq Y, \{w \in \Omega : y(w) \in D\} \in F$($y$ is measurable).

Remark 1.6.1. *If $C(w) \equiv Y$, then the definition of random operator with stochastic domain coincides with the definition of random operator.*

Lemma 1.6.1. *[41] Let $C : \Omega \longrightarrow 2^Y$ be measurable with $C(w)$ closed, covex and solid (i.e., int $C(w) \neq \emptyset$) for all $w \in \Omega$. We assume that there exists measurable $y_0 : \Omega \longrightarrow Y$ with $y_0 \in$ int $C(w)$ for all $w \in \Omega$. Let T be* *a continuous random operator with stochastic domain C such that for every $w \in \Omega, \{y \in C(w) : T(w)y = y\} \neq \emptyset$. Then T has a stochastic fixed point.*

Let y be a mapping of $J \times \Omega$ into X. y is said to be a stochastic process if for each $t \in J$, the function $y(t, \cdot)$ is measurable.

1.7 Some fixed point theorems

Fixed point theory plays an important role in our existence results, therefore we state the following fixed point theorems.

Theorem 1.7.1. *(**Schauder fixed point**) [47]*
Let B be a closed, convex and nonempty subset of a Banach space E.
Let $N : B \to B$ be a continuous mapping such that $N(B)$ is a relatively compact subset of E. Then N has at least one fixed point in B.

Theorem 1.7.2. *(Mönch) [3, 72] Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv} N(V) \quad \text{or} \quad V = N(V) \cup 0 \Longrightarrow \alpha(V) = 0$$

holds for every subset V of D, then N has a fixed point.

Chapter 2

Controllability of First Order Functional Differential Equations with Delay and Random Effect

2.1 Introduction

Differential delay equations, and functional differential equations, have been used in modeling scientific phenomena for many years. Often, it has been assumed that the delay is either a fixed constant or is given as an integral in which case it is called a distributed delay [51,52,57,81,90]. In 1806 Poisson [78] published one of the first papers on functional differential equations and studied a geometric problem leading to an example with a state-dependent delay (see also [89]). An extensive theory is developed for evolution equations [4, 42]. Uniqueness and existence results have been established recently for different evolution problems in the papers by Baghli and Benchohra for finite and infinite delay in [7,8,9]. On the other hand, the nature of a dynamic system in engineering or natural sciences depends on the accuracy of the information we have concerning the parameters that describe that system. If the knowledge about a dynamic system is precise then a deterministic dynamical system arises. Unfortunately in most cases the available data for the description and evaluation of parameters of a dynamic system are inaccurate, imprecise or confusing. In other words, evaluation of parameters of a dynamical system is not without uncertainties. When our knowledge about the parameters of a dynamic system are of statistical nature, that is, the information is probabilistic, the common approach in mathematical modeling of such systems is the use of random differential equations or stochastic differential equations. Random differential equations, as natural extensions of deterministic ones, arise in many applications and have been investigated by many mathematicians.We refer the reader to the monographs [28, 88], the papers [40, 82, 37, 36] and the references therein. We also refer the reader to recent results [37, 36, 70, 71]. There are real world phenomena with anomalous dynamics such as signals transmissions through strong magnetic fields, atmospheric diffusion of pollution, network traffic, the effect of speculations on the profitability of stocks in financial markets and so on where the classical models are not sufficiently good to describe these features.

Control theory, on the other hand, is that branch of application-oriented mathematics that deals with the basic principles underlying the analysis and design of control systems. To control an object implies the influence of its behavior so as to accomplish a desired goal. Controllability of linear and nonlinear systems represented by ordinary differential equations in finite-dimensional spaces has been extensively investigated. The problem of controllability of linear systems represented by differential equations in Banach spaces has been extensively studied by several authors [60]. Several papers have appeared on finite dimensional controllability of linear systems [60] and infinite dimensional systems in abstract spaces [34]. Of late the controllability of nonlinear systems in finite-dimensional spaces is studied by means of fixed point principles [13]. Several authors have extended the concept of controllability to infinite-dimensional spaces by applying semigroup theory [31, 77, 92, 95]. Controllability of nonlinear systems with different types of nonlinearity has been studied by many authors with the help of fixed point principles [14]. Naito [75]. discussed the controllability of nonlinear Volterra integrodifferential systems and in [73, 74]. He studied the controllability of semilinear systems where as Yamamoto and Park [91]. investigated the same problem for a parabolic equation with a uniformly bounded nonlinear term. A standard approach is to transform the controllability problem into a fixed point problem for an appropriate operator in a function space. Most of the above mentioned works require the assumption of compactness of the semigroups. Balachandran and Kim [11] pointed out that controllability results are only true for ordinary differential systems in finite dimensional spaces if the corresponding semigroup is compact. However, controllability results may be true for abstract differential systems in infinite dimensional spaces if the compactness of the corresponding operator semigroup is dropped. In this chapter we study the controllability of the following functional differential equation with delay and random parameters of the form:

$$y'(t,w) = Ay(t,w) + f(t, y_t(\cdot,w), w) + Bu(t,w), \text{ a.e. } t \in J := [0,b] \quad (2.1)$$

$$y(t,w) = \phi(t,w), \quad t \in (-\infty, 0], \quad (2.2)$$

where $w \in \Omega$, $f : J \times \mathcal{B} \times \Omega \to E$, $\phi : (-\infty, 0] \times \Omega \to E$ are given random functions.
$A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t), t \in J$, of bounded linear operators in a Banach space E, $\mathcal{B}$ is the phase space as defined in Chapter 1, and $(E, |\cdot|)$ is a real Banach space. The control function $u(t,w)$ is given in $L^2(J,U)$, a Banach space of admissible control functions with U as a Banach space. B is a bounded linear operator from U into E. For any function y defined on $(-\infty, b] \times \Omega$ and any $t \in J$ we denote by $y_t(\cdot,w)$ the element of $\mathcal{B} \times \Omega$ defined by $y_t(\theta,w) = y(t+\theta,w)$,
$\theta \in (-\infty, 0]$. Here $y_t(\cdot,w)$ represents the history of the state from time $-\infty$, up to the present time t.

2.2 Controllability Result

In this section, we study the controllability for the differential system $(2.1) - (2.2)$.

Definition 2.2.1. *The system* $(2.1) - (2.2)$ *is said to be controllable on the interval* J*, if for every initial function* $\phi \in \mathcal{B}$ *and* $y^1(w)$ *in* E*, there exists a control* u *in* $L^2(J,U)$*, such that the solution* $y(t,w)$ *of* $(2.1) - (2.2)$ *satisfies* $y(b,w) = y^1(w)$.

Definition 2.2.2. *Let* y *be a mapping of* $J \times \Omega$ *into* E. y *is said to be a stochastic process if for each* $t \in J, y(t,\cdot)$ *is measurable.*

Now we give our main existence result for problem $(2.1)-(2.2)$. Before starting and proving this result, we give the definition of the mild random solution.

Definition 2.2.3. *A stochastic process* $y : (-\infty, b] \times \Omega \to E$ *is said to be a random mild solution of problem* $(2.1) - (2.2)$ *if* $y(t,w) = \phi(t,w)$, $t \in (-\infty, 0]$ *and the restriction of* $y(\cdot, w)$ *to the interval* J *is continuous and satisfies the following integral equation :*

$$y(t,w) = T(t)\phi(0,w) + \int_0^t T(t-s)f(s, y_s(\cdot,w), w)ds + \int_0^t T(t-s)Bu(t,w)ds$$

We will need to introduce the following hypotheses which are be assumed there after

(H_1) $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t), t \in J$ which is compact for $t > 0$ in the Banach space E. Let $M = \sup\{\|T\,(t)\,\|_{B(E)} : t \geq 0\}$.

(H_2) The function $f : J \times \mathcal{B} \times \Omega \to E$ is Carathéodory.

(H_3) There exist functions $\psi : J \times \Omega \to \mathbb{R}^+$ and $p : J \times \Omega \to \mathbb{R}^+$ such that for each $w \in \Omega$, $\psi(\cdot, w)$ is a continuous nondecreasing function and $p(\cdot, w)$ integrable with:

$$|f(t,u,w)| \leq p(t,w)\ \psi(\|u\|_{\mathcal{B}}, w) \text{ for a.e. } t \in J \text{ and each } u \in \mathcal{B},$$

(H_4) There exists a random function $R : \Omega \longrightarrow \mathbb{R}^+/\{0\}$ such that:

$$M\,(1 + bMK)\,(\|\phi\|_{\mathcal{B}} + \psi(D_b, w)\|p\|_{L^1}) + bMK\,\left|y^1\,(w)\right| \leq R(w)$$

where

$$D_b := K_b R(w) + M_b \|\phi\|_{\mathcal{B}}.$$

(H_5) The linear operator $W : L^2(J, U) \to E$ defined by:

$$Wu = \int_0^b T(b-s)Bu(s,w)ds$$

has an pseudo inverse operator W^{-1} which takes values in $L^2(J,U) \to kerW$ and there exists a positive constant K such that $\|BW^{-1}\| \leq K$.

(H_6) For each $w \in \Omega, \phi(\cdot, w)$ is continuous and for each $t, \phi(t, \cdot)$ is measurable.

Theorem 2.2.1. *Suppose that hypotheses* $(H_1)-(H_6)$ *are valid, then the problem* $(2.1)-(2.2)$ *is controllable on* J.

Proof. Using (H_5), we define the control :

$$u(t,w)=W^{-1}\left(y^1(w)-T(b)\phi(0,w)-\int_0^b T(b-s)f(s,y_s(\cdot,w),w)ds\right)(t,w)$$

We shall show that when using the control $u(t,w)$, the operator defined by:

$$\begin{aligned}&(\Psi(w)y)(t)\\ =\ & T(t)\phi(0,w)+\int_0^t T(t-s)f(s,y_s(\cdot,w),w)ds\\ &+\int_0^t T(t-s)BW^{-1}\\ &\left(y^1(w)-T(b)\phi(0,w)-\int_0^b T(b-\tau)f(\tau,y_\tau(\cdot,w),w)d\tau\right)ds\end{aligned}$$

has a fixed point $y(t,w)$. This fixed point is a mild solution of the system $(2.1)-(2.2)$ and this implies that the system is controllable on J.

Let $Y=\{u\in C(J,E):u(0,w)=\phi(0,w)=0\}$ endowed with the uniform convergence topology and $N:\Omega\times Y\to Y$ be the random operator defined by:

$$\begin{aligned}&(N(w)y)(t)\\ =\ & T(t)\ \phi(0,w)+\int_0^t T(t-s)\ f(s,\overline{y}_s(\cdot,w),w),w)ds\\ &+\int_0^t T(t-s)BW^{-1}\\ &\left(y^1(w)-T(b)\phi(0,w)-\int_0^b T(b-\tau)f(\tau,\bar{y}_\tau(\cdot,w),w)d\tau\right)ds \quad (2.3)\end{aligned}$$

where $\bar{y}:(-\infty,b]\times\Omega\to E$ is such that $\bar{y}_0(\cdot,w)=\phi(\cdot,w)$ and $\bar{y}(\cdot,w)=y(\cdot,w)$ on J. Let $\bar{\phi}:(-\infty,b]\times\Omega\to E$ be the extension of ϕ to $(-\infty,b]$ such that $\bar{\phi}(\theta,w)=\phi(0,w)=0$ on J. Then we show that the mapping defined by (2.3) is a random operator. To do this, we need to prove that for any $y \in Y$, $N(\cdot)(y) : \Omega \to Y$ is a random variable. Then we prove that $N(\cdot)(y) : \Omega \to Y$ is measurable since the mapping $f(t, y, \cdot), t \in J, y \in Y$ is measurable by assumption (H_2) and (H_6).

Let $D : \Omega \longrightarrow 2^Y$ be defined by:

$$D(w) = \{y \in Y : \|y\| \leq R(w)\}.$$

$D(w)$ is bounded, closed, convex and solid for all $w \in \Omega$. Then D is measurable by Lemma 17 (see [58]).

