N° d'ordre :

REPUBLIQUE ALGERIENNE DEMOCRATIQUE & POPULAIRE

MINISTERE DE L'ENSEIGNEMENT SUPERIEUR & DE LA RECHERCHE SCIENTIFIQUE

UNIVERSITE DJILLALI LIABES
FACULTE DES SCIENCES EXACTES
SIDI BEL ABBES

THESE DE DOCTORAT EN SCIENCES

Présentée par

TELLI BENOUMRAN

Spécialité : Mathématiques
Option : Equations différentielles ordinaires

Intitulée

Une contribution à l'étude de certaines classes d'équations Et d'inclusions différentielles d'ordre fractionnaire

Soutenue le 18 / 05 / 2022
Devant le jury composé de :

Président : Mr. BENCHOHRA Mouffak, Prof, Université de SIDI BEL ABBES.

Examinateurs Mr. LAZREG Jamal Eddine, Prof, Université de SIDI BEL ABBES.
Mr. AZZOUZ Noureddine, MCA, Centre Universitaire el Bayadh.
Mr. MAAZOUZ Kadda, MCA, Université de Tiaret.

Directeur de thèse : Mr. SOUID Mohammed Said, Prof, Université de Tiaret.

Co-Directeur de thèse : Mr. Ouahab Abdelghani, Prof, Université de SIDI BEL ABBES.

Année universitaire : 2021 / 2022

Remerciements

Je remercie DIEU, le tout puissant, pour m'avoir accordé santé, courage et patience afin d'accomplir ce travail.

Je tiens à remercier vivement Monsieur le Professeur Souid Mohammed Said (Université de Tiaret) , mon directeur de thèse,pour ses précieux conseils ainsi que les encouragements qu'il m'a donnés durant la préparation de la thèse. C'est aussi grâce à lui que j'ai pu améliorer considérablement mes compétences rédactionnelles en théories des équations différentielles fractionnaires.

Je tiens ensuite à remercier tout spécialement Monsieur le Professeur Abdelghani Ouahab (Université Djillali Liabes SBA), mon co-directeur de thèse, pour ses conseils et son attention constante, sa sympathie et les discussions très intéressantes qu'il a menées pour me suggérer les voies de recherche.

Je remercie remercie vivement Monsieur le Professeur Benchohra Mouaffak (Université Djillali Liabes SBA) pour m'avoir fait l'honneur de prèsider le jury .

Un grand merci aux menbres du jury, Pr. Lazreg Jamal Eddine (Université Djillali Liabes SBA),Dr.Maazouz Kadda (Maitre de Conférences A ,à l'Université de Tiaret) et Dr.Azzouz Noureddine (Maitre de Conférences A,à l'Université de Elbayadh) de m'avoir fait l'honneur d'examiner ce travail.

Finalement, rien de tout cela n'aurait été possible sans mes parents, qui m'ont donné la possibilité de faire des études et qui m'ont toujours encouragé, et que je remercie chaleureusement.

Publications

1. **B. Telli** and M. S. Souid , Integrable Solutions for Implicit Fractional Order Differential Equations with Nonlocal Condition. *Transylvanian Journal of Interdisplinary Mathematics* **6** (2020), No. 2, 1-13.

2. **B. Telli** and M. S. Souid , L^1-Solutions of the Boundary Value Problem for Implicit Fractional Order Differential Equations . *Transylvanian Journal of Applied Analysis,* **27** (2021), No. 1, 121-128.

3. **B. Telli** and M. S. Souid, L^1-Solutions of the initial value problems for implicit differential equations with Hadamard fractional derivative . *Transylvanian Journal of Applied Analysis* **6** (2021), No. 1, 1-14.

4. S. Etemad, M.S. Souid, **B. Telli**, M.K.A. Kaabar, Sh. Rezapour , Investigation of the neutral fractional differential inclusions of Katugampola-type involving both retarded and advanced arguments via Kuratowski MNC technique. *Transylvanian Advances in Differential Equations* **1** (2021), No. 2, 1-20.

Dédicaces

Je dédie ce modeste travail
♡ ***À ma très chère femme***
Pour la patience et le soutien dont elle a fait preuve pendant toute la durée de cette thèse.

♡ ***À mes enfants****:*
Mohamed ,Missoum, Zine-elabiddine, Insaf et Zakaria

♡ ***À la petite famille de Mohamed :***
B. Soumia,Omran,Amine

♡ ***À la petite famille de Insaf :***
S.Noureddine

TELLI Benoumran

Abstract

In this thesis, we discuss the existence and uniqueness of integrable and continuous solutions for a class of initial and of boundary value problem for nonlinear implicit fractional differential equations and inclusions (NIFDE for short) with Riemann-Liouville fractional derivative, Hadamard fractional derivative, Caputo's derivative and Katugampola fractional derivative .All results in this study are established by means of fixed points theorems .

Keywords: Initial value problem, boundary value problems, Caputo's fractional derivative, Implicit fractional-order differential equation, fixed point, integrable solution, inclusion, local and nonlocal conditions.
(2010) Mathematics Subject Classifications: 26A33, 34A08, 34A37, 34A60,34K05

Résumé

Dans cette thèse, nous discutons l'existence et l'unicité des solutions intégrables et continues pour une classe de problèmes *à* valeurs initiales et aux limites pour des équations et des inclusions factionnaires implicites non linéaires (NIFDE pour le short) avec la dérivée fractionnaires au sens de Riemann-Liouville, la dérivée factionnaires de Hadamard, Caputo et de Katugampola. Tous les résultats de cette étude sont établis par l'approche de points fixes .

Mots clés: Problème *à* valeur initiale,problème aux limites ,la dérivée fractionnaires de Hadamard, de Caputo et la dérivée factionnaires de Katugampola,les équations et les inclusions factionnaires implicites non linéaires, point fixe,solution intégrable , inclusion, conditions non locales.

الملخص

في هذه الأطروحة نناقش وجود الحلول القابلة للمكاملةوالمستمرة و وحدانيتها لفئة مشكلة القيمة الإبتدائية و الحدية للمعادلات التفاضلية والتصنيفات الضمنية و الغير خطية مع المشتق الكسري لريمان-ليوفيل، هادامار، كابتو و المشتق الكسري كانو جامبولا سيتم الحصول على نتائجها عن طريق نظريات النقاط الصامدة.

الكلمات المفتاحية: مشكلة القيم الإبتدائية، مشكلة القيم الحدية، المشتق الكسري،معادلات تفاضلية كسرية، التأخير المحدود، فضاء باناخ، النقطة الصامدة، الحل المتكامل، شروط غير محلية.

Contents

Introduction

Fractional calculus is a generalization of differentiation and integration to arbitrary order (non-integer) fundamental operator D^{α}_{a+} where $a, \alpha \in \mathbb{R}$. Several approaches to fractional derivatives exist: Riemann-Liouville (RL), Hadamard, Grunwald-Letnikov (GL), Weyl and Caputo etc. The Caputo fractional derivative is well suitable to the physical interpretation of initial conditions and boundary conditions. We refer readers, for example, to the books [3, 22, 26, 54, 83, 90, 104, 101, 105, 81] and the references therein. In this thesis, we use the Riemann-Liouville fractional derivative, Hadamard fractional derivative, Caputo's derivative and Katugampola fractional derivative.

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications of differential equations of fractional order in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [26, 70, 94, 101, 105, 115]). There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas *et al.* [3], Kilbas *et al.* [83], Lakshmikantham *et al.* [90], and the papers by Agarwal *et al* [10, 11], Belarbi *et al.* [28], Benchohra *et al.* [32], and the references therein.

Fractional differential equations with nonlocal conditions have been discussed in ([8, 14, 58, 68, 49, 67, 99]) and references therein. Nonlocal conditions were initiated by Byszewski [45] when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems (C.P. for short). As remarked by Byszewski ([46, 47]), the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena.

The theory of functional differential equations has emerged as an important branch of nonlinear analysis. It is worthwhile mentioning that several important problems of the theory of ordinary and delay differential equations lead to investigations of functional differential equations of various types (see the books by Hale and Verduyn Lunel [69], Wu [127], and the references therein).

Differential delay equations, or functional differential equations, have been used in modelling scientific phenomena for many years. Often, it has been assumed that the delay is either a fixed constant or is given as an integral in which case is called distributed delay; see for instance the books ([69, 87, 127]), and the papers ([50]).

In the literature devoted to equations with finite delay, the state space is usually the space of all continuous function on $[-r, 0]$, $r > 0$ and $\alpha = 1$ endowed with the uniform norm topology, see the book of Hale and Lunel [69]. For detailed discussion and applications on this topic, we refer the reader to the book by Hale and Verduyn Lunel [69], Hino et al. [71] and Wu [127].

Differential inclusions are generalization of differential equations, therefore all problems considered for differential equations, that is, existence of solutions,continuation of solutions, dependence on initial conditions and parameters, are present in the theory of differential inclusions. Since a differential inclusion usually has many solutions starting at a given point, new issues appear, such as investigation of topological properties of the set of solutions, and selection of solutions with given properties. As a consequence, differential inclusions have been the subject of an intensive study of many researchers in the recent decades; see, for example, the monographs [23, 44, 63, 73, 76, 111, 118] and the papers of Bressan and Colombo [41, 42].

Implicit differential equations involving the regularized fractional derivative were analyzed by many authors, in the last year; see for instance [121, 6, 7, 2] and the references therein.

The problem of the existence of solutions of Cauchy-type problems for ordinary differential equations of fractional order and without delay in spaces of integrable functions was studied in some works [82, 109]. The similar problem in spaces of continuous functions was studied in [123].

To our knowledge, the literature on integral solutions for fractional differential equations is very limited. El-Sayed and Hashem [59] studies the existence of integral and continuous solutions for quadratic integral equations. El-Sayed and Abd El Salam considered L^p-solutions for a weighted Cauchy problem for differential equations involving the Riemann-Liouville fractional derivative.

Motivated by the above works, this thesis is devoted to the existence and uniqueness of integrable solutions for the nonlocal problem, for fractional order implicit differential equation,L^1-Solutions of the initial value problems for implicit differential equations with Hadamard fractional derivative, and Investigation of the neutral fractional differential inclusions of Katugampola-type involving both retarded and advanced arguments .

In the following we give an outline of our thesis organization, consisting of **5 chapters**.

The first chapter gives some notations, definitions, lemmas and fixed point theorems which are used throughout this thesis.

In Chapter 2, we study of the existence and uniqueness of integrable solutions for the nonlocal problem, for fractional order implicit differential equation

$$^cD^{\alpha}y(t) = f(t, y(t), ^cD^{\alpha}y(t)),\ t \in J := [0, T],\ 0 < \alpha \leq 1,$$

$$y(0) = y_0 - g(y),$$

where $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, $g : L^1(J, \mathbb{R}) \to \mathbb{R}$ is a continuous function , $y_0 \in \mathbb{R}$, and $^cD^{\alpha}$ is the Caputo fractional derivative.

In Chapter 3, we deal with the L^1-Solutions of the Boundary Value Problem for Implicit Fractional Order Differential Equations

$$^cD^\alpha y(t) = f(t, y, ^c D^\alpha y(t)),\ t \in J := [0, T],\ 1 < \alpha \leq 2,$$

$$y(0) = g(y),\ y(T) = y_T$$

where $^cD^\alpha$ is the Caputo fractional derivative,$y_T \in \mathbb{R}$, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, and $g : L^1(J, \mathbb{R}) \to \mathbb{R}$ is a continuous function.

In Chapter 4, we shall be concerned with L^1-Solutions of the initial value problems for implicit differential equations with Hadamard fractional derivative

$$^HD^\alpha_{1^+} y(t) = f(t, y(t), ^H D^\alpha_{1^+} y(t)),\ t \in J := [1, T],\ 0 < \alpha \leq 1,$$

$$^HI^{1-\alpha}_{1^+} y(1) = b,$$

where $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, $b, T \in \mathbb{R}$ with $T > 1$, and $^HD^\alpha_{1^+}$ is the Hadamard fractional derivative.

In Chapter 5, we study the Investigation of the neutral fractional differential inclusions of Katugampola-type involving both retarded and advanced arguments

$$^\varrho D^\xi_{n^+}(w(t) - q(t, w^t)) \in K(t, w^t), \qquad t \in \mathcal{J} := [n, m], \quad 1 < \xi \leq 2,$$

$$w(t) = \chi(t), \qquad t \in [n - s, n], \quad s > 0,$$

$$w(t) = \psi(t), \qquad t \in [m, m + \gamma], \quad \gamma > 0,$$

where a given function $K : \mathcal{J} \times C([-s, \gamma], \mathbb{R}]) \to \mathcal{P}(\mathbb{R})$ exists so that $\chi, \psi \in \mathcal{C}([n - s, m + \gamma], \mathbb{R}])$ via $\chi(n) = 0$ and $\psi(m) = 0$, and a given mapping $q : \mathcal{J} \times \mathcal{C}([-s, \gamma], \mathbb{R}]) \to \mathbb{R}$ exists such that $q(n, \chi^n) = 0$ and $q(m, \psi^m) = 0$. The element of $\mathcal{C}([-s, \gamma], \mathbb{R}])$, denoted by w^t, is defined as follows:

$$w^t(\tau) := w(t + \tau),\ \tau \in [-s, \gamma].$$

Chapter 1

Preliminaires

We introduce in this chapter notations, definitions, fixed point theorems and preliminary facts from multi-valued analysis . Also we give some fixed point theorems on the multivalued version which are used throughout this thesis.

1.1 Notations and definitions.

Let $C(J, \mathbb{R})$ be the Banach space of all continuous functions from $J := [0, T]$ into $\mathbb{R}$ with the usual norm

$$\|y\| = sup\{|y(t)| : 0 \leq t \leq T\}.$$

$L^1(J, \mathbb{R})$ denote the Banach space of functions $y : J \to \mathbb{R}$ that are measurable and Lebesgue integrable with the norm

$$\|y\|_{L_1} = \int_0^T |y(t)|dt.$$

Definition 1. *[52]. A map $f : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is said to be L^1-Carathéodory if*

(i) *the map $t \longmapsto f(t, x, y)$ is measurable for each $(x, y) \in \mathbb{R} \times \mathbb{R}$,*

(ii) *the map $(x, y) \longmapsto f(t, x, y)$ is continuous for almost all $t \in J$,*

(iii) *For each $q > 0$, there exists $\varphi_q \in L^1(J, \mathbb{R}_+)$ such that*

$$|f(t, x, y)| \leq \varphi_q(t)$$

for all $|x| \leq q$, $|y| \leq q$ and for a.e. $t \in J$.

The map f is said of Carathéodory if it satisfies (i) and (ii).

Definition 2. *An operator $T : E \longrightarrow E$ is called compact if the image of each bounded set $B \subset E$ is relatively compact i.e ($\overline{T(B)}$ is compact). T is called completely continuous operator if it is continuous and compact.*

Theorem 1. *(Diestel-Ruess-Schachermayer [55]). Let $p \in [1, \infty[$. Let $M \subset L^p(J, E)$ be countable and suppose there exists some $\nu \in L^p(J, \mathbb{R}_+)$ with $\|u(t)\|_E \leq \nu(t)$ a.e on J for all $u \in M$. If $M(t)$ is relatively compact in E for a.e $t \in J$, then M is weakly relatively compact in $L^p(J, E)$.*

Theorem 2. *(Kolmogorov compactness criterion [53]). Let $\Omega \subseteq L^p(J, \mathbb{R}),\ 1 \leq p < \infty$. If*

(i) *Ω is bounded in $L^p(J, \mathbb{R})$, and*

(ii) *$u_h \longrightarrow u$ as $h \longrightarrow 0$ uniformly with respect to $u \in \Omega$,i.e.,*

$$\lim_{h \to 0} \sup_{u \in \Omega} \|u_h - u\|_p = 0.$$

then Ω is relatively compact in $L^p(J, \mathbb{R})$,
where

$$u_h(t) = \frac{1}{h} \int_t^{t+h} u(s)ds.$$

1.2 Fractional Calculus.

1.2.1 Fractional integral and derivative of Riemann-Liouville

As we know, the integration of order n (n is integer) of the function h is given by

$$\int_a^x dt_1 \int_a^{t_1} dt_2 ... \int_a^{t_{n-1}} h(t_n)dt_n = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} h(t)dt \tag{1.1}$$

by using the function $\Gamma(.)$, we can give to this formula a sens when n is non integer.

Definition 3. *([83, 105]) The fractional (arbitrary) order integral of the function $h \in L^1([a,b],\mathbb{R})$ of order $\alpha > 0$ is defined by*

$$I_a^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)ds.$$

When $a = 0$,we write

$$I^\alpha h(t) = h(t) * \varphi_\alpha(t),$$

where

$$\varphi_\alpha(t) := \begin{cases} \dfrac{t^{\alpha-1}}{\Gamma(\alpha)} & for\ t > 0, \\ 0 & for\ t \leq 0, \end{cases}$$

and

$$\varphi_\alpha \to \delta(t)\ as\ \alpha \to 0,$$

where δ is the delta function.

Definition 4. *([83, 105]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ of function $h \in L^1([a,b],\mathbb{R})$, is given by*

$$(D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \Big(\frac{d}{dt}\Big)^n \int_a^t (t-s)^{n-\alpha-1} h(s)ds,$$

Here $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.
If $\alpha \in (0,1]$, then

$$(D_{a+}^\alpha h)(t) = \frac{d}{dt} I_{a+}^{1-\alpha} h(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-s)^{-\alpha} h(s)ds.$$

Definition 5. *([83]) The Caputo fractional derivative of order $\alpha > 0$ of function $h \in L^1([a,b],\mathbb{R})$ is given by*

$$({}^cD_{a+}^\alpha h)(t) = I_{a+}^{n-\alpha} h^{(n)}(t) = \int_a^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} h^{(n)}(s)ds,$$

Here $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.
If $\alpha \in (0,1]$, then

$$({}^cD_{a+}^\alpha h)(t) = \int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \frac{d}{ds} h(s)ds$$

.

Remark 1. *([83])The fractional derivative of Riemann-Liouville and the fractional derivative of Caputo are connected with each other by the following relation:*

$$({}^cD^\alpha_{a+}h)(t) = D^\alpha_{a+}\left[h(t) - \sum_{k=0}^{n-1}\frac{h^{(k)}(a)}{k!}(t-a)^k\right].$$

The following properties are some of the main ones of the fractional derivatives and integrals.

Lemma 1. *([83]) Let $\alpha > 0$, $h \in L^1([0,b],\mathbb{R})$,${}^cD^\alpha h \in L^1([0,b],\mathbb{R})$.*
Then the differential equation

$${}^cD^\alpha h(t) = 0$$

has solution

$$h(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1}t^{n-1},\ c_i \in \mathbb{R},\ i = 0,1,2,...,n-1,\ n = [\alpha]+1.$$

Lemma 2. *([83]) Let $\alpha > 0$,$h \in L^1([0,b],\mathbb{R})$,${}^cD^\alpha h \in L^1([0,b],\mathbb{R})$.*
Then

$$I^{\alpha c}D^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1}t^{n-1},$$

for some $c_i \in \mathbb{R},\ i = 0,1,2,...,n-1,\ n = [\alpha]+1$.

Proposition 1. *[83] Let $\alpha,\ \beta > 0$. Then we have*

(1) *$I^\alpha :\ L^1(J,\mathbb{R}) \to L^1(J,\mathbb{R})$, and if $f \in L^1(J,\mathbb{R})$, then*

$$I^\alpha I^\beta f(t) = I^\beta I^\alpha f(t) = I^{\alpha+\beta}f(t).$$

(2) *If $f \in L^p(J,\mathbb{R}),\ 1 \le p < +\infty$, then $\|I^\alpha f\|_{L^p} \le \frac{T^\alpha}{\Gamma(\alpha+1)}\|f\|_{L^p}$.*

(3) *The fractional integration operator I^α is linear.*

(4) *The fractional order integral operator I^α maps $L^1(J,\mathbb{R})$ into itself.*

(5) *When $\alpha = n \in \mathbb{N}, I^\alpha_0$ is the n-fold integration.*

(6) *The Caputo derivative of a constant is equal to zero.*

1.2.2 Fractional integral and derivative of Hadamard

We define space

$$AC_\delta^n([a,b],\mathbb{R}) := \{h : [a,b] \to \mathbb{R}, \delta^{n-1}(h) \in AC([a,b],\mathbb{R})\},$$

where $n \in \mathbb{N}$ and $AC([a,b],\mathbb{R})$ is the set of absolutely continuous functions on $[a,b]$ and $\delta := t\frac{d}{dt}$. As we know, the integration of order n (n is integer) of the function h is given by

$$\int_a^x \frac{dt_1}{t_1}\int_a^{t_1}\frac{dt_2}{t_2}...\int_a^{t_{n-1}} h(t_n)\frac{dt_n}{t_n} = \frac{1}{(n-1)!}\int_a^x (\ln\frac{x}{t})^{n-1}h(t)\frac{dt}{t} \tag{1.2}$$

by using the function $\Gamma(.)$, we can give to this formula a sens when n is non integer.