Let $w \in \Omega$ be fixed, then for any $y \in D(w)$, and by assumption (A_1), we get:

$$\begin{aligned}
\|\overline{y}_s\|_{\mathcal{B}} &\leq L(s)|\overline{y}(s)| + M(s)\|\overline{y}_0\|_{\mathcal{B}} \\
&\leq K_b|\overline{y}(s)| + M_b\|\phi\|_{\mathcal{B}}
\end{aligned}$$

and by (H_4) and (H_3), we have

$$\begin{aligned}
|(N(w)y)(t)| &\leq \|T(t)\||\phi(0,w)| + M\int_0^t |f(s,\overline{y}_s,w)|ds \\
&\quad + MK\int_0^t \left|y^1(w)\right| + \|T(b)\|\,|\phi(0,w)|\,ds \\
&\quad + MK\int_0^t\int_0^b \|T(\tau - s)\|\,|f(\tau,\overline{y}_\tau(\cdot,w),w)|\;d\tau ds \\
&\leq M\|\phi\|_{\mathcal{B}} + M\int_0^b p(s,w)\,\psi\left(\|\overline{y}_s\|_{\mathcal{B}},w\right)\,ds \\
&\quad + bMK\left|y^1(w)\right| + bM^2K\|\phi\|_{\mathcal{B}} \\
&\quad + bM^2K\int_0^b p(\tau,w)\,\psi\left(\|\overline{y}_\tau\|_{\mathcal{B}},w\right)\,d\tau. \\
&\leq M\left(1 + bMK\right)\|\phi\|_{\mathcal{B}} + bMK\left|y^1(w)\right| \\
&\quad + M\left(1 + bMK\right)\int_0^b p(s,w)\,\psi\left(\|\overline{y}_s\|_{\mathcal{B}},w\right)\,ds. \\
&\leq M\left(1 + bMK\right)\left(\|\phi\|_{\mathcal{B}} + \psi(D_b,w)\|p\|_{L^1}\right) + bMK\left|y^1(w)\right|.
\end{aligned}$$

Set

$$D_b := K_bR(w) + M_b\|\phi\|_{\mathcal{B}}.$$

Then, we have

$$|(N(w)y)(t)| \leq M\left(1+bMK\right)\left(\|\phi\|_{\mathcal{B}}+\psi(D_b,w)\|p\|_{L^1}\right)+bMK\left|y^1\left(w\right)\right|.$$

Thus

$$\|(N(w)y)\| \leq M\left(1+bMK\right)\left(\|\phi\|_{\mathcal{B}}+\psi(D_b,w)\|p\|_{L^1}\right)+bMK\left|y^1\left(w\right)\right| \leq R(w).$$

This implies that N is a random operator with stochastic domain D and $N(w) : D(w) \longrightarrow D(w)$ for each $w \in \Omega$.

Step 1: N is continuous.
Let y^n be a sequence such that $y^n \longrightarrow y$ in Y.Then

$$\begin{aligned}
& |(N(w)y^n)(t)-(N(w)y)(t)| \\
\leq\ & M\int_0^t |f(s,\overline{y}^n_s,w)-f(s,\overline{y}_s,w)|\,ds \\
& +kM\int_0^t\int_0^b \|T(b-\tau)\|\,|f(\tau,\overline{y}^n_\tau(\cdot,w)-f(\tau,\overline{y}_\tau,w)|d\tau ds \\
\leq\ & M\int_0^t |f(s,\overline{y}^n_s,w)-f(s,\overline{y}_s,w)|\,ds \\
& +bM^2K\int_0^b |f(\tau,\overline{y}^n_\tau(\cdot,w)-f(\tau,\overline{y}_\tau,w)|d\tau \\
\leq\ & M\left(1+bMK\right)\int_0^b |f(\tau,\overline{y}^n_\tau(\cdot,w)-f(\tau,\overline{y}_\tau,w)|d\tau
\end{aligned}$$

Since $f(s,\cdot,w)$ is continuous, we have by the Lebesgue dominated convergence theorem

$$\|f(\cdot,y^n_\cdot,w)-f(\cdot,y_\cdot,w)\|_{L^1}\to 0 \text{ as } n\to+\infty.$$

Thus N is continuous.

Step 2: We prove that for every $w \in \Omega$, $\{y \in D(w) : N(w)y = y\} \neq \emptyset$. For prove this we apply Schauder's theorem.

(a) N maps bounded sets into equicontinuous sets in $D(w)$.
Let $\tau_1, \tau_2 \in [0, b]$ with $\tau_2 > \tau_1$, $D(w)$ be a bounded set, and $y \in D(w)$. Then

$$
\begin{aligned}
& |(N(w)y)(\tau_2) - (N(w)y)(\tau_1)| \\
\leq\ & \|T(\tau_2) - T(\tau_1)\| \, \|\phi\|_{\mathcal{B}} \\
& + \int_0^{\tau_1} |[T(\tau_2 - s) - T(\tau_1 - s)] f(s, \overline{y}_s, w)\,| \, ds \\
& + \int_{\tau_1}^{\tau_2} |T(\tau_2 - s) f(s, \overline{y}_s, w)\,| \, ds \\
& + K \int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\| \left[\left| y^1(w) \right| + \|T(b)\| \, |\phi(0, w)| \right] ds \\
& + K \int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\| \\
& \times \int_0^b \|T(b - \tau)\| \, |f(\tau, \bar{y}_\tau(\cdot, w), w)| \, d\tau ds \\
& + K \int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\| \left[\left| y^1(w) \right| + \|T(b)\| \, |\phi(0, w)| \right] ds \\
& + K \int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\| \int_0^b \|T(b - \tau)\| \, |f(\tau, \bar{y}_\tau(\cdot, w), w)| \, d\tau ds \\
\leq\ & \|T(\tau_2) - T(\tau_1)\| \, \|\phi\|_{\mathcal{B}} \\
& + \psi(D_b, w) \int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\| \, p(s, w) ds \\
& + M\psi(D_b, w) \int_{\tau_1}^{\tau_2} p(s, w) ds \\
& + K \int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\| \left[\left| y^1(w) \right| + \|T(b)\| \, |\phi(0, w)| \right] ds \\
& + KM\psi(D_b, w) \int_0^{\tau_1} \left[\|T(\tau_2 - s) - T(\tau_1 - s)\| \int_0^b p(\tau, w) d\tau \right] ds \\
& + KM \\
& \times \int_{\tau_1}^{\tau_2} \left[\left| y^1(w) \right| + \|T(b)\| \, |\phi(0, w)| + M\psi(D_b, w) \int_0^b p(\tau, w) d\tau \right] ds
\end{aligned}
$$

The right-hand of the above inequality tends to zero as $\tau_2 - \tau_1 \to 0$, since $T(t)$ is uniformly continuous. As N is bounded and equicontinuous together with the Arzelà-Ascoli Theorem it suffices to show that the operator N maps $D(w)$ into a precompact set in E.

(b) Let $t \in [0, b]$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$.

For $y \in D(w)$ we define:

$$\begin{aligned} & (N_\epsilon(w)y)(t) \\ = \; & T(t)\phi(0,w) + T(\epsilon)\int_0^{t-\epsilon} T(t-s-\epsilon)f(s,y_s(\cdot,w),w)ds \\ & +T(\epsilon)\int_0^{t-\epsilon} T(t-s-\epsilon)BW^{-1} \\ & \left(y^1(w) - T(b)\phi(0,w) - \int_0^b T(b-s)f(\tau,y_\tau(\cdot,w),w)d\tau\right) ds. \end{aligned}$$

Since $T(t)$ is a compact operator, the set
$Z_\epsilon(t,w) = \{(N_\epsilon(w)y)(t) : y \in D(w)\}$ is pre-compact in E for every ϵ, $0 < \epsilon < t$. Moreover

$$\begin{aligned} & |(N(w)y)(t) - (N_\epsilon(w)y)(t)| \\ \leq \; & \int_{t-\epsilon}^t \||T(t-s)\|| \, |f(s,y_s,w)|ds \\ & + \; MK\int_{t-\epsilon}^t \left(\left|y^1(w)\right| + |T(b)| \, \|\phi\|_B + M\int_0^b |f(\tau,y_\tau(\cdot,w),w)| \, d\tau\right) ds \\ \leq \; & M\psi(D_b,w)\int_{t-\epsilon}^t p(s,w)ds \\ & +MK\int_{t-\epsilon}^t \left(\left|y^1(w)\right| + M\|\phi\|_B\right) ds \\ & +M^2K\psi(D_b,w)\int_{t-\epsilon}^t\int_0^b p(\tau,w)d\tau ds \\ \leq \; & M\psi(D_b,w)\int_{t-\epsilon}^t p(s,w)ds + MK\epsilon\left(\left|y^1(w)\right| + M\|\phi\|_B\right) \\ & +M^2K\epsilon\psi(D_b,w)\int_0^b p(\tau,w)d\tau \end{aligned}$$

Therefore the set $Z(t,w) = \{(N(w)y)(t) : y \in D(w)\}$ is precompact in E. A consequence of Steps $1-2$ and $(a),(b)$, we can conclude that $N(w) : D(w) \to D(w)$ is continuous and compact. From Schauder's Theorem, we deduce that $N(w)$ has a fixed point $y(w)$ in $D(w)$.
Since $\bigcap_{w\in\Omega} D(w) \neq \emptyset$, the hypothesis that a measurable selector of int$D$

exists holds. By Lemma 1.6.1, the random operator N has a stochastic fixed point $y^*(w)$, which is a random mild solution of the random problem $(1)-(2)$

2.3 An example

Consider the following functional partial differential equation:

$$\begin{aligned}\frac{\partial}{\partial t}z(t,x,w) &= \frac{\partial^2}{\partial x^2}z(t,x,w)+C_0(w)K(w)e^{-t}\int_{-\infty}^{0}\frac{\exp(z(t+s,x,w))}{1+s^2}ds\\ +Bu\,(t,w)\quad x &\in [0,\pi],\ t\in[0,b]\end{aligned} \tag{2.4}$$

$$z(t,0,w) = z(t,\pi,w)=0,\ t\in[0,b] \tag{2.5}$$

$$z(s,x,w) = z_0(s,x,w),\ s\in(-\infty,0],\ x\in[0,\pi] \tag{2.6}$$

where K and C_0 are a real-valued random variable.
Suppose that $E=L^2[0,\pi]$, (Ω,F,P) is a complete probability space. Take and define $A:E\to E$ by $Av=v''$ with domain

$$D(A)=\{v\in E, v, v'\text{are absolutely continuous},\ v''\in E,\ v(0)=v(\pi)=0\}.$$

Then

$$Av=\sum_{n=1}^{\infty}n^2(v,v_n)v_n, v\in D(A)$$

where $\omega_n(s)=\sqrt{\frac{2}{\pi}}\sin ns, n=1,2,\ldots$ is the orthogonal set of eigenvectors in A. It is well know (see [77]) that A is the infinitesimal generator of an analytic semigroup $T(t), t\geq 0$ in E and is given by

$$T(t)v=\sum_{n=1}^{\infty}\exp(-n^2t)(v,v_n)v_n, v\in E.$$

Since the analytic semigroup $T(t)$ is compact, there exists a positive constant M such that

$$\|T(t)\|_{B(E)}\leq M.$$

Let $\mathcal{B}=BUC(\mathbb{R}^-;E)$: the space of uniformly bounded continuous functions endowed with the following norm:

$$\|\phi\| = \sup_{s\leq 0} |\phi(s)|, \qquad \text{for } \phi \in \mathcal{B},$$

If we put $\phi \in BUC(\mathbb{R}^-; E), x \in [0, \pi]$ and $w \in \Omega$

$$y(t, x, w) = z(t, x, w), t \in [0, b]$$

$$\phi(s, x, w) = z_0(s, x, w),\ s \in (-\infty, 0],\ x \in [0, \pi], w \in \Omega$$

Set

$$f(t, \phi(x), w) = C_0(w)K(w)e^{-t} \int_{-\infty}^{0} \frac{\exp(z(t+s, x, w))}{1+s^2} ds,$$

with

$$\phi(s, x, w) = \exp(z(t+s, x, w)).$$

The function $f(t, \phi(x), w)$ is Carathéodory, and it satisfies (H_2) with

$$p(t; w) = |K(w)| \frac{\pi}{2} e^{-t} \text{ and } \phi(x; w) = |C_0(w)|\, e^x$$

The problem $(2.1) - (2.2)$ is an abstract formulation of the problem $(2.4) - (2.6)$, and conditions $(H_1) - (H_6)$ are satisfied. Theorem 2.4 implies that the random problem $(2.4) - (2.6)$ is controlable.