Definition 6. *([83, 105]).The Hadamard fractional integral of order $\alpha > 0$ of function $h \in L^1([1,T],\mathbb{R})$, is given by*

$${}^H I_{1^+}^\alpha h(t) := \frac{1}{\Gamma(\alpha)}\int_1^t (\ln\frac{t}{s})^{\alpha-1}h(s)\frac{ds}{s},$$

where $\Gamma(.)$ is the Euler gamma function defined by

$$\Gamma(\alpha) := \int_0^\infty t^{\alpha-1}e^{-t}dt, \alpha > 0.$$

Definition 7. *([83, 105]). The Hadamard fractional derivative of order $\alpha > 0$ of function $h \in L^1([1,T],\mathbb{R})$, is given by*

$${}^H D_{1^+}^\alpha h(t) := \frac{1}{\Gamma(n-\alpha)}\Big(t\frac{d}{dt}\Big)^n\int_1^t (\ln\frac{t}{s})^{n-\alpha-1}h(s)\frac{ds}{s},$$

here $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α.
If $\alpha \in (0,1]$, then

$${}^H D_{1^+}^\alpha h(t) := \frac{1}{\Gamma(1-\alpha)}\Big(t\frac{d}{dt}\Big)\int_1^t (\ln\frac{t}{s})^{-\alpha}h(s)\frac{ds}{s}.$$

Remark 2. *The function $f \in L^1([1,T],\mathbb{R})$, possesses the Hadamard derivative ${}^H D_{1^+}^\alpha f$ of order α, if ${}^H I_{1^+}^{1-\alpha} f \in AC^1([1,T],\mathbb{R})$.*

The following properties are some of the main ones of the fractional derivatives and integrals.

Lemma 3. *[83] If $\alpha,\beta > 0$ and $1 < t \le T < \infty$, then*

- $[{}^H I^{\alpha}_{1^+}(\ln(.))^{\beta-1}](t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\ln(t))^{\beta+\alpha-1}$.
- $[({}^H D^{\alpha}_{1^+}\ln(.))^{\alpha-1}](t) = 0$.

Proposition 2. *[83] Let $0 < \alpha \leq 1,\ 0 < \beta \leq 1$. Then we have*

(i) ${}^H I^{\alpha}_{1^+} :\ L^1(J,\mathbb{R}) \to L^1(J,\mathbb{R})$*, and if $f \in L^1(J,\mathbb{R})$, then*

$$ {}^H I^{\alpha}_{1^+}\ {}^H I^{\beta}_{1^+} f = \ {}^H I^{\beta}_{1^+}\ {}^H I^{\alpha}_{1^+} f(t) = \ {}^H I^{\alpha+\beta}_{1^+} f.$$

(ii) *If $f \in L^1(J,\mathbb{R})$, then*

$$ {}^H D^{\alpha}_{1^+}\ {}^H I^{\alpha}_{1^+} f = f.$$

(iii) *If $f \in L^1([a,b],\mathbb{R})$, then $\|\ {}^H I^{\alpha}_{1^+} f\|_{L_1} \leq \frac{K(\alpha)}{\Gamma(\alpha)}\|f\|_{L_1}$,*

where

$$K(\alpha) = \int_0^{\ln\frac{b}{a}} t^{\alpha-1}e^t dt.$$

(iv) *The fractional integration operator ${}^H I^{\alpha}_{1^+}$ is linear.*

(v) *The fractional order integral operator ${}^H I^{\alpha}_{1^+}$ maps L^1 into itself continuously.*

Theorem 3. *[80](Thm.3.1) The space $AC^n_\delta[a,b]$ consists of those and only those functions $g(t)$, which are represented in the form*

$$g(t) = \frac{1}{(n-1)!}\int_1^t (\ln\frac{t}{s})^{n-1}\varphi(s)ds + \sum_{k=0}^{n-1} c_k(\ln\frac{t}{a})^k,$$

where $\varphi \in L^1([1,T]$ and c_k (k=0,1,...,n-1) are arbitrary constants.

Lemma 4. *[83](Thm.3.2) If $y \in L^1([1,T],\mathbb{R})$ and ${}^H I^{1-\alpha}_{1^+} y \in AC^1([1,T],\mathbb{R})$ then*

$$ {}^H I^{\alpha}_{1^+}\ {}^H D^{\alpha}_{1^+} y(t) = y(t) - \frac{{}^H I^{1-\alpha}_{1^+} y(1)}{\Gamma(\alpha)}(\ln t)^{\alpha-1}.$$

1.2.3 Fractional integral and derivative of Katugampola

Let us define a space $X^l_c(n,m)$ $(c \in \mathbb{R}, 1 \leq l \leq \infty)$ of real-valued Lebesgue measurable functions, k on $[n,m]$ for which $\|k\|_{X^l_c(n,m)} < \infty$ where

$$\|k\|_{X^l_c} := (\int_n^m |v^c k(v)|^l \frac{dv}{v})^{\frac{1}{l}}, \qquad (1 \leq l < \infty),$$

$$\|k\|_{X_c^\infty} := ess \sup_{n \le v \le m} (|v^c k(v)|).$$

Specifically, if $c = \frac{1}{l}$, then the space $X_c^l(n,m)$ coincides with the $L^l[n,m]$ space. Let us define the following space:

$$AC^p[n,m] := \{q : [n,m] \to \mathbb{R}, \quad \delta^{p-1}(q) \in AC[n,m]\},$$

where $AC[n,m]$ is a set including functions with the absolute continuity property from $[n,m]$ into $\mathbb{R}$ with $\delta := t\frac{d}{dt}$.

Let $\mathcal{C}[n,m]$ be the Banach space of all continuous functions from $[n,m]$ into $\mathbb{R}$ with the usual norm

$$\|u\|_{[n,m]} := \sup\{|u(t)| : t \in [n,m]\}.$$

$$\mathcal{C}(I) := \mathcal{C}([n-s, m+\gamma], \mathbb{R}) \tag{1.3}$$

with a norm:

$$\|u\|_\infty := \sup\{|u(t)| : t \in [n-s, m+\gamma]\}.$$

Here we want to present the fractional integration, which generalizes both the Riemann-Liouville and Hadamard fractional integrals into a single form. New generalization is based on the observation that, for $p \in \mathbb{N}$,

$$\int_a^x t_1^{\varrho-1} dt_1 \int_a^{t_1} t_2^{\varrho-1} dt_2 ... \int_a^{t_{p-1}} t_p^{\varrho-1} h(t_p) dt_p = \frac{1}{(p-1)!} \int_a^x (\frac{x^\varrho - t^\varrho}{\varrho})^{p-1} t^{\varrho-1} h(t) dt \tag{1.4}$$

by using the function $\Gamma(.)$, we can give to this formula a sens when p is non integer.

Definition 8. *[77] Let $\xi > 0, \varrho > 0$. The Katugampola generalized integral of fractional integral order ξ for a function $z \in X_c^l(n,m)$ is displayed by*

$${}^\varrho I_{n^+}^\xi z(t) := \frac{1}{\Gamma(\xi)} \int_n^t (\frac{t^\varrho - v^\varrho}{\varrho})^{\xi-1} z(v) v^{\varrho-1} dv,$$

where the Euler gamma function is represented by $\Gamma(.)$ which is expressed as follows:

$$\Gamma(\xi) := \int_0^\infty t^{\xi-1} e^{-t} dt, \xi > 0.$$

Definition 9. *[78]Let $\xi > 0, \varrho > 0$.The Katugampola generalized derivative of order ξ for a given function $z \in X_c^l(n,m)$ is expressed as:*

$$\begin{aligned} {}^\varrho D_{n^+}^\xi z(t) &= \delta_\varrho^k \; {}^\varrho I_{n^+}^{k-\xi} z(t) \\ &= \frac{1}{\Gamma(k-\xi)} (t^{1-\varrho} \frac{d}{dt})^k \int_n^t (\frac{t^\varrho - v^\varrho}{\varrho})^{k-\xi-1} z(v) v^{\varrho-1} dv, \end{aligned}$$

in which $k = [\xi] + 1$ and $[\xi]$ represents the integer part of ξ and $\delta_\varrho^k := (t^{1-\varrho}\frac{d}{dt})^k$.

Theorem 4. *[78]Let $\xi > 0, \varrho > 0, k = [\xi] + 1$. Then for $t > n$*

1. $\lim_{\varrho \to 1} \ {}^{\varrho}I^{\xi}_{n^+} z(t) = I^{\xi}_{n} z(t)$,

2. $\lim_{\varrho \to 0} \ {}^{\varrho}I^{\xi}_{n^+} z(t) = {}^{H} I^{\xi}_{n^+} z(t)$,

3. $\lim_{\varrho \to 1} \ {}^{\varrho}D^{\xi}_{n^+} z(t) = (D^{\xi}_{n^+} z)(t)$

4. $\lim_{\varrho \to 0} \ {}^{\varrho}D^{\xi}_{n^+} z(t) = ({}^{H}D^{\xi}_{n^+} z)(t)$

Let us discuss some essential properties of the fractional derivatives and integrals as follows:

Lemma 5. *[102] Assume $\xi > 0, \varrho > 0$; then we have*

$$(\ {}^{\varrho}I^{\xi}_{n^+} {}^{\varrho}D^{\xi}_{n^+} z)(t) = z(t) + c_1 (\frac{t^{\varrho} - n^{\varrho}}{\varrho})^{\xi - 1} + c_2 (\frac{t^{\varrho} - n^{\varrho}}{\varrho})^{\xi - 2} + ... + c_k (\frac{t^{\varrho} - n^{\varrho}}{\varrho})^{\xi - k}, \quad (1.5)$$

where $c_j \in \mathbb{R}, j = 1, 2, ..., k$, and $k = [\xi] + 1$.

Lemma 6. *[102] If $x > n$, we have*

- $[{}^{\varrho}I^{\xi}_{n^+} (\frac{t^{\varrho} - n^{\varrho}}{\varrho})^{\gamma - 1}](x) = \frac{\Gamma(\gamma)}{\Gamma(\gamma + \xi)} (\frac{x^{\varrho} - n^{\varrho}}{\varrho})^{\gamma + \xi - 1}$,

- $[{}^{\varrho}D^{\xi}_{n^+} (\frac{t^{\varrho} - n^{\varrho}}{\varrho})^{\xi - 1}](x) = 0.$

1.3 Multi-valued analysis

If X is a Hausdorff topological space and K be a subset of X. We denote by:

$$\mathcal{P}(X) = \{K \subset X : K \neq \emptyset\},$$

$$\mathcal{P}_{cl}(X) = \{K \subset \mathcal{P}(X) : K \ is \ closed\},$$

$$\mathcal{P}_{cp}(X) = \{K \subset \mathcal{P}(X) : K \ is \ compact\},$$

If $(X, \|.\|)$ is a normed space. We denote by:

$$\mathcal{P}_{b}(X) = \{K \subset \mathcal{P}(X) : K \ is \ bounded\},$$

$$\mathcal{P}_{cv}(X) = \{K \subset \mathcal{P}(X) : K \ is \ convex\},$$

$$\mathcal{P}_{b,cl}(X) = \mathcal{P}_b(X) \cap \mathcal{P}_{cl}(X),$$

$$\mathcal{P}_{cv,cp}(X) = \mathcal{P}_{cv}(X) \cap \mathcal{P}_{cp}(X).$$

Let $A, B \in \mathcal{P}(X)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$ the Hausdorff distance between A and B given by:

$$H_d(A, B) = max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\},$$

where $d(A, b) = \inf_{a\in A} d(a, b)$ and $d(a, B) = \inf_{b\in B} d(a, b)$. As usual, $d(x, \emptyset) = +\infty$.

1.3.1 Definitions

Definition 10. *A set valued map (also called multivalued map) $F : X \to \mathcal{P}(Y)$ is an application which associate with any $x \in X$ a subset $F(x)$ which belongs to $\mathcal{P}(Y)$, where X and Y are two sets.*

Definition 11. *[24]*

1. *A multi-valued map $F : X \to \mathcal{P}(Y)$ is convex (closed)valued if $F(x)$ is convex (closed) for all $x \in X$.*

2. *F is bounded on bounded sets if $F(\mathcal{B}) = \cup_{x\in\mathcal{B}} F(x)$ is bounded in X for all $\mathcal{B} \in \mathcal{P}_b(X)$, i.e. $sup_{x\in\mathcal{B}}\{sup\{|y| : y \in F(x)\}\} < \infty$.*

3. *F is said to be completely continuous if $F(\mathcal{B})$ is relatively compact for every $\mathcal{B} \in \mathcal{P}_b(X)$.*

4. *A multi-valued map $G : X \to \mathcal{P}(X)$ has a fixed point if there exists $x \in X$ such that $x \in G(x)$. The set of fixed points of the multi-valued operator G will be denoted by $FixG$.*

Definition 12. *[24] Let X and Y be metric spaces. A set-valued map F from X to Y is characterized by its graph $Gr(F)$, the subset of the product space $X \times Y$ defined by*

$$Gr(F) := \{(x, y) \in X \times Y : y \in F(x)\}.$$

F is called closed graph if $Gr(F)$ is closed in $X \times Y$ (i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in F(x_n)$ imply $y_* \in F(x_*)$).

1.3.2 Continuity of set-valued maps

Let X ,Y be Hausdorff topological spaces and let $F : X \to \mathcal{P}(Y)$ be a multivalued map.

Definition 13. *F is called upper semi-continuous (u.s.c. for short) on $x_0 \in X$ if the set $F(x_0)$ is a nonempty subset of Y and for each open set U of Y containing $F(x_0)$, there exists an open neighborhood V of x_0 such that $F(V) \subset U$.*

Example 1. *Let $\Omega \neq \emptyset$ be a subset of a Banach space. Then the general u.s.c. $F : \Omega \to \mathcal{P}_{cv,cp}(\mathbb{R})$ is given by $F(\omega) = [\varphi(\omega), \psi(\omega)]$, where $\varphi : \Omega \to \mathbb{R}$ is l.s.c. , $\psi : \Omega \to \mathbb{R}$ is u.s.c. and $\varphi(\omega) \leq \psi(\omega)$ on Ω.*

Proposition 3. *[73] Let X, Y be Hausdorff topological spaces and let $F : X \to \mathcal{P}_{cl}(Y)$ be a multivalued map.*
If F is upper semicontinuous ,then $Gr(F)$ is closed in $X \times Y$. Conversely, if $Gr(F)$ is closed and $\overline{F(X)}$ is compact, then F is upper semicontinuous.

Proposition 4. *[73] Let X be Hausdorff topological space ,$Y \in T_4$ be normal space and let $F, G : X \to \mathcal{P}_{cl}(Y)$ are two upper semicontinuous mappings .Then*

a *the map $F \cup G : X \to \mathcal{P}_{cl}(Y)$ is upper semicontinuous ;*

b *the map $F \cap G : X \to \mathcal{P}_{cl}(Y)$ is upper semicontinuous .*

Proposition 5. *[73] Let X, Y be Hausdorff topological spaces and $F : X \to \mathcal{P}_{cp}(Y)$ upper semicontinuous . Then $F(A)$ is compact for each compact $A \subset X$.*

Definition 14. *If X is a set, a function d*

$$d : X \times X \to [0, +\infty]$$

is called a generalized metric space (gms) on X, provided that for $x, y, z \in X$,

- $d(x, y) = 0$ *if and only if* $x = y$
- $d(x, y) = d(y, x)$
- $d(x, z) \leq d(x, y) + d(y, z)$

Proposition 6. *[86] Let (X, d) metric space and H_d the Hausdorff distance. Then we have*

(1) *$(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space*

(2) *$(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space*

(3) *If$(X, d)$is a complete metric space, then $(\mathcal{P}_{cl}(X), H_d)$is a complete generalized metric space .*

(4) *If$(X, d)$is a complete metric space, then $(\mathcal{P}_{b,cl}(X), H_d)$ is a complete metric space.*

Definition 15. *Let $(X; d)$ be a separable metric space. A multivalued map $F : J \to \mathcal{P}_{cl}(X)$ is said to be measurable if, for each $y \in X$, the function*

$$t \longmapsto d(y, F(t)) = inf\{d(y, z) : z \in F(t)\}$$

is measurable.

Theorem 5. *([63]) Let X be a Banach separable space , $(T; \Sigma)$ a measurable space, $f : T \to X$ is $\Sigma-$ measurable and $\rho : T \to (0, +\infty)$ is $\Sigma-$ measurable.*
Then the multi-valued map

$$F : T \to \mathcal{P}_{cl}(X)$$

is formulated by $F(t) := \overline{B(f(t), \rho(t))}$ is $\Sigma-$ measurable.

Definition 16. *A multi-valued map $F(t) : J \times E \to \mathcal{P}(E)$ is said to be L^1-Carathéodory if*

(i) *$t \to F(t, x)$ is measurable for each $x, y \in E$;*

(ii) *$x \to F(t, x)$ is upper semicontinuous for almost all $t \in J$;*

(iii) *For each $q > 0$, there exists $\varphi_q \in L^1(J, \mathbb{R}_+)$ such that*

$$\|F(t, x)\|_{\mathcal{P}} = sup\{|f| : f \in F(t, x)\} \leq \varphi_q(t)$$

for all $|x| \leq q$ and for a.e. $t \in J$.

Definition 17. *Let X, Y be nonempty sets and $F : X \to \mathcal{P}(Y)$. The single-valued operator $f : X \to Y$ is called a selection of F if and only if $f(x) \in F(x)$, for each $x \in X$.*

Definition 18. *Let multi-valued map $F : J \times E \to \mathcal{P}(E)$.*
For each $x \in \mathcal{C}([n,m], E)$ define the set a selection of F by

$$S_{F,x} := \left\{ z \in L^1[n,m] : z(t) \in F(t, x(t)), (a.e.) \quad t \in \mathcal{J} \right\}.$$

Theorem 6. *(Kuratowski-Ryll Nardzewski ,[63]) Let $(X; d)$ be a complete separable metric space , $(T; \Sigma)$ a measurable space.Then every measurable multi-valued map $F : T \to \mathcal{P}_{cl}(X)$ has a measurable selection.*

For more details on multivalued maps and the proof of the known results cited in this section we refer interested reader to the books of Aubin and Cellina [23], Deimling [54], Gorniewicz [63], Hu and Papageorgiou [73], Smirnov [111], Tolstonogov [118], Djebali and al [56] and Graef and al [64] .

1.4 Some fixed point theorems

Definition 19. *([19]) Let (M, d) be a metric space. The map $T : M \longrightarrow M$ is said to be Lipschitzian if there exists a constant $k > 0$ (called Lipschitz constant) such that*

$$d(T(x), T(y)) \leq kd(x, y), \ for\ all\ x,\ y \in M$$

A Lipschitzian mapping with a Lipschitz constant $k < 1$ is called contraction.

Definition 20. *A multivalued operator $N : X \to \mathcal{P}_{b,cl}(X)$ is called:*
(a) *γ-Lipschitz if there exists $\gamma > 0$ such that*

$$H_d(N(x), N(y)) \leq \gamma d(x, y), \ for\ all\ x,\ y \in X;$$

(b) *a contraction if it is γ-Lipschitz with $\gamma < 1$.*

Theorem 7. *(Banach's fixed point theorem [65]). Let C be a nonempty closed subset of a Banach space X, then any contraction mapping T of C into itself has a unique fixed point.*

Theorem 8. *(Schauder fixed point theorem ([53]) Let E a Banach space and Q be a nonempty closed and convex subset of E and $T : Q \longrightarrow Q$ is compact, and continuous map. Then T has at least one fixed point in Q.*

Next we state two multi-valued fixed point theorems

Lemma 7. *[65](Nonlinear alternative for Kakutani maps) Let E be a Banach space, G a closed convex subset of E, $\mathcal{U}$ an open subset of G and $0 \in \mathcal{U}$. Suppose that $\mathcal{E} : \overline{\mathcal{U}} \longrightarrow \mathcal{P}_{cp,cv}(G)$ is a upper semi-continuous and compact map. Then either*

(i) *$\mathcal{E}$ has a fixed point in $\overline{\mathcal{U}}$, or*

(ii) *there is a $\omega \in \partial\overline{\mathcal{U}}$ and $\sigma \in (0, 1)$ with $w \in \sigma\mathcal{E}(w)$.*

Lemma 8. *[51]. (Nadler-Covitz)Let (X, d) be a complete metric space . If $\mathcal{E} : X \to \mathcal{P}_{cl}(X)$ is a contraction , then $Fix(\mathcal{E}) \neq \phi$.*

Chapter 2

Integrable Solutions for Implicit Fractional Order Differential Equations with Nonlocal Condition[1]

2.1 Introduction and Motivations

In the last years a lot contributions in the fixed point theory in Banach spaces. Especially, that theory turns out to be a very useful tool in existence of solutions continuous and integral for several types of differential equations for example see Benchohra *et al.* [35, 37], Lakshmikantham *et al.* [89], El-Sayed *et al.* [57, 60], Souid [112], Witthayarat *et al.* [126] and the references therein.

Benchohra *et al.* in [30] devoted to some existence of continuous solutions for the following nonlocal problem

$$^{c}D^{\alpha}y(t) = f(t, y(t)), \qquad t \in J := [0, T], \quad 0 < \alpha < 1,$$

$$y(0) + g(y) = y_0,$$

where $^{c}D^{\alpha}$ is the Caputo fractional derivative of order α, $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $g : C(J, \mathbb{R}) \to \mathbb{R}$ is a continuous function.

Benchohra *et al.* in [37] investigated the existence of integrable solutions for the

[1]**B. Telli** and M. S. Souid, Integrable Solutions for Implicit Fractional Order Differential Equations with Nonlocal Condition. *Transylvanian Journal of Interdisplinary Mathematics* **6** (2020), No. 2, 1-13.

following nonlocal boundary value problem

$$^{c}D^{\alpha}y(t) = f(t, y, ^{c}D^{\alpha}y(t)),\ t \in J := [0, T],\ 1 < \alpha \leq 2,$$

$$y(0) = g(y),\ y(T) = y_T$$

where $^{c}D^{\alpha}$ is the Caputo fractional derivative of order α, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function and $g : L^1(J, \mathbb{R}) \to \mathbb{R}$ is a continuous function.

Motivated by [30, 36] we are interested in this chapter with the existence and uniqueness of integrable solutions for the nonlocal problem, for fractional order implicit differential equation

$$^{c}D^{\alpha}y(t) = f(t, y(t), ^{c}D^{\alpha}y(t)),\ t \in J := [0, T],\ 0 < \alpha \leq 1, \tag{2.1}$$

$$y(0) = y_0 - g(y), \tag{2.2}$$

where $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, $g : L^1(J, \mathbb{R}) \to \mathbb{R}$ is a continuous function , $y_0 \in \mathbb{R}$, and $^{c}D^{\alpha}$ is the Caputo fractional derivative.

The nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena, details are found in [45, 46, 47].

The rest of this chapter is organized as follows: In subsection 2, we give two results, the first one is based on Theorem 8 and the second one on the contraction principle . An example is given in subsection 3 to demonstrate the application of our main results.

2.2 Existence of solutions

Let us start by defining what we mean by an integrable solution of the nonlocal problem (2.1) – (2.2).

Definition 21. *By a solution of the nonlocal problem* (2.1) – (2.2) *we mean a function* $y \in L^1(J, \mathbb{R})$ *that satisfies the condition* $y(0) = y_0 - g(y)$ *and the equation* $^{c}D^{\alpha}y(t) = f(t, y(t), ^{c}D^{\alpha}y(t))$ *on* J.

For the existence of solutions for the problem (2.1) – (2.2), we need the following auxiliary lemma.

Lemma 9. *The solution of the nonlocal problem* (2.1) – (2.2) *can be expressed by the integral equation*

$$y(t) = y_0 - g(y) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x_y(s) ds, \tag{2.3}$$

where $x_y \in L^1(J, \mathbb{R})$ *is the solution of the functional integral equation*

$$x_y(t) = f\left(t, y_0 - g(y) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x_y(s) ds, x_y(t)\right). \tag{2.4}$$

Proof. Let ${}^cD^\alpha y(t) = x_y(t)$ in equation (2.1), then

$$x_y(t) = f(t, y(t), x_y(t)) \tag{2.5}$$

Hence, we get equation (2.4).

And Lemma (2) implies that

$$\begin{aligned} y(t) &= y(0) + I^\alpha x_y(t) \\ &= y_0 - g(y) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x_y(s) ds. \end{aligned} \tag{2.6}$$

Hence, we get equation (2.3).

Inversely, we prove that equation (2.3) – (2.4) satisfies the nonlocal problem (2.1) – (2.2)

.

Differentiating (2.3), we get

$${}^cD^\alpha y(t) = x_y(t).$$

By (2.4) we have

$${}^cD^\alpha y(t) = f(t, y(t), {}^c D^\alpha y(t)).$$

A simple calculation give $y(0) = y_0 - g(y)$. This complete the proof of the equivalence between the nonlocal problem (2.1) – (2.2) and the integral equation (2.3) – (2.4) . □

Let us introduce the following assumptions:

(H1) $f : J \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is measurable in $t \in J$, for any $(u_1, u_2) \in \mathbb{R}^2$ and continuous in $(u_1, u_2) \in \mathbb{R}^2$, for almost all $t \in J$.

(H2) There exist two constants $k_1 > 0$ and $0 < k_2 < 1$ such that,,for every $t \in J$,and for every $(u_1, u_2) \in \mathbb{R}^2$,

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq k_1|u_1 - u_2| + k_2|v_1 - v_2|.$$

(H3) There exists a constant $k > 0$ such that, for every $y, y' \in L^1(J, \mathbb{R})$

$$|g(y) - g(y')| \leq k\|y - y'\|_{L_1}.$$

Our first result is based on Schauder fixed point Theorem .

Theorem 9. *Assume that the assumptions* $(H1) - (H3)$ *are satisfied. If*

$$\frac{b_1 T^\alpha}{(1 - b_2)\Gamma(\alpha + 1)} < 1, \tag{2.7}$$

then the nonlocal problem $(2.1) - (2.2)$ *has at least one solution* $y \in L^1(J, \mathbb{R})$*.*

Proof. We first notice that we have:
from assumptions **(H3)**,there exists a constant $M > 0$ such that, for every $y \in L^1(J, \mathbb{R})$

$$|g(y)| \leq M,$$

and from assumptions **(H2)**,there exist $a \in L^1(J, \mathbb{R})$, two constants $b_1 > 0$ and $0 < b_2 < 1$ such that,for every $t \in J$,and for every $(u_1, u_2) \in \mathbb{R}^2$, we have

$$|f(t, u_1, u_2)| \leq |a(t)| + b_1|u_1| + b_2|u_2|.$$

Transform the problem $(2.1) - (2.2)$ into a fixed point problem. Consider the operator

$$H : \ L^1(J, \mathbb{R}) \to L^1(J, \mathbb{R})$$

defined by:

$$(Hy)(t) = y_0 - g(y) + I^\alpha x_y(t), \tag{2.8}$$

where

$$x_y(t) = f(t, y_0 - g(y) + I^\alpha x_y(t), x_y(t)).$$

The operator H is well defined, indeed, for each $y \in L^1(J, \mathbb{R})$, From assumptions **(H1)** ,**(H2)**and **(H3)**, we obtain

$$\begin{aligned}\|Hy\|_{L_1} &= \int_0^T |Hy(t)|dt\\ &= \int_0^T |y_0 - g(y) + I^\alpha x_y(t)|dt\\ &\le T(|y_0| + M) + \int_0^T \Big(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|x_y(s)|ds\Big)dt \end{aligned}\tag{2.9}$$

and

$$\begin{aligned}|x_y(t)| &= |f(t, y(t), x_y(t))|\\ &\le |a(t)| + b_1|y(t| + b_2|x_y(t)|.\end{aligned}$$

Thus

$$|x_y(t)| \le \frac{|a(t)| + b_1|y(t|}{1-b_2}.\tag{2.10}$$

By replacing (2.10) in the inequality (2.9), we obtain

$$\begin{aligned}\|Hy\|_{L_1} &\le T(|y_0| + M) + \int_0^T \Big(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(\frac{|a(s)| + b_1|y(s|}{1-b_2})ds\Big)dt\\ &\le T(|y_0| + M) + \frac{T^\alpha}{(1-b_2)\Gamma(\alpha+1)}\|a\|_{L_1}\\ &\quad+\frac{b_1}{1-b_2}\int_0^T \Big(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|y(s)|ds\Big)dt\\ &\le T(|y_0| + M) + \frac{T^\alpha}{(1-b_2)\Gamma(\alpha+1)}\|a\|_{L_1} + \frac{b_1T^\alpha}{(1-b_2)\Gamma(\alpha+1)}\|y\|_{L_1} < +\infty.\end{aligned}\tag{2.11}$$

Clearly, the fixed point of the operator H are solutions of the problem $(2.1) - (2.2)$.
Let

$$r = \frac{T(|y_0| + M) + \frac{T^\alpha}{(1-b_2)\Gamma(\alpha+1)}\|a\|_{L_1}}{1 - \frac{b_1T^\alpha}{(1-b_2)\Gamma(\alpha+1)}}$$

and consider the set

$$B_r = \{y \in L^1(J, \mathbb{R}) : \|y\|_{L_1} \le r\}.$$

Clearly B_r is nonempty, bounded, convex and closed.

Now, we shall show that H satisfies the assumption of Theorem 8. The proof is given in following several steps.

Step 1. $HB_r \subset B_r$.

For each $y \in B_r$, from (2.7), and (2.11) we get

$$\begin{aligned}\|Hy\|_{L_1} &\leq T(|y_0| + M) + \frac{T^\alpha}{(1-b_2)\Gamma(\alpha+1)}\|a\|_{L_1} + \frac{b_1 T^\alpha}{(1-b_2)\Gamma(\alpha+1)}\|y\|_{L_1} \\ &\leq r.\end{aligned}$$

Then $HB_r \subset B_r$.

Step 2. H is continuous.

Assumption **(H1)**, **(H2)** and the hypothesis that g is continuous imply that H is continuous.

Step 3. H is compact.

We will show HB_r is relatively compact. Clearly HB_r is bounded in $L^1(J, \mathbb{R})$, then **(i)** of Theorem (2) is satisfied.

It remains to show

$$\lim_{h\to 0} \sup_{y\in B_r} \|(Hy)_h - (Hy)\|_1 = 0.$$

Let $y \in B_r$, then we have

$$\begin{aligned}\|(Hy)_h - (Hy)\|_{L^1} &= \int_0^T |(Hy)_h(t) - (Hy)(t)|dt \\ &= \int_0^T \left|\frac{1}{h}\int_t^{t+h}(Hy)(s)ds - (Hy)(t)\right| dt \\ &\leq \int_0^T \left|\frac{1}{h}\int_t^{t+h}(I^\alpha x_y(s) - I^\alpha x_y(t))ds\right| dt\end{aligned}$$

Since $y \in B_r \subset L^1(J, \mathbb{R})$ and assumption **(H2)** that implies $x_y = f(., y_0 - g(y) + I^\alpha x_y(.), x_y(.)) \in L^1(J, \mathbb{R})$ and by Proposition (1) (v), it follows that $I^\alpha x_y \in L^1(J, \mathbb{R})$, then we have

$$\frac{1}{h}\int_t^{t+h}(I^\alpha x_y(s) - I^\alpha x_y(t))ds \longrightarrow 0 \ as \ h \longrightarrow 0, \ t \in J, \forall y \in B_r$$

i.e.,

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} I^\alpha x_y(s)ds = I^\alpha x_y(t), \ t \in J, \forall y \in B_r.$$

Let $\epsilon > 0$,there exist $\delta > 0$ such that

$$\left|\frac{1}{h}\int_t^{t+h}(I^\alpha x_y(s) - I^\alpha x_y(t))ds\right| < \frac{\epsilon}{T}, \ t \in J, \forall y \in B_r, |h| < \delta.$$

Hence,

$$\begin{aligned}\|(Hy)_h - (Hy)\|_{L^1} &\leq \int_0^T \left| \frac{1}{h}\int_t^{t+h} (I^\alpha x_y(s) - I^\alpha x_y(t))ds \right| dt, \ \forall y \in B_r \\ &\leq \frac{\epsilon}{T}T, \ \forall y \in B_r, |h| < \delta, \\ &\leq \epsilon, \ \forall y \in B_r, |h| < \delta.\end{aligned}$$

Hence,

$$\sup_{y\in B_r} \|(Hy)_h - (Hy)\|_1 < \epsilon, \ |h| < \delta,$$

i.e.,

$$\lim_{h\to 0} \sup_{y\in B_r} \|(Hy)_h - (Hy)\|_1 = 0.$$

Thus

$$(Hy)_h \longrightarrow (Hy) \ as \ h \longrightarrow 0 \quad \text{uniformly with respect to } \ y \in B_r.$$

Then by Theorem 2, HB_r is relatively compact.
As a consequence of Steps 1 to 3 together with Theorem 2, we conclude that H is continuous and compact. As a consequence of Theorem 8 the problem $(2.1) - (2.2)$ has at least one solution in B_r. □

The following result is based on contraction principle.

Theorem 10. *Assume that conditions* ***(H1)*,*(H2)*** *and* ***(H3)*** *hold. If*

$$kT + \frac{k_1 T^\alpha}{(1-k_2)\Gamma(\alpha+1)} < 1, \tag{2.12}$$

then the problem $(2.1) - (2.2)$ *has a unique solution* $y \in L^1(J, \mathbb{R})$.

Proof. We shall use the Banach contraction principle to prove that H defined by (2.8) has a fixed point. Let $y, z \in L^1(J, \mathbb{R})$, and $t \in J$. Then we have,

$$|(Hy)(t) - (Hz)(t)| \leq |g(y) - g(z)| + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x_y(s) - x_z(s)| ds \tag{2.13}$$

then

$$|(Hy)(t) - (Hz)(t)| \leq k\|y - z\|_{L_1} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x_y(s) - x_z(s)| ds \tag{2.14}$$

On the other hand, we have for every $t \in J$

$$\begin{aligned} |x_y(t) - x_z(t)| &= |f(t, y(t), x_y(t)) - f(t, z(t), x_z(t))| \\ &\le k_1|y(t) - z(t)| + k_2|x_y(t) - x_z(t)|. \end{aligned}$$

Thus

$$|x_y(t) - x_z(t)| \le \frac{k_1}{1-k_2}|y(t) - z(t)| \tag{2.15}$$

By replacing (2.15) in the inequality 2.13, we obtain

$$\begin{aligned} \|(Hy) - (Hz)\|_{L_1} &\le T|g(y) - g(z)| + \int_0^T \Big(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x_y(s) - x_z(s)| ds \Big) dt \\ &\le Tk\|y - z\|_{L_1} + \int_0^T \Big(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x_y(s) - x_z(s)| ds \Big) dt \\ &\le Tk\|y - z\|_{L_1} + \frac{k_1}{1-k_2} \int_0^T \Big(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |y(s) - z(s)| ds \Big) dt \\ &\le Tk\|y - z\|_{L_1} + \frac{T^\alpha k_1}{(1-k_2)\Gamma(\alpha+1)} \|y - z\|_{L_1} \\ &\le (kT + \frac{k_1 T^\alpha}{(1-k_2)\Gamma(\alpha+1)}) \|y - z\|_{L_1}. \end{aligned}$$

Consequently by (2.12) H is a contraction. As a consequence of contraction principle, we deduce that H has a fixed point unique which is a solution of the problem $(2.1)-(2.2)$. $\square$

2.3 Example

Let us consider the following fractional nonlocal problem,

$${}^cD^\alpha y(t) = \frac{e^{-t}}{(e^t + 8)(1 + |y(t)| + |{}^cD^\alpha y(t)|)}, \; t \in J := [0, 1], \; \alpha \in (0, 1], \tag{2.16}$$

$$y(0) = \frac{3}{5} \int_0^T y(t) dt. \tag{2.17}$$

Set

$$f(t, y, z) = \frac{e^{-t}}{(e^t + 8)(1 + y + z)}, \; (t, y, z) \in J \times [0, +\infty) \times [0, +\infty).$$

Let $y, z \in [0, +\infty)$ and $t \in J$. Then we have

$$\begin{aligned}
|f(t,y_1,z_1) - f(t,y_2,z_2)| &= \left|\frac{e^{-t}}{e^t+8}\left(\frac{1}{1+y_1+z_1} - \frac{1}{1+y_2+z_2}\right)\right| \\
&\leq \frac{e^{-t}(|y_1-y_2|+|z_1-z_2|)}{(e^t+8)(1+y_1+z_1)(1+y_2+z_2)} \\
&\leq \frac{e^{-t}}{(e^t+8)}(|y_1-y_2|+|z_1-z_2|) \\
&\leq \frac{1}{9}|y_1-y_2| + \frac{1}{9}|z_1-z_2|.
\end{aligned}$$

Hence the condition **(H2)** holds with $k_1 = k_2 = \frac{1}{9}$.

Also we have

$$|g(x) - g(y)| \leq \frac{3}{5}\|x-y\|_{L_1}.$$

Hence **(H3)** is satisfied with $k = \frac{3}{5}$.

We shall check that condition (2.12) is satisfied with $T = 1$. Indeed

$$\begin{aligned}
kT + \frac{k_1 T^\alpha}{(1-k_2)\Gamma(\alpha+1)} &= \frac{3}{5} + \frac{\frac{1}{9}}{(1-\frac{1}{9})\Gamma(\alpha+1)} \\
&< \frac{3}{5} + \frac{\frac{1}{9}}{\frac{8}{9}\Gamma(\alpha+1)} \\
&= \frac{3}{5} + \frac{1}{8\Gamma(\alpha+1)} \\
&< \frac{3}{5} + \frac{1}{5\Gamma(\alpha+1)}
\end{aligned}$$

and

$$\frac{3}{5} + \frac{1}{5\Gamma(\alpha+1)} < 1 \Leftrightarrow \Gamma(\alpha+1) > \frac{1}{2}. \tag{2.18}$$

which is satisfied for each $\alpha \in (0,1]$. Then by Theorem 10, the problem (2.16) − (2.17) has a unique integrable solution on $L^1([0,1],\mathbb{R})$.

Chapter 3

L^1-Solutions of the Boundary Value Problem for Implicit Fractional Order Differential Equations[1]

3.1 Introduction

In this chapter we deal with the existence integrable solutions and uniqueness results to the following class of nonlocal problems

$$^cD^\alpha y(t) = f(t, y, ^c D^\alpha y(t)),\ t \in J := [0, T],\ 1 < \alpha \leq 2, \tag{3.1}$$

$$y(0) = g(y),\ y(T) = y_T \tag{3.2}$$

where $^cD^\alpha$ is the Caputo fractional derivative, $y_T \in \mathbb{R}$, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, and $g : L^1(J, \mathbb{R}) \to \mathbb{R}$ is a continuous function.

This chapter is organized as follows. In Section 2, we will recall briefly some basic definitions and preliminary facts which will be used throughout the following section. In Section 3, we give two results, the first one is based on Schauder's fixed point Theorem and the second one on contraction principle . An example is given in Section 4 to demonstrate the application of our main results.