Chapter 3

Controllability of First Order Functional Differential Equations with State-Dependent Delay and Random Effects

3.1 Introduction

Controllability has played a central role throughout the history of modern control theory. This is the qualitative property of control systems and is of particular importance in control theory. Controllability generally means that it is possible to steer dynamical system from an arbitrary initial state to the desired final state using the set of admissible controls.Controllability of linear and nonlinear systems represented by ordinary differential equations in finite-dimensional space has been extensively studied. Several authors have extended the concept to infinite-dimensional systems in Banach spaces with bounded operators see [75, 76, 87, 65]. Quinn and Carmichael [79] have shown that the controllability problem can be converted into a fixed point problem [44, 45, 46, 62]. Balachandran and Dauer have considered various classes of first and second order semilinear ordinary functional and neutral functional differential equations on Banach spaces in [35]. By means of fixed point arguments, Benchohra *et al* have studied many classes of functional differential equations and inclusions and proposed some controllability results

in [6,20,24,22,23,26]. An extensive theory is developed for evolution equations [1,4].
In this chapter we study the controllability of mild solutions of the following functional differential equation with delay and random effects (random parameters) of the form:

$$y'(t,w) = Ay(t,w) + f(t, y_{\rho(t,y_t)}(\cdot,w), w) + Bu\,(t,w)\,, \text{ a.e. } t \in J := [0,b] \quad (3.1)$$

$$y(t,w) = \phi(t,w), \quad t \in (-\infty, 0], \quad (3.2)$$

where $w \in \Omega$, $f : J \times \mathcal{B} \times \Omega \to E$, $\phi : (-\infty,0] \times \Omega \to E$ are given random functions, $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t),\ t \in J$, of bounded linear operators in a Banach space E, $\rho : J \times \mathcal{B} \to \mathbb{R}$, and $(E, |.|)$ is a real Banach space. The control function $u(t,w)$ is given in $L^2(J,U)$, a Banach space of admissible control functions with U as a Banach space. B is a bounded linear operator from U into E. For any function y defined on $(-\infty,b] \times \Omega$ and any $t \in J$ we denote by $y_t(\cdot,w)$ the element of $\mathcal{B} \times \Omega$ defined by $y_t(\theta,w) = y(t+\theta,w), \theta \in (-\infty,0]$. Here $y_t(\cdot,w)$ represents the history of the state from time $-\infty$, up to the present time t.

3.2 Controllability result

In this section, we study the controllability results for the differential system (3.1)-(3.2). Before starting and proving this result, we give the definition of controllability and the mild random solution.

Definition 3.2.1. *The System (3.1)-(3.2) is said to be controllable on the interval J, if for every initial function $\phi \in \mathcal{B}$ and $y^1(w)$ in E, there exists a control $u(\cdot,w)$ in $L^2(J,U)$ such that the solution $y(t,w)$ of (3.1)-(3.2) satisfies $y(b,w) = y^1(w)$*

Let y be a mapping of $J \times \Omega$ into E. y is said to be a stochastic process if for each $t \in J, y(t,\cdot)$ is measurable.

Definition 3.2.2. *A stochastic process $y : (-\infty,b] \times \Omega \to E$ is said to be random mild solution of problem (3.1)-(3.2) if $y(t,w) = \phi(t,w),\ t \in (-\infty,0]$ and the restriction of $y(\cdot,w)$ to the interval $[0,b]$ is continuous and satisfies* *the following integral equation:*

$$y(t,w) = T(t)\phi(0,w) + \int_0^t T(t-s)f(s, y_{\rho(s,y_s)}(\cdot,w),w)ds + \int_0^t T(t-s)Bu(t,w)ds. \tag{3.3}$$

Set

$$\mathcal{R}(\rho^-) = \{\rho(s,\varphi) : (s,\varphi) \in J \times \mathcal{B}, \rho(s,\varphi) \leq 0\}.$$

We always assume that $\rho : J \times \mathcal{B} \to (-\infty, b]$ is continuous. Additionally, we introduce following hypothesis:

(H_ϕ) The function $t \to \phi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $L^\phi : \mathcal{R}(\rho^-) \to (0,\infty)$ such that

$$\|\phi_t\|_{\mathcal{B}} \leq L^\phi(t)\|\phi\|_{\mathcal{B}} \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

Remark 3.2.1. The condition (H_ϕ), is frequently verified by functions continuous and bounded. For more details, see for instance [57].

Lemma 3.2.1. *([56], Lemma 2.4) If $y : (-\infty, b] \to E$ is a function such that $y_0 = \phi$, then*

$$\|y_s\|_{\mathcal{B}} \leq (M_b + L^\phi)\|\phi\|_{\mathcal{B}} + K_b \sup\{|y(\theta)|; \theta \in [0, max\{0,s\}]\},\ s \in \mathcal{R}(\rho^-) \cup J,$$

where $L^\phi = \sup\limits_{t \in \mathcal{R}(\rho^-)} L^\phi(t)$.

We will need to introduce the following hypotheses which are assumed there after:

(H_1) $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t), t \in J$ which is compact for $t > 0$. Let $M = \sup\{\|T(t)\|_{B(E)} : t \geq 0\}$,

(H_2) The function $f : J \times \mathcal{B} \times \Omega \to E$ is random Carathéodory,

(H_3) There exist functions $\psi : J \times \Omega \to \mathbb{R}^+$ and $p : J \times \Omega \to \mathbb{R}^+$ such that for each $w \in \Omega$, $\psi(\cdot,w)$ is a continuous nondecreasing function and $p(\cdot,w)$ integrable with:

$$|f(t,u,w)| \leq p(t,w)\ \psi(\|u\|_{\mathcal{B}}, w) \text{ for a.e. } t \in J \text{ and each } u \in \mathcal{B},$$

(H_4) There exists a function $l : J \times \Omega \longrightarrow \mathbb{R}^+$ with $l(\cdot, w) \in L^1(J, \mathbb{R}^+)$ for each $w \in \Omega$ such that for any bounded $B \subseteq E$.

$$\alpha(f(t, B, w)) \leq l(t, w)\alpha(B),$$

(H_5) There exists a random function $R : \Omega \longrightarrow \mathbb{R}^+ \backslash \{0\}$ such that:

$$M\left(1 + bMC\right)\left(\|\phi\|_{\mathcal{B}} + \psi\left((M_b + L^{\phi})\|\phi\|_{\mathcal{B}} + K_b R(w), w\right)\int_0^b p(s, w)ds\right)$$

$$+ bMC\left|y^1\left(w\right)\right| \leq R(w),$$

(H_6) The linear operator $W : L^2(J, U) \to E$ defined by:

$$Wu = \int_0^b T(b - s)Bu(s, w)ds$$

has an pseudo inverse operator W^{-1} which takes values in $L^2(J, U) \to kerW$ and there exists a positive constant C such that $\|BW^{-1}\| \leq C$,

(H_7) For each $w \in \Omega, \phi(\cdot, w)$ is continuous and for each $t, \phi(t, \cdot)$ is measurable.

Theorem 3.2.1. *Suppose that the hypotheses (H_ϕ) and $(H_1) - (H_7)$ hold with*

$$M\left(1 + MCb\right)\int_0^b l(s)K(s)ds < 1 \tag{3.4}$$

then the random problem (3.1)-(3.2) is controllable on J.

Proof. Using (H_6), we define the control :

$$u(t, w) = W^{-1}\left(y^1\left(w\right) - T(b)\phi(0, w) - \int_0^b T(b - s)f(s, y_{\rho(s, y_s)}(\cdot, w), w)ds\right)$$

We shall show that when using the control $u(t, w)$, the operator defined by:

$$\begin{aligned}
& (\Psi(w)y)(t) \\
= \;& T(t)\phi(0,w) + \int_0^t T(t-s)f(s, y_{\rho(s,y_s)}(\cdot,w), w)ds \\
& + \int_0^t T(t-s)BW^{-1} \\
& \times \left(y^1(w) - T(b)\phi(0,w) - \int_0^b T(b-\tau)f(\tau, y_{\rho(\tau,y_\tau)}(\cdot,w), w)d\tau \right) ds
\end{aligned}$$

has a fixed point. $y(t,w)$. This fixed point is a mild solution of the system (3.1)-(3.2) and this implies that the system is controllable on J.

Let $Y = \{u \in C(J,E) : u(0,w) = \phi(0,w) = 0\}$ endowed with the uniform convergence topology and $N : \Omega \times Y \to Y$ be the random operator defined by:

$$\begin{aligned}
& (N(w)y)(t) \\
= \;& T(t)\, \phi(0,w) + \int_0^t T(t-s)\; f(s, \overline{y}_{\rho(s,\overline{y}_s)}(\cdot,w), w), w)ds \\
& + \int_0^t T(t-s)BW^{-1} \\
& \times \left(y^1(w) - T(b)\phi(0,w) - \int_0^b T(b-\tau)f(\tau, \bar{y}_{\rho(\tau,\bar{y}_\tau)}(\cdot,w), w)d\tau \right) ds
\end{aligned} \tag{3.5}$$

where $\bar{y} : (-\infty, b] \times \Omega \to E$ is such that $\bar{y}_0(\cdot,w) = \phi(\cdot,w)$ and $\bar{y}(\cdot,w) = y(\cdot,w)$ on J. Let $\bar{\phi} : (-\infty,b] \times \Omega \to E$ be the extension of ϕ to $(-\infty,b]$ such that $\bar{\phi}(\theta,w) = \phi(0,w) = 0$ on J. Then we show that the mapping defined by (3.6) is a random operator. To do this, we need to prove that for any $y \in Y$, $N(\cdot)(y) : \Omega \longrightarrow Y$ is a random variable. Then we prove that $N(\cdot)(y) : \Omega \longrightarrow Y$ is measurable. as a mapping $f(t,y,\cdot),\ t \in J,\ y \in Y$ is measurable by assumptions (H_2) and (H_7).

Let $D : \Omega \longrightarrow 2^Y$ be defined by:

$$D(w) = \{y \in Y : \|y\| \leq R(w)\}.$$

The set $D(w)$ bounded, closed, convex and solid for all $w \in \Omega$. Then D is measurable by Lemma 17 in [58].

Let $w \in \Omega$ be fixed. If $y \in D(w)$, from Lemma 3.7 it follows that

$$\|\bar{y}_{\rho(t,\bar{y}_t)}\|_{\mathcal{B}} \leq (M_b + L^{\phi})\|\phi\|_{\mathcal{B}} + K_b R(w)$$

and for each $y \in D(w)$, by (H_3), (H_5) and (H_6), we have for each $t \in J$

$$\begin{aligned}
|(N(w)y)(t)| &\leq M\|\phi\|_{\mathcal{B}} + M\int_0^t |f(s,\overline{y}_{\rho(s,\overline{y}_s)},w)|ds \\
&\quad + MC\int_0^t \left|y^1(w)\right| + \|T(b)\|\,|\phi(0,w)|\,ds \\
&\quad + MC\int_0^t\int_0^b \|T(\tau-s)\|\left|f(\tau,\overline{y}_{\rho(\tau,y_\tau)}(\cdot,w),w)\right|\,d\tau ds \\
&\leq M\|\phi\|_{\mathcal{B}} + M\int_0^t p(s,w)\,\psi\left(\|\overline{y}_{\rho(s,\overline{y}_s)}\|_{\mathcal{B}},w\right)ds \\
&\quad + bMC\left|y^1(w)\right| + bM^2C\|\phi\|_{\mathcal{B}} \\
&\quad + bM^2C\int_0^b p(\tau,w)\,\psi\left(\|\overline{y}_{\rho(\tau,\overline{y}_\tau)}\|_{\mathcal{B}},w\right)\,d\tau \\
&\leq M\|\phi\|_{\mathcal{B}} \\
&\quad + M\int_0^t p(s,w)\,\psi\left((M_b+L^{\phi})\|\phi\|_{\mathcal{B}} + K_bR(w),w\right)ds \\
&\quad + bMC\left\|y^1(w)\right\| + bM^2C\|\phi\|_{\mathcal{B}} \\
&\quad + bM^2C\int_0^b p(\tau,w)\,\psi\left((M_b+L^{\phi})\|\phi\|_{\mathcal{B}} + K_bR(w),w\right)\,d\tau \\
&\leq M\|\phi\|_{\mathcal{B}} \\
&\quad + M\,\psi\left((M_b+L^{\phi})\|\phi\|_{\mathcal{B}} + K_bR(w),w\right)\int_0^b p(s,w)ds \\
&\quad + bMC\left|y^1(w)\right| + bM^2C\|\phi\|_{\mathcal{B}} \\
&\quad + bM^2C\,\psi\left((M_b+L^{\phi})\|\phi\|_{\mathcal{B}} + K_bR(w),w\right)\int_0^b p(s,w)ds
\end{aligned}$$

$$\begin{aligned}
&\leq M\left(1+bMC\right) \\
&\quad \times\left(\|\phi\|_{\mathcal{B}} + \psi\left((M_b+L^{\phi})\|\phi\|_{\mathcal{B}} + K_bR(w),w\right)\|p\|_{L^1}\right) \\
&\quad + bMC\left|y^1(w)\right| \\
&\leq R(w).
\end{aligned}$$

This implies that N is a random operator with stochastic domain D and $N(w) : D(w) \to D(w)$ for each $w \in \Omega$.