[1]**B. Telli** and M. S. Souid,L^1-Solutions of the Boundary Value Problem for Implicit Fractional Order Differential Equations . *Transylvanian Journal of Applied Analysis,* **27** (2021), No. 1, 121-128.

3.2 Existence of solutions

Let us start by defining what we mean by an integrable solution of the nonlocal problems (3.1) – (3.2).

Definition 22. *A function $y \in L^1(J, \mathbb{R})$ is said a solution of the nonlocal problems* (3.1)– (3.2) *if y satisfies* (3.1) *and* (3.2).

For the existence of solutions for the problem (3.1) – (3.2), we need the following auxiliary lemma.

Lemma 10. *Let $1 < \alpha \leq 2$ and let $x \in L^1(J, \mathbb{R})$. The nonlocal problems* (3.1) – (3.2) *is equivalent to the integral equation*

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^T G(t,s)x(s)ds + g(y) + \frac{(y_T - g(y))t}{T} \tag{3.3}$$

where x is the solution of the functional integral equation

$$x(t) = f\left(t, \frac{1}{\Gamma(\alpha)} \int_0^T G(t,s)x(s)ds + g(y) + \frac{(y_T - g(y))t}{T}, x(t)\right). \tag{3.4}$$

and $G(t,s)$ is the Green's function defined by:

$$G(t,s) = \begin{cases} (t-s)^{\alpha-1} - \frac{t(T-s)^{\alpha-1}}{T} & \textit{if } 0 \leq s \leq t \leq T \\ \frac{-t(T-s)^{\alpha-1}}{T}, & \textit{if } 0 \leq t \leq s \leq T. \end{cases} \tag{3.5}$$

Proof. Let ${}^cD^\alpha y(t) = x(t)$ in equation (3.1), then

$$x(t) = f(t, y(t), x(t)) \tag{3.6}$$

and Lemma (2) implies that

$$y(t) = c_0 + c_1 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)ds.$$

From (3.2), a simple calculation gives

$$c_0 = g(y),$$

and

$$c_1 = -\frac{1}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} x(s)ds + \frac{(y_T - g(y))}{T}.$$

Hence, we get equation (3.3).

Inversely, we prove that equation (3.3) satisfies the nonlocal problems (3.1) – (3.2).

Differentiating (3.3), we get

$$^cD^\alpha y(t) = x(t) = f(t, y(t),^c D^\alpha y(t)).$$

By (3.3) and (3.5) we have

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)ds - \frac{t}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}x(s)ds \\ &\quad + g(y) + \frac{(y_T - g(y))t}{T}. \end{aligned} \tag{3.7}$$

A simple calculation gives $y(0) = g(y)$ and $y(T) = y_T$. This complete the proof of the equivalence between the nonlocal problems (3.1)-(3.2) and the integral equation (3.3). □

Let

$$G_0 := \max\{|G(t,s)|, (t,s) \in J \times J\},$$

and let us introduce the following assumptions:

(B1) $f : J \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is measurable in $t \in J$, for any $(u_1, u_2) \in \mathbb{R}^2$ and continuous in $(u_1, u_2) \in \mathbb{R}^2$, for almost all $t \in J$.

(B2) There exist constants $k_1, k_2 > 0$ such that,for every $t \in J$, and for every $(u_1, u_2), (v_1, v_2) \in \mathbb{R}^2$,

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq k_1|u_1 - u_2| + k_2|v_1 - v_2|.$$

(B3) There exists a constant $k > 0$ such that, for every $y, y' \in L^1(J, \mathbb{R})$

$$|g(y) - g(y')| \leq k\|y - y'\|_{L_1}.$$

Our first result is based on Schauder fixed point Theorem.

Theorem 11. *Assume that the assumptions* (***B1***), (***B2***), (***B3***) *are satisfied. If*

$$\frac{G_0 T}{\Gamma(\alpha)}\frac{b_1}{1-b_2} < 1, \tag{3.8}$$

then the nonlocal problems (3.1) – (3.2) *has at least one solution* $y \in L^1(J, \mathbb{R})$.

Proof. We first notice that we have:
from assumptions **(B3)**,there exists a constant $M > 0$ such that, for every $y \in L^1(J, \mathbb{R})$

$$|g(y)| \leq M,$$

and from assumptions **(B2)**,there exist $a \in L^1(J, \mathbb{R})$, two constants $b_1 > 0$ and $0 < b_2 < 1$ such that,for every $t \in J$,and for every $(u_1, u_2) \in \mathbb{R}^2$, we have

$$|f(t, u_1, u_2)| \leq |a(t)| + b_1|u_1| + b_2|u_2|.$$

Transform the problem $(3.1) - (3.2)$ into a fixed point problem. Consider the operator

$$H : L^1(J, \mathbb{R}) \longrightarrow L^1(J, \mathbb{R})$$

defined by:

$$(Hy)(t) = \frac{1}{\Gamma(\alpha)} \int_0^T G(t, s)x_y(s)ds + g(y) + \frac{(y_T - g(y))t}{T}. \tag{3.9}$$

where

$$x_y(t) = f\left(t, \frac{1}{\Gamma(\alpha)} \int_0^T G(t, s)x_y(s)ds + g(y) + \frac{(y_T - g(y))t}{T}, x_y(t)\right).$$

and G is given by (3.5).
Clearly, the fixed point of the operator H are solutions of the problem $(3.1) - (3.2)$.

Let

$$r = \frac{\left(\left(\frac{G_0 T}{\Gamma(\alpha)}\right) \frac{1}{1-b_2}\right) \|a\|_{L_1} + T(M + |y_T|)}{1 - \frac{b_1}{1-b_2}\left(\frac{G_0 T}{\Gamma(\alpha)}\right)}.$$

Consider the set

$$B_r = \{y \in L^1(J, \mathbb{R}) : \|y\|_{L_1} \leq r\}.$$

Clearly, B_r is nonempty, bounded, convex and closed.
Now, we shall show that H satisfies the assumption of Schauder fixed point Theorem. The proof is given in following several steps.
Step 1. $HB_r \subset B_r$.

For each $y \in B_r$, from assumption **(B2)** and (3.8) we get

$$\begin{aligned}
\|Hy\|_{L_1} &= \int_0^T |Hy(t)|dt \\
&= \int_0^T \left| \frac{1}{\Gamma(\alpha)} \int_0^T G(t,s)x_y(s)ds + g(y) + \frac{(y_T - g(y))t}{T} \right| dt \\
&\leq \int_0^T \left| \frac{1}{\Gamma(\alpha)} \int_0^T G(t,s)x_y(s)ds - (\frac{t}{T} - 1)g(y) + \frac{t}{T}y_T \right| dt \\
&\leq \frac{G_0 T}{\Gamma(\alpha)} \|x_y\|_{L_1} + T(M + |y_T|) \\
&\leq T(M + |y_T|) + \left(\frac{G_0 T}{\Gamma(\alpha)} \right) \|x_y\|_{L_1} \qquad (3.10)
\end{aligned}$$

and

$$\begin{aligned}
|x_y(t)| &= |f(t, y(t), x_y(t))| \\
&\leq |a(t)| + b_1|y(t| + b_2|x_y(t)|.
\end{aligned}$$

Thus

$$|x_y(t)| \leq \frac{|a(t)| + b_1|y(t|}{1 - b_2}.$$

and

$$\|x_y\|_{L_1} \leq \frac{1}{1 - b_2}\|a\|_{L_1} + \frac{b_1}{1 - b_2}\|y\|_{L_1}. \qquad (3.11)$$

By replacing (3.11) in the inequality (3.10), we obtain

$$\begin{aligned}
\|Hy\|_{L_1} &\leq T(M + |y_T|) + \left(\frac{G_0 T}{\Gamma(\alpha)} \right) \left(\frac{1}{1 - b_2}\|a\|_{L_1} + \frac{b_1}{1 - b_2}\|y\|_{L_1} \right) \\
&\leq \left(\left(\frac{G_0 T}{\Gamma(\alpha)} \right) \frac{1}{1 - b_2} \right) \|a\|_{L_1} + T(M + |y_T|) + \left(\frac{G_0 T}{\Gamma(\alpha)} \right) \frac{b_1}{1 - b_2}\|y\|_{L_1} \\
&\leq r \qquad (3.12)
\end{aligned}$$

Then $HB_r \subset B_r$.

Step 2. H is continuous.

Assumption **(B1)**, **(B2)** and the hypothesis that g is continuous imply that H is continuous.

Step 3. H is compact.

Now, we will show that H is compact, this is HB_r is relatively compact. Clearly, HB_r is bounded in $L^1(J, \mathbb{R})$, i.e. condition **(i)** of Kolmogorov compactness criterion is satisfied.

It remains to show that

$$(Hy)_h \longrightarrow (Hy) \; as \; h \longrightarrow 0 \quad \text{uniformly with respect to} \;\; y \in B_r.$$

Let $y \in B_r$, then we have

$$\begin{aligned}
&\|(Hy)_h - (Hy)\|_{L^1} \\
= & \int_0^T |(Hy)_h(t) - (Hy)(t)| dt \\
= & \int_0^T \left| \frac{1}{h} \int_t^{t+h} (Hy)(s) ds - (Hy)(t) \right| dt \\
\leq & \int_0^T \left(\frac{1}{h} \int_t^{t+h} |(Hy)(s) - (Hy)(t)| ds \right) dt \\
\leq & \int_0^T \left(\frac{1}{h} \int_t^{t+h} \left| \left(\frac{1}{\Gamma(\alpha)} \int_0^T G(s,\tau) x_y(\tau) d\tau + g(y) \right. \right. \right. \\
& \left. + \frac{(y_T - g(y))s}{T} \right) - \left(\frac{1}{\Gamma(\alpha)} \int_0^T G(t,\tau) x_y(\tau) d\tau + g(y) \right. \\
& \left. \left. \left. + \frac{(y_T - g(y))t}{T} \right) \right| ds \right) dt.
\end{aligned}$$

Since $y \in B_r \subset L^1(J, \mathbb{R})$ and assumptions **(B1)**and **(B2)** that implies

$$x_y = f(., y(.), x_y(.)) \in L^1(J, \mathbb{R}),$$

hence

$$\frac{1}{\Gamma(\alpha)} \int_0^T G(., \tau) x_y(\tau) d\tau + g(y) + \frac{(y_T - g(y)).}{T} \in L^1(J, \mathbb{R}),$$

then we have

$$\begin{aligned}
&\frac{1}{h} \int_t^{t+h} \left| \left(\frac{1}{\Gamma(\alpha)} \int_0^T G(s,\tau) x_y(\tau) d\tau + g(y) + \frac{(y_T - g(y))s}{T} \right) \right. \\
&\left. - \left(\frac{1}{\Gamma(\alpha)} \int_0^T G(t,\tau) x_y(\tau) d\tau + g(y) + \frac{(y_T - g(y))t}{T} \right) \right| ds \longrightarrow 0,
\end{aligned}$$

as $h \longrightarrow 0, \; t \in J$.

Hence,

$$(Hy)_h \longrightarrow (Hy) \; as \; h \longrightarrow 0 \quad \text{uniformly with respect to} \;\; y \in B_r.$$

Then by the Kolmogorov compactness criterion, HB_r is relatively compact. As a consequence of Schauder's fixed point theorem the nonlocal problems $(3.1) - (3.2)$ has at least one solution in B_r. □

The following result is based on the Banach contraction principle.

Theorem 12. *Assume that the assumptions **(B1)**,**(B2)**,**(B3)** are satisfied. If*

$$T\left(\frac{G_0}{\Gamma(\alpha)}\frac{k_1}{1-k_2}+k\right)<1, \tag{3.13}$$

then the nonlocal problems (3.1) − (3.2) *has a unique solution* $y\in L^1(J,\mathbb{R})$.

Proof. We shall use the Banach contraction principle to prove that H defined by (3.9) has a fixed point. Let $y,z\in L^1(J,\mathbb{R})$, and $t\in J$. Then we have,

$$\begin{aligned}
|(Hy)(t)-(Hz)(t)| &= \Big|\frac{1}{\Gamma(\alpha)}\int_0^T G(t,s)x_y(s)ds-(\frac{t}{T}-1)g(y)+\frac{t}{T}y_T\\
&\quad -\frac{1}{\Gamma(\alpha)}\int_0^T G(t,s)x_z(s)ds-(\frac{t}{T}-1)g(z)+\frac{t}{T}y_T\Big|\\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^T \Big|G(t,s)(x_y(s)-x_z(s))\Big|ds\\
&\quad +|g(y)-g(z)|\\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^T \Big|G(t,s)(x_y(s)-x_z(s))\Big|ds+k\|y-z\|_{L_1}\\
&\leq \frac{G_0}{\Gamma(\alpha)}\|x_y-x_z\|_{L_1}+k\|y-z\|_{L_1}.
\end{aligned} \tag{3.14}$$

On the other hand, we have for every $t\in J$

$$\begin{aligned}
|x_y(t)-x_z(t)| &= |f(t,y(t),x_y(t))-f(t,z(t),x_z(t))|\\
&\leq k_1|y(t)-z(t)|+k_2|x_y(t)-x_z(t)|.
\end{aligned}$$

Thus

$$|x_y(t))-x_z(t)|\leq\frac{k_1}{1-k_2}|y(t)-z(t)| \tag{3.15}$$

and

$$\|x_y-x_z\|_{L_1}\leq\frac{k_1}{1-k_2}\|y-z\|_{L_1} \tag{3.16}$$

By replacing (3.16) in the inequality (3.14), we obtain

$$\begin{aligned}
|(Hy)(t)-(Hz)(t)| &\leq \frac{G_0}{\Gamma(\alpha)}\frac{k_1}{1-k_2}\|y-z\|_{L_1}+k\|y-z\|_{L_1}\\
&\leq \left(\frac{G_0}{\Gamma(\alpha)}\frac{k_1}{1-k_2}+k\right)\|y-z\|_{L_1}
\end{aligned}$$

Hence,

$$\|(Hy)-(Hz)\|_{L_1} \leq T\left(\frac{G_0}{\Gamma(\alpha)}\frac{k_1}{1-k_2}+k\right)\|y-z\|_{L_1}.$$

Consequently, by (3.13) H is a contraction. As a consequence of the Banach contraction principle, we deduce that H has a fixed point which is a solution of the problem (3.1) − (3.2). □

3.3 Example

Let us consider the following boundary value problem,

$$^cD^\alpha y(t) = \frac{e^{-t}}{(e^t+9)(1+|y(t)|+|^cD^\alpha y(t)|)},\ t\in J := [0,1],\ 1<\alpha\leq 2, \tag{3.17}$$

$$y(0) = \frac{3}{5}\int_0^T y(t)dt,\ y(1) = 2. \tag{3.18}$$

Set

$$f(t,y,z) = \frac{e^{-t}}{(e^t+9)(1+y+z)},\ (t,y,z)\in J\times[0,+\infty)\times[0,+\infty),$$

Let $y, z \in [0,+\infty)$ and $t\in J$. Then we have

$$\begin{aligned} |f(t,y_1,z_1)-f(t,y_2,z_2)| &= \left|\frac{e^{-t}}{e^t+9}\left(\frac{1}{1+y_1+z_1}-\frac{1}{1+y_2+z_2}\right)\right| \\ &\leq \frac{e^{-t}(|y_1-y_2|+|z_1-z_2|)}{(e^t+9)(1+y_1+z_1)(1+y_2+z_2)} \\ &\leq \frac{e^{-t}}{(e^t+9)}(|y_1-y_2|+|z_1-z_2|) \\ &\leq \frac{1}{10}|y_1-y_2|+\frac{1}{10}|z_1-z_2|. \end{aligned}$$

Hence, the condition **(B2)** holds with $k_1 = k_2 = \frac{1}{10}$. Also we have

$$|g(x)-g(y)| \leq \frac{3}{5}\|x-y\|_{L_1}.$$

Hence, **(B3)** is satisfied with $k = \frac{3}{5}$.

From (3.5), a simple calculation gives $G_0 = 1$. We shall check that condition (3.13) is satisfied with $T = 1$. Indeed

$$\begin{aligned} T\left(\frac{G_0}{\Gamma(\alpha)}\frac{k_1}{1-k_2}+k\right) &= \frac{1}{\Gamma(\alpha)} \times \frac{\frac{1}{10}}{1-\frac{1}{10}} + \frac{3}{5} \\ &= \frac{1}{\Gamma(\alpha)} \times \frac{1}{9} + \frac{3}{5} \\ &= \frac{1}{9\Gamma(\alpha)} + \frac{3}{5} \\ &< \frac{1}{5\Gamma(\alpha)} + \frac{3}{5}. \end{aligned}$$

and

$$\frac{1}{5\Gamma(\alpha)} + \frac{3}{5} < 1 \Leftrightarrow \Gamma(\alpha) > \frac{1}{2}. \tag{3.19}$$

which is satisfied for each $\alpha \in (1, 2]$. Then by Theorem 3.2, the problem (3.17) − (3.18) has a unique integrable solution in $L^1([0, 1], \mathbb{R})$.

Chapter 4

L^1-Solutions of the initial value problems for implicit differential equations with Hadamard fractional derivative[1]

4.1 Introduction

In this chapter we investigate of existence integrable solutions and uniqueness of the initial value problem, for fractional order implicit differential equation as follows:

$$^{H}D^{\alpha}_{1^+}y(t) = f(t, y(t), {}^{H}D^{\alpha}_{1^+}y(t)),\ t \in J := (1, T],\ 0 < \alpha \leq 1 \tag{4.1}$$

$$^{H}I^{1-\alpha}_{1^+}y(1) = b, \tag{4.2}$$

where $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, $b, T \in \mathbb{R}$ with $T > 1$, and $^{H}D^{\alpha}_{1^+}$ is the Hadamard fractional derivative.

This chapter is organized as follows. In Section 2, we will recall briefly some basic definitions and preliminary facts which will be used throughout the following section. In Section 3, we give two results, the first one is based on Schauder's fixed point theorem and

[1]**B. Telli** and M. S. Souid, L^1-Solutions of the initial value problems for implicit differential equations with Hadamard fractional derivative . *Transylvanian Journal of Applied Analysis* **6** (2021), No. 2, 1-14.

the second one on contraction principle . An example is given in Section 4 to demonstrate the application of our main results.

4.2 Existence of solutions

Let us start by defining what we mean by an integrable solution of the problem (4.1)−(4.2).

Definition 23. *A function* $y \in L^1(J, \mathbb{R})$ *is said to be a solution of IVP* $(4.1) - (4.2)$ *if* y *satisfies* (4.1) *and* (4.2).

Lemma 11. *Let* $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be a function such that* $f\left(., y(.), ^H D^{\alpha}_{1^+} y(.)\right) \in L^1(J, \mathbb{R})$ *for any* $y \in L^1(J, \mathbb{R})$. *A function* $y \in L^1(J, \mathbb{R})$ *is a solution of* $(4.1) - (4.2)$, *if and only if* y *satisfies the following integral equation:*

$$y(t) = \frac{b}{\Gamma(\alpha)}(\ln t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t (\ln \frac{t}{s})^{\alpha-1} x_y(s)\frac{ds}{s}, \tag{4.3}$$

where x_y *is the solution of the functional integral equation*

$$x_y(t) = f\left(t, \frac{b}{\Gamma(\alpha)}(\ln t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t (\ln \frac{t}{s})^{\alpha-1} x_y(s)\frac{ds}{s}, x_y(t)\right). \tag{4.4}$$

Proof. First we prove the necessity. Let $y \in L^1(J, \mathbb{R})$, be a solution of the problem $(4.1) - (4.2)$ and $^H D^{\alpha}_{1^+} y(t) = x_y(t))$ in equation (4.1), then

$$x_y(t) = f(t, y(t), x_y(t)) \tag{4.5}$$

Hence, we get equation (4.4).