Step 1: N is continuous.
Let y^n be a sequence such that $y^n \longrightarrow y$ in Y. Then

$$
\begin{aligned}
& |(N(w)y^n)(t) - (N(w)y)(t)| \\
\leq\ & M \int_0^t |f(s, \overline{y}^n_{\rho(s,\overline{y^n}_s)}, w) - f(s, \overline{y}_{\rho(s,\overline{y}_s)}, w)|\, ds \\
& + CM \int_0^t \int_0^b \|T(b-\tau)\|\, |f(\tau, \overline{y}^n_{\rho(s,\overline{y^n}_s)}(\cdot, w) - f(\tau, \overline{y}_{\rho(s,\overline{y}_s)}, w)| d\tau ds \\
\leq\ & M \int_0^t |f(s, \overline{y}^n_{\rho(s,\overline{y^n}_s)}, w) - f(s, \overline{y}_{\rho(s,\overline{y}_s)}, w)|\, ds \\
& + bM^2C \int_0^b |f(\tau, \overline{y}^n_{\rho(s,\overline{y^n}_s)}(\cdot, w) - f(\tau, \overline{y}_{\rho(s,\overline{y}_s)}, w)| d\tau \\
\leq\ & M\,(1 + bMC) \int_0^b |f(\tau, \overline{y}^n_{\rho(s,\overline{y^n}_s)}(\cdot, w) - f(\tau, \overline{y}_{\rho(s,\overline{y}_s)}, w)| d\tau
\end{aligned}
$$

Since $f(s, \cdot, w)$ is continuous, we have by the Lebesgue dominated convergence theorem

$$|(N(w)y^n)(t) - (N(w)y)(t)| \to 0 \text{ as } n \to +\infty.$$

Thus N is continuous.
Step 2: We prove that for every $w \in \Omega, \{y \in D(w) : N(w)y = y\} \neq \emptyset$. For this we apply the Mönch fixed point theorem.

(a) N maps bounded sets into equicontinuous sets in $D(w)$.
Let $\tau_1, \tau_2 \in [0, b]$ with $\tau_2 > \tau_1$, $D(w)$ be a bounded set as in Step 2, and $y \in D(w)$. Then

$$
\begin{aligned}
& |(N(w)y)(\tau_2) - (N(w)y)(\tau_1)| \\
\leq\ & \|T(\tau_2) - T(\tau_1)\| \, \|\phi\|_{\mathcal{B}} + \left| \int_0^{\tau_1} [T(\tau_2 - s) - T(\tau_1 - s)] f(s, \overline{y}_{\rho(s,\overline{y}_s)}, w) ds \right| \\
& + \left| \int_{\tau_1}^{\tau_2} T(\tau_2 - s) f(s, \overline{y}_{\rho(s,\overline{y}_s)}, w) ds \right| \\
& + C \int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\| \left[\left| y^1(w) \right| + \|T(b)\| \, |\phi(0,w)| \right] ds \\
& + C \int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\| \int_0^b \|T(b - \tau)\| \left| f(\tau, \bar{y}_{\rho(s,\overline{y}_s)}, w) \right| d\tau ds \\
& + C \int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\| \left[\left| y^1(w) \right| + \|T(b)\| \, |\phi(0,w)| \right] ds \\
& + C \int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\| \int_0^b \|T(b - \tau)\| \left| f(\tau, \bar{y}_{\rho(\tau,\overline{y}_\tau)}, w) \right| d\tau ds \\
\leq\ & \|T(\tau_2) - T(\tau_1)\| \, \|\phi\|_{\mathcal{B}} \\
& + \int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\| \, |f(s, \overline{y}_{\rho(s,\overline{y}_s)}, w)| \, ds \\
& + \int_{\tau_1}^{\tau_2} |T(\tau_2 - s) f(s, \overline{y}_{\rho(s,\overline{y}_s)}, w)| \, ds \\
& + C \int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\| \left[\left| y^1(w) \right| + \|T(b)\| \, |\phi(0,w)| \right] ds \\
& + C \int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\| \, \psi \left((M_b + L^\phi) \|\phi\|_{\mathcal{B}} + K_b R(w) \right) \\
& \times \int_0^b p(\tau, w) d\tau ds \\
& + CM \int_{\tau_1}^{\tau_2} \left| y^1(w) \right| + \|T(b)\| \, |\phi(0,w)| \\
& + M\psi \left((M_b + L^\phi) \|\phi\|_{\mathcal{B}} + K_b R(w) \right) \int_0^b p(\tau, w) d\tau ds \\
\leq\ & |T(\tau_2) - T(\tau_1)| \|\phi\|_{\mathcal{B}} + \psi \left((M_T + L^\phi) \|\phi\|_{\mathcal{B}} + K_T R(w) \right) \\
& \times \int_0^{\tau_1} |T(\tau_2 - s) - T(\tau_1 - s)| p(s,w) ds + M\psi \left((M_b + L^\phi) \|\phi\|_{\mathcal{B}} + K_b R(w), w \right) \\
& \times \int_{\tau_1}^{\tau_2} p(s,w) ds. \\
& + C \int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\| \left[\left| y^1(w) \right| + \|T(b)\| \, |\phi(0,w)| \right] ds \\
& + \int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\| \, \psi \left((M_b + L^\phi) \|\phi\|_{\mathcal{B}} + K_b R(w) \right) \\
& \times \int_0^b p(\tau, w) d\tau ds \\
& + CM \int_{\tau_1}^{\tau_2} \left| y^1(w) \right| + \|T(b)\| \, |\phi(0,w)| + M\psi \left((M_b + L^\phi) \|\phi\|_{\mathcal{B}} + K_b R(w) \right)
\end{aligned}
$$

The right-hand of the above inequality tends to zero as $\tau_2 - \tau_1 \to 0$, since $T(t)$ is uniformly continuous.

Next, let $w \in \Omega$ be fixed (therefore we do not write $'w'$ in the sequel) but arbitrary.

(b) Now let V be a subset of $D(w)$ such that $V \subset \overline{conv}\,(N(V) \cup \{0\})$. V is bounded and equicontinuous and therefore the function $t \to v(t) = \alpha(V(t))$ is continuous on $(-\infty, b]$.

Let us denote by

$$V(t) = \{h(t) : h \in V\}$$

and

$$V(J) = \{h(t) : h \in V,\ t \in J\}$$

By (H_4), Lemma 1.9, Theorem 1.14 and the properties of the measure α we have for each $t \in (-\infty, b]$

$$\begin{aligned}
v(t) &\leq \alpha\,(N(V))\,(t) \cup \{0\}) \\
&\leq \alpha\,(N(V(t)) \\
&\leq \alpha\Big(T(t)\,\phi(0) + \int_0^t T(t-s)\,f(s, \overline{y}_{\rho(s,\overline{y}_s)})\,ds + \int_0^t T(t-s)BW^{-1} \\
&\quad \times \left(y^1\,(w) - T(b)\phi(0,w) - \int_0^b T(b-\tau)f(\tau, \overline{y}_{\rho(\tau,\overline{y}_\tau)}, w)d\tau\right) ds\Big) \\
&\leq \alpha\Big(T(t)\,\phi(0)\Big) + \alpha\Big(\int_0^t T(t-s)\,f(s, \overline{y}_{\rho(s,\overline{y}_s)})\,ds\Big) \\
&\quad + \alpha\Big(\int_0^t T(t-s)BW^{-1} \\
&\quad \left(y^1\,(w) - T(b)\phi(0,w) - \int_0^b T(b-\tau)f(\tau, \overline{y}_{\rho(\tau,\overline{y}_\tau)}, w)d\tau\right) ds\Big) \\
&\leq M\int_0^t l(s)\alpha(\{\overline{y}_{\rho(s,\overline{y}_s)} : \overline{y} \in V\})ds + MC \\
&\quad \times \int_0^t \alpha\left(\left(y^1\,(w) - T(b)\phi(0,w) - \int_0^b T(b-\tau)f(\tau, \overline{y}_{\rho(\tau,\overline{y}_\tau)}, w)d\tau\right)\right) ds
\end{aligned}$$

$$\begin{aligned}
&\leq M\int_0^t l(s)K(s)\sup_{0\leq\tau\leq s}\alpha(V(\tau))ds\\
&\quad +MC\int_0^t (\alpha\left(y^1(w)-T(b)\phi(0,w)\right)\\
&\quad +\int_0^b \alpha\left(T(b-\tau)f(\tau,\bar{y}_{\rho(\tau,\bar{y}_\tau)},w)\right)d\tau)ds\\
&\leq M\int_0^t l(s)K(s)\sup_{0\leq\tau\leq s}\alpha(V(\tau))ds\\
&\quad +MC\int_0^t\int_0^b \alpha\left(T(b-\tau)f(\tau,\bar{y}_{\rho(\tau,\bar{y}_\tau)},w)\right)d\tau ds\\
&\leq M\int_0^t l(s)K(s)\alpha(V(s))ds\\
&\quad +M^2Cb\int_0^b l(\tau)\alpha(\{\overline{y}_{\rho(\tau,\overline{y}_\tau)}:\overline{y}\in V\})d\tau\\
&\leq M\int_0^t v(s)\,l(s)K(s)ds+M^2Cb\int_0^b l(\tau)K(\tau)\alpha(V(\tau))d\tau\\
&\leq M\int_0^b l(s)K(s)v(s)ds+M^2Cb\int_0^b l(\tau)K(\tau)v(\tau)d\tau.\\
&\leq M(1+MCb)\int_0^b l(s)K(s)v(s)ds.\\
&\leq M(1+MCb)\int_0^b l(s)K(s)\sup_{0\leq\tau\leq s}v(\tau)ds\\
&\leq M(1+MCb)\|v\|_\infty\int_0^b l(s)K(s)ds
\end{aligned}$$

$$\|v\|_\infty\leq M(1+MCb)\|v\|_\infty\int_0^b l(s)K(s)ds$$

Then

$$\|v\|_\infty(1-M(1+MCb)\int_0^b l(s)K(s)ds)\leq 0$$

By (3.4) it follows that $\|v\|_\infty=0$, this implies that $v(t)=0$ for each $t\in J$, and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzelà Theorem, V is relatively compact in $D(w)$. Applying now Theorem 3.8 we conclude that N has a fixed point $y(w)\in D(w)$. Since $\bigcap_{w\in\Omega}D(w)\neq\emptyset$, and a measurable selector of int$D$ exists, Lemma 1.9 implies that the random operator N has a stochastic fixed point $y^*(w)$, which is a mild solution of the random problem (3.1)-(3.2.

3.3 An example

Consider the following functional partial differential equation:

$$\begin{aligned}\frac{\partial}{\partial t}z(t,x,w) &= \frac{\partial^2}{\partial x^2}z(t,x,w) + C_0(w)b(t)\\ &\quad \times \int_{-\infty}^{0} F(z(t+\sigma(t,z(t+s,x,w)),x,w))ds + Bu\,(t,w)\\ x &\in [0,\pi],\ t\in[0,b],\ w\in\Omega \end{aligned} \tag{3.6}$$

$$z(t,0,w) = z(t,\pi,w) = 0,\ t\in[0,b], w\in\Omega \tag{3.7}$$

$$z(s,x,w) = z_0(s,x,w),\ s\in(-\infty,0],\ x\in[0,\pi],\ w\in\Omega, \tag{3.8}$$

where C_0 are a real-valued random variable, $b \in L^1(J;\mathbb{R}_+)$, $F:\mathbb{R}\to\mathbb{R}$ is continuous, $z_0 : (-\infty,0]\times[0,\pi]\times\Omega\to\mathbb{R}$ and $\sigma : J\times\mathbb{R}\to\mathbb{R}$ are given functions.

Suppose that $E = L^2[0,\pi]$. Take and define $A : E\to E$ by $Av = v''$ with domain

$$D(A) = \{v\in E, v, v'\text{are absolutely continuous},\ v''\in E,\ v(0) = v(\pi) = 0\}.$$

Then

$$Av = \sum_{n=1}^{\infty} n^2(v,v_n)v_n, v\in D(A)$$

where $v_n(s) = \sqrt{\frac{2}{\pi}}\sin ns, n = 1,2,\dots$ is the orthogonal set of eigenvectors in A. It is well know (see [77]) that A is the infinitesimal generator of an analytic semigroup $T(t), t\geq 0$ in E and is given by

$$T(t)v = \sum_{n=1}^{\infty}\exp(-n^2t)(v,v_n)v_n, v\in E.$$

Since the analytic semigroup $T(t)$ is compact, there exists a positive constant M such that

$$\|T(t)\|_{B(E)} \leq M.$$

Let $\mathcal{B} = BUC(\mathbb{R}^-; E)$: the space of uniformly bounded continuous functions endowed with the following norm:

$$\|\phi\| = \sup_{s\leq 0} |\phi(s)|, \qquad \text{for } \phi \in \mathcal{B}.$$

If we put $\phi \in BUC(\mathbb{R}^-; E), x \in [0, \pi]$ and $w \in \Omega$

$$y(t, x, w) = z(t, x, w),\ t \in [0, b]$$
$$\phi(s, x, w) = z_0(s, x, w),\ s \in (-\infty, 0],\ x \in [0, \pi],\ w \in \Omega.$$

Set

$$f(t, \phi(x), w) = C_0(w)b(t) \int_{-\infty}^{0} F(z(t + \sigma(t, z(t + s, x, w)), x, w)),$$

and

$$\rho(t, \phi)(x) = \sigma(t, z(t, x, w)).$$

Let $\phi \in \mathcal{B}$ be such that (H_ϕ) holds, and let $t \to \phi_t$ be continuous on $\mathcal{R}(\rho^-)$, and let f satisfied the conditions $(H_3), (H_4), (H_5)$. Assume that B is a bounded linear operator from U into E and the linear operator $W : L^2(J, U) \to E$ defined by:

$$Wu = \int_0^b T(b - s)Bu(s, w)ds$$

has an inverse operator W^{-1} which takes values in $L^2(J, U) \to kerW$.

Then the problem (3.1)-(3.2) is an abstract formulation of the problem $(3.7) - (3.9)$, and conditions $(H_1) - (H_7)$ are satisfied. Theorem 3.8 implies that the random problem $(3.7) - (3.9)$ is controllable.

Chapter 4

Controllability of Second Order Functional Differential Equations with Delay and Random Effect

4.1 Introduction

The problems of existence, uniqueness and other properties of solutions for the second order systems have much attention in the recent year. In many cases, it is advantageous to treat the second-order abstract differential equations directly rather than to convert them to first-order systems. A useful tool for the study of second-order equations is the theory of strongly continuous cosine families. We will make use of some of the basic ideas from cosine family theory [85]. Motivation for the second-order systems can be found in [12,53], when authors established sufficient conditions for controllability of second-order systems in Banach spaces for deterministic and stochastic systems using different fixed point theorems and strongly continuous cosine family with nonlinearity satisfying Lipschitz condition.

The main results are based upon Schauder's fixed theorem and random fixed point theorem combined with the family of cosine operators. The cosine function theory is related to abstract linear second order differential equations in the same manner that the semigroup theory of bounded linear operators is related to first order partial differential equations and it's equally appealing devoted their generality and simplicity. For basic concepts and applications of this theory, we refer to the reader to Fattorini [43], Travis and Weeb [85].

4.2 The constant delay case

4.2.1 Introduction

In this section, we study the controllability of the following functional differential equation with delay and random effect of the form:

$$y^{''}(t,w) = Ay(t,w) + f(t,y_t(\cdot,w),w) + Bu(t,w), \text{ a.e. } t \in J := [0,b], \quad (4.1)$$

$$y(t,w) = \phi(t,w);\ t \in (-\infty,0], y^{'}(0,w) = \varphi(w), w \in \Omega, \quad (4.2)$$

where $w \in \Omega$, $f : J \times \mathcal{B} \times \Omega \to E$, $\phi : (-\infty,0] \times \Omega \to E$ are given random functions,
$A : D(A) \subset E \to E$ is the infinitesimal generator of of a strongly continuous cosine family of bounded linear operators $(C(t))_{t\in\mathbb{R}}$ on E, and $(E, |\cdot|)$ is a real Banach space.The control function $u(\cdot,w)$ is given in $L^2(J,U)$, a Banach space of admissible control functions with U as a Banach space. B is a bounded linear operator from U into E. For any function y defined on $(-\infty,b] \times \Omega$ and any $t \in J$ we denote by $y_t(\cdot,w)$ the element of $\mathcal{B} \times \Omega$ defined by $y_t(\theta,w) = y(t+\theta,w), \theta \in (-\infty,0]$. Here $y_t(\cdot,w)$ represents the history of the state from time $-\infty$, up to the present time t.

4.2.2 Controllability result

Consider the space

$$\Lambda := \left\{y : (-\infty,b], y|_{(-\infty,0]} \in \mathcal{B} \text{ and } y|_J \in C\right\}.$$

Let $\|y\|$ be the seminorm in Λ defined by

$$\|y\|_\Lambda = \|\phi\|_{\mathcal{B}} + \|y\|_C .$$

Definition 4.2.1. *The system* $(4.1)-(4.2)$ *is said to be controllable on the interval* J*, if for every initial function* $\phi \in \mathcal{B}$*,* $\varphi \in E$ *and* $y^1(w) \in E$ *, there exists a control* $u(\cdot,w)$ *in* $L^2(J,U)$*, such that the solution* $y(t,w)$ *of* $(4.1)-(4.2)$ *satisfies* $y(b,w) = y^1(w)$.

Let y be a mapping of $J \times \Omega$ into E. y is said to be a stochastic process if for each $t \in J, y(t, \cdot)$ is measurable.

Now we give our main existence result for problem $(4.1)-(4.2)$. Before starting and proving this result, we give the definition of the mild random solution.

Definition 4.2.2. *A stochastic process* $y : (-\infty, b] \times \Omega \to E$ *is said to be a random mild solution of problem* $(4.1) - (4.2)$ *if* $y(t, w) = \phi(t, w)$, $t \in (-\infty, 0], y^{'}(0, w) = \varphi(w)$ *and the restriction of* $y(\cdot, w)$ *to the interval* J *is continuous and satisfies the following integral equation:*

$$\begin{aligned} y(t, w) &= C(t)\phi(0, w) + S(t)\varphi(w) + \int_0^t C(t-s) f(s, y_s(\cdot, w), w) ds \\ &\quad + \int_0^t C(t-s) Bu(t, w) ds \end{aligned}$$

Let

$$M = \sup \left\{ \|C(t)\|_{B(E)} : t \geq 0 \right\} \quad M^{'} = \sup \left\{ \|S(t)\|_{B(E)} : t \geq 0 \right\}$$

We will need to introduce the following hypotheses :

(H_1) $C(t)$ is compact for $t > 0$,

(H_2) The function $f : J \times \mathcal{B} \times \Omega \to E$ is random Carathéodory,

(H_3) There exist functions $\psi : J \times \Omega \to \mathbb{R}^+$ and $p : J \times \Omega \to \mathbb{R}^+$ such that for each $w \in \Omega$, $\psi(\cdot, w)$ is a continuous nondecreasing function and $p(\cdot, w)$ integrable with:

$$|f(t, u, w)| \leq p(t, w)\, \psi(\|u\|_{\mathcal{B}}, w) \text{ for a.e. } t \in J \text{ and each } u \in \mathcal{B},$$

(H_4) There exists a random function $R : \Omega \longrightarrow \mathbb{R}^+/\{0\}$ such that:

$$M(1 + bMK)(\|\phi\|_{\mathcal{B}} + \psi(D_b, w)\|p\|_{L^1}) + bMK\left|y^1(w)\right|$$

$$+M^{'}(1 + bMK)|\varphi| \leq R(w)$$

where

$$D_b := K_b R(w) + M_b \|\phi\|_{\mathcal{B}},$$

(H_5) The linear operator $W : L^2(J,U) \to E$ defined by:

$$Wu = \int_0^b C(b-s)Bu(s,w)ds$$

has an pseudo inverse operator W^{-1} which takes values in $L^2(J,U) \to kerW$ and there exists a positive constant K such that $\|BW^{-1}\| \le K$,

(H_6) For each $w \in \Omega, \phi(\cdot,w)$ is continuous and for each $t, \phi(t,\cdot)$ is measurable, and for each $w \in \Omega$, $\varphi(w)$ is measurable.

Theorem 4.2.1. *Suppose that the hypotheses* $(H_1)-(H_6)$ *are satisfied, then the problem* $(4.1)-(4.2)$ *is controllable on* J*.*

Proof. Using (H_5) and define the control :

$$\begin{aligned} & u(t,w) \\ = \; & W^{-1}\left(y^1(w) - C(b)\phi(0,w) - S(b)\varphi(w) - \int_0^b C(b-s)f(s,y_s(\cdot,w),w)ds\right) \end{aligned}$$

We shall show that when using the control $u(t,w)$, the operator $N : \Omega \times \Lambda \longrightarrow \Lambda$ defined by: $(N(w)y)(t) = \phi(t,w)$, if $t \in (-\infty,0]$, and for $t \in J$:

$$\begin{aligned} & (N(w)y)(t) \\ = \; & C(t)\phi(0,w) + S(t)\varphi(w) + \int_0^t C(t-s)\; f(s,y_s(\cdot,w),w),w)ds \\ & + \int_0^t C(t-s)BW^{-1} \\ & \times \left(y^1(w) - C(b)\phi(0,w) - S(b)\varphi(w) - \int_0^b C(b-\tau)f(\tau,y_\tau(\cdot,w),w)d\tau\right) ds \end{aligned} \tag{4.3}$$

has a fixed point $y(t,w)$. This fixed point is a mild solution of the system $(4.1)-(4.2)$ and this implies that the system is controllable on J. Then we show that the mapping defined by (4.3) is a random operator. To do this, we need to prove that for any $y \in \Lambda$, $N(\cdot)(y) : \Omega \longrightarrow \Lambda$ is a random variable. Then we prove that $N(\cdot)(y) : \Omega \longrightarrow \Lambda$ is measurable. As a mapping $f(t,y,\cdot), t \in J, y \in \Lambda$ is measurable by assumption (H_2) and (H_6). Let $D : \Omega \longrightarrow 2^\Lambda$ be defined by:

$$D(w) = \{y \in \Lambda : \|y\|_\Lambda \le R(w)\}.$$

$D(w)$ is bounded, closed, convex and solid for all $w \in \Omega$. Then D is measurable by Lemma 17 (see [58]).
Let $w \in \Omega$ be fixed, then for any $y \in D(w)$, and by assumption (A_1), we get:

$$\begin{aligned} \|y_s\|_{\mathcal{B}} &\leq L(s)|y(s)| + M(s)\|y_0\|_{\mathcal{B}} \\ &\leq K_b|y(s)| + M_b\|\phi\|_{\mathcal{B}} \end{aligned}$$

and by (H_3) and (H_4), we have

$$\begin{aligned} |(N(w)y)(t)| &\leq M\left\|\phi\right\|_{\mathcal{B}} + M^{'}\left|\varphi\right| + M\int_0^t |f(s,y_s,w)|ds \\ &+MK\int_0^t \left|y^1(w)\right| + M\left\|\phi\right\|_{\mathcal{B}} + M^{'}\left|\varphi\right| ds \\ &+MK\int_0^t\int_0^b \|C(\tau-s)\|\left|f(\tau,y_\tau(\cdot,w),w)\right|\,d\tau ds \\ &\leq M\|\phi\|_{\mathcal{B}} + M^{'}\left|\varphi\right| + M\int_0^b p(s,w)\,\psi\left(\|y_s\|_{\mathcal{B}},w\right)\,ds \\ &+bMK\left|y^1(w)\right| + bM^2K\|\phi\|_{\mathcal{B}} + bMM^{'}K\left|\varphi\right| \\ &+bM^2K\int_0^b p(\tau,w)\,\psi\left(\|y_\tau\|_{\mathcal{B}},w\right)\,d\tau. \\ &\leq M\left(1+bMK\right)\|\phi\|_{\mathcal{B}} + bMK\left|y^1(w)\right| + M^{'}\left(1+bMK\right)\left|\varphi\right| \\ &+M\left(1+bMK\right)\int_0^b p(s,w)\,\psi\left(\|y_s\|_{\mathcal{B}},w\right)\,ds. \\ &\leq M\left(1+bMK\right)\left(\|\phi\|_{\mathcal{B}} + \psi(D_b,w)\int_0^b p(s,w)\,\,ds\right) \\ &+bMK\left\|y^1(w)\right\| + M^{'}\left(1+bMK\right)\left|\varphi\right|. \end{aligned}$$

Set

$$D_b := K_bR(w) + M_b\|\phi\|_{\mathcal{B}}.$$

Then, we have

$$\begin{aligned} |(N(w)y)(t)| &\leq M\left(1+bMK\right)\left(\|\phi\|_{\mathcal{B}} + \psi(D_b,w)\int_0^b p(s,w)\,\,ds\right) \\ &+bMK\left|y^1(w)\right| + M^{'}\left|\varphi\right|\left(1+bMK\right). \end{aligned}$$

Thus

$$\begin{aligned}\|(N(w)y)\|_\Lambda &\leq M\left(1+bMK\right)\left(\|\phi\|_{\mathcal{B}}+\psi(D_b,w)\|p\|_{L^1}\right)\\ &\quad +bMK\left|y^1\left(w\right)\right|+M^{'}\left(1+bMK\right)|\varphi|\\ &\leq R(w).\end{aligned}$$

This implies that N is a random operator with stochastic domain D and $N(w) : D(w) \longrightarrow D(w)$ for each $w \in \Omega$.