Then (4.1) means the

$$^H D^{\alpha}_{1^+} y \in L^1(J, \mathbb{R}). \tag{4.6}$$

According to Definition (7)

$$^H D^{\alpha}_{1^+} y(t) = \left(t\frac{d}{dt}\right)(^H I^{1-\alpha}_{1^+} y)(t) \tag{4.7}$$

and hence by the Theorem (3), we have

$$y \in L^1(J, \mathbb{R}) \ \ and \ \ ^H I^{1-\alpha}_{1^+} y \in AC^1([1, T], \mathbb{R}).$$

Now we apply Lemma (4) to obtain

$$\begin{aligned} {}^{H}I_{1^+}^{\alpha}\ {}^{H}D_{1^+}^{\alpha}y(t) &= y(t) - \frac{{}^{H}I_{1^+}^{1-\alpha}y(1)}{\Gamma(\alpha)}(\ln t)^{\alpha-1} \\ &= y(t) - \frac{b}{\Gamma(\alpha)}(\ln t)^{\alpha-1}. \end{aligned} \tag{4.8}$$

Applying ${}^{H}I_{1^+}^{\alpha}$ to both sides of (4.1) yield

$$ {}^{H}I_{1^+}^{\alpha}\ {}^{H}D_{1^+}^{\alpha}y = {}^{H}I_{1^+}^{\alpha}f\left(.,y(.),{}^{H}D_{1^+}^{\alpha}y(.)\right). \tag{4.9}$$

From 4.8 and (4.9) we obtain

$$y(t) = \frac{b}{\Gamma(\alpha)}(\ln t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t (\ln\frac{t}{s})^{\alpha-1}x_y(s)\frac{ds}{s},$$

which is the equation (4.3).

Now we prove the sufficiency. Let $y \in L^1(J,\mathbb{R})$, satisfies $Eq.$(4.3). Applying the operator ${}^{H}D_{1^+}^{\alpha}$ to both sides of (4.3), it follows from Lemma 3, proposition 2, and Definition 7 that

$$\begin{aligned} {}^{H}D_{1^+}^{\alpha}y(t) &= {}^{H}D_{1^+}^{\alpha}(\frac{b}{\Gamma(\alpha)}(\ln t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t (\ln\frac{t}{s})^{\alpha-1}x_y(s)\frac{ds}{s}), \\ &= {}^{H}D_{1^+}^{\alpha}(\frac{b}{\Gamma(\alpha)}(\ln t)^{\alpha-1}) + {}^{H}D_{1^+}^{\alpha}(\frac{1}{\Gamma(\alpha)}\int_1^t (\ln\frac{t}{s})^{\alpha-1}x_y(s)\frac{ds}{s}) \\ &= x_y(t) \\ &= f\left(t,y(t),\ {}^{H}D_{1^+}^{\alpha}y(t)\right). \end{aligned} \tag{4.10}$$

Now we show that the initial condition 4.2 also holds. Multiply both sides of (4.3), by $(\ln t)^{1-\alpha}$, then

$$(\ln t)^{1-\alpha}y(t) = \frac{b}{\Gamma(\alpha)} + (\ln t)^{1-\alpha}\ {}^{H}I_{1^+}^{\alpha}f\left(.,y(.),\ {}^{H}D_{1^+}^{\alpha}y(.)\right)(t). \tag{4.11}$$

Since, $f\left(.,y(.),\ {}^{H}D_{1^+}^{\alpha}y(.)\right) \in L^1(J,\mathbb{R})$ for any $y \in L^1(J,\mathbb{R})$, implies that

$$\|f\left(.,y(.),\ {}^{H}D_{1^+}^{\alpha}y(.)\right)\|_{L_1} < \infty, \forall y \in L^1(J,\mathbb{R}).$$

Hence

$$\lim_{t\to 1}(\ln t)^{1-\alpha}\ {}^{H}I_{1^+}^{\alpha}f\left(.,y(.),\ {}^{H}D_{1^+}^{\alpha}y(.)\right)(t) = 0,$$

then, taking in (4.11) the limit as $t \to 1$, we obtain

$$\begin{aligned} \lim_{t\to 1}(\ln t)^{1-\alpha}y(t) &= \lim_{t\to 1}(\frac{b}{\Gamma(\alpha)} + (\ln t)^{1-\alpha}\ {}^{H}I_{1^+}^{\alpha}f\left(.,y(.),\ {}^{H}D_{1^+}^{\alpha}y(.)\right)(t)) \\ &= \frac{b}{\Gamma(\alpha)} + \lim_{t\to 1}(\ln t)^{1-\alpha}\ {}^{H}I_{1^+}^{\alpha}f\left(.,y(.),\ {}^{H}D_{1^+}^{\alpha}y(.)\right)(t) \\ &= \frac{b}{\Gamma(\alpha)}. \end{aligned}$$

which gives

$$^{H}I_{1^+}^{1-\alpha}y(1) = b.$$

□

Leu us introduce the following assumptions:

$(C1)$ $f : J \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is measurable in $t \in J$, for any $(u_1, u_2) \in \mathbb{R}^2$ and continuous in $(u_1, u_2) \in \mathbb{R}^2$, for almost all $t \in J$.

($C2$) There exist constants $k_1, k_2 > 0$ such that,for every $t \in J$,and for every $(u_1, u_2), (v_1, v_2) \in \mathbb{R}^2$,

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq k_1|u_1 - u_2| + k_2|v_1 - v_2|.$$

Our first result is based on Schauder fixed point theorem.

Theorem 13. *Suppose that the assumptions* $(C1)$ *and* $(C2)$ *hold true . If*

$$\frac{b_1 K(\alpha)}{\Gamma(\alpha)} + b_2 < 1, \tag{4.12}$$

then the IVP $(4.1) - (4.2)$ *has at least one solution* $y \in L^1(J, \mathbb{R})$.

Proof. We first notice that we have:
from assumptions **(C2)**,there exist $a \in L^1(J, \mathbb{R})$, two constants $b_1 > 0$ and $0 < b_2 < 1$ such that,for every $t \in J$,and for every $(u_1, u_2) \in \mathbb{R}^2$, we have

$$|f(t, u_1, u_2)| \leq |a(t)| + b_1|u_1| + b_2|u_2|.$$

Transform the problem $(4.1) - (4.2)$ into a fixed point problem. Consider the operator

$$H : \ L^1(J, \mathbb{R}) \to L^1(J, \mathbb{R})$$

defined by:

$$(Hu)(t) = \frac{b}{\Gamma(\alpha)}(\ln t)^{\alpha-1} +^{H} I_{1^+}^{\alpha}u(t), t \in J, \tag{4.13}$$

and

$$S : L^1(J, \mathbb{R}) \to L^1(J, \mathbb{R}),$$

defined by:

$$(Su)(t) = f\left(t, (Hu)(t), u(t)\right), \ t \in J \tag{4.14}$$

The operator S is well defined, indeed, for each $u \in L^1(J,\mathbb{R})$, from assumptions $(C1)$, $(C2)$ we obtain

$$\begin{aligned}
\|Su\|_{L_1} &= \int_1^T |Su(t)|dt \\
&= \int_1^T |f(t,(Hu)(t),u(t))|\,dt \\
&= \int_1^T \left| f\left(t, \frac{b}{\Gamma(\alpha)}(\ln t)^{\alpha-1} + ^H I_{1^+}^{\alpha} u(t), u(t)\right)\right| dt \\
&\leq \int_1^T \left[|a(t)| + b_1 \left|\frac{b}{\Gamma(\alpha)}(\ln t)^{\alpha-1} + ^H I_{1^+}^{\alpha} u(t)\right| + b_2|u(t)| \right] dt \\
&\leq \|a\|_{L_1} + b_1 \int_1^T \frac{|b|}{\Gamma(\alpha)}(\ln t)^{\alpha-1} dt \\
&\quad + b_1 \int_1^T |^H I_{1^+}^{\alpha} u(t)|dt + b_2 \int_1^T |u(t)|dt \\
&\leq C + \|a\|_{L_1} + \frac{b_1 K(\alpha)}{\Gamma(\alpha)}\|u\|_{L_1} + b_2\|u\|_{L_1} \\
&\leq C + \|a\|_{L_1} + \left(\frac{b_1 K(\alpha)}{\Gamma(\alpha)} + b_2\right)\|u\|_{L_1} < +\infty, \qquad (4.15)
\end{aligned}$$

where $C := b_1|b| \int_1^T \frac{1}{\Gamma(\alpha)}(\ln t)^{\alpha-1}dt$.

Let

$$r := \frac{C + \|a\|_{L_1}}{1 - \left(\frac{b_1 K(\alpha)}{\Gamma(\alpha)} + b_2\right)}$$

and consider the set

$$B_r = \{u \in L^1(J,\mathbb{R}) : \|u\|_{L_1} \leq r\}.$$

Clearly B_r is nonempty, bounded, convex and closed.

Now, we will show that $SB_r \subset B_r$, indeed, for each $u \in B_r$, from (4.12) and (4.15) we get

$$\begin{aligned}
\|Su\|_{L_1} &\leq C + \|a\|_{L_1} + \left(\frac{b_1 K(\alpha)}{\Gamma(\alpha)} + b_2\right)\|u\|_{L_1} \\
&\leq r.
\end{aligned}$$

Then $SB_r \subset B_r$.

Assumption $(C1)$ implies that S is continuous.

Now, we will show that S is compact, this is SB_r is relatively compact. Clearly SB_r is bounded in $L^1(J,\mathbb{R})$, i.e condition **(i)** of Kolmogorov compactness criterion is satisfied.

It remains to show $(Su)_h \longrightarrow (Su)$ *as* $h \longrightarrow 0$ uniformly with respect to $u \in B_r$.

Let $u \in B_r$, then we have

$$\begin{aligned}
\|(Su)_h - (Su)\|_{L^1} &= \int_1^T |(Su)_h(t) - (Su)(t)|dt \\
&= \int_1^T \left| \frac{1}{h}\int_t^{t+h} (Su)(s)ds - (Su)(t) \right| dt \\
&\leq \int_1^T \left(\frac{1}{h}\int_t^{t+h} |(Su)(s) - (Su)(t)|ds \right) dt \\
&\leq \int_1^T \frac{1}{h}\int_t^{t+h} |f(s,(Hu)(s),u(s)) \\
&\quad - f(t,(Hu)(t),u(t))|dsdt.
\end{aligned}$$

Since $u \in B_r \subset L^1(J,\mathbb{R})$,$Hu \in L^1(J,\mathbb{R})$ and assumption **(C2)** that implies $f(.,(Hu)(.),u(.)) \in L^1(J,\mathbb{R})$, then we have

$$\frac{1}{h}\int_t^{t+h} |f(s,(Hu)(s),u(s)) - f(t,(Hu)(t),u(t))|ds \longrightarrow 0$$
$$\text{as} \;\; h \longrightarrow 0, t \in J,$$

Hence $(Su)_h \longrightarrow (Su)$ *as* $h \longrightarrow 0$ uniformly with respect to $u \in B_r$. Then by Kolmogorov compactness criterion, SB_r is relatively compact. As a consequence of Schauder's fixed point theorem the S has at least one a fixed point $u_\star$ in B_r (the set of fixed points of operator S is non empty).

On then , in the spirit of Lemma (11), on could define the function $y_\star$ as

$$y_\star(t) := (Hu_\star)(t), \;\; t \in J \tag{4.16}$$

From $(4.13), (4.14)$ and (4.16) we obtain

$$\begin{aligned}
y_\star(t) &= (Hu_\star)(t), \;\; t \in J \\
&= \frac{b}{\Gamma(\alpha)}(\ln t)^{\alpha-1} + {}^H I_{1^+}^{\alpha} u_\star(t), \;\; t \in J
\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}
u_\star(t) &= (Su_\star)(t), \;\; t \in J \\
&= f(t,(Hu_\star)(t),u_\star(t)), \;\; t \in J \\
&= f(t,y_\star(t),u_\star(t)), \;\; t \in J \\
&= f\left(t, \frac{b}{\Gamma(\alpha)}(\ln t)^{\alpha-1} + {}^H I_{1^+}^{\alpha} u_\star(t), u_\star(t)\right), \;\; t \in J
\end{aligned} \tag{4.18}$$

According to (4.17), (4.18), $y_\star$ satisfies the (4.3) where $u_\star$ is the solution of 4.4. As a consequence of Lemma (11), we deduce that $y_\star$ is a solution of the problem (4.1) − (4.2). □

The following result is based on the Banach contraction principle.

Theorem 14. *Assume that conditions **(C1)**,**(C2)** hold. If*

$$\frac{K(\alpha)k_1}{\Gamma(\alpha)} + k_2 < 1, \tag{4.19}$$

then the IVP (4.1) − (4.2) *has a unique solution* $y \in L^1(J, \mathbb{R})$.

Proof. We shall use the Banach contraction principle to prove that S defined by (4.13) has a fixed point. Let $u, v \in L^1(J, \mathbb{R})$, and $t \in J$.
In view of hypothesis **(C3)** and (iii) of the Proposition (2) one can write ,

$$\begin{aligned} |(Su)(t) - (Sv)(t)| &= |f(t, (Hu)(t), u(t)) - f(t, (Hv)(t), v(t))|, \; t \in J, \\ &\leq k_1|(Hu)(t) - (Hv)(t)| + k_2|u(t) - v(t)| \\ &\leq k_1|({}^H I^\alpha_{1^+} u)(t) - ({}^H I^\alpha_{1^+} v)(t)| + k_2|u(t) - v(t)| \\ &\leq k_1|{}^H I^\alpha_{1^+}(u - v)(t)| + k_2|u(t) - v(t)| \end{aligned}$$

Integration of $[1, T]$ yields

$$\begin{aligned} \|(Su) - (Sv)\|_{L_1} &= \int_1^T |(Su)(t) - (Sv)(t)|dt \\ &\leq k_1 \int_1^T |{}^H I^\alpha_{1^+}(u - v)(t)|dt + k_2 \int_1^T |u(t) - v(t)|dt \\ &\leq \frac{K(\alpha)k_1}{\Gamma(\alpha)}\|u - v\|_{L_1} + k_2\|u - v\|_{L_1} \\ &\leq \left(\frac{K(\alpha)k_1}{\Gamma(\alpha)} + k_2\right)\|u - v\|_{L_1} \end{aligned}$$

Consequently by (4.19) ,S is a contractible and hence , in view of the Banach contraction principle , il follows that S has a unique fixed point $u_\star \in L^1(J, \mathbb{R})$. In view of Lemma (11), we deduce that

$$y_\star := Hu_\star,$$

is a solution unique of the problem (4.1) − (4.2). □

4.3 Example

Let us consider the following fractional initial value problem,

$$ {}^{H}D_{1^+}^{\frac{1}{2}} y(t) = \frac{1}{t(t+8)(1+|y(t)|+|{}^{H}D^{\alpha}y(t)|)}, \ t \in J := (1, e], \tag{4.20} $$

$$ {}^{H}I_{1^+}^{1-\alpha} y(1) = 1. \tag{4.21} $$

Set

$$ f(t, y, z) = \frac{1}{t(t+8)(1+y+z)}, \ (t, y, z) \in J \times [0, +\infty) \times [0, +\infty). $$

Let $y, z \in [0, +\infty)$ and $t \in J$. Then we have

$$ \begin{aligned} |f(t, y_1, z_1) - f(t, y_2, z_2)| &= \left| \frac{1}{t(t+8)} \left(\frac{1}{1+y_1+z_1} - \frac{1}{1+y_2+z_2} \right) \right| \\ &\leq \frac{|y_1 - y_2| + |z_1 - z_2|}{t(t+8)(1+y_1+z_1)(1+y_2+z_2)} \\ &\leq \frac{1}{t(t+8)} (|y_1 - y_2| + |z_1 - z_2|) \\ &\leq \frac{1}{9}|y_1 - y_2| + \frac{1}{9}|z_1 - z_2|. \end{aligned} $$

Hence the condition **(C2)** holds with $k_1 = k_2 = \frac{1}{9}$.

Thus condition

$$ \begin{aligned} \frac{K(\alpha)k_1}{\Gamma(\alpha)} + k_2 &= \frac{K(\alpha)\frac{1}{9}}{\Gamma(\alpha)} + \frac{1}{9} \\ &< \frac{2e}{9\Gamma(\frac{1}{2})} + \frac{1}{9} \cong 0,4519169 \\ &< 1. \end{aligned} $$

is satisfied . Then by Theorem 14, the problem $(4.20) - (4.21)$ has a unique integrable solution in $L^1((1, e], \mathbb{R})$.

Chapter 5

Investigation of the neutral fractional differential inclusions of Katugampola-type involving both retarded and advanced arguments [1]

5.1 Introduction and Motivations

In [39], Boumaaza and Benchohra investigated the following FDI:

$$
\begin{aligned}
{}^{\varrho}_{c}D^{\xi}_{n^+}(k(t)) &\in K(t, k^t), \quad t \in \mathcal{J} := [n, m], \quad 1 < \xi \leq 2, \\
k(t) &= \chi(t), \qquad t \in [n - r, n], \, r > 0, \\
k(t) &= \psi(t), \qquad t \in [m, m + \gamma], \, \gamma > 0,
\end{aligned}
\tag{5.1}
$$

where ${}^{\varrho}_{c}D^{\xi}_{n^+}$ is a modified Caputo formulation of the Erdélyi-Kober fractional derivative of order $1 < \xi \leq 2$. In 2016, Agarwal et al. [12] extended their study to a set-valued

[1] S. Etemad, M.S. Souid, **B. Telli**, M.K.A. Kaabar, Sh. Rezapour , Investigation of the neutral fractional differential inclusions of Katugampola-type involving both retarded and advanced arguments via Kuratowski MNC technique. *Transylvanian Advances in Differential Equations* **1** (2021), No. 2, 1-20.

version of the functional FDI subject to retarded-advanced arguments as

$$D^{\xi}k(t) \in K(t, k^t), \quad t \in \mathcal{J} := [1, e], \;\; 1 < \xi < 2,$$

$$k(t) = \phi(t), \qquad t \in [1 - r, 1], \; r > 0,$$

$$k(t) = \psi(t), \qquad t \in [e, e + \gamma], \; \gamma > 0, \tag{5.2}$$

where $K : \mathcal{J} \times C([-r, \gamma], \mathbb{R}) \to \mathcal{P}(\mathbb{R})$ is a multifunction. Regarding to the existence solutions for this FDI they focused on some standard fixed-point methods.

Stimulated by aforesaid researches, this research work investigates the existence solutions for the neutral fractional functional differential inclusions of Katugampola-type which involves retarded and advanced arguments as follows:

$${}^{\varrho}D^{\xi}_{n^+}(w(t) - q(t, w^t)) \in K(t, w^t), \qquad t \in \mathcal{J} := [n, m], \;\; 1 < \xi \leq 2, \tag{5.3}$$

$$w(t) = \chi(t), \qquad t \in [n - s, n], \;\; s > 0, \tag{5.4}$$

$$w(t) = \psi(t), \qquad t \in [m, m + \gamma], \;\; \gamma > 0, \tag{5.5}$$

where a given function $K : \mathcal{J} \times C([-s, \gamma], \mathbb{R}) \to \mathcal{P}(\mathbb{R})$ exists so that $\chi, \psi \in \mathcal{C}([n - s, m + \gamma], \mathbb{R})$ via $\chi(n) = 0$ and $\psi(m) = 0$, and a given mapping $q : \mathcal{J} \times \mathcal{C}([-s, \gamma], \mathbb{R}) \to \mathbb{R}$ exists such that $q(n, \chi^n) = 0$ and $q(m, \psi^m) = 0$. The element of $\mathcal{C}([-s, \gamma], \mathbb{R})$, denoted by w^t, is defined as follows:

$$w^t(\tau) := w(t + \tau), \;\; \tau \in [-s, \gamma].$$

Unlike the previous research works, we here implement our theoretical techniques on a generalized inclusion version of the neutral functional system via generalized derivative attributed to Katugampola for the first time. Due to the importance of such neutral systems, we prefer to extract the existence solutions with the help of a generalized operator which covers some previous results by assuming special kernels. This chapter is divided into the following sections. In section 2 which are needed to obtain our results in the other sections. Two interesting results will be obtained in section 3 in relation to the set-valued analogue of Nonlinear alternative for Kakutani maps and multi-valued maps named as Nadler-Covitz. In section 4, an application example will be provided to validate and apply our obtained results.