Step 1: N is continuous.

Let y^n be a sequence such that $y^n \longrightarrow y$ in Y. Then

$$\begin{aligned}&|(N(w)y^n)(t)-(N(w)y)(t)|\\ \leq\ & M\int_0^t |f(s,y_s^n,w)-f(s,y_s,w)|\,ds\\ &+KM\int_0^t\int_0^b \|C(b-\tau)\|\,|f(\tau,y_\tau^n(\cdot,w)-f(\tau,y_\tau,w)|d\tau ds\\ \leq\ & M\int_0^t |f(s,y_s^n,w)-f(s,y_s,w)|\,ds\\ &+bM^2K\int_0^b |f(\tau,y_\tau^n(\cdot,w)-f(\tau,y_\tau,w)|d\tau\\ \leq\ & M\left(1+bMK\right)\int_0^b |f(\tau,y_\tau^n(\cdot,w)-f(\tau,y_\tau,w)|d\tau\end{aligned}$$

Since $f(s,\cdot,w)$ is continuous, we have by the Lebesgue dominated convergence theorem

$$\|f(\cdot,y_\cdot^n,w)-f(\cdot,y_\cdot,w)\|_{L^1}\to 0 \text{ as } n\to+\infty.$$

Thus N is continuous.

Step 2: We prove that for every $w \in \Omega, \{y \in D(w) : N(w)y = y\} \neq \emptyset$. For prove this we apply Schauder's theorem.

(a) N maps bounded sets into equicontinuous sets in $D(w)$.
Let $\tau_1, \tau_2 \in [0, b]$ with $\tau_2 > \tau_1$, $D(w)$ be a bounded set as in Step 2, and

$y \in D(w)$. Then

$$\begin{aligned}
& |(N(w)y)(\tau_2) - (N(w)y)(\tau_1)| \\
\leq\ & \|C(\tau_2) - C(\tau_1)\|_{B(E)} \|\phi\|_{\mathcal{B}} + \|S(\tau_2) - S(\tau_1)\|_{B(E)} |\varphi| \\
& + \int_0^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)} |f(s, y_s, w)|\, ds \\
& + \int_{\tau_1}^{\tau_2} \|C(\tau_2 - s)\|_{B(E)} |f(s, y_s, w)|\, ds \\
& + K \int_0^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)} \\
& \times \left[\left| y^1(w) \right| + \|C(b)\|_{B(E)} \|\phi\|_{\mathcal{B}} + \|S(b)\|_{B(E)} |\varphi| \right] ds \\
& + K \int_0^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)} \\
& \times \int_0^b \|C(b - \tau)\|_{B(E)} |f(\tau, y_\tau(\cdot, w), w)|\, d\tau ds \\
& + K \int_{\tau_1}^{\tau_2} \|C(\tau_2 - s)\|_{B(E)} \\
& \times \left[\left| y^1(w) \right| + \|C(b)\|_{B(E)} \|\phi\|_{\mathcal{B}} + \|S(b)\|_{B(E)} |\varphi| \right] ds \\
& + K \int_{\tau_1}^{\tau_2} \|C(\tau_2 - s)\|_{B(E)} \\
& \times \int_0^b \|C(b - \tau)\|_{B(E)} |f(\tau, y_\tau(\cdot, w), w)|\, d\tau ds \\
\leq\ & \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)} \|\phi\|_{\mathcal{B}} + \|S(\tau_2) - S(\tau_1)\|_{B(E)} |\varphi| \\
& + \psi(D_T, w) \int_0^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)} p(s, w) ds \\
& + M\psi(D_T, w) \int_{\tau_1}^{\tau_2} p(s, w) ds \\
& + K \int_0^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)} \\
& \times \left[\left| y^1(w) \right| + \|C(b)\|_{B(E)} \|\phi\|_{\mathcal{B}} + \|S(b)\|_{B(E)} |\varphi| \right] ds \\
& + KM\psi(D_b, w) \int_0^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)} \int_0^b p(\tau, w) d\tau ds \\
& + KM \int_{\tau_1}^{\tau_2} (\left| y^1(w) \right| + \|C(b)\|_{B(E)} \|\phi\|_{\mathcal{B}} + \|S(b)\|_{B(E)} |\varphi| \\
& + M\psi(D_b, w) \int_0^b p(\tau, w) d\tau) ds
\end{aligned}$$

The right-hand of the above inequality tends to zero as $\tau_2 - \tau_1 \to 0$, since $C(t), S(t)$ are a strongly continuous operator and the compactness of $C(t), S(t)$ for $t > 0$, implies the continuity in the uniform operator topology see [85,84].

(b) Let $t \in [0,b]$ be fixed and let $y \in D(w)$: by assumptions $(H_3), (H_5)$ and since $C(t)$ is compact, the set

$$\left\{ \int_0^t C(t-s) f(s, y_s(\cdot,w), w) ds + \int_0^t C(t-s) Bu(t,w) ds \right\}$$

is precompact in E, then the set

$$\left\{ \begin{array}{c} C(t)\phi(0,w) + S(t)\varphi(w) + \int_0^t C(t-s) f(s, y_s(\cdot,w), w) ds \\ + \int_0^t C(t-s) Bu(t,w) ds \end{array} \right\}$$

is precompact in E.

A consequence of Steps $1-2$ and $(a),(b)$, we can conclude that $N(w) : D(w) \to D(w)$ is continuous and compact. From Schauder's theorem, we deduce that $N(w)$ has a fixed point $y(w)$ in $D(w)$. Since $\bigcap_{w\in\Omega} D(w) \neq \emptyset$, the hypothesis that a measurable selector of int$D$ exists holds. By Lemma 1.6.1, the random operator N has a stochastic fixed point $y^*(w)$, which is a random mild solution of the random problem $(4.1)-(4.2)$

4.2.3 An example

Consider the functional partial differential equation of second order

$$\frac{\partial^2}{\partial t^2} z(t,x,w) = \frac{\partial^2}{\partial x^2} z(t,x,w) + f(t, z(t,x,w), w) + Bu\,(t,w) \;\; x \in [0,\pi];\; t \in [0,b], \tag{4.4}$$

$$z(t,0,w) = z(t,\pi,w) = 0;\; t \in [0,b], w \in \Omega, \tag{4.5}$$

$$z(t,x,w) = \phi(t,w), \frac{\partial}{\partial t} z(0,x,w) = v(x,w);\; t \in (-\infty, 0],\; w \in \Omega, \tag{4.6}$$

where $f : J \times \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$ is a given function. Let $E = L^2[0,\pi]$, The operator $A : E \longrightarrow E$ by $Av = v''$ with domain $D(A) = \{v \in E; v, v''$ are absolutely continuous, $v'' \in E, v(0) = v(\pi) = 0\}$.

It is well known that A is the infinitesimal generator of a strongly continuous cosine function $(C(t))_{t\in\mathbb{R}}$ on E, respectively. Moreover, A has discrete spectrum, the eigenvalues are $-n^2, n \in IN$ with corresponding normalized eigenvectors

$$z_n(\tau) := \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin(n\tau),$$

and the following properties hold:

(a) $\{z_n : n \in IN\}$ is an orthonormal basis of E,

(b) If $y \in E$, then $Ay = -\sum_{n=1}^{\infty} n^2 \langle y, z_n\rangle z_n$,

(c) For $y \in E$,$C(t)y = \sum_{n=1}^{\infty} cos(nt) \langle y, z_n\rangle z_n$, and the associated sine family is

$$S(t)y = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} \langle y, z_n\rangle z_n,$$

which implies that the operator $S(t)$ is compact for all $t > 0$ and that

$$\|C(t)\| = \|S(t)\| \leq 1, \text{ for all } t \geq 0.$$

(d) If we denote the group of translations on E defined by

$$\Phi(t)y(\zeta, w) = \widetilde{y}(\zeta + t, w),$$

where $\widetilde{y}$ is the extension of y with period 2π, then

$$C(t) = \frac{1}{2}(\Phi(t) + \Phi(-t)); A = D,$$

where D is the infinitesimal generator of the group on

$$X = \{y(\cdot, w) \in H^1(0, \pi) : y(0, w) = y(\pi, w) = 0\}.$$

Assume that B is a bounded linear operator from U into E and the linear operator $W : L^2(J, U) \to E$ defined by:

$$Wu = \int_0^b C(b - s)Bu(s, w)ds,$$

Then the problem $(4.1) - (4.2)$ is an abstract formulation of the problem (4.4)-(4.6). If conditions $(H_1) - (H_6)$ are satisfied, theorem 4.3 implies that the problem (4.4)-(4.6) is controllable.

4.3 The state-dependent delay case

4.3.1 Introduction

We consider the following problem:

$$y^{''}(t,w) = Ay(t,w) + f(t, y_{\rho(t,y_t)}(\cdot,w),w) + Bu(t,w), \quad \text{a.e. } t \in J := [0,b] \tag{4.7}$$

$$y(t,w) = \phi(t,w), \quad t \in (-\infty,0],\ y^{'}(0,w) = \varphi(w) \in E, \tag{4.8}$$

where $w \in \Omega$, $f : J \times \mathcal{B} \times \Omega \to E$, $\phi : (-\infty,0] \times \Omega \to E$ are given random functions, $A : D(A) \subset E \to E$ as in problem $(4.7) - (4.8)$, $\mathcal{B}$ is the phase space to be specified later, $\rho : J \times \mathcal{B} \to \mathbb{R}$, and $(E, |.|)$ is a real Banach space. The control function $u(\cdot,w)$ is given in $L^2(J,U)$, a Banach space of admissible control functions with U as a Banach space. B is a bounded linear operator from U into E. For any function y defined on $(-\infty,b] \times \Omega$ and any $t \in J$ we denote by $y_t(\cdot,w)$ the element of $\mathcal{B} \times \Omega$ defined by $y_t(\theta,w) = y(t+\theta,w), \theta \in (-\infty,0]$. Here $y_t(\cdot,w)$ represents the history of the state from time $-\infty$, up to the present time t.

4.3.2 Controllability result

In this section, we give our main controllability result for problem $(4.7) - (4.8)$.Before starting and proving this result, we give definitions of controllability and random mild solution.

Definition 4.3.1. *The system* $(4.7) - (4.8)$ *is said to be controllable on the interval* J*, if for every initial function* $\phi \in \mathcal{B}$*,* $\varphi \in E$ *and* $y^1(w) \in E$*, there exists a control* $u(\cdot,w)$ *in* $L^2(J,U)$*, such that the solution* $y(t,w)$ *of* $(4.7) - (4.8)$ *satisfies* $y(b,w) = y^1(w)$.

Definition 4.3.2. *A stochastic process* $y : (-\infty,b] \times \Omega \to E$ *is said to be a random mild solution of problem* $(4.7) - (4.8)$ *if* $y(t,w) = \phi(t,w)$,
$t \in (-\infty,0], y^{'}(0,w) = \varphi(w)$ *and the restriction of* $y(\cdot,w)$ *to the interval* J *is continuous and satisfies the following integral equation:*

$$\begin{aligned} y(t,w) &= C(t)\phi(0,w) + S(t)\varphi(w) + \int_0^t C(t-s)f(s, y_{\rho(s,y_s)}(\cdot,w),w)ds \\ &+ \int_0^t C(t-s)Bu(t,w)ds \end{aligned}$$

Set

$$\mathcal{R}(\rho^-) = \{\rho(s,\varphi) : (s,\varphi) \in J \times \mathcal{B}, \rho(s,\varphi) \leq 0\}.$$

We always assume that $\rho : J \times \mathcal{B} \to (-\infty, b]$ is continuous. Additionally, we introduce following hypothesis:

(H_ϕ) The function $t \to \phi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $L^\phi : \mathcal{R}(\rho^-) \to (0, \infty)$ such that

$$\|\phi_t\|_{\mathcal{B}} \leq L^\phi(t)\|\phi\|_{\mathcal{B}} \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

Remark 4.3.1. The condition (H_ϕ), is frequently verified by functions continuous and bounded. For more details, see for instance [57].