5.2 Existence of solutions

This section investigates the existence solutions of (5.3)-(5.5), by considering the Banach space $\mathcal{C}(I) := \mathcal{C}([n-s, m+\gamma], \mathbb{R})$ introduced in (1.3), with a norm:

$$\|u\|_\infty := \sup\{|u(t)| : t \in [n-s, m+\gamma]\}.$$

Definition 24. *A function $w \in \mathcal{C}(I)$ is named as a solution of* (5.3)-(5.5) *if $z \in L^1([n,m], \mathbb{R})$ exists subject to $z(t) \in K(t, w^t)$, (a.e.) on $[n,m]$ so that ${}^{\varrho}D^{\xi}_{n^+}(w(t) - q(t, w^t)) = z(t)$ on $\mathcal{J}$, $w(t) = \chi(t)$, on $[n-s, n], w(n) = 0$, and $w(t) = \psi(t)$, on $[m, m+\gamma], w(m) = 0$.*

For the existence of solutions for the problem $(5.3) - (5.5)$, we need the following auxiliary lemma .

Lemma 12. *Assume that $z : \mathcal{J} \to \mathbb{R}$ is an integrable function. A function $w \in \mathcal{C}(I)$ is a solution for a fractional equation, expressed as follows:*

$$w(t) = \begin{cases} \chi(t), & t \in [n-s, n], s > 0, \\ q(t, w^t) + \displaystyle\int_n^m F(t,v) z(v) v^{\varrho-1} dv, & t \in \mathcal{J}. \\ \psi(t), & t \in [m, m+\gamma], \gamma > 0. \end{cases} \tag{5.6}$$

iff w is a solution of the following problem given by:

$${}^{\rho}D^{\xi}_{n^+}(w(t) - q(t, w^t)) = z(t),\ t \in \mathcal{J} := [n,m] \quad 1 < \xi \leq 2, \tag{5.7}$$

$$w(t) = \chi(t),\ t \in [n-s, n], s > 0 \tag{5.8}$$

$$w(t) = \psi(t),\ t \in [m, m+\gamma], \gamma > 0, \tag{5.9}$$

where

$$F(t,v) = \frac{1}{\Gamma(\xi)} \begin{cases} (\dfrac{t^\varrho - v^\varrho}{\varrho})^{\xi-1} - (\dfrac{t^\varrho - n^\varrho}{m^\varrho - n^\varrho})^{\xi-1} (\dfrac{m^\varrho - v^\varrho}{\varrho})^{\xi-1},\ n \leq\ v \leq t \leq m, \\ -(\dfrac{t^\varrho - n^\varrho}{m^\varrho - n^\varrho})^{\xi-1} (\dfrac{m^\varrho - v^\varrho}{\varrho})^{\xi-1},\ n \leq t \leq\ v \leq m. \end{cases} \tag{5.10}$$

Proof. From (1.5), we have

$$w(t)-q(t,w^t)=\frac{1}{\Gamma(\xi)}\int_n^t\left(\frac{t^\varrho-v^\varrho}{\varrho}\right)^{\xi-1}z(v)v^{\varrho-1}dv+c_1\left(\frac{t^\varrho-n^\varrho}{\varrho}\right)^{\xi-1}+c_2\left(\frac{t^\varrho-n^\varrho}{\varrho}\right)^{\xi-2} \quad (5.11)$$

Using $w(n)=q(n,\chi^n)=w(m)=q(m,\psi^m)=0$, we find that $c_2=0$ and

$$c_1=-\left(\frac{m^\varrho-n^\varrho}{\varrho}\right)^{1-\xi}\frac{1}{\Gamma(\xi)}\int_n^m\left(\frac{m^\varrho-v^\varrho}{\varrho}\right)^{\xi-1}z(v)v^{\varrho-1}dv$$

By substituting the value of c_1 and c_2 in (5.11), we obtain

$$\begin{aligned}
w(t) &= q(t,w^t)+\frac{1}{\Gamma(\xi)}\int_n^t\left(\frac{t^\varrho-v^\varrho}{\varrho}\right)^{\xi-1}z(v)v^{\varrho-1}dv \\
&\quad -\left(\frac{t^\varrho-n^\varrho}{\varrho}\right)^{\xi-1}\left(\frac{m^\varrho-n^\varrho}{\varrho}\right)^{1-\xi}\frac{1}{\Gamma(\xi)}\int_n^m\left(\frac{m^\varrho-v^\varrho}{\varrho}\right)^{\xi-1}z(v)v^{\varrho-1}dv \\
&= q(t,w^t)+\frac{1}{\Gamma(\xi)}\int_n^t\left[\left(\frac{t^\varrho-v^\varrho}{\varrho}\right)^{\xi-1}-\left(\frac{t^\varrho-n^\varrho}{m^\varrho-n^\varrho}\right)^{\xi-1}\left(\frac{m^\varrho-v^\varrho}{\varrho}\right)^{\xi-1}\right]z(v)v^{\varrho-1}dv \\
&\quad +\frac{1}{\Gamma(\xi)}\int_t^m\left[-\left(\frac{t^\varrho-n^\varrho}{m^\varrho-n^\varrho}\right)^{\xi-1}\left(\frac{m^\varrho-v^\varrho}{\varrho}\right)^{\xi-1}\right]z(v)v^{\varrho-1}dv \\
&= q(t,w^t)+\int_n^m F(t,v)z(v)v^{\varrho-1}dv
\end{aligned}$$

where $F(t,v)$ is given by (5.10). On the contrary, if w satisfies equation (5.6), then equations (5.7)-(5.9) hold obviously and the argument is ended. □

Corollary 1. *Let $K:\mathcal{J}\times\mathcal{C}[-s,\gamma]\to\mathcal{P}(\mathbb{R})$ is Carathéodory.A function $w\in\mathcal{C}(I)$ is a solution for inclusion problem* (5.3)-(5.5) *iff*

$$w(t)=\begin{cases}\chi(t), & t\in[n-s,n], s>0,\\[2mm] q(t,w^t)+\displaystyle\int_n^m F(t,v)z(v)v^{\varrho-1}dv, & t\in\mathcal{J}.\\[2mm] \psi(t), & t\in[m,m+\gamma],\gamma>0.\end{cases}$$

where $z\in L^1([n,m],\mathbb{R})$ with

$$z(t)\in K(t,w^t),(a.e.)\ \ t\in\mathcal{J}$$

Remark 3. *The function $t\mapsto\displaystyle\int_n^m|F(t,v)|v^{\varrho-1}dv$ is continuous on $[n,m]$, and hence is bounded. Thus, we assume:*

$$\widetilde{F}:=\sup\left\{\int_n^m|F(t,v)|v^{\varrho-1}dv,\quad t\in[n,m]\right\}.$$

Let us assume the following:

(A1) The multivalued map $K : \mathcal{J} \times \mathcal{C}[-s, \gamma] \to \mathcal{P}(\mathbb{R})$ is Carathéodory;

(A2) There exist $l \in L^{\infty}(\mathcal{J}, \mathbb{R}_+)$ and constants $0 < c_1$ and $c_2 \geq 0$ provided

$$\|K(t,u)\|_{\mathcal{P}} := \sup\{|v| : v \in K(t,u)\} \leq l(t)(c_1\|u\|_{[-s,\gamma]} + c_2),$$

for any $u \in \mathcal{C}[-s, \gamma]$ and a.e $t \in \mathcal{J}$;

(A3) there exists $\ell_1 > 0$, such that

$$H_d(K(t,x), K(t,\overline{x})) \leq \ell_1\|x - \overline{x}\|_{[-s,\gamma]}, \qquad \forall x, \overline{x} \in \mathcal{C}[-s,\gamma]; t \in \mathcal{J}$$

and

$$d(0, K(0,u^t)) \leq \ell_1, \qquad \forall u \in \mathcal{C}[n,m]; (a.e.) \;\; t \in \mathcal{J}$$

;

(A4) the function q is continuous , and for any bounded set $\mathcal{M}$ in $\mathcal{C}[n,m]$, the set $\{t \mapsto q(t, w^t) : w \in \mathcal{M}\}$ is equicontinous in $\mathcal{C}[n,m]$;

(A5) There exist constants $0 \leq d_1 < 1$ and $d_2 \geq 0$ such that

$$|q(t,h)| \leq d_1\|h\|_{[-s,\gamma]} + d_2, t \in \mathcal{J}, h \in \mathcal{C}[-s,\gamma].$$

(A6) There exists a nonnegative constant $\mathcal{L}$ such that:

$$|q(t,h) - q(t,\overline{h})| \leq \mathcal{L}\|h - \overline{h}\|_{[-s,\gamma]},$$

for every $h, \overline{h} \in \mathcal{C}[-s,\gamma], t \in \mathcal{J}$.

by setting $l^* := ess\sup_{t\in\mathcal{J}} l(t)$

5.2.1 The Convex Case

Now, we state and prove our existence result for problem (5.3)-(5.5) based on a nonlinear alternative for Kakutani maps. Here K is assumed to have convex and compact values.

Theorem 15. *Assume that our assumptions **(A1)-(A2)** and **(A4)-(A5)** are settled. If*

$$d_1 + l^* c_1 \widetilde{F} < 1, \tag{5.12}$$

then the problem (5.3)-(5.5) *has at least one solution* $w \in \mathcal{C}(I)$.

Proof. Let the operator $\mathcal{E} : \mathcal{C}(I) \longrightarrow \mathcal{P}(\mathcal{C}(I))$ defined by

$$\mathcal{E}(w) = \left\{ \hbar \in \mathcal{C}(I) : \hbar(t) = \left\{ \begin{array}{ll} \chi(t), & t \in [n-s, n], \\ q(t, w^t) + \displaystyle\int_n^m F(t,v) z(v) v^{\varrho - 1} dv, & t \in \mathcal{J},\ z \in S_{K,w} \\ \psi(t), & t \in [m, m+\gamma] \end{array} \right. \right\}, \tag{5.13}$$

where

$$S_{K,w} := \left\{ z \in L^1[n,m] : z(t) \in K(t, w^t), (a.e.)\ \ t \in \mathcal{J} \right\}.$$

By Corollary 1 it is clear that the fixed points of $\mathcal{E}$ are solutions of the problem (5.3)-(5.5). Let the constant R be such that

$$R \geq \max\{(d_1 + l^* c_1 \widetilde{F})R + d_2 + l^* \widetilde{F} c_2, \|\chi\|_{[n-s,n]}, \|\psi\|_{[m,m+\gamma],\mu+1}\} \tag{5.14}$$

where $\mu := \frac{d_2 + l^* \widetilde{F} c_2}{1 - d_1 - l^* c_1 \widetilde{F}}$ and define

$$G := \{w \in \mathcal{C}(I) : \|w\|_\infty\} \leq R\}.$$

It is clear that G is a bounded, closed and convex subset of $\mathcal{C}(I)$.

We shall show that $\mathcal{E}$ satisfies the assumptions of Lemma 7.

Step I:$\mathcal{E}(\omega)$ is convex for all $w \in \mathcal{C}(I)$.

Indeed, if $\hbar_1, \hbar_2$ belong to $\mathcal{E}(w)$,there exists $z_1, z_2 \in S_{K,w}$ such that for all $t \in \mathcal{J}$ we obtain

$$\hbar_j(t) = q(t, w^t) + \int_n^m F(t,v) z_j(v) v^{\varrho-1} dv, \quad j = 1, 2.$$

Suppose that $0 \leq \gamma \leq 1$. Then ,for each $t \in \mathcal{J}$,we obtain

$$(\gamma \hbar_1 + (1-\gamma)\hbar_2)(t) = q(t, w^t) + \int_n^m F(t,v)(\gamma z_1(v) + (1-\gamma) z_2(v)) v^{\varrho-1} dv.$$

Since $S_{K,w}$ is convex due to the convexity values of K, we get

$$\gamma \hbar_1 + (1-\gamma)\hbar_2 \in \mathcal{E}(w)$$

and our first claim is verified.

STEP II: For each $w \in G$,$\mathcal{E}(w) \in \mathcal{P}(G)$

Let $w \in G$. Then ,for each $\hbar \in \mathcal{E}(w)$.

It $t \in [n-s, n]$,then

$$|\hbar(t)| \leq \|\chi\|_{[n-s,n]} \leq R,$$

and it $t \in [m, m+\gamma]$,then

$$|\hbar(t)| \leq \|\psi\|_{[m,m+\gamma]} \leq R,$$

and it $t \in \mathcal{J}$,then for each $\hbar \in \mathcal{E}(w)$, there exists $z \in S_{K,w}$ such that

$$\hbar(t) = q(t, w^t) + \int_n^m F(t,v) z(v) v^{\varrho-1} dv.$$

By (**A2**) ,(**A5**) , we have

$$\begin{aligned}
|\hbar(t)| &\leq |q(t,w^t)| + \int_n^m |F(t,v)||z(v)| v^{\rho-1} dv \\
&\leq d_1 \|w^t\|_{[-s,\gamma]} + d_2 + \int_n^m |F(t,v)| l(v)(c_1\|w^v\|_{[-s,\gamma]} + c_2) v^{\rho-1} dv \\
&\leq d_1 \|w\|_\infty + d_2 + \int_n^m |F(t,v)| l(v)(c_1\|w\|_\infty + c_2) v^{\rho-1} dv \\
&\leq d_1 \|w\|_\infty + d_2 + l^*(c_1\|w\|_\infty + c_2) \int_n^m |F(t,v)| v^{\rho-1} dv \\
&\leq (d_1 + l^* c_1 \widetilde{F})\|w\|_\infty + d_2 + l^* \widetilde{F} c_2 \\
&\leq (d_1 + l^* c_1 \widetilde{F}) R + d_2 + l^* \widetilde{F} c_2 \\
&\leq (d_1 + l^* c_1 \widetilde{F}) R + (1 - d_1 - l^* c_1 \widetilde{F}) R \\
&\leq R,
\end{aligned}$$

from which it follows that for each $t \in [n-s, m+\gamma]$, we have $|\hbar(t)| \leq R$, which implies that $\|\hbar\|_\infty \leq R$, and so $\mathcal{E}(w) \in \mathcal{P}(G)$.

STEP III $\mathcal{E}$ maps bounded sets into bounded sets in $\mathcal{C}(I)$.

Let B_r be bounded set of G, as in Step 2 we have

$$\mathcal{E}(B_r) \subset \mathcal{E}(G) \subset G.$$

STEP IV: $\mathcal{E}$ maps bounded sets in G into equicontinous sets .

We consider B_r is bounded set in G, for arbitrary $t_*, t^* \in \mathcal{J}$, with $t_* < t^*$, let $w \in B_r$,

$\hbar \in \mathcal{E}(w)$.Then, there exists $z \in S_{K,w}$ such that

$$\begin{aligned} |\hbar(t^*) - \hbar(t_*)| &= |q(t^*, w^{t^*}) - q(t_*, w^{t_*})| + \int_n^m |F(t^*, v) - F(t_*, v)||z(v)|v^{\varrho-1}dv \quad (5.15) \\ &\leq |q(t^*, w^{t^*}) - q(t_*, w^{t_*})| + l^*(c_1 r + c_2) \int_n^m |F(t^*, v) - F(t_*, v)|v^{\varrho-1}dv. \end{aligned}$$

By (**A4**), we have $|q(t^*, w^{t^*}) - q(t_*, w^{t_*})| \to 0$, as $t_* \to t^*$. As a result, as $t_* \to t^*$, the inequality (5.15) goes to zero, which proves that $\mathcal{E}(B_r)$ is equicontinous. As consequence of Step 3 to Step 4, together withe the Arzela-Ascoli theorem, we can conclude that $\mathcal{E}$ is completely continuous multi-valued operator.

STEP IV :$\mathcal{E}$ has a closed graph.
Assume that $w_k \to w_*$, $\hbar_k \in \mathcal{E}(w_k)$, and $\hbar_k \to \hbar_*$. We need to show that $\hbar_* \in \mathcal{E}(w_*)$. Now, $\hbar_k \in \mathcal{E}(w_k)$ implies that there exists $z_k \in S_{K,w_k}$ provided for $t \in \mathcal{J}$,

$$\hbar_k(t) = q(t, w_k^t) + \int_n^m F(t, v) z_k(v) v^{\varrho-1}dv.$$

Let us verify that some $z_* \in S_{K,w_*}$ can be chosen so that

$$\hbar_*(t) = q(t, w_*^t) + \int_n^m F(t, v) z_*(v) v^{\varrho-1}dv$$

for all $t \in \mathcal{J}$. Since $K(t, \cdot)$ is up-semi-con, so for each $\epsilon > 0$, $k_0(\epsilon) \geq 0$ exists provided for all $k \geq k_0$, we get:

$$z_k(t) \in K(t, w_k^t) \subset K(t, w_*^t) + \epsilon\mathbb{BALL}(0, 1), \quad (a.e.)\ t \in \mathcal{J}.$$

On the other side, due to the compactness values of K, a subsequence $z_{k_r}(\cdot)$ exists that

$$z_{k_r}(t) \to \upsilon_*(t) \quad \text{as} \quad r \to \infty,\ a.e.\ t \in \mathcal{J};$$

and thus $z_*(t) \in K(t, w_*^t)$ for almost all $t \in \mathcal{J}$. Further, it is apparent that

$$|z_{k_r}(t)| \leq l(t)(c_1 R + c_2).$$

According to the theorem of Lebesgue dominated convergence, it is derived that $z_* \in L^1(\mathcal{J})$ which yields $z_* \in S_{K,w_*}$. In conclusion,

$$\hbar_*(t) = q(t, w_*^t) + \int_n^m F(t, v) z_*(v) v^{\varrho-1}dv, \quad t \in \mathcal{J}.$$

So $\hbar_* \in \mathcal{E}(w_*)$.

STEP V:A priori bounds on solutions.

let $w \in \mathcal{C}(I)$ be such that $w \in \sigma\mathcal{E}(w)$ for all $\sigma \in (0,1)$.Then, there exists $z \in S_{K,w}$ such that for each $t \in \mathcal{J}$, we have

$$w(t) = \sigma q(t, w^t) + \sigma \int_n^m F(t,v)z(v)v^{\varrho-1}dv. \tag{5.16}$$

From (**A2**) and (**A5**), we get:

$$\begin{aligned}
|w(t)| &\leq |q(t,w^t)| + \int_n^m |F(t,v)||z(v)|v^{\varrho-1}dv \\
&\leq d_1\|w^t\|_{[-s,\gamma]} + d_2 + \int_n^m |F(t,v)|l(v)(c_1\|w^v\|_{[-s,\gamma]} + c_2)v^{\varrho-1}dv \\
&\leq d_1\|w\|_\infty + d_2 + \int_n^m |F(t,v)|l(v)(c_1\|w\|_\infty + c_2)v^{\varrho-1}dv \\
&\leq d_1\|w\|_\infty + d_2 + l^*(c_1\|w\|_\infty + c_2)\int_n^m |F(t,v)|v^{\varrho-1}dv \\
&\leq (d_1 + l^*c_1\widetilde{F})\|w\|_\infty + d_2 + l^*\widetilde{F}c_2.
\end{aligned}$$

Then

$$\|w\|_\infty \leq (d_1 + l^*c_1\widetilde{F})\|w\|_\infty + d_2 + l^*\widetilde{F}c_2,$$

i.e.

$$(1 - d_1 - l^*c_1\widetilde{F})\|w\|_\infty \leq d_2 + l^*\widetilde{F}c_2.$$

Thus, by (5.12), we have

$$\|w\|_\infty \leq \frac{d_2 + l^*\widetilde{F}c_2}{1 - d_1 - l^*c_1\widetilde{F}} := \mu$$

Set

$$\mathcal{U} := \{w \in \mathcal{C}(I) : \|w\|_\infty < \mu + 1\}.$$

From the choice of $\mathcal{U}$, there is no $w \in \partial\mathcal{U}$ such that $w \in \sigma\mathcal{E}(w)$ for some $\sigma \in (0,1)$.

At last, from the above steps and Lemma 7, we deduce that $\mathcal{E} : \overline{U} \longrightarrow \mathcal{P}_{cp,cv}(G)$ has a fixed point $w \in \overline{U}$ which is a solution of (5.3)-(5.5) and the argument is ended.

$\square$

5.2.2 The Non-convex Case

We now prove an existence result for (5.3)-(5.5) with non-convex valued right hand side. Our considerations are based on the fixed point Lemma for contraction multivalued maps given by Covitz and Nadler. Let us suppose that K has compact values.

Theorem 16. *Assume that our assumptions, i.e.* ***(A1)*** $-$ ***(A6)*** *are satisfied. If we have*

$$\left(\mathcal{L}+\ell_1\widetilde{F}\right)<1, \tag{5.17}$$

then the problem (5.3)-(5.5) *has at least one solution* $w \in \mathcal{C}(I)$.

Proof. For each $w \in \mathcal{C}(I)$, the set $S_{K,w}$ is nonempty since,
by **(A1)**, $K(.,w^{(.)}) : \mathcal{J} \to \mathcal{P}_{cl}(\mathbb{R})$ is measurable; and by this fact that $\mathbb{R}$ is separable,then by Theorem 6 de Kuratowski-Ryel-Naradzewski $K(.,w^{(.)})$ has a measurable selection $z : \mathcal{J} \to \mathbb{R}$. By **(A2)** ,we have

$$|z(t)| \leq p(t)$$

where $p(t) := l(t)(c_1\|\omega^t\|_{[-s,\gamma]} + c_2)$ a.e $t \in \mathcal{J}$; and since $l \in L^{\infty}(\mathcal{J},\mathbb{R}_+)$, by Hölder's Inequality $p \in L^1(\mathcal{J},\mathbb{R}_+)$, thus $z \in L^1(\mathcal{J},\mathbb{R})$. Consequently $z \in S_{K,w}$.
We shall prove that $\mathcal{E}$ given by (5.13) satisfies the assumptions of Lemma 8. The proof will be given in two steps.

STEP I: $\mathcal{E}(w) \in \mathcal{P}_{cl}(\mathcal{C}(I))$ for all $w \in \mathcal{C}(I)$.

let $(\hbar_k)_{k\geq 0} \subset \mathcal{E}(w)$ be such that $\hbar_k \to \hbar_*$ in $\mathcal{C}(I)$. Then, there exists $z_k \in S_{K,w}$ so that for any $t \in \mathcal{J}$,

$$\hbar_k(t) = q(t,w^t) + \int_n^m F(t,v)z_k(v)v^{\varrho-1}dv.$$

From **(A1)** and by this fact that K has compact values, we need to move to a subsequence in order to deduce that $z_k \to z_*$ weakly in $L^1_W(\mathcal{J},\mathbb{R})$ which is a space furnished with the weak topology. As a result, by a simple approach, it is verified that z_k converges strongly to z_* and so $z_* \in S_{K,w}$. Hence, for any $t \in \mathcal{J}$,

$$\hbar_k(t) \to \hbar_*(t) = q(t,w^t) + \int_n^m F(t,v)z_*(v)v^{\varrho-1}dv.$$

Thus $\hbar_* \in \mathcal{E}(w)$ and $\mathcal{E}(w) \in \mathcal{P}_{cl}(\mathcal{C}(I))$ for all $w \in \mathcal{C}(I)$.

STEP II: There exists $\beta < 1$ such that $\mathcal{H}_d(\mathcal{E}(w), \mathcal{E}(\overline{w})) \leq \beta\|w - \overline{w}\|_\infty$ for all $w, \overline{w} \in \mathcal{C}(I)$.

Let $w, \overline{w} \in \mathcal{C}(I)$ and $\hbar_1 \in \mathcal{E}(w)$. Then, there exists $z_1(t) \in K(t, w^t)$ such that for each $t \in \mathcal{J}$,

$$\hbar_1(t) = q(t, w^t) + \int_n^m F(t, v) z_1(v) v^{\varrho - 1} dv.$$

From **(A3)**, we obtain the following:

$$\mathcal{H}_d(K(t, w^t), K(t, \overline{w}^t)) \leq \ell_1 \|w^t - \overline{w}^t\|_{[-s,\gamma]}.$$

Thus, there exists $\theta \in K(t, \overline{w}^t)$ such that

$$|z_1(t) - \theta| \leq \ell_1 \|w^t - \overline{w}^t\|_{[-s,\gamma]}, \ t \in \mathcal{J}.$$

At this moment, consider $\mathcal{H} : \mathcal{J} \to \mathcal{P}_{cp}(\mathbb{R})$ which is expressed as:

$$\mathcal{H}(t) = \{\theta \in \mathbb{R} : |z_1(t) - \theta| \leq \ell_1 \|w^t - \overline{w}^t\|_{[-s,\gamma]}\}.$$

Since $\mathcal{U}(t) = \mathcal{H}(t) \cap K(t, \overline{w}^t)$ is measurable (see Proposition [48], III.4), there exists a function z_2 which is measurable selection for $\mathcal{U}$. So , $z_2(t) \in K(t, \overline{w}^t)$ and for each $t \in \mathcal{J}$,

$$|z_1(t) - z_2(t)| \leq \ell_1 \|w^t - \overline{w}^t\|_{[-s,\gamma]} \qquad .$$

Now, introduce

$$\hbar_2(t) = q(t, \overline{w}^t) + \int_n^m F(t, v) z_2(v) v^{\varrho - 1} dv.$$

In that case, for $t \in \mathcal{J}$,

$$\begin{aligned}
|\hbar_1(t) - \hbar_2(t)| &\leq |q(t, w^t) - q(t, \overline{w}^t)| + \int_n^m |F(t, v)||z_1(v) - z_2(v)| v^{\varrho - 1} dv \\
&\leq \mathcal{L}\|w^t - \overline{w}^t\|_{[-s,\gamma]} + \int_n^m |F(t, v)| \ell_1 \|w^v - \overline{w}^v\|_{[-s,\gamma]} v^{\varrho - 1} dv \\
&\leq \mathcal{L}\|w^t - \overline{w}^t\|_{[-s,\gamma]} + \int_n^m |F(t, v)| \ell_1 \|w - \overline{w}\|_\infty v^{\varrho - 1} dv \\
&\leq \mathcal{L}\|w - \overline{w}\|_\infty + \ell_1 \widetilde{F} \|w - \overline{w}\|_\infty \\
&\leq \left(\mathcal{L} + \ell_1 \widetilde{F}\right) \|w - \overline{w}\|_\infty
\end{aligned}$$

Therefore, we have

$$\|\hbar_1 - \hbar_2\|_\infty \leq \left(\mathcal{L} + \ell_1 \widetilde{F}\right) \|w - \overline{w}\|_\infty.$$

According to the analogous relation and interchanging the roles of w and $\overline{w}$, we arrive at

$$\mathcal{H}_d(\mathcal{E}(w), \mathcal{E}(\overline{w})) \leq \left(\mathcal{L} + \ell_1 \widetilde{F}\right) \|w - \overline{w}\|_\infty.$$

Therefore, by (5.17), $\mathcal{E}$ is a contraction, and according to Lemma 8, $\mathcal{E}$ possesses a fixed point w that is a solution of (5.3)-(5.5). Thus, the argument is fully completed. □

5.3 Application

In this section we give two examples to our results introduced above in Theorem 15 and Theorem 16.

Example 2. *Let us consider the following neutral fractional problem:*

$$\begin{cases} {}^{\frac{1}{2}}D^{\frac{3}{2}}_{n^+}(w(t) - q(t, w^t)) \in K(t, w^t), \qquad t \in \mathcal{J} := [1, 2], \\ w(t) = \chi(t), \;\; t \in [0, 1], \\ w(t) = \psi(t), \;\; t \in [2, 3]. \end{cases} \tag{5.18}$$

Set $K(t, w^t) = [k_1(t, w^t), k_2(t, w^t)]$, *where*

$$k_1 : \mathcal{J} \times \mathcal{C}([-1, 1], \mathbb{R}) \to \mathbb{R}$$

is formulated by $k_1(t, u) = 0$ *and*

$$k_2 : \mathcal{J} \times \mathcal{C}([-1, 1], \mathbb{R}) \to \mathbb{R}$$

is formulated as $k_2(t, u) = \dfrac{1}{2(t+2)}(\|u\|_{[-1,1]}) + 1)$. *Let*

$$q(t, u) = \frac{\|u\|_{[-1,1]}}{2(1 + \|u\|_{[-1,1]})}.$$

and

$$\varrho = \frac{1}{2}, \quad \xi = \frac{3}{2}, \quad s = \gamma = 1$$

It is obvious that K *has compact and convex values.*

Also, $K(\cdot, u) : \mathcal{J} \to \mathcal{P}_{cl}(\mathbb{R})$ *is measurable for any* $u \in \mathcal{C}([-1, 1], \mathbb{R})$.

For each $t \in [1, 2]$, $k_1(t, \cdot)$ *is lower semi-continuous, and* $k_2(t, \cdot)$ *is upper semi-continuous.*

This express that $u \to K(t,u)$ is upper semicontinuous for almost all $t \in \mathcal{J}$.

*Therefore, **(A1)** is verified*

For each $u \in \mathcal{C}[-s,\gamma]$, $t \in \mathcal{J}$; we have

$$\|K(t,u)\|_{\mathcal{P}} := \sup\{|v| : v \in K(t,u)\} \leq \frac{1}{2(t+2)}(\|u\|_{[-1,1]}+1)$$

*Therefore, **(A2)** is verified , with $l(t) = \dfrac{1}{2(t+2)}$, $c_1 = c_2 = 1$, and $l^* = \dfrac{1}{6}$.*

The function q is continuous , and for each $u \in \mathcal{C}[-s,\gamma]$, $t \in \mathcal{J}$; we have

$$|q(t,u)| = \frac{\|u\|_{[-1,1]}}{2(1+\|u\|_{[-1,1]})} \leq \frac{1}{2}.$$

*Therefore, the conditions **(A4)** – **(A5)** are verified , with, $d_1 = 0$ and $d_2 = \dfrac{1}{2}$.*

For each $t \in \mathcal{J}$; we have

$$\begin{aligned}
\int_n^m |F(t,v)|v^{\varrho-1}dv &\leq \frac{1}{\Gamma(\xi)}\int_n^t (\frac{t^\varrho - v^\varrho}{\varrho})^{\xi-1}v^{\varrho-1}dv \\
&\quad +(\frac{t^\varrho - n^\varrho}{m^\varrho - n^\varrho})^{\xi-1}\frac{1}{\Gamma(\xi)}\int_n^m (\frac{m^\varrho - v^\varrho}{\varrho})^{\xi-1}v^{\varrho-1}dv \\
&\leq \frac{1}{\Gamma(\xi)}\int_n^t (\frac{t^\varrho - v^\varrho}{\varrho})^{\xi-1}v^{\varrho-1}dv \\
&\quad +\frac{1}{\Gamma(\xi)}\int_n^m (\frac{m^\varrho - v^\varrho}{\varrho})^{\xi-1}v^{\varrho-1}dv \\
&\leq \frac{1}{\Gamma(\xi+1)}(\frac{t^\varrho - n^\varrho}{\varrho})^{\xi} + \frac{1}{\Gamma(\xi+1)}(\frac{m^\varrho - n^\varrho}{\varrho})^{\xi} \\
&\leq \frac{2}{\Gamma(\xi+1)}(\frac{m^\varrho - n^\varrho}{\varrho})^{\xi}.
\end{aligned}$$

Therefore, we get

$$\widetilde{F} \leq \frac{2}{\Gamma(\xi+1)}(\frac{m^\varrho - n^\varrho}{\varrho})^{\xi}.$$

The condition (5.12) *is satisfied. Indeed, we have*

$$\begin{aligned}
d_1 + l^* c_1 \widetilde{F} &\leq 0 + \frac{1}{6} \times 1 \times \frac{2}{\Gamma(\frac{5}{2})}(\frac{2^{\frac{1}{2}}-1}{\frac{1}{2}})^{\frac{3}{2}} \\
&\leq \frac{1}{3\Gamma(\frac{5}{2})}(\frac{2^{\frac{1}{2}}-1}{\frac{1}{2}})^{\frac{3}{2}} \simeq 0,189070603 < 1.
\end{aligned}$$

Hence all conditions of Theorem 15 are satisfied. It follows that the problem (5.18) *has at least one solution $w \in \mathcal{C}(I)$.*

Example 3. *Let us consider the following neutral fractional problem:*

$$\begin{cases} \frac{1}{2}D^{\frac{3}{2}}_{n^+}(w(t) - q(t, w^t)) \in K(t, w^t), \qquad t \in \mathcal{J} := [1, 2], \\ \\ w(t) = \chi(t), \ t \in [0, 1], \\ \\ w(t) = \psi(t), \ t \in [2, 3]. \end{cases} \tag{5.19}$$

Set $K(t, w^t) = K_1(t, w^t) \cup K_2(t, w^t)$, $K_1(t, w^t) = [k_1(t, w^t), k_2(t, w^t)]$, *and* $K_2(t, w^t) = [k_3(t, w^t), k_4(t, w^t)]$, *where*

$$k_1 : \mathcal{J} \times \mathcal{C}([-1, 1], \mathbb{R}) \to \mathbb{R}$$

is formulated by $k_1(t, u) = 0$ *and*

$$k_2 : \mathcal{J} \times \mathcal{C}([-1, 1], \mathbb{R}) \to \mathbb{R}$$

is formulated as $k_2(t, u) = \dfrac{1}{2(t+4)}(\|u\|_{[-1,1]}) + 1).$

$$k_3 : \mathcal{J} \times \mathcal{C}([-1, 1], \mathbb{R}) \to \mathbb{R}$$

is formulated as $k_3(t, u) = \dfrac{1}{2(t+3)}(\|u\|_{[-1,1]}) + 1).$

$$k_4 : \mathcal{J} \times \mathcal{C}([-1, 1], \mathbb{R}) \to \mathbb{R}$$

is formulated as $k_4(t, u) = \dfrac{1}{2(t+2)}(\|u\|_{[-1,1]}) + 1).$ *Let*

$$q(t, u) = \frac{\|u\|_{[-1,1]}}{2(1 + \|u\|_{[-1,1]})}.$$

It is obvious that K *has compact and nonconvex values.*

Also, $K(\cdot, u) : \mathcal{J} \to \mathcal{P}_{cl}(\mathbb{R})$ *is measurable for any* $u \in \mathcal{C}([-1, 1], \mathbb{R})$.

By Proposition (4),*this express that* $u \to K(t, u)$ *is upper semicontinuous for almost all* $t \in \mathcal{J}$.

Therefore, ***(A1)*** *is verified .*

For each $u \in \mathcal{C}[-s, \gamma]$, $t \in \mathcal{J}$; *we have*

$$\|K(t, u)\|_{\mathcal{P}} := \sup\{|\upsilon| : \upsilon \in K(t, u)\} \leq \frac{1}{2(t+2)}(\|u\|_{[-1,1]} + 1)$$

*Therefore, **(A2)** is verified , with* $l(t) = \dfrac{1}{2(t+2)}$, $c_1 = c_2 = 1$.
The function q *is continuous , and for each* $u \in \mathcal{C}[-s,\gamma]$, $t \in \mathcal{J}$*; we have*

$$|q(t,u)| = \frac{\|u\|_{[-1,1]}}{2(1+\|u\|_{[-1,1]})} \leq \frac{1}{2}.$$

*Therefore, the conditions **(A4)** − **(A5)** are verified , with,* $d_1 = 0$ *and* $d_2 = \dfrac{1}{2}$.
For each $u, \overline{u} \in \mathcal{C}[-s,\gamma]$, $t \in \mathcal{J}$*; we have*

$$\begin{aligned} H_d(K(t,u), K(t,\overline{u})) &\leq \frac{1}{2(t+2)}\|u-\overline{u}\|_{[-s,\gamma]}, ; \\ H_d(K(t,u), K(t,\overline{x})) &\leq \frac{1}{4}\|u-\overline{u}\|_{[-s,\gamma]}, ; \end{aligned}$$

*Therefore, **(A3)** is verified , with* $\ell_1 = \frac{1}{4}$.
For each $u, \overline{u} \in \mathcal{C}[-s,\gamma]$, $t \in \mathcal{J}$*; we have*

$$\begin{aligned} |q(t,u) - q(t,\overline{u})| &\leq \frac{1}{2}|\|u\|_{[-s,\gamma]} - \|\overline{u}\|_{[-s,\gamma]}| \\ &\leq \frac{1}{2}\|u-\overline{u}\|_{[-s,\gamma]} \end{aligned}$$

*Therefore, **(A6)** is verified , with* $\mathcal{L} = \frac{1}{2}$.
We have

$$\widetilde{F} \leq \frac{2}{\Gamma(\xi+1)}(\frac{m^{\varrho} - n^{\varrho}}{\varrho})^{\xi}.$$

The condition (5.17) *is satisfied. Indeed, we have*

$$\left(\mathcal{L} + \ell_1 \widetilde{F}\right) \leq \frac{1}{2} + \frac{1}{4} \times \frac{2}{\Gamma(\frac{5}{2})}(\frac{2^{\frac{1}{2}} - 1}{\frac{1}{2}})^{\frac{3}{2}} \simeq 0,5354507382 < 1.$$

Hence all conditions of Theorem 16 are satisfied. It follows that the problem (5.19) *has at least one solution* $w \in \mathcal{C}(I)$.

Conclusion and Perspectives

In this thesis, We have established the existence and uniqueness of integral solutions for a class of initial value problem, local and nonlocal conditions and of boundary value problem for nonlinear implicit fractional differential equations involving the Caputo and the Hadamard fractional derivative. Our results will be obtained by means of fixed points theorems.
Also in this work we have discussed the existence of continuous solutions for our proposed fractional boundary value problem (FBVP) has been successfully investigated for the neutral fractional differential inclusions of Katugampola fractional derivative (KaFrD) which involves retarded and advanced arguments. Two cases have been discussed throughout our investigation via fixed point theorems for convex and non-convex multifunctions. An application in the format of a simulative example of the neutral functional FBVP has been provided to validate our obtained results. This research work sheds the light on the importance of studying neutral fractional problem with its application in science and engineering.

All results obtained in this thesis can be considered as a contribution to this emerging field.

In the future, we will generalize this approach for the resolution the problems in $L^{p(x)}$ space.

Bibliography

[1] S. Abbas and M. Benchohra. Darboux problem for implicit impulsive partial hyperbolic fractional order differential equations. *Electronic Journal of Differential Equations*, 2011(150):1–14, 2011.

[2] S. Abbas and M. Benchohra. A global uniqueness result for fractional order implicit differential equations. *Commentationes Mathematicae Universitatis Carolinae*, 53(4):605–614, 2012.

[3] S. Abbas, M. Benchohra, and G.M. N'Guérékata. *Topics in fractional differential equations*, volume 27. Springer Science and Business Media, 2012.

[4] S. Abbas, M. Benchohra, and G.M. N'Guerekata. Advanced fractional differential and integral equations. 2015.

[5] S. Abbas, M. Benchohra, and S. Sivasundaram. Dynamics and ulam stability for hilfer type fractional differential equations. *Nonlinear Studies*, 23(4), 2016.

[6] S. Abbas, M. Benchohra, and J.J. Trujillo. Fractional order impulsive hyperbolic implicit differential equations with state-dependent delay. *submitted*, 64(6):1616–1626, 2012.

[7] S. Abbas, M. Benchohra, and A.N. Vityuk. On fractional order derivatives and darboux problem for implicit differential equations. *Fractional Calculus and Applied Analysis*, 15(2):168–182, 2012.

[8] S.A. Abd-Salam and A.M.A. El-Sayed. On the stability of a fractional-order differential equation with nonlocal initial condition. Electron. J. Qual. Theory Differ. Equat, 29:1–8, 2008.

[9] M.S. Abdo, M. Shah, M.H. Wahash, and S.K. Panchal. On a comprehensive model of the novel coronavirus (covid-19) under mittag-leffler derivative. *Chaos, Solitons and Fractals*, 135:109867, 2020.

[10] R. Agarwal, M. Belmekki, and M. Benchohra. A survey on semilinear differential equations and inclusions involving riemann-liouville fractional derivative. *Advances in Difference Equations*, 2009:1–47, 2009.

[11] R.P. Agarwal, M. Benchohra, and S. Hamani. A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions. *Acta Applicandae Mathematicae*, 109(3):973–1033, 2010.