Lemma 4.3.1. *([56], Lemma 2.4) If $y : (-\infty, b] \to E$ is a function such that $y_0 = \phi$, then*

$$\|y_s\|_{\mathcal{B}} \leq (M_b + L^\phi)\|\phi\|_{\mathcal{B}} + K_b \sup\{|y(\theta)|; \theta \in [0, max\{0,s\}]\},\ s \in \mathcal{R}(\rho^-) \cup J,$$

where $L^\phi = \sup\limits_{t\in\mathcal{R}(\rho^-)} L^\phi(t)$.

We will need to introduce the following hypotheses which are be assumed there after:

(H_1) $C(t)$ is compact for $t > 0$,

(H_2) The function $f : J \times \mathcal{B} \times \Omega \to E$ is random Carathéodory,

(H_3) There exist functions $\psi : J \times \Omega \to \mathbb{R}^+$ and $p : J \times \Omega \to \mathbb{R}^+$ such that for each $w \in \Omega$, $\psi(\cdot, w)$ is a continuous nondecreasing function and $p(\cdot, w)$ integrable with:

$$|f(t,u,w)| \leq p(t,w)\ \psi(\|u\|_{\mathcal{B}}, w) \text{ for a.e. } t \in J \text{ and each } u \in \mathcal{B},$$

(H_4) There exists a function $l : J \times \Omega \longrightarrow \mathbb{R}^+$ with $l(\cdot, w) \in L^1(J, \mathbb{R}^+)$ for each $w \in \Omega$ such that for any bounded $B \subseteq E$

$$\alpha(f(t, B, w)) \leq l(t, w)\alpha(B),$$

(H_5) There exists a random function $R : \Omega \longrightarrow \mathbb{R}^+ \backslash \{0\}$ such that:

$$M\left(1 + bM\lambda\right)\left(\|\phi\|_{\mathcal{B}} + \psi((M_b + L^{\phi})\|\phi\|_{\mathcal{B}} + K_b R(w), w)\int_0^b p(s, w)\ ds\right)$$
$$+ bM\lambda\left\|y^1\left(w\right)\right\| + M^{'}\left(1 + bM\lambda\right)|\varphi| \leq R(w),$$

(H_6) The linear operator $W : L^2(J, U) \to E$ defined by:

$$Wu = \int_0^b C(b - s)Bu(s, w)ds$$

has an pseudo inverse operator W^{-1} which takes values in $L^2(J, U)/kerW$ and there exists a positive constant λ such that $\|BW^{-1}\| \leq \lambda$,

(H_7) For each $w \in \Omega, \phi(\cdot, w)$ is continuous and for each $t, \phi(t, \cdot)$ is measurable and for each $w \in \Omega, \varphi\left(w\right)$ is measurable.

Theorem 4.3.1. *Suppose that the hypotheses* (H_{ϕ}) *and* $(H_1) - (H_7)$ *hold with*

$$M\left(1 + M\lambda b\right)\int_0^b l(s)K(s)ds < 1 \tag{4.9}$$

then the random problem $(4.7) - (4.8)$ *is controllable on* J.

Proof. Using (H_6), we define the control :

$$\begin{aligned} & u(t, w) \\ = \ & W^{-1}\left(y^1\left(w\right) - C(b)\phi(0, w) - S(b)\varphi(w) - \int_0^b C(b - s)f(s, y_{\rho(s, y_s)}(\cdot, w), w)ds\right) \end{aligned}$$

We shall show that when using the control $u(t, w)$, the operator $N : \Omega \times \Lambda \longrightarrow \Lambda$ defined by:

$(N(w)y)(t) = \phi(t,w)$, if $t \in (-\infty, 0]$, and for $t \in J$:

$$\begin{aligned}
&(N(w)y)(t) \\
= \;& C(t)\phi(0,w) + S(t)\varphi(w) + \int_0^t C(t-s)\, f(s, y_{\rho(s,y_s)}(\cdot, w), w), w)ds \\
&+ \int_0^t C(t-s)BW^{-1} \\
&\left(y^1(w) - C(b)\phi(0,w) - S(b)\varphi(w) - \int_0^b C(b-\tau)f(\tau, y_{\rho(\tau,y_\tau)}(\cdot,w),w)d\tau \right) ds
\end{aligned} \tag{4.10}$$

has a fixed point $y(t,w)$. This fixed point is a mild solution of the system $(4.7) - (4.8)$ and this implies that the system is controllable.

Then we show that the mapping defined by (4.10) is a random operator. To do this, we need to prove that for any $y \in \Lambda$, $N(\cdot)(y) : \Omega \longrightarrow \Lambda$ is a random variable. Then we prove that $N(\cdot)(y) : \Omega \longrightarrow \Lambda$ is measurable. as a mapping $f(t,y,\cdot)$, $t \in J, y \in \Lambda$ is measurable by assumption (H_2) and (H_6).

Let $D : \Omega \longrightarrow 2^\Lambda$ be defined by:

$$D(w) = \{y \in \Lambda : \|y\|_\Lambda \leq R(w)\}.$$

$D(w)$ is bounded, closed, convex and solid for all $w \in \Omega$. Then D is measurable by Lemma 17 (see [58]).
Let $w \in \Omega$ be fixed, If $y \in D(w)$, from Lemma (4.7) follows that

$$\left\| y_{\rho(t,y_t)} \right\|_{\mathcal{B}} = (M_b + L^\phi)\|\phi\|_{\mathcal{B}} + K_b R(w)$$

and For each $y \in D(w)$, by (H_3) and (H_5), we have for each $t \in J$.

$$
\begin{aligned}
|(N(w)y)(t)| \leq\; & M\|\phi\|_{\mathcal{B}} + M^{'}|\varphi| + M\int_0^t |f(s, y_{\rho(s,y_s)}, w)|ds \\
& + M\lambda\int_0^t \left|y^1(w)\right| + M\|\phi\|_{\mathcal{B}} + M^{'}|\varphi|\, ds \\
& + M\lambda\int_0^t\int_0^b \|C(\tau - s)\|\left|f(\tau, y_{\rho(\tau,y_\tau)}(\cdot, w), w)\right|\, d\tau ds \\
\leq\; & M\|\phi\|_{\mathcal{B}} + M^{'}|\varphi| + M\int_0^b p(s,w)\,\psi\left(\|y_{\rho(s,y_s)}\|_{\mathcal{B}}, w\right)\, ds \\
& + bM\lambda\left|y^1(w)\right| + bM^2K\|\phi\|_{\mathcal{B}} + bMM^{'}K|\varphi| \\
& + bM^2\lambda\int_0^b p(\tau,w)\,\psi\left(\|y_{\rho(\tau,y_\tau)}\|_{\mathcal{B}}, w\right)\, d\tau. \\
\leq\; & M(1 + bM\lambda)\|\phi\|_{\mathcal{B}} + bM\lambda\left|y^1(w)\right| + M^{'}(1 + bM\lambda)|\varphi| \\
& + M(1 + bM\lambda)\int_0^b p(s,w)\,\psi\left(\|y_{\rho(s,y_s)}\|_{\mathcal{B}}, w\right)\, ds. \\
\leq\; & M(1 + bM\lambda) \\
& \times\left(\|\phi\|_{\mathcal{B}} + \psi((M_b + L^{\phi})\|\phi\|_{\mathcal{B}} + K_bR(w), w)\int_0^b p(s,w)\, ds\right) \\
& + bM\lambda\left|y^1(w)\right| + M^{'}(1 + bM\lambda)|\varphi|.
\end{aligned}
$$

This implies that N is a random operator with stochastic domain D and $N(w) : D(w) \to D(w)$ for each $w \in \Omega$.

Step 1: N is continuous.
Let y^n be a sequence such that $y^n \longrightarrow y$ in Λ . Then

$$
\begin{aligned}
& |(N(w)y^n)(t) - (N(w)y)(t)| \\
\leq\; & M\int_0^t |f(s, y^n_{\rho(s,y^n_s)}, w) - f(s, y_{\rho(s,y_s)}, w)|\, ds \\
& + \lambda M\int_0^t\int_0^b \|C(b - \tau)\|\,|f(\tau, y^n_{\rho(s,y^n_s)}(\cdot, w) - f(\tau, y_{\rho(s,y_s)}, w)| d\tau ds
\end{aligned}
$$

$$\begin{aligned}
&\leq M\int_0^t \left|f(s,y^n_{\rho(s,y^n_s)},w)-f(s,y_{\rho(s,y_s)},w)\right| ds \\
&\quad +bM^2\lambda\int_0^b \left|f(s,y^n_{\rho(s,y^n_s)},w)-f(s,y_{\rho(s,y_s)},w)\right| ds \\
&\leq M\left(1+bM\lambda\right)\int_0^b \left|f(s,y^n_{\rho(s,y^n_s)},w)-f(s,y_{\rho(s,y_s)},w)\right| ds
\end{aligned}$$

Since $f(s,\cdot,w)$ is continuous, we have by the Lebesgue dominated convergence theorem

$$||(N(w)y^n)(t)-(N(w)y)(t)||_{BC}\to 0 \text{ as } n\to+\infty.$$

Thus N is continuous.

Step 2: We prove that for every $w\in\Omega,\{y\in D(w):N(w)y=y\}\neq\emptyset$. For this we apply the Mönch fixed point theorem.

(a) N maps bounded sets into equicontinuous sets in $D(w)$.
Let $\tau_1,\tau_2\in[0,b]$ with $\tau_2>\tau_1$, $D(w)$ be a bounded set as in Step 2, and $y\in D(w)$. Then

$$\begin{aligned}
&\left|(N(w)y)(\tau_2)-(N(w)y)(\tau_1)\right| \\
\leq\ & \|C(\tau_2)-C(\tau_1)\|_{B(E)}\|\phi\|_{\mathcal{B}}+\|S(\tau_2)-S(\tau_1)\|_{B(E)}|\varphi| \\
&+\int_0^{\tau_1}\|C(\tau_2-s)-C(\tau_1-s)\|_{B(E)}f(s,y_{\rho(s,y_s)},w)ds \\
&+\int_{\tau_1}^{\tau_2}\|C(\tau_2-s)\|_{B(E)}f(s,y_{\rho(s,y_s)},w)ds \\
&+\lambda\int_0^{\tau_1}\|C(\tau_2-s)-C(\tau_1-s)\|_{B(E)}\left[\left|y^1(w)\right|+\|C(b)\|\left|\phi(0,w)\right|\right]ds \\
&+\lambda\int_0^{\tau_1}\|C(\tau_2-s)-C(\tau_1-s)\|_{B(E)} \\
&\times\int_0^b\|C(b-\tau)\|_{B(E)}\left|f(\tau,y_{\rho(s,y_s)},w)\right|d\tau ds \\
&+\lambda\int_{\tau_1}^{\tau_2}\|C(\tau_2-s)\|_{B(E)}\left[\left|y^1(w)\right|+\|C(b)\|\left|\phi(0,w)\right|\right]ds \\
&+\lambda\int_{\tau_1}^{\tau_2}\|C(\tau_2-s)\|_{B(E)}\int_0^b\|C(b-\tau)\|_{B(E)}\left|f(\tau,y_{\rho(\tau,y_\tau)},w)\right|d\tau ds
\end{aligned}$$

$$
\begin{aligned}
\leq\ & |C(\tau_2) - C(\tau_1)| \|\phi\|_{\mathcal{B}} + \|S(\tau_2) - S(\tau_1)\|_{B(E)} |\varphi| \\
& + \int_0^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)} f(s, y_{\rho(s,y_s)}, w) ds \\
& + \int_{\tau_1}^{\tau_2} \|C(\tau_2 - s)\|_{B(E)} f(s, y_{\rho(s,y_s)}, w) ds \\
& + \lambda \int_0^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)} \left[|y^1(w)| + \|C(b)\|_{B(E)} |\phi(0,w)| \right] ds \\
& + \lambda \int_0^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)} \psi \left((M_b + L^{\phi}) \|\phi\|_{\mathcal{B}} + K_b R(w) \right) \\
& \times \int_0^b p(\tau, w) d\tau ds \\
& + \lambda M \int_{\tau_1}^{\tau_2} |y^1(w)| + \|C(b)\|_{B(E)} |\phi(0,w)| \\
& + M \psi \left((M_b + L^{\phi}) \|\phi\|_{\mathcal{B}} + K_b R(w) \right) \int_0^b p(\tau, w) d\tau ds \\
\leq\ & \|C(\tau_2) - C(\tau_1)\|_{B(E)} \|\phi\|_{\mathcal{B}} + \|S(\tau_2) - S(\tau_1)\|_{B(E)} |\varphi| \\
& + \psi \left((M_b + L^{\phi}) \|\phi\|_{\mathcal{B}} + K_b R(w) \right) \\
& \times \int_0^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)} p(s, w) ds \\
& + M \psi \left((M_b + L^{\phi}) \|\phi\|_{\mathcal{B}} + K_b R(w), w \right) \int_{\tau_1}^{\tau_2} p(s, w) ds. \\
& + \lambda \int_0^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)} \\
& \times \left[|y^1(w)| + \|C(b)\|_{B(E)} |\phi(0,w)| \right] ds \\
& + \int_0^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)} \psi \left((M_b + L^{\phi}) \|\phi\|_{\mathcal{B}} + K_b R(w) \right) \\
& \times \int_0^b p(\tau, w) d\tau ds \\
& + \lambda M \int_{\tau_1}^{\tau_2} |y^1(w)| + \|C(b)\|_{B(E)} |\phi(0,w)| \\
& + M \psi \left((M_b + L^{\phi}) \|\phi\|_{\mathcal{B}} + K_b R(w) \right) \int_0^b p(\tau, w) d\tau ds
\end{aligned}
$$