[12] R.P. Agarwal, S.K. Ntouyas, B. Ahmad, and A.K. Alzahrani. *Advances in Difference Equations*, 2016(1):1–15, 2016.

[13] R.P. Agarwal, Y. Zhou, and Y. He. Existence of fractional neutral functional differential equations. *Comput. Math. Appl.*, 59(3):1095–1100, 2010.

[14] B. Ahmad and J.R. Graef. Coupled systems of nonlinear fractional differential equations with nonlocal boundary conditions. Panamer. Math. J., 19:29–39, 2009.

[15] B. Ahmad and J.J. Nieto. Existence of solutions for nonlocal boundary value problems of higher-order nonlinear fractional differential equations. *Abstrac. Appl. Anal*, 2009(1):1–9, 2009.

[16] B. Ahmad and S.K. Ntouyas. Initial value problems of fractional order hadamard-type functional differential equations. *Electron. J. Differ. Equ.*, 77(9):627–637, 2015.

[17] B. Ahmad and S. Sivasundaram. Theory of fractional differential equations with three-point boundary conditions. *Communications in Applied Analysis*, 12(4):479–484, 2008.

[18] R.A. Almeida. A caputo fractional derivative of a function with respect to another function. *Commun. Nonlinear Sci. Numer. Simul.*, 44:460–481, 2017.

[19] S. Almezel, Q.H. Ansari, and M.A. Khamsi. *Topics in fixed point theory*, volume 5. Springer, 2014.

[20] H. Alrabaiah, I. Ahmad, K. Shah, and G.U. Rahman. Qualitative analysis of nonlinear coupled pantograph differential equations of fractional order with integral boundary conditions. *Bound. Value Probl.*, 2020:1–13, 2020.

[21] J.C. Alvárez. Measure of noncompactness and fixed points of nonexpansive condensing mappings in locally convex spaces. *Rev. Real. Acad. Cienc. Exact. Fis. Natur. Madrid*, 79(1-2):53–66, 1985.

[22] G.A. Anastassiou. *Advances on fractional inequalities.* Springer Science and Business Media, 2011.

[23] J.P. Aubin and A. Cellina. *Differential inclusions: set-valued maps and viability theory*, volume 264. Springer Science and Business Media, 2012.

[24] J.P. Aubin and H. Frankowska. *Set-valued analysis.* Springer Science and Business Media, 2009.

[25] Z. Bai. On positive solutions of a nonlocal fractional boundary value problem. *Nonlinear Ana*, 72(1):916–924, 2010.

[26] D. Baleanu, K. Diethelm, E. Scalas, and J.J. Trujillo. *Fractional calculus: models and numerical methods*, volume 3. World Scientific, 2012.

[27] D. Baleanu, S. Etemad, and S. Rezapour. A hybrid caputo fractional modeling for thermostat with hybrid boundary value conditions. *Boundary Value Problems*, 2020(1):1–16, 2020.

[28] A. Belarbi, M. Benchohra, and A. Ouahab. Uniqueness results for fractional functional differential equations with infinite delay in fréchet spaces. *Applicable Analysis*, 85(12):1459–1470, 2006.

[29] M. Benchohra, A. Cabada, and D. Seba. An existence result for nonlinear fractional differential equations on banach spaces. *Boundary Value Problems*, 2009:1–11, 2009.

[30] M. Benchohra, S. Hamani, and S.K. Ntouyas. Boundary value problems for differential equations with fractional order. *Surveys in Mathematics and its Applications*, 3:1–12, 2008.

[31] M. Benchohra, S. Hamani, and S.K. Ntouyas. Boundary value problems for differential equations with fractional order and nonlocal conditions. *Nonlinear Analysis: Theory, Methods and Applications*, 71(7):2391–2396, 2009.

[32] M. Benchohra, J. Henderson, S.K. Ntouyas, and A. Ouahab. Existence results for fractional order functional differential equations with infinite delay. *Journal of Mathematical Analysis and Applications*, 338(2):1340–1350, 2008.

[33] M. Benchohra and J.E. Lazreg. Existence and uniqueness results for nonlinear implicit fractional differential equations with boundary conditions. *Rom. J. Math. Comput. Sci*, 4(1):60–72, 2014.

[34] M. Benchohra, J. R. Graef, and S. Hamani. Existence results for boundary value problems with non-linear fractional differential equations. *Applicable analysis*, 87(7):851–863, 2008.

[35] M. Benchohra and M.S. Souid. Integrable solutions for implicit fractional order functional differential equations with infinite delay. *Archivum Mathematicum*, 51(2):67–76, 2015.

[36] M. Benchohra and M.S. Souid. L^1-solutions of boundary value problems for implicit fractional order differential equations. *Surveys in Mathematics and its Applications*, 10, 2015.

[37] M. Benchohra and M.S. Souid. L^1-solutions for implicit fractional order differential equations with nonlocal conditions. *Filomat*, 30(6):1485–1492, 2016.

[38] H.F. Bohnenblust and S. Karlin. On a theorem of ville. *Contributions to the Theory of Games*, 1:155–160, 1950.

[39] M. Boumaaza, M. Benchohra, and C. Tunç. Erdelyi–kober fractional differential inclusions in banach space with retarded and advanced arguments. *Discuss. Math., Differ. Incl. Control Optim*, 40(1):75–92, 2020.

[40] A. Boutiara, K. Guerbati, and M. Benbachir. Caputo-hadamard fractional differential equation with three-point boundary conditions in banach spaces. *Aims Mathematics*, 5(1):259–272, 2019.

[41] A. Bressan and G. Colombo. Generalized baire category and differential inclusions in banach spaces. *Journal of differential equations*, 76(1):135–158, 1988.

[42] A. Bressan and G. Colombo. Boundary value problems for lower semicontinuous differential inclusions. *Funkcial. E.*, 36:359–373, 1993.

[43] H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer Science and Business Media, 2010.

[44] R.F. Brown, M. Furi, L. Górniewicz, and B. Jiang. *Handbook of Topological Fixed Point Theory*. Springer-Verlag, 2005.

[45] L. Byszewski. Theorem about the existence and uniqueness of a solution of a nonlocal abstract cauchy problem in a banach space. *Appl. Anal*, 40(1):11–19, 1991.

[46] L. Byszewski. Theorems about existence and uniqueness of solutions of a semilinear evolution nonlocal cauchy problem. *J. Math. Anal Appl*, 162(5):494–505, 1991.

[47] L. Byszewski. Existence and uniqueness of mild and classical solutions of semilinear functional-differential evolution nonlocal cauchy problem. *J. Math. Anal Appl*, pages 25–33, 1995.

[48] C. Castaing and M. Valadier. *Convex analysis and measurable multifunctions*, volume 580. Springer, 2006.

[49] L. Chen and Z. Fan. On mild solutions to fractional differential equations with nonlocal conditions. *Electron. J. Qual. Theory Differ. Equ.*, 53:1–13, 2011.

[50] C. Corduneanu and V. Lakshmikantham. Equations with unbounded delay: a survey. *Nonlinear Analysis: Theory, Methods and Applications*, 4(5):831–877, 1980.

[51] H. Covitz and S.B. Nadler. Multi-valued contraction mappings in generalized metric spaces. *Isr. J. Math.*, 8(17):5–11, 1970.

[52] Ruth F Curtain and Anthony J Pritchard. *Functional analysis in modern applied mathematics.* Academic press, 1977.

[53] K. Deimling. *Nonlinear Functional Analysis.* Springer-Verlag, 1985.

[54] K. Deimling. *Multivalued differential equations.* de Gruyter, 2011.

[55] J. Diestel, W.M. Ruess, and W. Schachermayer. On weak compactness in $l^1(\mu, x)$. *Proceedings of the American Mathematical Society*, 118(2):447–453, 1993.

[56] S. Djebali, L. Górniewicz, and A. Ouahab. Solution sets for differential equations and inclusions. In *Solution Sets for Differential Equations and Inclusions.* de Gruyter, 2012.

[57] A. El-Sayed and S.A. Abd El-Salam. l^p-solution of weighted cauchy-type problem of a diffreintegral functional equation. *Int. J. Nonlinear Sci*, 5(3):281–288, 2008.

[58] A. El-Sayed and F. Gaafar. Stability of a nonlinear non-autonomous fractional order systems with different delays and non-local conditions. *Advances in Difference Equations*, 2011(1):1–8, 2011.

[59] A. El-Sayed and H. Hashem. Integrable and continuous solutions of a nonlinear quadratic integral equation. *Electronic Journal of Qualitative Theory of Differential Equations*, 2008(25):1–10, 2008.

[60] A.M.M. El-Sayed and H.H.G. Hashem. Integrable and continuous solutions of a nonlinear quadratic integral equation. *Electron. J. Qual. Theory Differ. Equ*, 2008(25):1–10, 2008.

[61] S. Etemad, S. Rezapour, and M.E. Samei. On a fractional caputo-hadamard inclusion problem with sum boundary value conditions by using approximate endpoint property. *Math. Model. Appl. Sci.*, 43(1):9719–9734, 2020.

[62] K.M. Furati, M.D. Kassim, and N.E. Tatar. Existence and uniqueness for a problem involving hilfer fractional derivative. *Comput. Math. Appl*, 64(6):1616–1626, 2012.

[63] L. Górniewicz and L. Górniewicz. *Topological fixed point theory of multivalued mappings*, volume 4. Springer, 2006.

[64] J.R. Graef, J. Henderson, and A. Ouahab. *Impulsive differential inclusions.* de Gruyter, 2013.

[65] A. Granas and J. Dugundji. *Fixed point theory.* Springer Science and Business Media, 2013.

[66] D. Guo, V. Lakshmikantham, and X. Liu. *Nonlinear integral equations in abstract spaces*, volume 373. Springer Science and Business Media, 2013.

[67] Z. Guo and M. Liu. A cauchy problem for some fractional abstract differential equation with non local conditions. *Nonlinear Anal.*, 70(5):1873–1876, 2009.

[68] Z. Guo and M. Liu. Existence and uniqueness of solutions for fractional order integrodifferential equations with nonlocal initial conditions. *Panamer. Math. J*, 21:51–61, 2011.

[69] J.K. Hale and S.M.V. Lunel. *Introduction to Functional Differential Equations.* Springer-Verlag, New York., 1993.

[70] R. Hilfer. *Applications of Fractional Calculus in Physics*, volume 204. North-Holland Mathematics Studies. Elsevier Science B.V., Amsterdam, 2006.

[71] Y. Hino, S. Murakami, and T. Naito. *Functional differential equations with infinite delay.* Springer, 2006.

[72] C. Horvath. Measure of non-compactness and multivalued mappings in complete metric topological vector spaces. *Journal of mathematical analysis and applications*, 108(2):403–408, 1985.

[73] S. Hu and N. Papageorgiou. Handbook of multivalued analysis. *Mathematics and Its-Applications*, 419, 1997.

[74] B. Józef and S. Kishin. On some measures of noncompactness in the space of continuous functions. *Nonlinear Analysis: Theory, Methods and Applications*, 68(2):377–383, 2008.

[75] M. Kaabar. Novel methods for solving the conformable wave equation. *Journal of New Theory*, (31):56–85, 2020.

[76] M.I. Kamenskii, V.V. Obukhovskii, and P. Zecca. Condensing multivalued maps and semilinear differential inclusions in banach spaces. In *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*. de Gruyter, 2011.

[77] U.N. Katugampola. New approach to a generalized fractional integral. *Applied Mathematics and Computation*, 218(3):860–865, 2011.

[78] U.N. Katugampola. A new approach to generalized fractional derivatives. *Bull. Math. Anal. Appl.*, 6:1–15, 2011.

[79] S.A. Khan, K. Shah, G. Zaman, and F. Jarad. Existence theory and numerical solutions to smoking model under caputo-fabrizio fractional derivative. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 29(1):013128, 2019.

[80] A. Kilbas. Hadamard-type fractional calculus. *Korean Math. Soc*, 38(6):1191–1204, 2001.

[81] A.A. Kilbas, O. Marichev, and S. Samko. *Fractional integrals and derivatives (theory and applications)*. Gordon and Breach, Switzerland, 1993.

[82] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo. Nonlinear differential equations of fractional order in a space of integrable functions. *Dokl. Ross. Akad. Nauk*, 374(4):445–449, 2000.

[83] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo. *Theory and applications of fractional differential equations*, volume 204. elsevier, 2006.

[84] M. Kirane and B.T. Torebek. Extremum principle for the hadamard derivatives and its application to nonlinear fractional partial differential equations. *Fractional Calculus and Applied Analysis*, 22(2):358–378, 2018.

[85] W.A. Kirk and B. Sims. Handbook of metric fixed point theory. *Australian Mathematical Society GAZETTE*, 29(2), 2002.

[86] M. Kisielewicz. Differential inclusions and optimal control. 44, 1991.

[87] V. Kolmanovskii and A. Myshkis. *Introduction to the theory and applications of functional differential equations*, volume 463. Springer Science and Business Media, 2013.

[88] E. Lakhel and M.A. McKibben. Existence of solutions for fractional neutral functional differential equations driven by fbm with infinite delay. *Stochastics*, 90(3):313–329, 2018.

[89] V. Lakshmikantham and J.V. Devi. Theory of fractional differential equations in a banach space. *Eur. J. Pure Appl. Math*, 1(1):38–45, 2008.

[90] V. Lakshmikantham, S. Leela, and J. Vasundhara. Theory of fractional dynamic systems, cambridge academic publishers. 2009.

[91] V. Lakshmikantham and A.S. Vatsala. General uniqueness and monotone iterative technique for fractional differential equations. *Appl. Math. Lett*, 21:823–834, 2008.

[92] Y. Luchko. Wave–diffusion dualism of the neutral-fractional processes. *Journal of Computational Physics*, 293:40–52, 2015.

[93] L. Ma. Comparison theorems for caputo-hadamard fractional differential equations. *Fractals*, 27(3):1950036, 2019.

[94] F. Mainardi. *Fractional Calculus and Waves in Linear Viscoelasticity. An introduction to mathematical models.* World Scientific, 2010.

[95] F. Martínez, I. Martínez, M.K.A. Kaabar, R. Ortíz-Munuera, and S. Paredes. Note on the conformable fractional derivatives and integrals of complex-valued functions of a real variable. *IAENG Int. J. Appl. Math.*, 50(3):1–7, 2020.

[96] F. Martínez, I. Martínez, M.K.A. Kaabar, and S. Paredes. New results on complex conformable integral. *AIMS Mathematics*, 5(6):7695–7710, 2020.

[97] M.M. Matar, M.I. Abbas, J. Alzabut, M.K.A. Kaabar, S. Etemad, and S. Rezapour. Investigation of the p-laplacian nonperiodic nonlinear boundary value problem via generalized caputo fractional derivatives. *Adv. Differ. Equ.*, 2021(1):1–18, 2021.

[98] H. Mönch. Boundary value problems for nonlinear ordinary differential equations of second order in banach spaces. *Nonlinear Analysis: Theory, Methods and Applications*, 4(5):985–999, 1980.

[99] G.M. N'Guérékata. A cauchy problem for some fractional differential equations. *Commun. Math. Anal.*, 7:1–11, 2009.

[100] M. Niezabitowski and A. Babiarz. Controllability problem of fractional neutral systems-a survey. *Math. Probl. Eng.*, 2017, 2015.

[101] K.B. Oldham and J. Spanier. *The Fractional Calculus.* Elsevier, 1974.

[102] D.S. Oliveira and E. Capelas. Hilfer katugampola fractional derivative. *J. Comput. Appl. Math.*, 37:3672–3690, 2018.

[103] D. O'regan and D. Precup. Fixed point theorems for set-valued maps and existence principles for integral inclusions. *J. Math. Anal. Appl.*, 245(2):594–612, 2000.

[104] M.D. Ortigueira. *Fractional Calculus for Scientists and Engineers.* Springer Science and Business Media, 2011.

[105] I. Podlubny. *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications.* Elsevier, 1998.

[106] A. Ravi, M. Benchohra, and S. Hamani. Boundary value problems for fractional differential equations. 16(3):401–411, 2009.

[107] K. Schumacher. Existence and continuous dependence for functional-differential equations with unbounded delay. *Archive for Rational Mechanics and Analysis*, 67(4):315–335, 1978.

[108] N.P. Semenchuk. A class of differential equations of nonintegral order. *Differentsial'nye Uravneniya*, 18(10):1831–1833, 1982.

[109] N.P. Semenchuk. On one class of differential equations of noninteger order. *Differents. Uravn.*, 10:1831–1833, 1982.

[110] J. Singh, D. andKumar, Z. Hammouch, and A. Atangana. A fractional epidemiological model for computer viruses pertaining to a new fractional derivative. *Applied Mathematics and Computation*, 316:504–515, 2018.

[111] G.V. Smirnov. *Introduction to the theory of differential inclusions*, volume 41. American Mathematical Soc., 2002.

[112] M.S. Souid. l^1-solutions of boundary value problems for implicit fractional order differential equations with integral conditions. *International Journal of Advanced Research in Mathematics*, 11:8–17, 5 2018.

[113] M.S. Souid. l^1-solutions of boundary value problems for implicit fractional order differential equations with integral conditions. *International Journal of Advanced Research in Mathematics*, 11:8–17, 5 2018.

[114] X. Su and L. Liu. Existence of solution for boundary value problem of nonlinear fractional differential equation. *Applied Mathematics-A Journal of Chinese Universities*, 22(3):291–298, 2007.

[115] V.E. Tarasov. *Fractional dynamics: applications of fractional calculus to dynamics of particles, fields and media.* Springer Science and Business Media., 2011.

[116] S. Thabet, S. Etemad, and S. Rezapour. On a coupled caputo conformable system of pantograph problems. *Turk. J. Math*, 45(1):496–519, 2021.

[117] S. Thabet and S. Etemad, S.and Rezapour. On a new structure of the pantograph inclusion problem in the caputo conformable setting. *Boundary Value Problems*, 2020(1):1–21, 2020.

[118] A. Tolstonogov. *Differential Inclusions in a Banach Space*, volume 524. Springer Science and Business Media, 2012.

[119] A. Vityuk. Existence of solutions for differential inclusions with partial derivatives of fractional order. *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, (8):13–19, 1997.

[120] A. Vityuk and A. Golushkov. Darboux problem for a differential equation with fractional derivative. *Nonlinear Oscillations*, 8(4):450–462, 2005.

[121] A. Vityuk and A. Mikhailenko. On one class of differential equations of fractional order. *Nonlinear Oscillations*, 3(11):307–319, 2008.

[122] A. Vityuk and A. Mikhailenko. On one class of differential equations of fractional order. *Nonlinear Oscillations*, 11(3):307–319, 2008.

[123] A.N. Vityuk. Existence of solutions of partial differential inclusions of fractional order. *Izv.Vyssh. Uchebn. Ser. Mat*, 8:13–19, 1997.

[124] A.N. Vityuk and A.V. Golushkov. The darboux problem for a differential equation containing a fractional derivative. *Nonlinear Oscil*, 8:450–462, 2005.

[125] D. Vivek, K. Kanagarajan, and E.M. Elsayed. Nonlocal initial value problems for implicit differential equations with hilfer-hadamard fractional derivative. *Nonlinear Analysis: Modelling and Control*, 23(3):341–360, 2018.

[126] U. Witthayarat, K. Wattanawitoon, and P. Kumam. Iterative scheme for system of equilibrium problems and bregman asymptotically quasi-nonexpansive mappings in banach spaces. *Journal of Information and Optimization Sciences*, 37(3):321–342, 2016.

[127] J. Wu. *Theory and applications of partial functional differential equations*, volume 119. Springer Science and Business Media, 2012.

[128] C. Yu and G. Gao. Existence of fractional differential equations. *Journal of Mathematical Analysis and Applications*, 310(1):26–29, 2005.

[129] S. Zhang. Positive solutions for boundary-value problems of nonlinear fractional differential equations. *Electronic Journal of Differential Equations (EJDE)[electronic only]*, 2006:Paper–No, 2006.

[130] H. Zhou, J. Alzabut, and L. Yang. On fractional langevin differential equations with anti-periodic boundary conditions. *The European Physical Journal Special Topics*, 226(16):3577–3590, 2017.