The right-hand of the above inequality tends to zero as $\tau_2 - \tau_1 \to 0$, since $C(t), S(t)$ are a strongly continuous operator and the compactness of $C(t), S(t)$ for $t > 0$, implies the continuity in the uniform operator topology see [85, 84]. Next, let $w \in \Omega$ be fixed (therefore we do not write $'w'$ in the sequel) but arbitrary.

(b) Now let V be a subset of $D(w)$ such that $V \subset \overline{conv}\,(N(V) \cup \{0\})$. V is bounded and equicontinuous and therefore the function $t \to v(t) = \alpha(V(t))$ is continuous on $(-\infty, b]$. By (H_4), Lemma 1.9, theorem 1.14 and the properties of the measure α we have for each $t \in (-\infty, b]$

$$\begin{aligned}
v(t) &\leq \alpha\left(N(V)\right)(t) \cup \{0\}) \\
&\leq \alpha\left(N(V(t)\right) \\
&\leq \alpha\left(C(t)\phi(0,w)\right) + \alpha\left(S(t)\varphi(w)\right) \\
&\quad + \alpha\left(\int_0^t C(t-s)\ f(s, y_{\rho(s,y_s)}\left(\cdot, w\right), w), w) ds\right) \\
&\quad + M\lambda \int_0^t \alpha\left(y^1\left(w\right) - C(b)\phi(0,w) - S(b)\varphi(w)\right) \\
&\quad + \alpha\left(\int_0^b C(b-\tau) f(\tau, y_{\rho(\tau,y_\tau)}(\cdot, w), w) d\tau\right) ds \\
&\leq M \int_0^t \alpha\ \Big(f(s, y_{\rho(s,y_s)}\left(\cdot, w\right), w), w)\Big)\, ds \\
&\quad + M\lambda \int_0^t \int_0^b \alpha\left(C(b-\tau) f(\tau, y_{\rho(\tau,y_\tau)}(\cdot, w), w)\right) d\tau ds \\
&\leq M \int_0^t l(s)\alpha(\left\{y_{\rho(s,y_s)} : y \in V\right\}) ds \\
&\quad + M\lambda \int_0^t \int_0^b \alpha\left(C(b-\tau) f(\tau, y_{\rho(\tau,y_\tau)}(\cdot, w), w)\right) d\tau ds \\
&\leq M \int_0^t l(s)K(s) \sup_{0\leq\tau\leq s} \alpha(V(\tau)) ds \\
&\quad + M^2\lambda \int_0^t \int_0^b \alpha\left(f(\tau, y_{\rho(\tau,y_\tau)}, w)\right) d\tau ds \\
&\leq M \int_0^t l(s)K(s)\alpha(V(s)) ds + M^2\lambda b \int_0^b l(\tau)\alpha(\left\{y_{\rho(\tau,y_\tau)} : y \in V\right\}) d\tau
\end{aligned}$$

$$\begin{aligned}
&\leq M\int_0^t v(s)\, l(s)K(s)ds + M^2\lambda b\int_0^b l(\tau)K(\tau)\alpha(V(\tau))d\tau\\
&\leq M\int_0^t v(s)\, l(s)K(s)ds + M^2\lambda b\int_0^b l(\tau)K(\tau)\alpha(V(\tau))d\tau\\
&= M\int_0^t l(s)K(s)v(s)ds + M^2\lambda b\int_0^b l(\tau)K(\tau)v(\tau)d\tau.\\
&\leq M\int_0^b l(s)K(s)v(s)ds + M^2\lambda b\int_0^b l(\tau)K(\tau)v(\tau)d\tau.\\
&\leq M\,(1+M\lambda b)\int_0^b l(s)K(s)v(s)ds.\\
&\leq M\,(1+M\lambda b)\int_0^b l(s)K(s)\sup_{0\leq\tau\leq s} v(\tau)ds\\
&\leq M\,(1+M\lambda b)\,\|v\|_\infty\int_0^b l(s)K(s)ds
\end{aligned}$$

$$\|v\|_\infty \leq M\,(1+M\lambda b)\,\|v\|_\infty\int_0^b l(s)K(s)ds$$

Then

$$\|v\|_\infty\,(1-M\,(1+M\lambda b)\int_0^b l(s)K(s)ds)\leq 0.$$

By (4.9) it follows that $\|v\|_\infty = 0$, this implies that $v(t) = 0$ for each $t \in J$, and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzelà theorem, V is relatively compact in $D(w)$. Applying now Theorem 1.14 we conclude that N has a fixed point $y(w) \in D(w)$. Since $\bigcap_{w\in\Omega} D(w) \neq \emptyset$, and a measurable selector of int$D$ exists, Lemma 1.6.1 implies that the random operator N has a stochastic fixed point $y^*(w)$, which is a mild solution of the random problem $(4.7) - (4.8)$.

4.3.3 An example

Take $E = L^2[0,\pi]; \mathcal{B} = C_0 \times L^2(g,E)$ and define $A : E \to E$ by $A\omega = \omega''$ with domain

$D(A) = \{\omega \in E; \omega, \omega'$are absolutely continuous, $\omega'' \in E$, $\omega(0) = \omega(\pi) = 0\}$.

It is well known that A is the infinitesimal generator of a strongly continuous cosine function $(C(t))_{t\in\mathbb{R}}$ on E, respectively. Moreover, A has discrete spectrum, the eigenvalues are $-n^2, n \in \mathbb{N}$ with corresponding normalized eigenvectors

$$z_n(\tau) := (\frac{2}{\pi})^{\frac{1}{2}} \sin n\tau,$$

and the following properties hold.

(a) $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of E.

(b) If $y \in E$, then $Ay = -\sum_{n=1}^{\infty} n^2 < y, z_n > z_n$.

(c) For $y \in E, C(t)y = \sum_{n=1}^{\infty} \cos(nt) < y, z_n > z_n$, and the associated sine family is

$$S(t)y = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} < y, z_n > z_n$$

which implies that the operator $S(t)$ is compact, for all $t \in J$ and that

$$\|C(t)\| = \|S(t)\| \leq 1, \text{ for all } t \in \mathbb{R}.$$

(d) If Φ denotes the group of translations on E defined by

$$\Phi(t)y(\xi, w) = \tilde{y}(\xi + t, w),$$

where $\tilde{y}$ is the extension of y with period 2π. Then

$$C(t) = \frac{1}{2}(\Phi(t) + \Phi(-t)); A = D^2,$$

where D is the infinitesimal generator of the group Φ on

$$X = \{y(., w) \in H^1(0, \pi) : y(0, w) = x(\pi, w) = 0\}.$$

For more details, see [43].

Consider the functional partial differential equation of second order

$$\begin{aligned} \frac{\partial^2}{\partial t^2} z(t, x, w) &= \frac{\partial^2}{\partial x^2} z(t, x, w) + C_0(w) \int_{-\infty}^{0} a(s-t) z(s - \rho_1(t)\rho_2(|z(t)|), x, w) ds \\ +Bu(t) \quad x &\in [0, \pi], \; t \in J, w \in \Omega, \end{aligned} \tag{4.11}$$

$$z(t,0,w) = z(t,\pi,w) = 0,\ t \in J, w \in \Omega \tag{4.12}$$

$$z(t,x,w) = \phi(t,w), \quad \frac{\partial z(0,x,w)}{\partial t} = v(x,w),\ t \in (-\infty, 0],\ x \in [0,\pi], w \in \Omega, \tag{4.13}$$

where C_0 is a real-valued random variable, $J := [0,+\infty)$,
$\rho_i : [0,\infty) \to [0,\infty),\ a : \mathbb{R} \to \mathbb{R}$ be continuous, $u(\cdot,w)$ is given in $L^2(J,U)$, a Banach space of admissible control functions with U as a Banach space. B is a bounded linear operator from U into E and

$$L_f = \left(\int_{-\infty}^{0} \frac{a^2(s)}{g(s)} ds \right)^{\frac{1}{2}} < \infty.$$

We define the functions $f : J \times \mathcal{B} \times \Omega \to E, \rho : J \times \mathcal{B} \to \mathbb{R}$ by

$$f(t,\psi(x),w) = C_0(w) \int_{-\infty}^{0} a(s)\psi(s,x)ds,$$

$$\rho(s,\psi) = s - \rho_1(s)\rho_2(|\psi(0)|).$$

We have $\|f(t,.,.)\|_{\mathcal{B}} \leq L_f$.

Let $\phi \in \mathcal{B}$ be sush that (H_ϕ) holds, and let $t \to \phi_t$ be continuous on $\mathcal{R}(\rho^-)$.

Then the problem $(4.7)-(4.8)$ is an abstract formulation of the problem $(4.11) - (4.13)$. If the conditions $(H_1) - (H_7)$ are satisfied, Theorem 4.8 implies that the problem $(4.11) - (4.13)$ is controllable.

Chapter 5

Conclusion and Perspective

In this thesis, we have presented some results to the theory of controllability of mild solutions of some classes of semilinear functional differential equations on infinite intervals with delay and random effects in a Banach space. The results are based on the semigroup theory, Cosine families theory, measure of noncompactness and the argument of fixed points. Some appropriate fixed point theorems have been used; in particular we have used Schauder's theorem and Mönch theorem.

It would be interesting, for a future research, to study the controllability of random mild solution for the Functional evolution equations with infinite delay and random effects of the form

$$\begin{aligned} y'(t,w) &= A(t)y(t,w) + f(t, y_t(\cdot, w), w) + Bu(t,w) \quad \text{a.e. } t \in J := [0,b] \\ y(t,w) &= \phi(t,w), \quad t \in (-\infty, 0], \end{aligned}$$

$$\begin{aligned} y'(t,w) &= A(t)y(t,w) + f(t, y_{\rho(t,y_t)}(., w), w) + Bu(t,w), \quad \text{a.e. } t \in J := [0,b] \\ y(t,w) &= \phi(t,w), \quad t \in (-\infty, 0], \end{aligned}$$

$$\begin{aligned} y^{''}(t,w) &= A(t)y(t,w) + f(t, y_t(\cdot, w), w) + Bu(t,w), \text{ a.e. } t \in J := [0,b], \\ y(t,w) &= \phi(t,w);\ t \in (-\infty, 0], y^{'}(0,w) = \varphi(w), w \in \Omega, \end{aligned}$$

and

$$\begin{aligned} y^{''}(t,w) &= A(t)y(t,w) + f(t, y_{\rho(t,y_t)}(\cdot,w),w) + Bu(t,w), \quad \text{a.e. } t \in J := [0,b], \\ y(t,w) &= \phi(t,w), \quad t \in (-\infty, 0], \; y^{'}(0,w) = \varphi(w) \in E, \end{aligned}$$

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